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Bachelor's Thesis

The Haag-Ruelle Theory of Scattering in diffrent frameworks

Haag-Ruelle Streutheorie in verschiedenen Formulierungen

prepared by

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Now we see only a dim likeness of things. It is as if we were seeing them in a mirror. But someday we will see clearly. We will see face to face. What I know now is not complete. But someday I will know completely, just as God knows me completely.

(1 Corinthians 13:12, New International Reader's Version*)*

Preface

This thesis is the physical sequel of my mathematical bachelor thesis [Spr13], done in winter semester 2012/13. The subject of the first thesis evolved in some discussion with Prof. Dorothea Bahns about my mathematical preferences and the possibility to extend such a thesis to a bipartite thesis in mathematics and physics. It mainly followed the chapters IX.8, X.7 in [RS75] and XI.16 of [RS79] as suggested by Prof. Witt.
In further discussions with Prof. Rehren on the conception of this sequel, three possible parts arose. First – basing on the results of [Spr13] – to derive theorem 2.20, which was formally given as remark without a proof, second to give the parallel Haag-Ruelle theory in the Haag-Kastler framework according to the paper [BS05] and third to develope extended asymptotic relations in LSZ framework. The third – optional part – was not realized in this thesis.

I firstly thank Prof. Witt, who supported me with knowledge and literature on important techniques and who helped me acquire the mathematical basement in the first thesis. Thanks a lot!

My special thanks go to Prof. Rehren, who in several proceedings had plenty time to answer my questions on various topics touched by the thesis and beyond, who gave reasonable suggestions, remarks and corrections on computations, structure and verbalisations and supplied me with several papers, articles and textbooks on qft. Many, many thanks for your patience and help!

I also thank my wife for her love, patience and faith that she shared with me during bustling and restless times. Your courage and ease helped me a lot not to loose track.

Last but not least I want to thank Prof. Bahns, who with her first impulses marked the path for my thesis and feeded my eagerness for mathematical physics. Thank you very much!

Contents

List of Symbols

Operators and relations

Symbol	Meaning
$f \lesssim g$	There is a positive constant C with $f \leq C \cdot g$.
$\langle x \rangle$	The smoothed modulus $\sqrt{1 + \|x\|^2}$ of the vector x.
$\langle -, - \rangle$	The scalar product of a Hilbert space. ⚠ linear in second, anti-linear in first entry.
$\|\cdot\|$	The norm of a Hilbert space
$\overline{A}$	The closure of the set A.
$\overline{f}$	The complex conjugated of the function f.
$\widehat{f}$	Fourier transform of the function f
$f^{\wedge(x_1)\vee(x_2)}$	The partially Fourier transform w.r.t. x_1 and anti Fourier transform w.r.t. x_2 of f. The function might still depend on a x_3 part.
$f_{\|D}$	The restriction of the function f to domain D.
$\tilde{f}$	The momentum function of a regular positive energy solution (p.7).
$\widehat{S}$	Fourier transform of the tempered distribution S
$\tilde{x}$	Co-variant vector of $x = (x_1, x_2, x_3, x_4)$ with respect to Minkowski-metric, i.e., $\tilde{x} = (x_1, -x_2, -x_3, -x_4)$
$\Re f$, $\Im f$	Real- and imaginary part of the complex function f
$(a \mp b)$	The interval $(a - b, a + b)$.
$[a \mp b]$	The interval $[a - b, a + b]$.

Sets and objects

Symbol	Meaning
$\mathscr{H}$	A complex Hilbert space.
$\mathscr{L}(\mathscr{H})$	The bounded linear operators on $\mathscr{H}$.
$\mathscr{S}(\mathbb{R}^n)$	The Schwartz functions on $\mathbb{R}^n$.
$\mathscr{D}(\mathbb{R}^n)$	The smooth functions with compact support on $\mathbb{R}^n$.
$\mathcal{A}(\mathcal{O})$	The local algebra on $\mathcal{O}$ (p.13).
$\mathcal{O}$	An open bounded spacetime region.
$\mathcal{O}(f)_{x\to c}$	The Landau class of f as $x \to c$. Mostly $c = \pm\infty$ and will be omitted (Big O notation).
$\mathfrak{S}_n$	Symmetric group of $\{1, 2, \dots, n\}$
$\mathscr{P}_+^\uparrow$	The restricted Poincaré group.
$\mathscr{P}_n$	The set of all partitions of $\{1, \dots, n\}$.
V_+	The forward light cone (p.3).
H_m	$p \in \mathbb{R}^4;\ p_0 > 0,\ p_0^2 - \vec{p}^2 = m^2$. The shell of mass m.
$B_r(A)$	The union of balls of radius r around each point in A.
E_Σ	The energy-momentum projection valued measure (p.3).
$đE_\lambda$	The projection valued measure corrected by constant $(2\pi)^2$.
$\mathrm{d}\Omega_m$	The up to scalar multiplication unique Lorentz invariant measure on H_m.
$\mathscr{W}_n, \mathscr{W}_{n,T}$	The Wightman distributions (p.5).
$W_n, W_{n,T}$	The reduced Wightman distributions (p.5).
$A_t(f), A(f)$	The time t and time zero creation operator (pp.16).
Ω	The vacuum (p.3).
Γ_f	The velocity support of f (p.18).
$\delta_{n,m}$	The Kronecker delta.
$\omega(\vec{p})$	The energy momentum relation (p.7).
$g_t(z)$	Short for $g(\frac{z-t}{\lvert t\rvert^\kappa})/\lvert t\rvert^\kappa$, as in p.16.

1 Introduction

Quantum field theory describes in a special relativistic framework elementary particles in Heisenberg picture as excited states of the underlying field. In this thesis we will deal with large time limits of a finite number of interacting bosonic particles in scattering processes. We will show, that there is an interpretation of such large time limits as freely propagating particles, described by the Klein-Gordon equation. These interpretations are the content of scattering theory.

We will give a short review on the Haag-Ruelle theory in the framework of the Gårding-Wightman axioms, which can be looked up in [RS75, Section IX.8, X.7] and [RS79, Section XI.16] or [Spr13]. We will then complete the covariance proof in this framework following [Jos65], which incorporates the fundamental physical statement on scattering.
We will then turn to another approach on Haag-Ruelle theory in the context of algebraic quantum field theory. We will encode the physical terms of the Gårding-Wightman axioms in algebraic statements on local algebras. In this setting we will try to develop a scattering theory without the strong restrictions on the mass spectrum made in the Gårding-Wightman setting, where we follow the sketchy and conceptual description in [BS05].

We use a stability condition stated by Herbst [Her71] and also used by Dybalski [Dyb05][1], this will be a fundamental improvement compared to the work done before. Due to an error in computation, which was found only very late in the process of this thesis, we will still have to consider the absence of massless particles, or rather the vacuum should be an isolated point in the mass-spectrum, because we lacked time to recapture the techniques needed for this generalization. The theorems 3.15 and 3.16 are still valid in the presence of massless particles as it was shown by Dybalski.

1 Which is the source for Buchholz and Summer's article in the relevant section, though Buchholz supplied Dybalski with some results of his work on that field

We will then derive the creation operator of the scattering theory and will show fundamental properties. Thereafter we use the stationary phase method to derive asymptotic commutativity, which will be key to everything following. The next important step is to derive the lemma we stated as clustering property, mostly this is only stated for asymptotic states. This is the only time we need the restriction of absence of massless particles. The proceeding in the work of Dybalski is different from ours and involves an important lemma by Buchholz, where the problems we faced in our computation are relocated in deriving the strong limit of $A_t(f)^* A_t(f)\Omega$.

With the clustering property theorem 3.15, the existence and properties of asymptotic states, is proven in an easy way. With proposition 3.17 we can still use the clustering property for the proof of theorem 3.16, the covariance of the asymptotic states.

2 Review on Haag-Ruelle Theory in Gårding-Wightman setting

In the first three sections of this chapter I will give a short review on the contents of [Spr13] being important for the proof of theorem 2.20.

2.1 Definitions

A Hermitian scalar quantum field theory is a quadruple $(\mathscr{H}, U(\cdot), \phi(\cdot), D)$ satisfying the following eight axioms.

Definition 2.1 (Property 1, relativistic invariance of states)**.** $\mathscr{H}$ is a separable Hilbert space with $\dim \mathscr{H} \neq 1$ and $U : \mathscr{P}_+^\uparrow \to \mathscr{U}(\mathscr{H})$ is a strongly continuous unitary representation of the restricted Poincaré group[1].

Definition 2.2 (Property 2, spectral condition)**.** The projection valued measure corresponding to our representation $U(a) := U(a, \mathbb{1})$ of $\mathbb{R}^4$, i.e., the projection valued-measure E_Σ such that

$$\langle \vartheta, U(a)\psi \rangle = \int_{\mathbb{R}^4} e^{ia\cdot\tilde{\lambda}} \mathrm{d} \langle \vartheta, E_\lambda \psi \rangle \tag{2.1}$$

has support in the closed forward light cone $\overline{V^+}$. This is the closure of the set $V^+ = \{x \in \mathbb{R}^4 | x \cdot \tilde{x} > 0, x_1 > 0\}$.

Definition 2.3 (Property 3, existence and uniqueness of the vacuum)**.** There exists a unique 1-dimensional subspace of fix points of $U(a) := U(a, \mathbb{1})$, $\forall a \in \mathbb{R}^4$. Call some unit vector of this subspace the vacuum and denote it by Ω.

1 see [Cor84] chapter 17.7-8 (p.706ff) for a definition and representation properties

Definition 2.4 (Property 4, invariant domains of fields)**.** D is a dense subspace of $\mathscr{H}$, and ϕ is a map from $\mathscr{S}(\mathbb{R}^4)$ to the closable operators on $\mathscr{H}$ satisfying

1. For each $f \in \mathscr{S}(\mathbb{R}^4)$, $D \subseteq \mathrm{dom}(\phi(f))$, $D \subseteq \mathrm{dom}((\phi(f))^*)$ and $(\phi(f))^*_{|D} = (\phi(\bar{f}))_{|D}$,
2. $\Omega \in D$ and $\phi(f)D \subseteq D \forall f \in \mathscr{S}(\mathbb{R}^4)$,
3. $\forall \psi \in D$, the map $f \mapsto \phi(f)\psi$ is linear.

Definition 2.5 (Property 5, regularity of fields)**.** For any ψ and ϑ in D the map $f \mapsto \langle \psi, \phi(f)\vartheta \rangle$ is a tempered distribution.

Definition 2.6 (Property 6, Poincaré invariance of fields)**.** Let $(a, \Lambda) \in \mathscr{P}_+^\uparrow$, $f \in \mathscr{S}(\mathbb{R}^4)$ and $\psi \in D$. Then $U(a,\Lambda)D \subseteq D$ and one has

$$U(a,\Lambda)\phi(f)U(a,\Lambda)^{-1}\psi = \phi\left((a,\Lambda)\circ f\right)\psi, \tag{2.2}$$

where $\mathscr{P}_+^\uparrow$ acts naturally on $\mathscr{S}(\mathbb{R}^4)$ by

$$(a,\Lambda)\circ f(x) = f(\Lambda^{-1}(x-a)).$$

Definition 2.7 (Property 7, microscopic causality or local commutativity)**.** If $f, g \in \mathscr{S}(\mathbb{R}^4)$ have spacelike separated support, i.e., $\forall x \in \mathrm{supp} f, \forall y \in \mathrm{supp} g$ one has $(x-y)(x-y) < 0$. Then for every $\psi \in D$

$$[\phi(f), \phi(g)]\psi = 0. \tag{2.3}$$

Definition 2.8 (Property 8, cyclicity of the vacuum)**.** The set

$$D_0 = \Big\langle \{\phi(f_1)\cdots\phi(f_n)\Omega | n \in \mathbb{N}_0, f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^4)\} \Big\rangle_{\mathbb{C}-\mathrm{span}}$$

is dense in $\mathscr{H}$.

For a detailed discussion of the physical content of the axioms and some of their direct consequences see [Spr13, Chapter 2] or [RS75, Section IX.8].
Now define the functionals

$$\mathscr{W}_n(f_1, \ldots, f_n) = \langle \Omega, \phi(f_1)\cdots\phi(f_n)\Omega \rangle\,,$$

which we will denote the Wightman distributions. By nuclear theorem this multilinear functional extends to a unique tempered distribution on $\mathscr{S}(\mathbb{R}^{4n})$ which we will denote by $\mathscr{W}_n$ also.

