

Dynamics of inelastically colliding spheres with Coulomb friction: Relaxation of translational and rotational energy

Olaf Herbst, Martin Huthmann, Annette Zippelius

Abstract We investigate the free cooling of inelastic rough spheres in the presence of Coulomb friction. Depending on the coefficients of normal restitution ϵ and Coulomb friction μ , we find qualitatively different asymptotic states. For nearly complete normal restitution (ϵ close to 1) and large μ , friction does not change the cooling properties qualitatively compared to a constant coefficient of tangential restitution. In particular, the asymptotic state is characterized by a constant ratio of rotational and translational energies, both decaying according to Haff's law. However, for small ϵ and small μ , the dissipation of rotational energy is suppressed, so that the asymptotic state is characterized by constant rotational energy while the translational energy continues to decay as predicted by Haff's law. Introducing either surface roughness for grazing collisions or cohesion forces for collisions with vanishing normal load, causes the rotational energy to decay according to Haff's law again in the asymptotic long-time limit with, however, an intermediate regime of approximately constant rotational energy.

Key words Kinetic theory, Rough spheres, Coulomb friction

1 Introduction

Impact properties of small grains have been measured by several groups [1,2]. The experimental data is frequently parametrized using a simple model introduced by Walton [3]. The model involves three parameters; the first one, ϵ characterizes the incomplete restitution of the normal component of the relative velocity of the contact point, denoted by \mathbf{g} . The second one, Coulomb's coefficient of friction μ describes the reduction of the tangential component of \mathbf{g} due to sliding, while the third parameter, β_0 accounts for the incomplete restitution of the tangential component of \mathbf{g} for sticking contacts. All three parameters have been measured experimentally for various materials

making a well calibrated model available for theoretical investigations.

2 Binary collisions

We have previously investigated the free cooling of rough spheres [4,5] and needles [6] using the formalism of a Pseudo-Liouville operator. In this paper we extend the analysis to include Coulomb friction. We briefly recall the collision rules for two spheres of equal diameter a , mass m and moment of inertia I . The unit-vector from the center of sphere two (\mathbf{r}_2) to the center of sphere one (\mathbf{r}_1) is denoted by $\hat{\mathbf{n}} := (\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|$. Center-of-mass velocities and angular velocities before collision are denoted by $\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Post-collisional quantities are primed. The relative velocity of the contact point is given by $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2 + \frac{a}{2}\hat{\mathbf{n}} \times (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$. The relative velocity after collision is given by

$$\hat{\mathbf{n}} \cdot \mathbf{g}' = -\epsilon(\mathbf{g}, \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \mathbf{g}) \quad \text{with } \epsilon(\mathbf{g}, \hat{\mathbf{n}}) \in [0, 1] \quad (1)$$

$$\hat{\mathbf{n}} \times \mathbf{g}' = -\beta(\mathbf{g}, \hat{\mathbf{n}})(\hat{\mathbf{n}} \times \mathbf{g}) \quad \text{with } \beta(\mathbf{g}, \hat{\mathbf{n}}) \in [-1, 1] \quad (2)$$

where $\epsilon(\mathbf{g}, \hat{\mathbf{n}})$ and $\beta(\mathbf{g}, \hat{\mathbf{n}})$ are the coefficients of restitution, which in general depend on \mathbf{g} and $\hat{\mathbf{n}}$. We assume ϵ to be constant and allow β to depend on the angle γ between \mathbf{g} and $\hat{\mathbf{n}}$ in order to account for the different energy loss mechanisms of sliding and sticking contacts. The impact angle satisfies $\gamma \in [\frac{\pi}{2}, \pi]$, so that $\cos \gamma = \hat{\mathbf{n}} \cdot \mathbf{g}/|\mathbf{g}| < 0$.

The two constitutive Eqs. (1), (2) plus the conservation laws for linear and angular momenta determine the post-collisional velocities

$$\begin{aligned} \mathbf{v}'_1 &= \mathbf{v}_1 + \Delta \mathbf{v}, & \mathbf{v}'_2 &= \mathbf{v}_2 - \Delta \mathbf{v}, \\ \boldsymbol{\omega}'_1 &= \boldsymbol{\omega}_1 + \Delta \boldsymbol{\omega}, & \boldsymbol{\omega}'_2 &= \boldsymbol{\omega}_2 + \Delta \boldsymbol{\omega} \end{aligned} \quad (3)$$

where

$$\begin{aligned} \Delta \mathbf{v} &= -\frac{(1+\epsilon)}{2}(\hat{\mathbf{n}} \cdot \mathbf{v}_{12})\hat{\mathbf{n}} - \eta \hat{\mathbf{n}} \times \left(\mathbf{v}_{12} \times \hat{\mathbf{n}} + \frac{a}{2}\boldsymbol{\omega}_{12} \right) \\ \Delta \boldsymbol{\omega} &= \frac{2\eta}{qa} \left(\hat{\mathbf{n}} \times \mathbf{v}_{12} + \frac{a}{2}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\omega}_{12}) \right) \end{aligned} \quad (4)$$

with $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$, $\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$, $\eta = \eta(\gamma) := q(1 + \beta(\gamma))/(2(1+q))$, and $q := (4I)/(ma^2)$ ($q = 0.4$ for homogeneous spheres).