Definition 2.9 (truncated expectation)**.** Let $C_1, \ldots, C_n$ be operator valued distributions. We define the truncated expectation $\langle \Omega, C_1, \ldots, C_n \Omega \rangle_T$ by the following recursive formula

$$\langle \Omega, C_1(f_1) \cdots C_n(f_n) \Omega \rangle = \sum_{P \in \mathscr{P}_n} \prod_{J \in P} \left\langle \Omega, C_{J_1}(f_{J_1}) \cdots C_{J_{k(J)}}(f_{J_{k(J)}}) \Omega \right\rangle_T . \tag{2.4}$$

Especially define $\langle \Omega, \phi(f_1) \cdots \phi(f_n) \Omega \rangle_T = \mathscr{W}_{n,T}(f_1, \ldots, f_n)$ and call this truncated vacuum expectation values (TVEV).

2.2 Preliminary facts

An abundance of theorems is necessary for the following brief discussion of the results on the Haag-Ruelle Scattering.

Definition 2.10. A distribution $H \in \mathscr{S}'(\mathbb{R}^{kn})$ is called $\mathbb{R}^k$-translation invariant or only $\mathbb{R}^k$ invariant iff $H(f) = H(a \circ f)$ for every $a \in \mathbb{R}^k$ with $a \circ f(x_1, \ldots, x_n) = f(x_1 - a, \ldots, x_n - a)$ the natural continuous representation of $\mathbb{R}^k$ on $\mathscr{S}(\mathbb{R}^{kn})$.

Proposition 2.11. *Let $H \in \mathscr{S}'(\mathbb{R}^{4n})$ be $\mathbb{R}^4$ invariant. Then there exists a unique tempered distribution $h \in \mathscr{S}(\mathbb{R}^{4n-4})$, which we will also refer as $\mathfrak{A}(H)$, such that*

$$H(f) = \int_{\mathbb{R}^4} h(f_{(x)}) dx \qquad , \textit{where}$$
$$f_{(x)}(\xi_1, \ldots, \xi_{n-1}) = f(x, x - \xi_1, x - \xi_1 - \xi_2, \ldots, x - \xi_1 - \cdots - \xi_{n-1}).$$

We apply this on the Wightman distributions and the TVEV and denote $\mathfrak{A}(\mathscr{W}_n)$ by W_n and $\mathfrak{A}(\mathscr{W}_{n,t})$ by $W_{n,T}$.

From the Bochner-Schwartz theorem and a classification of Lorentz invariant polynomially bounded measures supported in $\overline{V}_+$ one can develop the Källén-Lehmann representation of W_2.

Theorem 2.12 (Källén-Lehmann representation)**.** *Let W_2 be the two point function of a field obeying the Wightman axioms and also the property that $\langle\Omega, \phi(f)\Omega\rangle = 0$ for all $f \in \mathscr{S}(\mathbb{R}^4)$. Then there exists a measure ρ of at most polynomially growth such that for every $f \in \mathscr{S}(\mathbb{R}^4)$ one has:*

$$W_2(f) = \int_0^\infty \int_{H_m} \widehat{f} \mathrm{d}\Omega_m \mathrm{d}\rho(m)$$

The Källén-Lehmann representation of the free field of mass m_0 is – up to a normalization constant – $\rho(m) = \delta(m - m_0)$. $\mathrm{d}\Omega_m$ is the up to scalar multiplication unique Lorentz invariant measure on H_m.
With the Källén-Lehmann representation one is ready to state two more properties one needs to establish a scattering theory.

Definition 2.13 (Property 9, upper and lower mass gap)**.** Let P_μ be the generators of the subgroup $U(a, \mathbb{1})$ and E_Σ be the projection valued measure corresponding to our representation of $\mathbb{R}^4$, i.e., it fulfills (2.1). Then for some $m > 0$ and some $\epsilon > 0$ one has the following strengthening of Property 2

$$\mathrm{supp} E_\Sigma \subseteq \{0\} \cup H_m \cup \overline{V_{m+\epsilon,+}}$$

further one has that the set $\mathbb{S}$ of eigenvectors of $P_1^2 - P_2^2 - P_3^2 - P_4^2$ to the eigenvalue m^2 is non empty and there is a cyclic vector for the action of $U(a, \mathbb{1})$ on $\mathbb{S}$.

$\mathbb{S}$ should describe the single particle states of spinless bosons of mass m.

Definition 2.14 (Property 10, coupling of the vacuum to the one particle states)**.** The spectral weight $\mathrm{d}\rho$ of the Källén-Lehmann representation (Theorem 2.12) has the form:

$$\mathrm{d}\rho(s) = \delta(s - m) + \tilde{\rho}(s)$$

where $\tilde{\rho}$ has support in $[m + \epsilon, \infty]$

An important tool for the main theorem are the following estimates on regular wave packets.

Definition 2.15 (regular wave packets and regular positive energy solutions)**.** We call a solution $\phi(x,t)$ of the Klein-Gordon equation

$$\partial_t^2 \phi(t,\vec{x}) = \Delta\phi(t,\vec{x}) - m^2\phi(t,\vec{x}) \tag{2.5}$$

a regular wave packet for the Klein-Gordon equation or just regular wave packet if there are functions $\tilde{\phi}^+, \tilde{\phi}^- \in \mathscr{D}(\mathbb{R}^3)$

$$\phi^{\wedge(\vec{x})}(t,\vec{p}) = e^{-i\sqrt{\vec{p}^2+m^2}}\tilde{\phi}^+(\vec{p}) + e^{i\sqrt{\vec{p}^2+m^2}}\tilde{\phi}^-(\vec{p})$$

Additionally for $m = 0$ we require that $0 \notin \mathrm{supp}\phi^+$ and $0 \notin \mathrm{supp}\phi^-$. For simplicity of notation we will sometimes write $\omega_m(\vec{p})$ for $\sqrt{\vec{p}^2 + m^2}$ and mostly drop the m.
If $\tilde{\phi}^- = 0$ we call ϕ a regular positive energy wave packet or a *regular positive energy solution*. We then write $\tilde{\phi}$ instead of $\tilde{\phi}^+$. We will call supp $\tilde{\phi}$ the momentum support of ϕ.

Theorem 2.16 ([RS79, XI.17 & Corollary, p.43f])**.** *If ϕ is a regular wave packet for the Klein-Gordon equation with $m \neq 0$, then there are constants $C, d > 0$ such that the following estimates hold true*

$$\begin{aligned} a) \qquad & \int \phi(x_0,\vec{x})\mathrm{d}\vec{x} \leq C(1+|x_0|)^{3/2}, \\ b) \qquad & |\phi(x_0,\vec{x})| \leq d(1+|x_0|)^{-3/2}. \end{aligned}$$

2.3 The main Theorem

Having a (Hermitian scalar) quantum field A obeying the eight Wightman axioms and the properties nine and ten one can construct a scattering theory by the following process. First define a new operator valued distribution B by

$$B(g) = \check{A}\left(h(p^2)\hat{g}(p)\right)$$

for some chosen function $h \in C_0^\infty(\mathbb{R},[0,1])$ with $\mathrm{supp}(h) \subseteq (0, m^2+\epsilon)$, which is constantly 1 on some neighbourhood of m^2. By some simple analysis one can show, that this distribution B is smooth in t, i.e. one can naturally apply test functions of the form $\frac{\partial^m}{\partial t}\delta(t-t_0)\cdot f(\vec{x})$, where $f \in \mathscr{S}(\mathbb{R}^3)$ such that the following definition makes sense.

Definition 2.17. Let $f \in C^\infty(\mathbb{R}^4)$ with $f(\cdot,t), \partial_t f(\cdot,t) \in \mathscr{S}(\mathbb{R}^3)$ for all $t \in \mathbb{R}$, then define the symbol $\overset{\leftrightarrow}{\partial}$ by

$$(f \overset{\leftrightarrow}{\partial} B)(t) = \dot{B}(f(\cdot,t),t) - B(\partial_t f(\cdot,t),t)$$

This construction allows the formulation of the main theorem of Haag-Ruelle theory.

Theorem 2.18. *Let A be a Hermitian scalar quantum field theory obeying the Gårding Wightman axioms (Properties 1-8) and also Property 9 and Property 10. Define B in dependence of h as above. For any regular wave packets $f^{(1)}, \dots, f^{(n)}$, the strong limits*

$$s - \lim_{t \to \pm\infty} \left(f^{(1)} \overset{\leftrightarrow}{\partial} B \right)(t) \cdots \left(f^{(n)} \overset{\leftrightarrow}{\partial} B \right)(t)\Omega \equiv \eta_{\substack{\text{in}\\ \text{out}}} \left(f^{(1)}, \dots, f^{(n)} \right) \tag{2.6}$$

exist.

We now want to extract some major part of the proof - exercised in [Spr13, Thm. 4.24] - as a lemma.

Lemma 2.19. *Let B be the operator valued distribution defined above, g_i be regular wave packets for the Klein-Gordon equation, p_i be polynomials in space and D_i constant coefficient differential operators. Define the operators*

$$C_i(t) = \int_{x^0=t} p(\vec{x}) g_i(x) D_i B(x) \mathrm{d}x^3$$

then there is a constant C such that

$$|\langle \Omega, C_1(t) \cdots C_m(t)\Omega \rangle_T| \leq C(1 + |t|)^{-3/2(m-2)}$$

2.4 Asymptotic freeness

We now want to derive the covariance and the Fock-structure of the ranges of $\eta_{\substack{\text{in}\\ \text{out}}}$.

Theorem 2.20. *Let again A be a Hermitian scalar quantum field theory obeying the Gårding Wightman axioms (Properties 1-8) and also Property 9 and Property 10.*

1. *Define $\mathscr{H}_{\text{in}}$ and $\mathscr{H}_{\text{out}}$ to be the closed span of η_{in} and η_{out} respectively. Then these two subspaces are invariant under $U(\cdot)$ the representation of $\mathscr{P}_+^\uparrow$ of the theory.*
2. *There are operator valued distributions ϕ_{in} and ϕ_{out} such that $(\mathscr{H}_{\text{in}}, U(\cdot), \phi_{\text{in}})$ and $(\mathscr{H}_{\text{out}}, U(\cdot), \phi_{\text{out}})$ are unitary equivalent to the free field of mass m and one has:*

$$\eta_{\substack{\text{in}\\\text{out}}}\left(f^{(1)},\dots,f^{(n)}\right) = \left(f^{(1)}\overset{\leftrightarrow}{\partial}\phi_{\substack{\text{in}\\\text{out}}}\right)\cdots\left(f^{(n)}\overset{\leftrightarrow}{\partial}\phi_{\substack{\text{in}\\\text{out}}}\right)\Omega$$

Proof. Let 'ex' be either 'in' or 'out'. It is clearly sufficient to satisfy that the action of $U(\cdot)$ is proper on the total subspace $\operatorname{im}(\eta_{\text{ex}})$ since $U(\cdot)$ is a unitary operator. It is also clear, that the subgroup generated by translations and rotations satisfy such a proper acting,

$$\begin{aligned}
U(x,V)\eta_{\text{ex}}(\underline{f}) &= \lim_{t\to\pm\infty}\left(\prod_{k=0}^{n} U(x,V)(\dot{B}(f^{(k)}(\cdot,t),t) - B(\partial_t f^{(k)}(\cdot,t),t))U^{-1}(x,V)\right)\Omega\\
&= \lim_{t\to\pm\infty}\left(\prod_{k=0}^{n}\dot{B}((\vec{x},V)\circ f^{(k)}(\cdot,t),t+x_0) - B((\vec{x},V)\circ\partial_t f^{(k)}(\cdot,t),t+x_0)\right)\Omega\\
&= \lim_{\tau\to\pm\infty}\left(\prod_{k=0}^{n}\dot{B}((\vec{x},V)\circ f^{(k)}(\cdot,\tau-x_0),\tau) - B(\partial_t(\vec{x},V)\circ f^{(k)}(\cdot,\tau-x_0),\tau)\right)\Omega\\
&= \lim_{t\to\pm\infty}\left(\prod_{k=0}^{n}\dot{B}((x,V)\circ f^{(k)}(\cdot,t),t) - B(\partial_t(x,V)\circ f^{(k)}(\cdot,t),t)\right)\Omega = \eta_{\text{ex}}(\underline{(x,V)\circ f})
\end{aligned}$$

which follows by Property 6 on A which is passed to B. So it remains to show, that such a proper acting holds also for boosts. We want to show the corresponding identity

$$U(\Lambda)\eta_{\text{ex}}(\underline{f}) = \eta_{\text{ex}}(\underline{\Lambda\circ f}) \qquad\iff\qquad \eta_{\text{ex}}(\underline{f}) = U(\Lambda^{-1})\eta_{\text{ex}}(\underline{\Lambda\circ f})$$