Sliding contacts are governed by Coulomb friction, giving rise to an impact angle dependent coefficient of tangential restitution

$$\beta^*(\gamma) = -1 - \frac{1+q}{q}(1+\epsilon)\mu \cot \gamma \quad (5)$$

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Sliding, however, only occurs for impact angles $\gamma < \gamma_0$. For higher values one expects sticking which is governed by a constant coefficient of tangential restitution $-1 < \beta_0 \leq 1$ such that β_0 is strictly > -1 . In agreement with Walton [3] we assume that either sliding or sticking occurs in any single collision, never both. We require $\beta(\gamma)$ to be continuous and obtain for the limiting angle $\gamma_0 = \gamma_0(\epsilon, \beta_0, \mu)$

$$\cot \gamma_0 = -\frac{q}{1+q} \frac{1+\beta_0}{1+\epsilon} \frac{1}{\mu} \quad (6)$$

so that

$$\beta(\gamma) = \min \left\{ \beta_0, -1 - \frac{1+q}{q} (1+\epsilon) \mu \cot \gamma \right\} \quad (7)$$

As shown in Fig.(1), $\mu \rightarrow 0$ corresponds to smooth spheres and $\mu \rightarrow \infty$ amounts to constant tangential restitution.

3

Free Cooling of the many particle system

3.1

Analytical Theory

We consider a system of N classical particles confined to a 3-dimensional volume V interacting through a hard-core potential. The time evolution of a dynamic variable $A(t) = A(\{\mathbf{r}_k(t), \mathbf{v}_k(t), \boldsymbol{\omega}_k(t)\})$ is determined by a pseudo-Liouville operator \mathcal{L}_+ for $t > 0$

$$A(t) = \exp(i\mathcal{L}_+ t) A(0) . \quad (8)$$

The pseudo-Liouville operator \mathcal{L}_+ consists of two parts $\mathcal{L}_+ = \mathcal{L}_0 + \mathcal{L}'_+$. The first one, \mathcal{L}_0 describes the free streaming of particles

$$\mathcal{L}_0 = -i \sum_k \mathbf{v}_k \cdot \nabla_{\mathbf{r}_k} , \quad (9)$$

and the second one, $\mathcal{L}'_+ = \frac{1}{2} \sum_{k \neq l} T_+^{kl}$ describes hard-core collisions of two particles

$$T_+^{kl} = i(\mathbf{v}_{kl} \cdot \hat{\mathbf{r}}_{kl}) \Theta(-\mathbf{v}_{kl} \cdot \hat{\mathbf{r}}_{kl}) \delta(|\mathbf{r}_{kl}| - a) (b_+^{kl} - 1) . \quad (10)$$

The operator b_+^{kl} replaces the linear and angular momenta of two particles k and l before collision by the corresponding ones after collision, according to Eqs. (3). $\Theta(x)$ is the

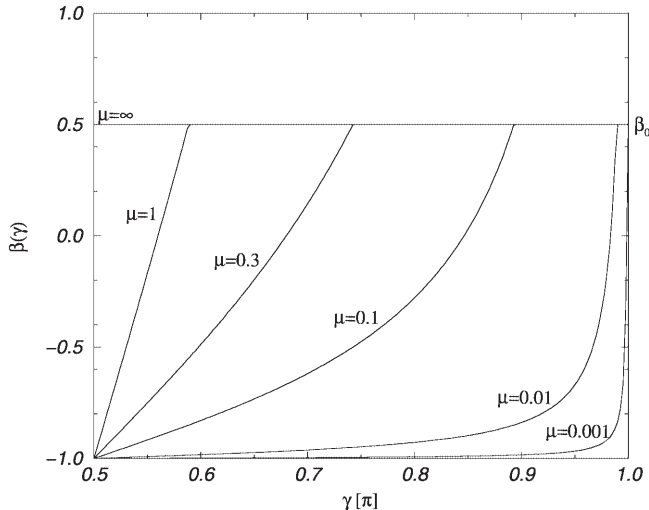


Fig. 1. $\beta(\gamma)$ for different values of μ ; $\epsilon = 0.5$, $\beta_0 = 0.5$

Heaviside step-function, and we have introduced the notation $\mathbf{r}_{kl} = \mathbf{r}_k - \mathbf{r}_l$ and $\hat{\mathbf{r}}_{kl} = \mathbf{r}_{kl}/|\mathbf{r}_{kl}|$. Equation (10) has a simple interpretation. The factor $\mathbf{v}_{kl} \cdot \hat{\mathbf{r}}_{kl}$ gives the flux of incoming particles. The Θ - and δ -functions specify the conditions for a collision to take place. A collision between particles k and l happens only if the two particles are approaching each other which is ensured by $\Theta(-\mathbf{v}_{kl} \cdot \hat{\mathbf{r}}_{kl})$. At the instant of a collision the distance between the two particles has to vanish expressed by $\delta(|\mathbf{r}_{kl}| - a)$. Finally, $(b_+^{kl} - 1)$ generates the change of linear and angular momenta according to Eq. (3).

The ensemble average of a dynamic variable is defined by

$$\begin{aligned} \langle A \rangle_t &= \int d\Gamma \rho(0) A(t) = \int d\Gamma \rho(t) A(0) \\ &= \int \prod_k (d^3 r_k d^3 v_k d^3 \omega_k) \rho(t) A(0) . \end{aligned} \quad (11)$$

Here $\rho(t) = \exp(-i\mathcal{L}'_+ t) \rho(0)$ is the N -particle distribution function, whose time development is governed by the adjoint \mathcal{L}'_+ of the time evolution operator \mathcal{L}_+ . Differentiating equation (11) with respect to time we get

$$\begin{aligned} \frac{d}{dt} \langle A \rangle_t &= \int d\Gamma \rho(0) \frac{d}{dt} A(t) = \int d\Gamma \rho(0) i\mathcal{L}_+ A(t) \\ &= \int d\Gamma \rho(0) \exp(i\mathcal{L}_+ t) i\mathcal{L}_+ A(0) \\ &= \int d\Gamma \rho(t) i\mathcal{L}_+ A(0) = \langle i\mathcal{L}_+ A \rangle_t \end{aligned} \quad (12)$$

We are interested in the average translational and rotational energies per particle

$$E_{\text{tr}} = \frac{1}{N} \sum_i \frac{m}{2} \mathbf{v}_i^2 \quad (13)$$

$$E_{\text{rot}} = \frac{1}{N} \sum_i \frac{I}{2} \boldsymbol{\omega}_i^2 \quad (14)$$

as well as the total kinetic energy $E = E_{\text{tr}} + E_{\text{rot}}$.

We assume a *homogeneous cooling state* (HCS) and approximate the N -particle distribution function by a Gaussian

$$\rho(t) \propto \prod_{k < l} \Theta(|\mathbf{r}_{kl}| - a) \exp \left\{ - \left(\frac{E_{\text{tr}}}{T_{\text{tr}}(t)} + \frac{E_{\text{rot}}}{T_{\text{rot}}(t)} \right) \right\} \quad (15)$$

where the product of Heaviside functions accounts for the excluded volume. The state of the system depends on time only through the average translational and rotational kinetic energies. Hence its full time dependence (within the HCS approximation) is determined by two coupled differential equations for $T_{\text{tr}}(t)$ and $T_{\text{rot}}(t)$

$$\frac{3}{2} \frac{d}{dt} T_{\text{tr}}(t) = \frac{d}{dt} \langle E_{\text{tr}} \rangle_t = \langle i\mathcal{L}_+ E_{\text{tr}} \rangle_t \quad (16)$$

$$\frac{3}{2} \frac{d}{dt} T_{\text{rot}}(t) = \frac{d}{dt} \langle E_{\text{rot}} \rangle_t = \langle i\mathcal{L}_+ E_{\text{rot}} \rangle_t \quad (17)$$

The expectation values on the right hand side can be calculated for the HCS state. We obtain

$$\begin{aligned} \frac{1}{\nu} \frac{d}{dt} T_{\text{tr}}(t) = & -T_{\text{tr}}^{3/2} \left\{ \frac{1 - \epsilon^2}{4} - \eta_c^2 \frac{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}}}{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cos^2 \gamma_0} \right. \\ & + \frac{\eta_c}{2} \left(\frac{\sin \gamma_0}{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cos^2 \gamma_0} \right. \\ & \left. \left. + \frac{\arctan \left(\sqrt{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cot \gamma_0} \right)}{\sqrt{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cos \gamma_0}} \right) \right\} \quad (18) \end{aligned}$$

$$\begin{aligned} \frac{1}{\nu} \frac{d}{dt} T_{\text{rot}}(t) = & +T_{\text{tr}}^{3/2} \frac{\eta_c}{2q} \left\{ \frac{2\eta_c(1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}})}{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cos^2 \gamma_0} - \frac{T_{\text{rot}}}{T_{\text{tr}}} \right. \\ & \left. \times \left(\frac{\sin \gamma_0}{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cos^2 \gamma_0} + \frac{\arctan \left(\sqrt{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cot \gamma_0} \right)}{\sqrt{1 + \frac{T_{\text{rot}}}{qT_{\text{tr}}} \cos \gamma_0}} \right) \right\} \end{aligned}$$