Let $\Lambda = \exp\left(-c\cdot(x^s\partial_0 - x^0\partial_s)\right)$ then the claim is

$$\left\|\eta_{\text{ex}}(\underline{f}) - U(\Lambda(c)^{-1})\eta_{\text{ex}}(\underline{\Lambda(c)\circ f})\right\| = 0 \tag{2.7}$$

Now we will observe $\left.\frac{\mathrm{d}}{\mathrm{d}c}\right|_{c=0}$ we achieve

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}c}\right|_{c=0} U(\Lambda(c)^{-1})\eta_{\text{ex}}(\underline{\Lambda(c)\circ f})\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}c}\right|_{c=0}\lim_{t\to\pm\infty}\prod_{k=0}^{n}\int_{x^0=t}\partial_0 B(\Lambda^{-1}x)f^{(k)}(\Lambda^{-1}x) - B(\Lambda^{-1}x)\partial_0 f^{(k)}(\Lambda^{-1}x)\mathrm{d}x^3\Omega
\end{aligned}$$

Since the estimates in the main theorem can be chosen uniformly in dependence on c with respect to t we can interchange limiting and derivation and apply product rule. So we are interested in

$$\left(\partial_c B \overleftrightarrow{\partial} f\right)(t) := \left.\frac{\mathrm{d}}{\mathrm{d}c}\right|_{c=0} \int_{x^0=t} \partial_0 B(\Lambda^{-1}x) f(\Lambda^{-1}x) - B(\Lambda^{-1}x)\partial_0 f(\Lambda^{-1}x)\mathrm{d}x^3$$

we now suppress the non mixing components in the calculation therefor one has

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}c}\right|_{c=0} \int_{x^0=t} \partial_0 B \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} f \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \\
&\qquad - B \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \partial_0 f \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \mathrm{d}x^3 \\
= &\left.\frac{\mathrm{d}}{\mathrm{d}c}\right|_{c=0} \int_{x^0=t} \cosh c \dot{B} \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} f \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \\
&\qquad - \sinh c (\partial_s B) \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} f \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \\
&\qquad - \cosh c B \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \dot{f} \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \\
&\qquad + \sinh c B \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} (\partial_s f) \begin{pmatrix} x^0\cosh c - x^s \sinh c \\ x^s \cosh c - x^0 \sinh c \end{pmatrix} \mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s \ddot{B} f - x^0(\partial_s \dot{B}) f - x^s \dot{B}\dot{f} - x^0 \dot{B}(\partial_s f) - (\partial_s B) f \\
&\qquad + x^s \dot{B}\dot{f} + x^0(\partial_s B)\dot{f} + x^s B\ddot{f} + x^0 B(\partial_s \dot{f}) + B(\partial_s f)\mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s \ddot{B} f - 2(\partial_s B) f + x^s B\ddot{f}\mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s \ddot{B} f - 2(\partial_s B) f + x^s B(\Delta - m^2) f\mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s f(\partial_0^2 + m^2)B - 2(\partial_s B) f + (\Delta x^s B) f\mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s f(\partial_0^2 + m^2)B - 2(\partial_s B) f + (\Delta x^s B) f\mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s f(\partial_0^2 + m^2)B - 2(\partial_s B) f + 2(\partial_s B) f + (\Delta B) f\mathrm{d}x^3 \\
= &\int_{x^0=t} -x^s f(\Box + m^2)B\mathrm{d}x^3
\end{aligned}$$

In order to show (2.7), we expand the limiting term through product rule.

$$\left\| \frac{\mathrm{d}}{\mathrm{d}c}\eta(t) \right\|^2 = \sum_i \langle C_{i_1}(t)\cdots C_{i_n}(t)\Omega, C_{i_{n+1}}(t)\cdots C_{i_{2n}}(t)\Omega \rangle$$
$$= \sum_i \left\langle \Omega, C^*_{i_n}(t)\cdots C^*_{i_1}(t) C_{i_{n+1}}(t)\cdots C_{i_{2n}}(t)\Omega \right\rangle$$

Where each C_i is either a $\partial_c B \overset{\leftrightarrow}{\partial} f$ or a $B \overset{\leftrightarrow}{\partial} f$ and in each summand exactly two derivatives occur. Now each such scalar product is expanded by equation (2.4) into a sum of products of truncated expectations. The summands with only quadratic truncated expectations disappear since $B\Omega$ is a solution of the Klein-Gordon equation and two Operators of the form $\partial_c B \overset{\leftrightarrow}{\partial} f^{(k)}$ appear i.e.

$$\left\langle \Omega, C_i(t) \left(\partial_c B \overset{\leftrightarrow}{\partial} f^{(k)} \right)(t)\Omega \right\rangle_T = \left\langle C_i^*(t)\Omega, \int_{x_0=t} -x^s f^{(k)}(\Box + m^2) B\Omega \mathrm{d}x^3 \right\rangle = 0$$
$$\left\langle \Omega, \left(\partial_c B \overset{\leftrightarrow}{\partial} f^{(k)} \right)(t) C_i(t)\Omega \right\rangle_T = \left\langle \int_{x_0=t} -x^s \overline{f^{(k)}}(\Box + m^2) B\Omega \mathrm{d}x^3, C_i(t)\Omega \right\rangle = 0$$

in at least one factor.
Therefore summands with factors, where at least one TVEV's of at least degree 3 appears, are the only contributions to our term, but from lemma 2.19 these contributions do vanish in great time limits, which shows that the term in (2.7) is constant, since the cases $c \neq 0$ can be achieved by changing the functions $\underline{f}$ via $\Lambda(c)$. Obviously the equation (2.7) is true for $c = 0$ and this yields part 1 of the theorem.

Now consider Φ_m the free field of mass m on some auxiliary Hilbert space $\mathfrak{H}$, which might be the usual Fock-space representation. Then of course $\mathfrak{H}_{\mathrm{ex}} = \mathfrak{H}$. Then we define the linear map $\Omega_{\mathrm{ex}} \colon \mathfrak{H}_{\mathrm{ex}} \to \mathscr{H}_{\mathrm{ex}}$ (the Møller operator) which continues

$$\left(f^{(1)} \overset{\leftrightarrow}{\partial} \Phi_m \right) \cdots \left(f^{(n)} \overset{\leftrightarrow}{\partial} \Phi_m \right) \Omega \mapsto \eta_{\mathrm{ex}} \left(f^{(1)}, \ldots, f^{(n)} \right)$$

Our claim is that Ω_{ex} is a unitary operator[2]. We consider the time limit of

$$\left\langle \eta_{\mathrm{ex}} \left(f^{(1)}, \ldots, f^{(n)} \right), \eta_{\mathrm{ex}} \left(g^{(1)}, \ldots, g^{(m)} \right) \right\rangle$$

2 In deed we show, that Ω_{ex} defined on the total set is isometric, from this follows that its extension is well-defined and from definition of $\mathscr{H}_{\mathrm{ex}}$ unitary

in form of TVEV's and obtain that by lemma 2.19 again only the summands with only quadratic TVEV's give a non vanishing contribution. By property 10 the two point function of B which is the restriction of A on mass m agrees with the two point function of the free field of mass m so Ω_{ex} is unitary. If we now construct the field A_{ex} as the pullback of the field Φ_m under Ω_{ex} one directly has the unitary equivalence to the free field and with the proof of part 1 of the theorem one has the corresponding transformation laws. The deserved identity is obvious from definition of Ω_{ex}. □

3 Scattering in algebraic quantum field theory

3.1 Local algebras

Instead of considering a complete theory of a quantum field obeying all the Wightman axioms stated in chapter 2 one can often use the term of local algebras[1] to achieve the results one wants a quantum field to yield. The content of the theory of local algebras is an algebra valued assignment (indeed a net)

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

where $\mathcal{O}$ is some open bounded region of spacetime. $\mathcal{A}(\mathcal{O})$ is a *-subalgebra of the closed operators on $\mathscr{H}$, which we will consider to be a C^*-subalgebra. An element $A \in \mathcal{A}(\mathcal{O})$ will be called a *local operator*.

This algebra should be thought as physical operations in the given space time region or – coming from the Gårding-Wightman case – as the field A smeared with functions f having support in $\mathcal{O}$. The usage of bounded operators instead of unbounded operators as one would suggest from the Gårding-Wightman axioms marks a distinction between the two axiomatic systems. There are certain mathematical difficulties in the step going from the algebra generated by the $A(f)$ to a bounded algebra. The physical reason, why bounded operators are sufficient, is due to a relation of the norm of an operator to the energy needed to carry out the physical operation described by the operator, which should always be finite.

1 Local is meant in a topological sense not in the algebraic definition of rings concerning maximal ideals

3.1.1 The Haag-Kastler axioms

Properties A-C describe very the same conditions on $U(\cdot)$ and $\mathscr{H}$ concerning vacuum and mass spectrum as the Gårding Wightman axioms and can be adopted.

Definition 3.1 (Property D, additivity property)**.** Having two open bounded space time regions $\mathcal{O}_1$ and $\mathcal{O}_2$ the algebra assigned by $\mathcal{O}_1 \cup \mathcal{O}_1$ is generated by $\mathcal{A}(\mathcal{O}_1) \cup \mathcal{A}(\mathcal{O}_1)$ as an algebra.

Definition 3.2 (Property E, Poincaré invariance)**.** The algebras transform as sets by

$$U(a,\Lambda)^{-1}\mathcal{A}(\mathcal{O})U(a,\Lambda) = \mathcal{A}((a,\Lambda)\circ\mathcal{O})$$

Definition 3.3 (Property F, microscopic causality)**.** Having two spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ any two elements of the corresponding algebras commute.

Then there are some versions of completeness properties growing in strength.

Definition 3.4 (Property G1, completeness)**.** Taking the union of all local algebras yields a dense set in $\mathcal{L}(\mathscr{H})$, i.e. its commutant[2] is the trivial algebra.

Definition 3.5 (Property G2, Time-slice-axiom)**.** Taking the union of all local algebras in a Time-slice yields a dense set in the above sense. It means the union over all

$$\mathcal{O} \subseteq \mathcal{O}_{t,\epsilon} = \{x \in \mathbb{R}^4 | \|x^0 - t\| \leq \epsilon\}.$$

Definition 3.6 (Property G3, Reeh-Schlieder property)**.** Taking any nonempty bounded open region $\mathcal{O}$ one has that $\mathcal{A}(\mathcal{O})\Omega$ is dense in the Hilbert space concerning norm topology.

This finishes the basic axioms of algebraic quantum field theory, though one can express the most of it in a more abstract and algebraic way, where only the existence of a representation on some Hilbert space analogously to the representation on $\mathscr{H}$ is needed, the group acting of $\mathscr{P}_+^\uparrow$ is then encoded in algebraic properties of the abstract assignment of C*-algebra.

2 The commutant of an algebra is the set of all operators in $\mathcal{L}(\mathscr{H})$ commutating with all operators of the algebra

3.1.2 Stability condition and isolated vacuum

Since we only consider large time limits in scattering theory it is clear, that we can only observe stable particles as scattering states. And the easiest stability condition would be definition 2.13. In our following computation we will assume some weaker stability condition instead.

Definition 3.7. Let $m > 0$ and $\mathscr{H}_1$ be a subspace of $\mathscr{H}$ such that $U_{|\mathscr{H}_1}$ is a representation[3] of mass m, let $\eta > 0$. Denote the projection onto $\mathscr{H}_1$ by P_1 and by $E^{(\mathrm{m})}_{\Sigma}$ the spectral projection of the mass operator with $(a \mp b)$ denoting the interval $(a-b, a+b)$. Then $\forall \mathcal{O} \subset \mathbb{R}^4$ open bounded, call an operator $A \in \mathcal{A}(\mathcal{O})$ with $P_1 A\Omega \neq 0$, regular, if $\exists C > 0$,

$$\|E^{(\mathrm{m})}_{(m\mp\mu)}(1-P_1)A\Omega\| \leq C\mu^{\eta} \tag{3.1}$$

for all μ sufficiently small.