where $\nu = 16/3\sqrt{\pi/ma^2n_0}g(a)$ sets the time scale, $\eta_c = \eta_0 \sin \gamma_0 = [q\{1 + \beta_0\}/\{2(1 + q)\}] \sin \gamma_0$, $g(a)$ denotes the pair correlation at contact, and $n_0 = N/V$. γ_0 is a function of ϵ , μ , β_0 , and the mass distribution q within the spheres, see Eq. (6). Details of the calculation are given in the appendix.

Rotational energy is conserved only in the perfectly smooth case characterized by ($\beta_0 = -1$) or ($\mu = 0$). Translational energy is conserved only if in addition $\epsilon = 1$. The total energy is conserved if both translational and rotational energies are conserved or in the perfectly rough case ($\mu = \infty \wedge \beta_0 = +1$) with complete normal restitution $\epsilon = 1$. For all other values of the parameters ϵ , β_0 , and μ the translational and rotational energies decrease with time. In the limit $\gamma_0 \rightarrow \frac{\pi}{2}$ (corresponding to $\mu \rightarrow \infty$, e.g. constant coefficients of restitution) Eqs. (18) reduce to the ones obtained by [4, 5].

3.2 Simulations

Simulations are performed using an event-driven algorithm where the particles follow an undisturbed translational and rotational motion until a collision occurs. In a collision, the particles' velocities just after contact are computed using the velocities just before contact as stated in Eqs. (3). To accelerate the simulations we use the algorithm of Lubachevsky [7] and a linked cell structure, which allows us to look for collision partners in the neighborhood of a given particle only.

To obtain a well-defined initial configuration we start the simulation on a regular lattice with random velocities chosen from a Boltzmann distribution and zero angular velocities. To equilibrate the system we choose $\epsilon = 1$ and $\beta_0 = -1$ corresponding to perfectly *smooth* spheres and let the simulation run for 200 collisions per particle. Then ϵ , μ , and β_0 are switched to their desired values.

To circumvent the problem of inelastic collapse, i.e. the time between two collisions becomes too short to be resolved properly, we use the t_c -model [8]: if the time between a collision and the preceding one for at least one

particle is smaller than a critical value t_c , the collisions parameters are set to their elastic values.

We are not primarily interested in phenomena like shear and cluster instabilities, but want to investigate how friction effects the cooling properties in the rapid flow regime. Hence we simulate dilute systems and aim at good statistics. We perform simulations of 3250 particles with a volume fraction $\rho = \frac{4\pi}{3}a^3\frac{N}{V} = 0.101$. For all plots we introduce the dimensionless temperatures $T = T_{\text{tr}}/T_{\text{tr}}(0)$ and $R = T_{\text{rot}}/T_{\text{tr}}(0)$ and a dimensionless time $\tau = \nu\sqrt{T_{\text{tr}}(0)}t$. The pair correlation function at contact, $g(a)$ is computed using the Carnahan-Starling formula in 3D [9]:

$$g(a) = \frac{1 - \rho/2}{(1 - \rho)^3}. \quad (19)$$

All data presented here corresponds to initially non-rotating ($R(0) = 0$, $T(0) = 1$) homogeneous spheres ($q = 0.4$).

3.3 Comparison of analytical theory and numerical simulations

The most surprising result for the model with impact angle dependent tangential restitution is a transition which separates two phases with different asymptotic decays of translational and rotational energies. For large μ and nearly complete normal restitution (ϵ close to 1), we observe cooling properties which are very similar to those obtained in the model with constant tangential restitution corresponding to the limit of infinite μ . The asymptotic state is characterized by a constant ratio of rotational and translational energy, both decaying according to Haff's law [10]. The time dependence of the translational and rotational energies for a typical set of parameters ($\epsilon = 0.9$, $\mu = 0.5$, and $\beta_0 = 0.5$) is shown in Fig. 2. The numerical solution of Eqs. (18) is compared to simulations and good agreement is found for the whole cooling range. For short times there is a linear change of both temperatures, whereas in the asymptotic state both temperatures decay like t^{-2} according to Haff's law. In the asymptotic state the ratio of rotational and translational temperature is constant in time.