We make the following assumption on the spectrum of the momentum-energy operator, i.e. we assume that the vacuum is isolated.

Definition 3.8 (Isolated vacuum)**.** Let E_{Σ} be the projection valued measure corresponding to the representation of $\mathbb{R}^4$ on $\mathscr{H}$. Then we call the vacuum isolated if $\exists \epsilon > 0$

$$E_{B_\epsilon(0)} = E_{\{0\}}$$

3.2 Asymptotic creation operators

Let $1/(1+\eta) < \kappa < 1$ and g some Schwartz-function, which integrates to 1 and whose Fourier transform has compact support, say within $[-1, 1]$. This will deal as a cutoff function similar to the construction in the Wightman setting.

Let A be a regular operator as in the stability condition and f be a regular positive energy solution, recall definition 2.15. Denote $U(x)AU(-x)$ by $A(x)$ and define for $t \neq 0$ the operator

$$A_t(f) = \int g\left(\frac{x_0 - t}{|t|^{\kappa}}\right) f(x)A(x)\frac{\mathrm{d}x}{|t|^{\kappa}}$$

3 We do not demand this representation to be irreducible

For simplicity of notation denote $g\left(\frac{x_0-t}{|t|^\kappa}\right)/|t|^\kappa$ by $g_t(x_0)$. By stretching the function g we will effectively compress its Fourier transform and end up with a cutoff in energy-momentum space in great time limit.

3.2.1 One-particle states

Proposition 3.9. *The operator $A_t(f)$ acts on Ω in creating a particle in $\mathscr{H}_1$ for great time limits, i.e.*

$$s-\lim_{t\to\pm\infty} A_t(f)\Omega = P_1 A(f)\Omega := P_1 \int_{\mathbb{R}^3} f(0,\vec{x})A(0,\vec{x})\Omega \mathrm{d}\vec{x}$$

Furthermore we have $A_t(f)^*\Omega = 0$ for $|t| > m^{-1/\kappa}$, even for a general A, i.e. it need neither be regular nor local.

Proof. We remember

$$U(x) = \int e^{-ix\cdot\tilde{\lambda}} \mathrm{d}E_\lambda = \int e^{ix_0\lambda_0 - i\vec{x}\cdot\vec{\lambda}} \mathrm{d}E_\lambda = (2\pi)^2 E^{\vee(\lambda_0)\wedge(\vec{\lambda})}(x)$$

Having this in mind, we now compute $\lim_{t\to\pm\infty} A_t(f)\Omega$ using the notation $đE_\lambda = (2\pi)^2 \mathrm{d}E_\lambda$

$$\begin{aligned}
\int g_t(x_0) f(x) U(x) A\Omega \mathrm{d}x &= \int g\left(\frac{x_0-t}{|t|^\kappa}\right) e^{-ix_0\sqrt{\vec{p}^2+m^2}} \tilde{f}(\vec{p}) U^{\vee(\vec{x})}(x_0,\vec{p}) A\Omega \frac{\mathrm{d}x_0}{|t|^\kappa} \,\mathrm{d}\vec{p} \\
&= \int e^{it\xi} \check{g}(|t|^k\xi) \tilde{f}(\vec{p}) U^{\wedge(x_0)\vee(\vec{x})}(\xi + \sqrt{\vec{p}^2+m^2}, \vec{p}) A\Omega \mathrm{d}\xi \mathrm{d}\vec{p} \\
&= \int e^{it\xi} \check{g}(|t|^k\xi) \tilde{f}(\vec{p}) \; đE_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A\Omega \\
&= \int e^{it\xi} \check{g}(|t|^k\xi) \tilde{f}(\vec{p})(1-P_1) \; đE_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A\Omega \\
&\quad + \int e^{it\xi} \check{g}(|t|^k\xi) \tilde{f}(\vec{p}) P_1 \; đE_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A\Omega
\end{aligned}$$

We start with estimating the first term, respect that E_μ is a (projection-valued) measure, and $\mathrm{supp}\check{g} \subseteq [-1,1]$ and moreover $\mathrm{supp}\tilde{f}$ is also compact.

$$\begin{aligned}
&\left\| \int e^{it\xi} \check{g}(|t|^k\xi) \tilde{f}(\vec{p})(1-P_1) \; đE_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A\Omega \right\| \\
&\leq \sup \left| \check{g}(|t|^k\xi) \tilde{f}(\vec{p}) \right| \cdot \left\| \int_{(m\mp|t|^{-\kappa})} (1-P_1) \; đE^{(\mathrm{m})}_\mu A\Omega \right\|
\end{aligned}$$

$$\leq C \cdot \|E^{(\mathrm{m})}_{(m \mp |t|^{-\kappa})}(1-P_1)A\Omega\| \leq C'|t|^{-\kappa\cdot\eta} \xrightarrow{t\to\pm\infty} 0$$

Since $U(\cdot)$ provides a $m > 0$ representation by restriction onto $\mathscr{H}_1$, we have that $\operatorname{supp} P_1 E_\mu A\Omega \subseteq H_m$ and since it is a (finite vector-valued) measure the expression is invariant under a change of the integrand, leaving the values on the support invariant[4]. The second term therefore equals

$$\begin{aligned}\int_{\mathbb{R}^4} \frac{\tilde{f}(\vec{p})}{\sqrt{2\pi}} P_1 \, \text{đ}E_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A\Omega &= \int_{\mathbb{R}^4} \frac{\tilde{f}(\vec{p})}{\sqrt{2\pi}} P_1 \, \text{đ}E_{\xi,\vec{p}} A\Omega \\ &= \int_{\mathbb{R}^3} e^{-i0\cdot\sqrt{m^2+\vec{p}^2}} \tilde{f}(\vec{p}) P_1 \left(\int_{\mathbb{R}} \frac{e^{i0\cdot\xi}}{\sqrt{2\pi}} \hat{U}(\xi,\vec{p}) A\Omega \mathrm{d}\xi \right) \mathrm{d}\vec{p} \\ &= P_1 \int_{\mathbb{R}^3} f(0,\vec{x}) U(0,\vec{x}) A\Omega \mathrm{d}\vec{x}\end{aligned}$$

Before proving the second part, first note that $A(x)^* = (U(x)AU(-x))^* = U(x)A^*U(-x)$ and then we analogously compute $\lim_{t\to\pm\infty} A_t(f)^*\Omega$

$$\begin{aligned}\int \overline{g_t(x_0) f(x)} U(x) A^* \Omega \mathrm{d}x &= \int \overline{g}\left(\frac{x_0 - t}{|t|^\kappa}\right) \overline{\tilde{f}(-\vec{p})} e^{ix_0\sqrt{\vec{p}^2+m^2}} U^{\vee(\vec{x})}(x_0,\vec{p}) A^*\Omega \, \frac{\mathrm{d}x_0}{|t|^\kappa} \, \mathrm{d}\vec{p} \\ &= \int e^{it\xi} \overline{\check{g}(-|t|^k\xi) \tilde{f}(-\vec{p})} U^{\wedge(x_0)\vee(\vec{x})}(\xi - \sqrt{\vec{p}^2+m^2},\vec{p}) A^*\Omega \mathrm{d}\xi \mathrm{d}\vec{p} \\ &= \int e^{it\xi} \overline{\check{g}(-|t|^k\xi) \tilde{f}(-\vec{p})} \, \text{đ}E_{\xi-\sqrt{\vec{p}^2+m^2},\vec{p}} A^*\Omega \overset{|t|\gg 0}{=} 0\end{aligned}$$

Since the support of $\check{g}$ shrinks the support of the integrand for $|t| > m^{-1/\kappa}$ to $\xi \leq |t|^{-\kappa} < m$ and therefore $\xi - \sqrt{\vec{p}^2+m^2} < 0$ and E_λ is not supported in regions of negative energy. □

3.2.2 Asymptotic commutativity

Before continuing with multi-particle states, we have to derive an important lemma on asymptotic properties of solutions to Klein-Gordon with disjoint momentum support using stationary phase method. We outline here that the results we achieve by stationary phase method can be vastly generalized.

4 Contrary to a general distribution, where the change should leave the values on a neighbourhood of the support invariant

Definition 3.10 (Velocity support)**.** Let f be a regular positive energy solution. Define the velocity support Γ_f by

$$\Gamma_f = \left\{ \nabla\omega(\vec{p}) = \frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2}} \middle| \vec{p} \in \operatorname{supp} \tilde{f} \right\}$$

Lemma 3.11. *Let f be a regular positive energy solution, $a(x)$ a polynomially bounded function and $A \subseteq \mathbb{R}^3$ be a closed set with $A \cap \Gamma_f = \emptyset$. Then for every $p \in [1, \infty]$*

$$\|a(t) \cdot f(t)\|_{t \cdot A, p} := \|a(t, \cdot) f(t, \cdot)_{|t \cdot A}\|_p$$

is rapidly decreasing in t.

Proof.

$$f(t, \vec{v}t) = \int_{\mathbb{R}^3} e^{it\vec{v}\cdot\vec{p}} e^{-it\sqrt{\vec{p}^2+m^2}} \tilde{f}(\vec{p}) \mathrm{d}\vec{p} = \int_{\mathbb{R}^3} e^{it(\vec{v}\cdot\vec{p} - \sqrt{\vec{p}^2+m^2})} \tilde{f}(\vec{p}) \mathrm{d}\vec{p}$$

Introduce $h_{\vec{v}}(\vec{p}) = \vec{v} \cdot \vec{p} - \sqrt{\vec{p}^2 + m^2}$, we then have that $\nabla h_{\vec{v}}(\vec{p}) = \vec{v} - \frac{\vec{p}}{\sqrt{\vec{p}^2+m^2}}$. And since Γ_f and A are disjoint for $\vec{p} \in \operatorname{supp} \tilde{f}$ we have $\nabla h_{\vec{v}}(\vec{p}) \in A - \Gamma_f$, which is closed and does not contain 0, $\exists \epsilon > 0$ such that $\|\nabla h_{\vec{v}}(\vec{p})\| > \epsilon$. Let $u \in \mathscr{D}(\mathbb{R}^3)$ and $h \in C^\infty(\mathbb{R}^3)$. Observe the following calculation[5]

$$\begin{aligned} \int u(\vec{p}) \cdot e^{ith(\vec{p})} \mathrm{d}\vec{p} &= \sum_j \int \frac{u \cdot (\partial_j \overline{h})}{\|\nabla h\|^2} (\partial_j h) e^{ith} \mathrm{d}\vec{p} \\ &= (-it)^{-1} \int \sum_j \left(\partial_j \frac{u \cdot (\partial_j \overline{h})}{\|\nabla h\|^2} \right) e^{ith} \mathrm{d}\vec{p} \end{aligned}$$

Using this equation, by induction one sees that for all k there are polynomials $P_{j,\alpha}^{\beta,\gamma}$ with only finitely many of them different from 0, such that

$$\begin{aligned} &\int u(\vec{p}) \cdot e^{ith(\vec{p})} \mathrm{d}\vec{p} \\ =&(-it)^{-k} \int \sum_{j \geq k} \|\nabla h\|^{-2j} \sum_{\substack{\alpha,\beta,\gamma \\ |\beta|+|\gamma| \leq j}} \partial^\alpha u (\nabla h)^\beta (\nabla \overline{h})^\gamma P_{j,\alpha}^{\beta,\gamma} \left(\left\{ \partial^\mu h, \partial^\mu \overline{h} \middle| |\mu| \geq 2 \right\} \right) e^{ith} \mathrm{d}\vec{p} \end{aligned}$$

5 Idea from [Hör83, Thm 7.7.1]

Plugging in our $h_{\vec{v}}$ and $\tilde{f}$, fortunately the unknown (but existent) polynomials $P_{j,\alpha}^{\beta,\gamma}$ do not depend on $\vec{v}$ so in order to estimate $|f(t,\vec{v}t)|$ we can take

$$C_{j,\alpha}^{\beta,\gamma} = \sup\left\{ \left|P_{j,\alpha}^{\beta,\gamma}\right| \left(\left\{ \partial^\mu h_{\vec{v}}(\vec{p}), \partial^\mu \overline{h_{\vec{v}}(\vec{p})} \middle| \, |\mu| \geq 2 \right\}\right) \middle| \, \vec{p} \in \operatorname{supp} \tilde{f} \right\}$$

independent on $\vec{v}$. Therefore we get, suprema taken over the support of $\tilde{f}$:

$$\begin{aligned}
|f(t,\vec{v}t)| = \left| \int e^{ith_{\vec{v}}} \tilde{f} \mathrm{d}\vec{p} \right| &\leq t^{-k} \lambda(\operatorname{supp} \tilde{f}) \sum_{j \geq k} \sum_{\substack{\alpha,\beta,\gamma \\ |\beta|+|\gamma| \leq j}} \sup |\partial^\alpha u| \|\nabla h_{\vec{v}}\|^{|\beta|+|\gamma|-2j} C_{j,\alpha}^{\beta,\gamma} \\
&\lesssim t^{-k} \sum_{j \geq k} \sum_{l \leq j} C_{j,l} \sup \|\nabla h_{\vec{v}}\|^{l-2j} \leq t^{-k} \sum_{j \geq k} \sum_{l \leq j} C_{j,l} \sup \|\nabla h_{\vec{v}}\|^{l-2j} \left\| \frac{\nabla h_{\vec{v}}}{\epsilon} \right\|^{j-l} \\
&\leq t^{-k} \sum_{j \geq k} C_j \sup \|\nabla h_{\vec{v}}\|^{-l} \leq t^{-k} \sum_{j \geq k} C_j \sup \|\nabla h_{\vec{v}}\|^{-j} \left\| \frac{\nabla h_{\vec{v}}}{\epsilon} \right\|^{j-k} \\
&\lesssim t^{-k} \sup \|\nabla h_{\vec{v}}\|^{-k}
\end{aligned}$$

We already have $\sup \|\nabla h_{\vec{v}}\|^{-k} \leq \epsilon^{-k}$, which yields an $\mathscr{L}^\infty$ estimate for f. Since Γ_f is compact, we also have the estimate $\sup \|\nabla h_{\vec{v}}\|^{-k} \leq (\|v\| - C)^{-k}$ for $\|v\| > C + 2$. Since q is polynomially bounded, there is a $N \in \mathbb{N}_0$ such that for $|t| > 1$

$$q(t,\vec{v}t) \lesssim (t \cdot \langle \vec{v} \rangle)^N = t^N (1 + \|\vec{v}\|^2)^{N/2}$$

Now estimating on $\vec{v} \in B_{C+2}(0)$, we get

$$a(t,\vec{v}t) f(t,\vec{v}t) \lesssim t^{N-k} (1 + (C+2)^2)^{N/2} \epsilon^{-k}$$

also we have outside this ball

$$a(t,\vec{v}t) f(t,\vec{v}t) \lesssim t^{N-k} \frac{\langle \vec{v} \rangle^N}{(\|\vec{v}\| - C)^k}$$

For $k > N$ this terms are bounded, thus we find an overall $\mathscr{L}^\infty$ estimate. For $k > N+3$ both parts are integrable. Due to the stretching of $\vec{v}$ by t, we receive t^{N+3-k} as decay only. By $\mathscr{L}^p$ interpolation this yields the claim.[6] □

6 Note that we haven't make use of any specific property of $\sqrt{\vec{p}^2 + m^2}$, it is therefore valid also for general C^∞ propagators.

From this fundamental Lemma, which has the physical interpretation as a momentum-velocity relation for regular positive energy solutions, we will derive using microscopic causality an asymptotic commutativity relation.

Lemma 3.12 (Asymptotic commutativity)**.** *Let f_1, f_2 be regular positive energy solutions with disjoint momentum support, let further $k_1, k_2 \in \mathscr{S}(\mathbb{R})$, $q_1(t,x)$ and $q_2(t,x)$ be polynomially bounded and $A_1, A_2 \in \mathcal{A}(\mathcal{O})$. Then*

$$\left\| \left[\int q_1(t,x) k_1 \left(\frac{x_0 - t}{|t|^\kappa} \right) f_1(x) A_1(x) \frac{\mathrm{d}x}{|t|^\kappa}, \int q_2(t,x) k_2 \left(\frac{y_0 - t}{|t|^\kappa} \right) f_2(y) A_2(y) \frac{\mathrm{d}y}{|t|^\kappa} \right] \right\|$$

is rapidly decreasing in $|t|$. The same holds true if f_1 and/or f_2 is replaced by its complex conjugated.

Proof. Let K_1 and K_2 be disjoint compact neighbourhoods of the Γ_{f_i}. Now choose an $\epsilon > 0$ with

$$L_i = \{(x_0, x_0 \cdot \vec{v}) | 1 - \epsilon \le x_0 \le 1 + \epsilon, \vec{v} \in K_i\}$$

spacelike separated, i.e. there is a $\eta > 0$ with

$$\sup_{x \in L_1 - L_2} (x_0^2 - \vec{x}^2) < -\eta^2 \qquad \Longrightarrow \qquad \sup_{x \in t(L_1 - L_2)} (x_0^2 - \vec{x}^2) < -t^2 \eta^2$$

Especially for all $|t| \gg 0$ we have that $B_{\mathrm{diam}(\mathcal{O})}(t \cdot L_i)$ are spacelike separated. And by microscopic causality we obtain for $|t| \gg 0$ that $[A_1(x), A_2(y)] = 0$ if $x \in tL_1$ and $y \in tL_2$. We can thus estimate the commutator expression by:

$$2\|A_1\|\|A_2\| \left(\int_{\mathbb{R}^4 \setminus tL_1} |q_1(t,x) k_{1,t}(x_0) f_1(x)| \,\mathrm{d}x \cdot \int_{\mathbb{R}^4} |q_2(t,x) k_{2,t}(y_0) f_2(y)| \,\mathrm{d}y \right.$$
$$\left. + \iint_{\mathbb{R}^4} |q_1(t,x) k_{1,t}(x_0) f_1(x)| \,\mathrm{d}x \cdot \int_{\mathbb{R}^4 \setminus tL_2} |q_2(t,x) k_{2,t}(y_0) f_2(y)| \,\mathrm{d}y \right)$$

W.l.o.g. we estimate only the first summand. We split the first factor into two parts. First the integral over the region outside of the $[t \mp t\epsilon]$ time slice, then the part inside the time slice without L_1.

Let $q_i(t,x) \lesssim t^N \langle x_0 \rangle^N \langle \vec{x} \rangle^N$ for $|t| > 1$. In applying theorem 2.16 and lemma 3.11 with $A = \overline{\mathbb{R}^3 \setminus K_1}$ we can estimate the spatial integration of the part outside the time slice, we get for $|t| \gg 0$

$$\begin{aligned}
&\int_{\mathbb{R}\setminus[t\mp t\epsilon]} Ct^N k_1\left(\frac{x_0-t}{|t|^\kappa}\right)\left(\int_{\{x_0\}\times x_0K_1} d(1+|x_0|)^{-3/2}\langle x_0\rangle^N\langle\vec{x}\rangle^N\,\mathrm{d}\vec{x}+\mathcal{O}(|x_0|^{-q})\right)\frac{\mathrm{d}x_0}{|t|^\kappa}\\
&\lesssim \int_{\mathbb{R}\setminus[t\mp t\epsilon]} t^N k_1\left(\frac{x_0-t}{|t|^\kappa}\right)\langle x_0\rangle^{N-3/2}\left(x_0^3\lambda(K_1)\cdot(1+\sup_{\vec{v}\in K_1}\|x_0\cdot\vec{v}\|^2)^{N/2}\right)\frac{\mathrm{d}x_0}{|t|^\kappa}\\
&\lesssim \int_{\mathbb{R}\setminus[t\mp t\epsilon]} t^N k_1\left(\frac{x_0-t}{|t|^\kappa}\right)\langle x_0\rangle^{2N+3/2}\frac{\mathrm{d}x_0}{|t|^\kappa}\\
&\leq t^N\int_{|z|>\epsilon|t|^{1-\kappa}} k_1(z)\,\langle|t|^\kappa z+t\rangle^{2N+3/2}\,\mathrm{d}z\\
&\lesssim \langle t\rangle^{3N+3/2}\int_{|z|>\epsilon|t|^{1-\kappa}} k_1(z)\,\langle z\rangle^{2N+3/2}\,\mathrm{d}z
\end{aligned}$$

Since k_1 is a Schwartz function the integral decreases rapidly in $\epsilon|t|^{1-\kappa}$, i.e. in $|t|$.

Applying lemma 3.11 with $A=\overline{\mathbb{R}^3\setminus K_1}$ and Hölder inequality with respect to the x_0 part, we get the estimate $\|k_1\|_1\cdot C_k t^{N-k}(1-\epsilon)^{-k}$ for the part within the time slice for any $k\in\mathbb{N}_0$. We altogether find, that for all $k\in\mathbb{N}_0$ we have that

$$\int_{\mathbb{R}^4\setminus tL_1}|k_{1,t}(x_0)f_1(x)|\,\mathrm{d}x\lesssim t^{-k}$$

for all $|t|\gg 0$.

Parallel to the estimate for the part outside of the time slice, we can estimate the second factor by

$$\begin{aligned}
\int_{\mathbb{R}^4} q_2(t,x)k_2\left(\frac{y_0-t}{|t|^\kappa}\right)f_2(y)\frac{\mathrm{d}y}{|t|^\kappa} &\lesssim \langle t\rangle^{3N+3/2}\int_{\mathbb{R}} k_2(z)\,\langle z\rangle^{2N+3/2}\,\mathrm{d}z\\
&\leq C'\langle t\rangle^{3N+3/2}\int_{\mathbb{R}} k_2(z)\,\langle z\rangle^{2N+3/2}\,\mathrm{d}z\\
&\lesssim \langle t\rangle^{3N+3/2}
\end{aligned}$$

which yields the claim. Since the estimates used properties of $|f(x)|$ they are also valid for the complex conjugated. □

The result on the estimate of the second factor we state as proposition

Proposition 3.13. *Let f be a regular positive energy solution, let further $k\in\mathscr{S}(\mathbb{R})$, $A\in\mathcal{A}(\mathcal{O})$*

and $q(t,x) \lesssim \langle t\rangle^N \langle x_0\rangle^N \langle\vec{x}\rangle^N$. *Then*

$$\left\| \int q(t,x) k\left(\frac{x_0 - t}{|t|^\kappa}\right) f(x) A_1(x) \frac{\mathrm{d}x}{|t|^\kappa} \right\| \lesssim \langle t\rangle^{3N+3/2}$$

3.2.3 Clustering property

We want to derive a formula for large time scalarproducts of multiple creation operators coming from regular positive energy solutions with disjoint momentum support acting on the vacuum, which we eventually want to prove to be strongly converging to multi-particle states.