For small μ and small ϵ the rotational energy remains constant in time (after an initial increase for small initial $R(0)$). The translational energy decays according to Haff's law. An example is shown in Fig. 3 for $\epsilon = 0.3$ and $\mu = 0.2$. This, at first surprising result can be explained quite easily: Coulomb's law of friction yields only very small friction for small normal loads. So, when the spheres lose large amounts of their translational energy (small ϵ) but only a tiny bit of their rotational energy (small μ) the system develops towards a state in which hardly any more rotational energy is lost because the grains only suffer impacts with very small normal load. Hence the *asymptotic* state resembles that of smooth spheres: The system consists of a finite fraction of rotating particles at rest. To discuss this state analytically we expand Eqs. (18) for small μ , which implies γ_0 close to π . We set $\gamma_0 = \pi - \delta$ and expand to leading order in δ

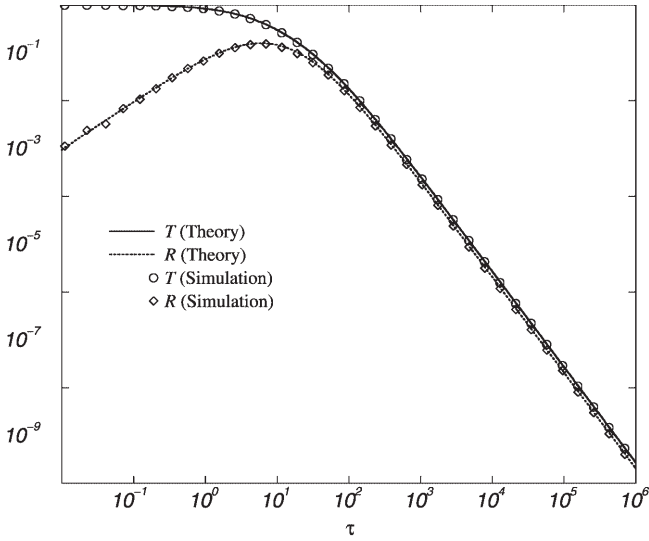


Fig. 2. Decay of translational and rotational energy for a set of parameter values ($\epsilon = 0.9$, $\mu = 0.5$, $\beta_0 = 0.5$) in the Haff regime; results of the approximate analytical theory are compared to data from simulations

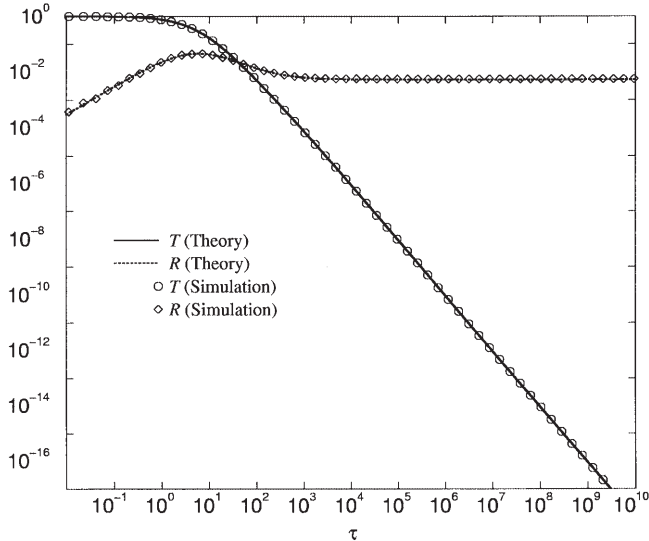


Fig. 3. Comparison of analytical theory and simulations for a set of parameters $\epsilon = 0.3$, $\mu = 0.2$, and $\beta_0 = 0.5$ such that the rotational energy survives and only the translational energy decays like t^{-2} (Haff's law)

$$\frac{1}{\nu} \frac{d}{dt} T_{\text{tr}}(t) = -T_{\text{tr}}^{3/2} \left(\frac{1 - \epsilon^2}{4} \right) \quad (20)$$

$$\frac{1}{\nu} \frac{d}{dt} T_{\text{rot}}(t) = -\frac{\pi\delta(1 + \beta_0)}{8(1 + q)} \frac{T_{\text{tr}} T_{\text{rot}}}{\sqrt{T_{\text{tr}} + T_{\text{rot}}/q}} \quad (21)$$

In leading order, the translational energy is decoupled from the rotational energy and equal to the one for smooth spheres. The solution of the second equation is given by

$$\sqrt{T_{\text{rot}}(t)} = \text{const} + \frac{4(1 + \beta_0)\pi\delta\sqrt{q}}{\nu(1 - \epsilon^2)^2(1 + q)} \frac{1}{t} \quad (22)$$