Lemma 3.14. *Let* $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m \in \mathscr{S}(\mathbb{R})$ *with compactly supported Fourier-transform and* $q_1(t,x), \ldots q_n(t,x), p_1(t,x), \ldots, p_m(t,x)$ *be polynomials in* x *with polynomially bounded coefficients in* t. $e_1, \ldots, e_n$ *be regular positive energy solutions with pairwise disjoint momentum support, and* $f_1, \ldots, f_m$ *also. Let further* $A_1, \ldots A_n$ *and* $B_1, \ldots B_m$ *be local operators.*
Denote by $\mu_{i,t}(z)$ *again* $\mu_i(\frac{z-t}{|t|^\kappa}) \cdot |t|^{-\kappa}$ *and* $\nu_{i,t}$ *respectively and define*

$$\tilde{B}_{i,t}(f_i) = \int q_i(t,x_i)\nu_{i,t}(y_i^{(0)}) f_i(y_i) B_i(y_i) \mathrm{d}y_i$$

and analogously $\tilde{A}_{i,t}(e_i)$. *Then we have for all* $q \in \mathbb{N}_0$

$$\begin{aligned}
&\left\langle \left(\prod_{i=1}^n \tilde{A}_{i,t}(e_i)\right)\Omega, \left(\prod_{j=1}^m \tilde{B}_{j,t}(f_j)\right)\Omega \right\rangle \\
=&\delta_{m,n} \sum_{\sigma\in\mathfrak{S}_m} \prod_{i=1}^m \left\langle \tilde{A}_{i,t}(e_i)\Omega, \tilde{B}_{\sigma(i),t}(f_{\sigma(i)})\Omega \right\rangle + \mathcal{O}(|t|^{-q}) \qquad \text{as } t \to \pm\infty
\end{aligned}$$

Proof. Assume $n \neq m$, w.l.o.g. $m = n + k$.
We can find open neighbourhoods V_i, U_i of supp $\tilde{f}_i$ with $\overline{V_i} \subseteq U_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$. Then split the functions $\tilde{e}_j$ into sums of test functions having either support in a U_i or outside of all V_i. Since the creation operator is linear we can consider each case after another and in the latter case – assume w.l.o.g. that $j = 1$ – due to the fact that commutators $[\tilde{A}_{1,t}(e_1)^*, \tilde{B}_{i,t}(f_i)]$ are rapidly decreasing if supp $\tilde{e}_1 \cap$ supp $\tilde{f}_i \subseteq$ supp $\tilde{e}_1 \cap V_i = \emptyset$, and by

proposition 3.13 the other operators are polynomially bounded in norm, we receive

$$\left\langle \prod_{j=1}^{n} \tilde{A}_{j,t}(e_j)\Omega, \prod_{i=1}^{n+k} \tilde{B}_{i,t}(f_i)\Omega \right\rangle = \left\langle \prod_{j=2}^{n} \tilde{A}_{j,t}(e_j)\Omega, \tilde{A}_{1,t}(e_1)^* \prod_{i=1}^{n+k} \tilde{B}_{i,t}(f_i)\Omega \right\rangle$$
$$\equiv \left\langle \prod_{j=2}^{n} \tilde{A}_{j,t}(e_j)\Omega, \prod_{i=1}^{n+k} \tilde{B}_{i,t}(f_i) A_{1,t}(e_1)^*\Omega \right\rangle \qquad \mod \mathcal{O}(|t|^{-q})$$

which yields no contribution since $\tilde{A}_{n,t}(e_n)^*\Omega$ is 0 after finite time from proposition 3.9. The expression does not really apply to the proposition, the only difference is the factor $q_i(t,x)$, being a polynomial in x, which under Fourier transform yields a differential operator, which does not change the argumentation via disjoint support properties of the integrand and the measure.
On the other hand, if there is a i such that supp $\tilde{e}_j \cap U_i = \emptyset$ for all j we can formulate the same argument swapping sides, since $[\tilde{A}_{j,t}(e_n)^*, \tilde{B}_{i,t}(f_i)]$ will rapidly decay. So for a contributing term in every U_i there lies the support of a $\tilde{e}_j$, but this is only possible if $k = 0$. We conclude the claim for $k \neq 0$. Now assume $n = m$. We can further consider that supp $\tilde{e}_i \subseteq U_i$ in our computation.

In the next computation step, we want to use methods of tempered distributions, especially we will make use of the Fourier transform of functions of the type

$$\langle \Omega, U(x_1)A_1 \cdots U(x_n)A_n\Omega \rangle$$

Since all operators are bounded, this is a $\mathcal{L}^\infty \subseteq \mathscr{S}'(\mathbb{R}^{8n})$ function and therefore has a Fourier transform as a tempered distribution. We will formally denote its Fourier transform by

$$\text{đ}^n \left\langle \Omega, E_{\xi_1,\vec{p}_1}A_1 \cdots E_{\xi_n,\vec{p}_n}A_n\Omega \right\rangle$$

This is somewhat misleading, because it is **not** a measure. The reason why we use this expression is due to the following calculation. Let $(\gamma_n)_{n\in\mathbb{N}_0}$ be an orthonormal basis, then

$$\langle \Omega, U(x_1)A_1 \cdots U(x_n)A_n\Omega \rangle = \sum_{\alpha\in\mathbb{N}_0^{n-1}} \langle \Omega, U(x_1)A_1\gamma_{\alpha_1} \rangle \cdots \left\langle \gamma_{\alpha_{n-1}}, U(x_n)A_n\Omega \right\rangle$$

The sum as tempered distributions is convergent. Here the Fourier-transform of each summand is a measure, but the sum need not to be, since measures are even dense in $\mathscr{S}'$, so far away from being closed. In the following calculation we are able to use common support properties of these summands, which obviously carry over to the sum. If $f(p_1,\dots,p_n)$ is supported within $p_i \in \Sigma$, then we can also do the following calculation,

$$\begin{aligned}
&\int f(p_1,\dots,p_n) đ^n \left\langle \Omega, E_{p_1}A_1 \cdots E_{p_n}A_n\Omega \right\rangle \\
&= \sum_{\alpha\in\mathbb{N}_0^{n-1}} \int f(p_1,\dots,p_n) đ \left\langle \Omega, E_{p_1}A_1\gamma_{\alpha_1}\right\rangle \cdots đ \left\langle \gamma_{\alpha_{n-1}}, E_{p_n}A_n\Omega\right\rangle \\
&= \sum_{\alpha\in\mathbb{N}_0^{n-1}} \int f(p_1,\dots,p_n) đ \left\langle \gamma_{\alpha_{i-1}}, E_{p_i}E_\Sigma A_i\gamma_{\alpha_i}\right\rangle đ \left\langle \Omega, E_{p_1}A_1\gamma_{\alpha_1}\right\rangle \cdots đ \left\langle \gamma_{\alpha_{n-1}}, E_{p_n}A_n\Omega\right\rangle \\
&= \int f(p_1,\dots,p_n) đ^n \left\langle \Omega, E_{p_1}A_1 \cdots E_{p_i}E_\Sigma A_i \cdots E_{p_n}A_n\Omega \right\rangle
\end{aligned}$$

With this techniques we start computing, note that we collect the $q_j(t,x_j), p_i(t,x_i)$ in a single polynomial, $P_t(x)$, which Fourier transforms to a differential operator $P_t(D)$.

$$\begin{aligned}
&\left\langle \Omega, \left(\prod_{i=1}^n \tilde{A}_{i,t}(e_i)^*\tilde{B}_{i,t}(f_i)\right)\Omega \right\rangle \\
&= \iint \Bigg\langle \Omega, \prod_{i=1}^n \overline{q_i(t,x_{2i-1})\mu_{i,t}(x^{(0)}_{2i-1})e_i(x_{2i-1})} p_i(t,x_{2i})\nu_{i,t}(x^{(0)}_{2i})f_i(x_{2i}) \\
&\qquad\qquad U(x_{2i-1})A_i^*U(x_{2i}-_{2i-1})B_iU(-x_{2i})\Omega \Bigg\rangle \mathrm{d}x^{2n} \\
&= \iint \left\langle \Omega, P_t(x)\prod_{i=1}^n \overline{\mu_{i,t}\left(\sum_{j=1}^{2i-1} y_j^{(0)}\right) e_i\left(\sum_{j=1}^{2i-1} y_j\right)} \nu_{i,t}\left(\sum_{j=1}^{2i} y_j^{(0)}\right) f_i\left(\sum_{j=1}^{2i} y_j\right) U(y_{2i-1})A_i^*U(y_{2i})B_i\Omega \right\rangle \mathrm{d}x^{2n} \\
&= \iint P_t(D)\prod_{i=1}^n \overline{\check{\mu}_{i,t}\left(\xi_{2i-1}-\xi_{2i}+\omega(\vec{p}_{2i}-\vec{p}_{2i-i})\right) \tilde{e}_i\left(\vec{p}_{2i}-\vec{p}_{2i-1}\right)} \check{\nu}_{i,t}\left(\xi_{2i}-\xi_{2i+1}-\omega(\vec{p}_{2i}-\vec{p}_{2i+1})\right) \tilde{f}_i\left(\vec{p}_{2i}-\vec{p}_{2i+1}\right) \\
&\qquad\qquad đ^{2n} \left\langle \Omega, \left(\prod_{i=1}^n E_{\xi_{2i-1},\vec{p}_{2i-1}}A_i^*E_{\xi_{2i},\vec{p}_{2i}}B_i\right)\Omega \right\rangle
\end{aligned}$$

First it is clear that we can restrict ourselves to $(\xi_1,\vec{p}_1) = 0$ because of the support of the distribution due to the vacuum as bra vector. By induction we can now conclude that for $(\xi_i,\vec{p}_i)_{i=2}^{2n}$ in the support of the integrand, we find

$$\vec{p}_{2i} \in \sum_{j=1}^{i-1}(\operatorname{supp}\tilde{e}_j - \operatorname{supp}\tilde{f}_j) + \operatorname{supp}\tilde{e}_i \qquad \vec{p}_{2i-1} \in \sum_{j=1}^{i-1}(\operatorname{supp}\tilde{e}_j - \operatorname{supp}\tilde{f}_j)$$

Denote the maximal norm of $\sum_{j=1}^{n-1}(\text{supp } \tilde{e}_j - \text{supp } \tilde{f}_j)$ by r. For t sufficiently large we can show that[7] $|\xi_{2i-1}| \leq 2r$, using the support properties of $\partial_t^{a_i}\check{\mu}_{i,t}$, $\partial_t^{b_i}\check{\nu}_{i,t}$ and the mean value theorem. We though can change the distribution to

$$đ^{2n} \left\langle \Omega, \left(\prod_{i=1}^{n} E_{\xi_{2i-1},\vec{p}_{2i-1}} E_{B_{2r}(0)} A_i^* E_{\xi_{2i},\vec{p}_{2i}} B_i \right) \right\rangle \tag{3.2}$$

Let η be a function of compact support with $\{\eta(\epsilon \cdot x + k) | k \in \epsilon \cdot \mathbb{Z}^3\}$ is a family of partitions of unity, and let $A = \text{diam supp } \eta$. Now given $\epsilon > 0$ split the $\tilde{e}_i$ and $\tilde{f}_i$ with respect to this partition. Again by lemma 3.12 the summands with disjoint support in any $\tilde{e}_i$ and $\tilde{f}_i$ part decay rapidly in $|t|$ and yield no contribution. We find that $r \leq 2A\epsilon$ for every remaining summand.

Choosing a sufficiently small partition parameter ϵ – for a sufficiently large $|t_0|$ – we can change the measure in the way described in (3.2). Now since the vacuum is an isolated point in the energy momentum spectrum, we can take $E_0 = |\Omega\rangle\langle\Omega|$ since ϵ was sufficiently small. The error we obtained by neglecting the contribution of the summands with disjoint support in the $\tilde{e}_i$ and $\tilde{f}_i$ parts is again rapidly decreasing in $|t|$.
Now plugging in this identity and doing the inverse Fourier-transform, we receive

$$\prod_{i=1}^{m} \langle \Omega, \tilde{A}_{i,t}(e_i)^* \tilde{B}_{i,t}(f_i)\Omega \rangle + \mathcal{O}(|t|^{-q})$$

Since for every permutation $\sigma \neq \text{id}$ we have a i with $\text{supp}\tilde{e}_i \cap \text{supp}\tilde{f}_{\sigma(i)} = \emptyset$, the remaining summands are rapidly decreasing, we conclude.

$$\sum_{\sigma \in \mathfrak{S}_m} \prod_{i=1}^{m} \left\langle \Omega, \tilde{A}_{i,t}(e_i)^* \tilde{B}_{\sigma(i),t}(f_{\sigma(i)})\Omega \right\rangle + \mathcal{O}(|t|^{-q})$$

For a general collection of $e_1, \dots, e_n$ and $f_1, \dots, f_m$ we get – because of linearity of the expressions – the deserved result by recombining the partitions we made. □

7 The constant 2 is by no way optimal, one can also show that this is still true for all c with $0 < q < c$ for some fixed $q \in (0, 1)$

3.3 Scattering

3.3.1 Multi-particle states

It is reasonable to think of $A_t(f)$ as a creation operator in large time limit and of $A_t(f)^*$ as an annihilation operator, which is suggested by the clustering property and for sufficiently small κ can indeed be shown, see [Dyb05, Proposition 3.2]. To create multi-particle states we have to act several times by creation operators, of which we know scalarproducts by the clustering property.

Theorem 3.15. *Let $A_1, \dots A_n$ be regular operators and $f_1, \dots, f_n$ be regular positive energy solutions with disjoint momentum support. Then there exist the strong limits*

$$s - \lim_{t \to \pm\infty} A_{1,t}(f_1) \cdots A_{n,t}(f_n)\Omega$$

and they do neither depend on the ordering of the $A_{i,t}(f_i)$, nor on the concrete A_i and f_i, but only on the corresponding one-particle states, i.e.

$$\eta_{\substack{\text{in}\\\text{out}}}\left(P_1 A_1(f_1)\Omega \otimes \cdots \otimes P_1 A_n(f_n)\Omega\right) := s - \lim_{t \to \pm\infty} A_{1,t}(f_1) \cdots A_{n,t}(f_n)\Omega$$

is well defined and isometric, thus they extend to isometries $\Omega_{\substack{\text{in}\\\text{out}}} : \mathscr{F}_s(\mathscr{H}_1) \to \mathscr{H}_{\substack{\text{in}\\\text{out}}}$, where $\mathscr{H}_{\substack{\text{in}\\\text{out}}}$ is defined as the range of $\Omega_{\substack{\text{in}\\\text{out}}}$, which are denoted as the Møller operators. We call the limits asymptotic states.

Proof. In order to prove this we use an analogue to Cook's method and compute

$$\begin{aligned} &\partial_t A_{1,t}(f_1) \cdots A_{n,t}(f_n)\Omega \\ &= \sum_{k=1}^{n} \left(\prod_{j=1}^{n} (\partial_t^{\delta_{j,k}} A_{j,t}) \right) \Omega \end{aligned}$$

We use the clustering property, lemma 3.14, to estimate the summands in norm, we achieve

$$\left\| \left(\prod_{j=1}^{n} (\partial_t^{\delta_{j,k}} A_{j,t}) \right) \Omega \right\| = \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} \left\langle \partial_t^{\delta_{j,k}} A_{j,t})\Omega, \partial_t^{\delta_{\sigma(j),k}} A_{\sigma(j),t})\Omega \right\rangle + \mathcal{O}(|t|^{-q})$$

Since for $\sigma \neq \text{id}$ we have $\operatorname{supp}\tilde{f}_i \cap \operatorname{supp}\tilde{f}_{\sigma(i)} = \emptyset$ for at least one i, these terms are rapidly decreasing. Furthermore from proposition 3.9 the $A_{j,t}\Omega$ are strongly converging and are therefore bounded in norm for $t \to \pm\infty$. We are left with estimating $\|\partial_t A_{k,t}\Omega\|$

$$
\begin{aligned}
&\partial_t \int g_t(x_0) f_k(x) U(x) A_k \Omega \mathrm{d}x \\
=&\partial_t \int g\left(\frac{x_0 - t}{|t|^\kappa}\right) e^{-ix_0\sqrt{\vec{p}^2+m^2}} \tilde{f}_k(\vec{p}) U^{\vee(\vec{x})}(x_0, \vec{p}) A_k \Omega \,\frac{\mathrm{d}x_0}{|t|^\kappa}\, \mathrm{d}\vec{p} \\
=&\partial_t \int e^{it\xi} \check{g}(|t|^k \xi) \tilde{f}_k(\vec{p}) U^{\wedge(x_0)\vee(\vec{x})}(\xi + \sqrt{\vec{p}^2 + m^2}, \vec{p}) A_k \Omega \mathrm{d}\xi \mathrm{d}\vec{p} \\
=&\int \partial_t e^{it\xi} \check{g}(|t|^k \xi) \tilde{f}_k(\vec{p}) \; đE_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A_k \Omega \\
=&\int \xi e^{it\xi} \left(i\check{g}\left(\xi|t|^\kappa\right) + \kappa \operatorname{sgn}(t) |t|^{\kappa-1} \check{g}'(\xi|t|^\kappa) \right) \tilde{f}(\vec{p}) \; đE_{\xi+\sqrt{\vec{p}^2+m^2},\vec{p}} A\Omega
\end{aligned}
$$

The integrated functions vanish for $\xi = 0$ and therefore we can apply $(1 - P_1)$ on the p.v. measure. Moreover the braced part of the function is supported within $(-|t|^{-\kappa}, |t|^{-\kappa})$, which yields a bound for ξ, and is bounded with respect to t for $|t| > 1$. Now using the inequality (3.1) and Hölder again yields.

$$
\begin{aligned}
&\lesssim |t|^{-\kappa} \int_{-|t|^{-\kappa}}^{|t|^{-\kappa}} \left\| đE_{(\xi_n+\sqrt{\vec{p}_n^2+m^2},\vec{p}_n)} (1 - P_1) A_n \Omega \right\| \lesssim |t|^{-\kappa} \|E^{(m)}_{(m \mp |t|^{-\kappa})} (1 - P_1) A_n \Omega\| \\
&\lesssim |t|^{-\kappa-\kappa\cdot\eta} = |t|^{-\kappa(\eta+1)} \in \mathcal{L}^1(\mathbb{R})
\end{aligned}
$$

This finishes the Cook's method proof for the existence of strong limits.

The independence on the ordering is obvious from lemma 3.12. To achieve independence from the A_i and f_i, we use independence on the ordering and assume then w.l.o.g. some other A'_n and f'_n with $P_1 A'_n(f'_n)\Omega = P_1 A_n(f_n)\Omega$. Now using the clustering property, we get

$$
\begin{aligned}
&\lim_{t\to\pm\infty} \|A_{1,t}(f_1) \cdots A_{n,t}(f_n)\Omega - A_{1,t}(f_1) \cdots A'_{n,t}(f'_n)\Omega\|^2 \\
=&\prod_{i=1}^{n-1} \|P_1 A_i(f_i)\Omega\|^2 \cdot \Big(\|P_1 A_n(f_n)\Omega\|^2 - \langle P_1 A_n(f_n)\Omega, P_1 A'_n(f'_n)\Omega \rangle \\
&\qquad - \langle P_1 A'_n(f'_n)\Omega, P_1 A_n(f_n)\Omega \rangle + \|P_1 A'_n(f'_n)\Omega\|^2 \Big)
\end{aligned}
$$

Which is zero and by the clustering property again, we obtain isometry of the mapping η_{ex}, which yields the claim. □

3.3.2 Covariance

The theory established so far is strictly depending on the choice of the inertial frame. We will first consider ourselves with the issue of non boosting transformations, and then derive complete Poincaré invariance of 1-particle states. Finally we want to derive the following theorem

Theorem 3.16. *The set of asymptotic states is invariant under Poincaré transformations and the states transform under the transformations. I.e.*

$$\begin{aligned} &U(\Lambda)\eta_{\substack{\text{in}\\\text{out}}}\big(P_1 A_1(f_1)\Omega\otimes\cdots\otimes P_1 A_n(f_n)\Omega\big)\\ &=\eta_{\substack{\text{in}\\\text{out}}}\big(U(\Lambda)P_1 A_1(f_1)\Omega\otimes\cdots\otimes U(\Lambda)P_1 A_n(f_n)\Omega\big)\end{aligned}$$

I.e. the Møller operators intertwine the representation of $\mathscr{P}_+^\uparrow$.

But first we have to show, that the tools we developed so far are available in a Poincaré invariant manner. Therefore we consider transformed regular positive energy solutions.

Proposition 3.17. *If f is a regular positive energy solution, so for all $(a,\Lambda)\in\mathscr{P}_+^\uparrow$ is $(a,\Lambda)\circ f$. The momentum support is bijectively mapped under (a,Λ) transformation.*

Proof. The subgroup $\mathbb{R}^4\rtimes SO(3)$ is trivial, one has

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3}e^{-i\vec{x}\vec{p}}f(x_0-a_0,V^{-1}\vec{x}-\vec{a})\mathrm{d}\vec{x}&=e^{-i(x_0-a_0)\omega(V^{-1}\vec{p})}\tilde{f}(V^{-1}\vec{p})\cdot e^{-i\vec{a}(V^{-1}\vec{p})}\\ &=e^{-ix_0\omega(\vec{p})}\left(e^{i(a_0\omega(\vec{p})-\vec{a}(V^{-1}\vec{p}))}\tilde{f}(V^{-1}\vec{p})\right)\end{aligned}$$

So we compute what's $\tilde{f}_\Lambda$ for $f(\Lambda^{-1}x)$, if Λ is a boost in $s-$direction. We compute the full Fourier-transform, we use the notation $\vec{p}=(p_s,\vec{p}_r)$, where $\vec{p}_r$ is the remaining part of the spatial vector not in $s-$component, furthermore use $m_{\vec{p}_r}^2=\vec{p}_r^2+m^2$.

$$\begin{aligned}\hat{f}(\mathbf{p})=&\sqrt{2\pi}\delta(p_0+\omega(\vec{p}))\tilde{f}(\vec{p})\\ \Rightarrow\qquad f(\Lambda^{-1}x)^{\wedge(x)}(\mathbf{p})=&\sqrt{2\pi}\delta(\cosh\alpha p_0+\sinh\alpha p_s+\sqrt{(\cosh\alpha p_s+\sinh\alpha p_0)^2+m_{\vec{p}_r}^2})\\ &\cdot\tilde{f}(\cosh\alpha p_s+\sinh\alpha p_0,\vec{p}_r)\end{aligned}$$

We solve the equation coming from the delta distribution

$$(\cosh\alpha p_0 + \sinh\alpha p_s)^2 = (\cosh\alpha p_s + \sinh\alpha p_0)^2 + m_{\vec{p}_r}^2$$
$$p_0^2 = p_s^2 + m_{\vec{p}_r}^2 \qquad \Longrightarrow \qquad p_0 = \pm\omega(\vec{p})$$

Where from the original equation one sees, that the negative sign is to be obtained. The derivative of the expression at the solution is

$$\frac{1}{\phi(\alpha,\vec{p})} := \cosh\alpha + \sinh\alpha \frac{\cosh\alpha p_s + \sinh\alpha p_0}{\sqrt{(\cosh\alpha p_s + \sinh\alpha p_0)^2 + m_{\vec{p}_r}^2}}$$
$$= \frac{\cosh\alpha(\cosh\alpha p_0 + \sinh\alpha p_s) - \sinh\alpha(\cosh\alpha p_s + \sinh\alpha p_0)}{\cosh\alpha p_0 + \sinh\alpha p_s}$$
$$= \frac{p_0}{\cosh\alpha p_0 + \sinh\alpha p_s} = \frac{\omega(\vec{p})}{\cosh\alpha\omega(\vec{p}) - \sinh\alpha p_s} > 0$$

The positivity comes from $\cosh\alpha > |\sinh\alpha|$ and the fact that $\omega(\vec{p}) > |p_s|$. This we can insert in the above equation yielding

$$f(\Lambda^{-1}x)^{\wedge(x)}(\mathbf{p}) = \sqrt{2\pi}\delta(p_0 + \omega(\vec{p}))\tilde{f}(\cosh\alpha p_s + \sinh\alpha p_0, \vec{p}_r)\phi(\alpha,\vec{p})$$
$$\Rightarrow \qquad \widetilde{\Lambda\circ f}(\vec{p}) = e^{-ix_0\omega(\vec{p})}\tilde{f}(\cosh\alpha p_s - \sinh\alpha\omega(\vec{p}), \vec{p}_r)\phi(\alpha,\vec{p})$$

□

Using this, we can immediately deduce the following corollary.

Corollary 3.18. *The lemmata 3.11, 3.12 and 3.14 are still valid for the assertions made in transformed frames of references.*
I.e. we can interchange the x_0 dependence of the smearing functions by $(\Lambda x)_0$.

Proof for lemma 3.11. In the case of lemma 3.11 one has to generalize time evolution. One takes the cone $\bigcup_{t\in\mathbb{R}}\{t\} \times t\cdot A$. Intersecting the cone with the hyperplanes $(\Lambda x)_0 = t$ gives the correct surface of definition for the L^p estimate, which will correspond[8] to a $f(\Lambda x)_{|_{t,t\cdot A'}}$. □

8 up to length contraction, which changes the integrals by a constant factor

Proof for lemma 3.12 and 3.14. Respect that

$$\|[A,B]\| = \|U(\Lambda)[A,B]U(\Lambda^{-1})\| = \|[U(\Lambda)AU(\Lambda^{-1}),U(\Lambda)BU(\Lambda^{-1})]\|$$

$$\left\langle \prod_{i=1}^{n} A_i\Omega, \prod_{j=1}^{m} B_j\Omega \right\rangle = \left\langle U(\Lambda)\left(\prod_{i=1}^{n} A_i\right)U(\Lambda^{-1})\Omega, U(\Lambda)\left(\prod_{j=1}^{m} B_j\right)U(\Lambda^{-1})\Omega \right\rangle$$
$$= \left\langle \left(U(\Lambda)\prod_{i=1}^{n} A_i U(\Lambda^{-1}\right))\Omega, \left(\prod_{j=1}^{m} U(\Lambda)B_jU(\Lambda^{-1})\right)\Omega \right\rangle$$

Furthermore with the following calculation we are able to reduce the statements to the lemma.

$$U(\Lambda)\int q(t,x)k_t\left((\Lambda x)_0\right)f(x)A(x)\mathrm{d}xU(\Lambda^{-1})$$
$$= \int q(t,\Lambda^{-1}y)k_t\left(y_0\right)f(\Lambda^{-1}y)U(\Lambda)U(\Lambda^{-1}y)AU(-\Lambda^{-1}y)U(\Lambda^{-1})\mathrm{d}y$$
$$= \int (\Lambda\circ q)(t,y)k_t\left(y_0\right)(\Lambda\circ f)(y)\left(U(\Lambda)AU(\Lambda^{-1})\right)(y)\mathrm{d}y$$

□

We are now ready to prove the covariance theorem.