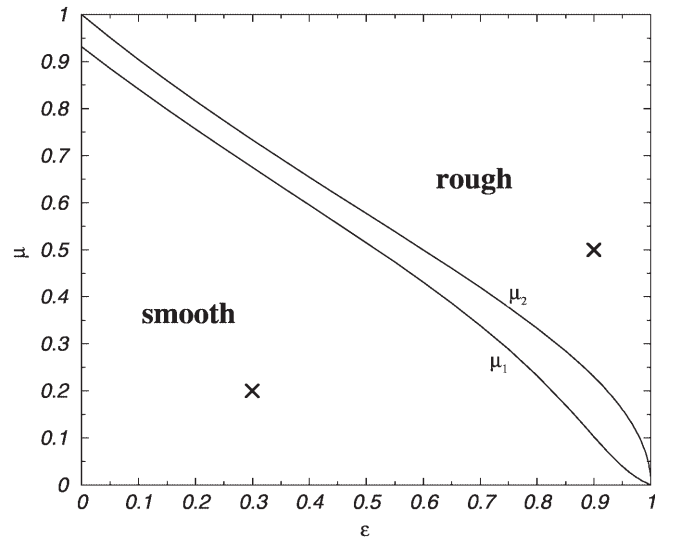


Fig. 4. Transition lines $\mu_1(\epsilon, \beta_0)$ and $\mu_2(\epsilon)$ for $\beta_0 = 0.5$. While μ_2 is independent of β_0 , μ_1 has a weak dependence on β_0 . The intermediate phase lying between μ_1 and μ_2 is largest for $\beta_0 = 1$, and $\mu_1 \rightarrow \mu_2$ as $\beta_0 \rightarrow 1$. For $\beta_0 = 1$ the curve for μ_1 lies less than 0.01 below the one for $\beta_0 = 0.5$. We have marked the two sets of parameters used in Fig. 2 and Fig. 3

The constant is difficult to evaluate because it depends on the time scale of the crossover to the asymptotic regime and on the values of T_{tr} and T_{rot} on this time scale. Analytical theory and simulation agree well in this range of parameters, too. These two regimes with qualitatively different long time behavior are indicated in Fig. 4 as rough and smooth. Why there are *two* critical values μ_1 and μ_2 will be explained in the following discussion.

To locate the transition between these two phases, we investigate the following question: For which range of parameters do Eqs. (18) allow for a solution with a constant ratio of rotational to translational energy? We plug the ansatz $k = T_{\text{rot}}/T_{\text{tr}} = R/T$ into Eqs. (18) and use $k = \frac{dT_{\text{rot}}/dt}{dT_{\text{tr}}/dt}$. Introducing $x := \sqrt{1 + \frac{k}{q}}$ we obtain a function $f(x)$ whose zeros are possible solutions for an asymptotic state

$$f(x) = 2\eta_0^2 \frac{x^2}{1 + x^2 \cot^2 \gamma_0} \left(1 - \frac{1}{q^2(x^2 - 1)} \right) + \eta_0 \frac{1 - q}{q} \times \left(\frac{1}{1 + x^2 \cot^2 \gamma_0} + \frac{\arctan(x \cot \gamma_0)}{x \cot \gamma_0} \right) - \frac{1 - \epsilon^2}{2} \quad (23)$$

In the limit $\mu \rightarrow \infty$ ($\gamma_0 \rightarrow \frac{\pi}{2}$), the equation $f(x_0) = 0$ reduces to a quadratic one for which exactly one positive zero exists. We obtain $k = X + \sqrt{X^2 + 1}$ where

$$X = \frac{q}{2\eta_0^2} \left(\frac{1 - \epsilon^2}{4} + \eta_0^2 \frac{1 - q^2}{q^2} - \eta_0 \frac{1 - q}{q} \right) \quad (24)$$

in agreement with [4, 5, 11, 12]. For $0 < \mu < \infty$ and $\beta_0 \neq -1$ we get $0 < |\cot \gamma_0| < \infty$. For $x > 1$, consider $f(x)$ in the limits $x \rightarrow 1$ and $x \rightarrow \infty$.

$$f(x) \xrightarrow{x \rightarrow 1} -\infty \quad (25)$$

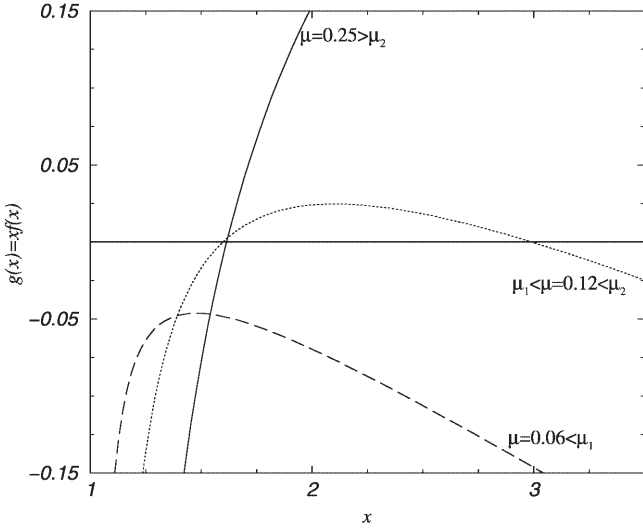


Fig. 5. Plot of the function $g(x)$ whose zeros are possible asymptotic states for different values of μ . ($\epsilon = 0.9$, $\beta_0 = 0.5$). In this case $\mu_1 = 0.10237$ and $\mu_2 = 0.22942$. For $\mu \gtrsim \mu_2$, $g(x \rightarrow \infty) \rightarrow \pm\infty$