Proof of Theorem 3.16. We now want to show covariance of the construction operator for the subgroup of rotations and translations, $\mathbb{R}^3 \rtimes SO(3)$, i.e.

$$U(a,V)\left(\prod_{i=1}^{n} A_{i,t}(f_i)\Omega\right) = \prod_{i=1}^{n}(U(a,V)A_iU(a,V)^{-1})_t(V\circ f_i)\Omega$$

Or equivalently

$$\left(\prod_{i=1}^{n} A_{i,t}(f_i)\Omega\right) = U(a,V)^{-1}\prod_{i=1}^{n}(U(a,V)A_iU(a,V)^{-1})_t(V\circ f_i)\Omega$$
$$= U(V^{-1})U(-a)\prod_{i=1}^{n}\int g_t(x_i^{(0)})f_i(V^{-1}x_i)U(a)U(x_i)U(V)A_iU(V^{-1})U(-x_i)U(-a)\mathrm{d}x_i\Omega$$
$$= U(V^{-1})\prod_{i=1}^{n}\int g_t(x_i^{(0)})f_i(V^{-1}x_i)U(V)U(V^{-1}x_i)A_iU(-V^{-1}x_i)U(V^{-1})\mathrm{d}x_i\Omega$$
$$= \prod_{i=1}^{n}\int g_t((Vy_i)^{(0)})f_i(y_i)U(y_i)A_iU(-y_i)\mathrm{d}y_i\Omega$$

From this expression it is clear, that rotations and translations obey the equation, since they leave the time invariant.

Now we consider Λ in a one-parameter group – say boosts in s direction and we start with a one-particle state. Consider a suitable regular positive energy solution f_Λ such that

$$\tilde{f}_\Lambda(\vec{p}) = \tilde{f}(\cosh\alpha p_s - \sinh\alpha\omega(\vec{p}), \vec{p}_r)$$

and the operator $U(\Lambda)AU(\Lambda)^{-1}$: We start with

$$\begin{aligned}
&U(\Lambda^{-1})(U(\Lambda)AU(\Lambda^{-1}))_t(f_\Lambda)\Omega = \int g_t(x_0) f_\Lambda(x) U(\Lambda^{-1}x) A\Omega \mathrm{d}x \\
&= \int \Lambda^{-1} \circ (g_t(x_0) f_\Lambda(x)) U(x) A\Omega \mathrm{d}x \\
&= \int (g_t(x_0) f_\Lambda(x))^{\wedge(x)}((\Lambda^{-1})^t \mathbf{p}) \text{đ}E_{-p_0,\vec{p}} A\Omega \\
&= \int \check{g}_t(\sinh\alpha p_s - \cosh\alpha p_0 - \omega(\cosh\alpha p_s - \sinh\alpha p_0, \vec{p}_r)) \\
&\quad \cdot \tilde{f}_\Lambda(\cosh\alpha p_s - \sinh\alpha p_0, \vec{p}_r) \text{đ}E_{-p_0,\vec{p}} A\Omega \\
&= \int \check{g}_t(\cosh\alpha\xi + \sinh\alpha p_s - \omega(\cosh\alpha p_s + \sinh\alpha\xi, \vec{p}_r)) \\
&\quad \cdot \tilde{f}_\Lambda(\cosh\alpha p_s + \sinh\alpha\xi, \vec{p}_r) \text{đ}E_{\xi,\vec{p}} A\Omega \\
&= \int \check{g}_t(\cosh\alpha\xi + \sinh\alpha p_s - \omega(\cosh\alpha p_s + \sinh\alpha\xi, \vec{p}_r)) \\
&\quad \cdot \tilde{f}(\cosh\alpha(\cosh\alpha p_s + \sinh\alpha\xi) - \sinh\alpha\omega(\cosh\alpha p_s + \sinh\alpha\xi, \vec{p}_r), \vec{p}_r) \text{đ}E_{\xi,\vec{p}} A\Omega
\end{aligned}$$

Observing the integrand on the hypersurface $\xi = \omega(\vec{p})$ one discovers

$$\begin{aligned}
&\omega(\cosh\alpha p_s + \sinh\alpha\omega(\vec{p}), \vec{p}_r) \\
&= \sqrt{\cosh^2\alpha p_s^2 + 2\sinh\cosh\alpha p_s\omega(\vec{p}) + \sinh^2\alpha(p_s^2 + m_{\vec{p}_r}^2) + m_{\vec{p}_r}^2} \\
&= \sqrt{\cosh^2\alpha\omega(\vec{p})^2 + 2\sinh\cosh\alpha p_s\omega(\vec{p}) + \sinh^2\alpha p_s^2} = \cosh\alpha\omega(\vec{p}) + \sinh\alpha p_s
\end{aligned}$$

Therefore the integrand equals $\tilde{f}(\vec{p})/\sqrt{2\pi}$ on the mass shell. Using the $1 = (1 - P_1) + P_1$ identity, one achieves, that the P_1 part equals $P_1 A(f)\Omega$, as requested. Since $\tilde{f}$ and $\check{g}_t$ are bounded it suffices to show, that the support w.r.t the mass shrinks to m as $|t| \to \infty$, to receive the deserved identity for a one-particle state through estimating the remaining part towards zero. Define $\vec{q} = (\cosh\alpha p_s + \sinh\alpha\xi, \vec{p}_r)$ and $\zeta = \cosh\alpha\xi + \sinh\alpha p_s$, let $|t| \gg 0$

such that $\zeta - \omega(\vec{q}) \in (-\epsilon, \epsilon) \supseteq \mathrm{supp}\tilde{g}_t$. Now one achieves

$$\begin{aligned} \zeta &\in (\omega(\vec{q}) - \epsilon, \omega(\vec{q}) + \epsilon) \\ \zeta^2 &\in (\omega(\vec{q})^2 - 2\epsilon\omega(\vec{q}) + \epsilon^2, \omega(\vec{q})^2 + 2\epsilon\omega(\vec{q}) + \epsilon^2) \\ \zeta^2 - \omega(\vec{q})^2 = \xi^2 - \omega(\vec{p})^2 &\in (-2\epsilon\omega(\vec{q}) + \epsilon^2, +2\epsilon\omega(\vec{q}) + \epsilon^2) \end{aligned}$$

Defining $\mu = \zeta - \omega(\vec{q})$ we can observe

$$|\cosh\alpha(\omega(\vec{q}) - \mu) - \sinh\alpha\omega(\vec{q})| \geq e^{-|\alpha|}\omega(\vec{q}) - \cosh\alpha\mu$$

Since $\tilde{f}$ has compact support, we get with this expression a bound for $\omega(\vec{q})$, which – using 3.1 – yields that

$$\begin{aligned} &\lim_{t\to\pm\infty} U(\Lambda^{-1})(U(\Lambda)AU(\Lambda^{-1}))_t(f_\Lambda)\Omega = P_1 A(f)\Omega \\ \Rightarrow \qquad &\frac{\mathrm{d}}{\mathrm{d}\alpha} \lim_{t\to\pm\infty} U(\Lambda^{-1})(U(\Lambda)AU(\Lambda^{-1}))_t(f_\Lambda)U(\Lambda)\Omega = 0 \end{aligned}$$

Because the clustering property yields us a uniformal estimate, we can interchange time limiting and differentiating. We now want to derive the statement for multi-particle states. So observe

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\alpha} U(\Lambda^{-1}) \prod_{i=1}^{n} \left(U(\Lambda)AU(\Lambda^{-1})\right)_{i,t} (f_{i,\Lambda})\Omega \\ =&\frac{\mathrm{d}}{\mathrm{d}\alpha} \prod_{i=1}^{n} \int g_t((\Lambda y_i)^{(0)}) f_{i,\Lambda}(\Lambda y_i) U(y_i) A_i U(-y_i) \mathrm{d}y_i \Omega \\ =&\sum_{j=1}^{n} \prod_{i=1}^{n} \int \left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^{\delta_{i,j}} g_t((\Lambda y_i)^{(0)}) f_{i,\Lambda}(\Lambda y_i) U(y_i) A_i U(-y_i) \mathrm{d}y_i \Omega \end{aligned}$$

By lemma 3.14 we can estimate the norm of a summand by a constant times the derivative of a single particle state. Since the derivative of a single particle state vanish for large times, the expression is constant. For $\Lambda = 1$ we have.

$$\lim_{t\to\pm\infty} U(\Lambda^{-1}) \prod_{i=1}^{n} \left(U(\Lambda)AU(\Lambda^{-1})\right)_{i,t} (f_{i,\Lambda})\Omega = \lim_{t\to\pm\infty} \prod_{i=1}^{n} A_{i,t}(f_i)\Omega$$

And since the left hand side is constant, this is true for every Λ, a boost in s-direction. After combining the results on the generating subgroups, this yields the claim. □

4 Conclusions

We have constructed the scattering theory in two different formulations of quantum field theory. First in the formulation of the Gårding-Wightman axioms, describing the field as an operator valued distribution. Here the construction needed an abundance of mathematical theorems about tempered distributions, Lorentz-invariant measures, multiplicators and more, all developed or cited in [Spr13]. We though had strong restrictions on the mass spectrum an upper and lower mass gap but there was no necessity in taking regular wave packets with disjoint momentum support.
The second approach was the formulation of algebraic quantum field theory with the Haag-Kastler axioms, describing the fields as local algebras. Here we could use a more general setting, without a strictly mass gap. We used the stability condition of Herbst [Her71] and an isolated vacuum as the assumptions on our field. The particles of the asymptotic states should have disjoint velocity support here.

It seems that the approach of the Haag-Kastler axioms is simpler, since we did not need an extended knowledge about tempered distributions to achieve strong results. The key ingredient here was the disjointness of the velocity supports. With this we could easily show asymptotic commutativity and the clustering property, which became our primary tool for everything following thereafter. This assumption might also extend the applications of the Gårding-Wightman framework.

The developed theorem 3.15 states that in the algebraic framework particles behave like free particles for large times. With theorem 3.16 we see, that the formulation is Poincaré invariant, as it should be. The same statement is derived by theorem 2.20 on the scattering theory in the Gårding-Wightman framework.
Of course one would not expect asymptotic completeness (i.e. $\mathscr{H}_{\text{in}} = \mathscr{H}_{\text{out}} = \mathscr{H}$), since we are only describing a single kind of particle yet. In the algebraic framework this can be resolved, if one defines the analogue construction for various particles. Thus one can

construct asymptotic states of a mixture of different particles in the previously defined way.

Assuming asymptotic completeness, the S-matrix can be defined by

$$S = \Omega_{\text{out}}^{-1} \Omega_{\text{in}}$$

This operator inherits the whole physical statements on scattering. Displaying it as an integral kernel acting on multi-particle wave functions, one can read off the scattering amplitudes whose absolute squares give rise to scattering probabilities and cross sections. Since S is a closed unitary operator and the wave functions with disjoint velocity support are dense one can not see the restriction of disjoint velocity support any more, thus this restriction is only due to the construction and has no physical relevance.

All this confirms that both formulations we made are physically meaningful, as this is, what we observe in collider experiments, especially the Fock-space interpretation of the asymptotic states is a key part.

In general the isolated vacuum condition can be relaxed, which was shown in [Dyb05]. We outline here, that the stability condition, i.e. regularity of the operators was only needed for existence of the time limits, the clustering property etc. only needed locality. Therefore a more general approach might be possible. Dybalski also gave an example of a generalized free field, where the regularity condition was not satisfied by a single operator, but of course a scattering theory could be developed.

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