$$\lim_{x \rightarrow \infty} f(x) = \frac{2\eta_0^2}{\cot^2 \gamma_0} - \frac{1 - \epsilon^2}{2} = \frac{(1 + \epsilon)^2}{2} \mu^2 - \frac{1 - \epsilon^2}{2} \quad (26)$$

From Eqs. (25), (26) we see that at least one solution x_0 for $f(x_0) = 0$ and hence for a constant asymptotic ratio $R/T = k = q(x_0^2 - 1)$ exists if

$$\mu > \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} =: \mu_2(\epsilon) \quad (27)$$

This critical value is independent of β_0 . To find out if there is more than one solution, and if there are solutions even if Eq. (27) is violated we take a look at $g(x) := xf(x)$. Since $x > 1$, x_0 is a zero of $f(x)$ if and only if x_0 is a zero of $g(x)$. For a given β_0 there are three qualitatively different shapes of $g(x)$ depending on μ (see Fig. 5). For small μ ($\mu < \mu_1$), $g(x)$ has no zero. For $\mu_1 < \mu < \mu_2$, $g(x)$ has two zeros, and for $\mu > \mu_2$, $g(x)$ has one zero. When $g(x)$ has two zeros only the *smaller* one serves as an asymptotic ratio, and only if the initial value $R(0)/T(0)$ is smaller than the *greater* zero the system behaves like in the regime where no solution for an asymptotic ratio exists, that means the rotational energy survives.

The critical lines $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ are shown in Fig. 4 for $\beta_0 = 0.5$. μ_2 is given by Eq. (27) and μ_1 is evaluated numerically. To conclude, we observe three different phases: 1) a rough phase with a constant ratio of translational and rotational energies, b) a smooth phase with constant rotational energy and c) an intermediate phase where the asymptotic state is determined by the initial value of $T_{\text{tr}}(0)/T_{\text{rot}}(0)$. The intermediate phase is largest for $\beta_0 = 1$, and $\mu_1 \rightarrow \mu_2$ as $\beta_0 \rightarrow 1$.

For $\beta_0 = 0.5$ the asymptotic ratio R/T (zero of Eq. (23)) is shown in Fig. 6 as a function of ϵ for various μ , $0.01 \leq \mu \leq \infty$. For $\mu > \mu_1(\epsilon = 0)$ the asymptotic state is characterized by a constant ratio of rotational and translational energies for *all* values of ϵ . When μ is decreased to smaller values, we find an asymptotically constant

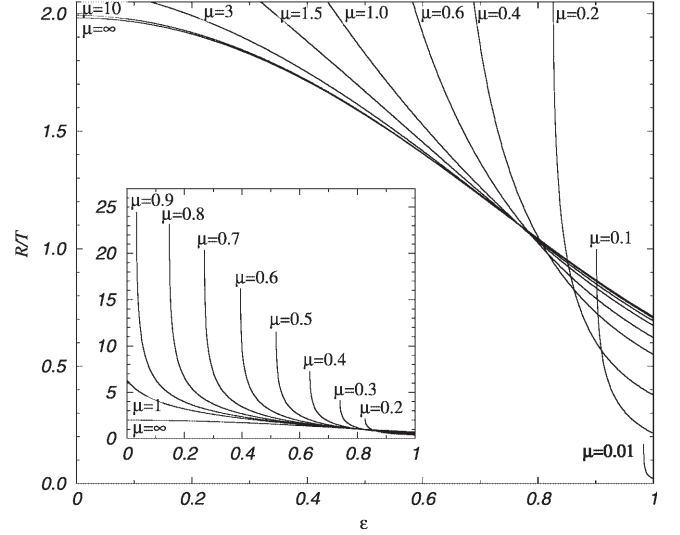


Fig. 6. Asymptotic ratio R/T as a function of ϵ for different values of μ , $\beta_0 = 0.5$. For large μ there exists a constant ratio for all $0 \leq \epsilon \leq 1$. For small μ , however, there is a critical ϵ such that for smaller ϵ the rotational energy survives while the translational energy decays like t^{-2}

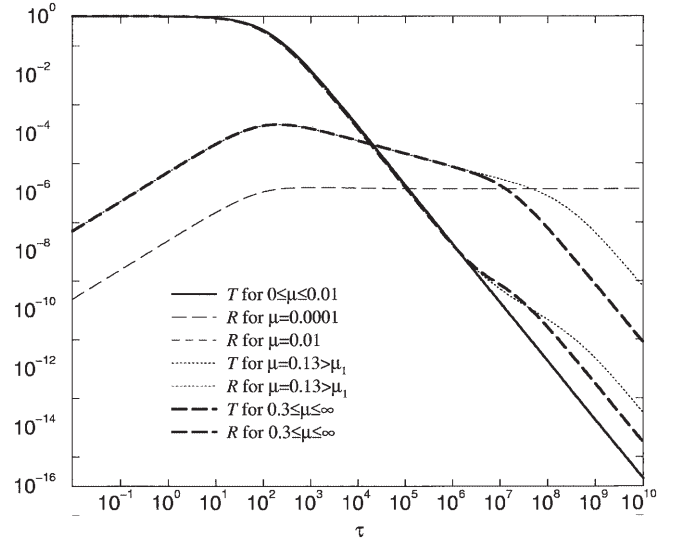


Fig. 7. Crossover between the two phases with different long time asymptotics as a function of μ for fixed $\epsilon = 0.97$ and $\beta_0 = -0.99$. $\mu_1 = 0.12340$. (Analytical data.)

ratio only for sufficiently large ϵ . We have chosen $R(0) = 0$ so that there is only one transition at μ_1 and we do not observe the intermediate phase, in which the asymptotic state depends on the choice of the initial condition for $R(0)$.

In Fig. 7 we show the crossover between the two phases with different long time asymptotics for the rotational energy. The coefficients of normal and tangential restitution are fixed ($\epsilon = 0.97$, $\beta_0 = -0.99$) while μ is varied over the full range $0 < \mu < \infty$. For small μ the translational energy is almost the same as that for smooth spheres and the rotational energy is constant for long times. For $\mu > \mu_1$ the rotational energy decays, the sooner the larger μ . The decay of the translational energy is slowest for $\mu \rightarrow \mu_1^+$.

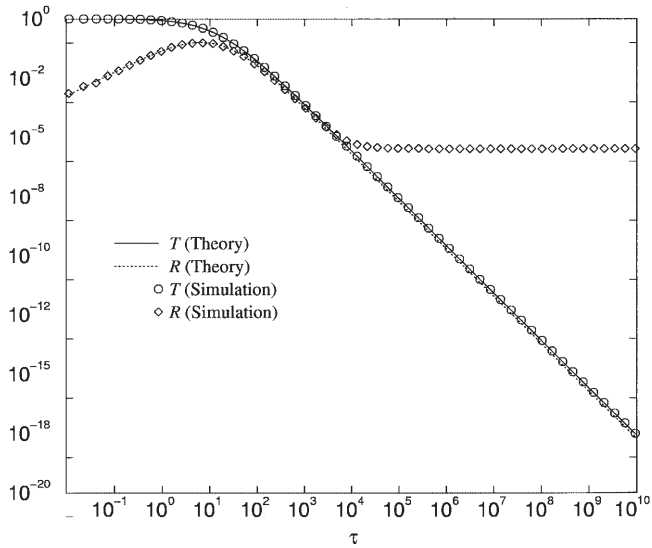


Fig. 8. Intermediate regime of parameters ($\epsilon = 0.87$, $\mu = 0.25$, $\beta_0 = 0.43$) corresponding to cellulose acetate; theory predicts a constant ratio for translational and rotational energies, whereas simulations show a time persistent rotational energy. ($\mu_1 = 0.14349$, $\mu_2 = 0.26366$.)

As $\mu \rightarrow \infty$ it approaches the curve for constant tangential restitution.

Deviations between the approximate analytical theory and simulations are observed in the parameter regime close to the transition lines. In particular, the parameters μ and ϵ can be chosen such that the analytical theory predicts a Haff type decay of the rotational energy, whereas the simulations reveal constant R . In Fig. 8 we show results of a simulation for the parameters of cellulose acetate spheres as measured by Foerster et al. [1]. Looking at single grains in the simulation, one finds extremely non-Gaussian states, in the sense that few particles rotate with high angular velocities and dominate the average rotational energy. In Fig. 9 we plot a histogram of the rotational velocities for a snapshot taken at $\tau = 10^6$. It reveals clearly that the rotational energy is dominated by few particles with high rotational velocities. Snapshots taken at other instants of time show that the identity of particles with high rotational velocities is conserved. The HCS approximation is not expected to hold in such a state which is strongly non-Gaussian and has a nearly log normal distribution of the angular velocity.

4

Modification of Coulomb's law for low load

The persistence of rotational energy during cooling can be traced back to Coulomb's law which predicts vanishing frictional losses for grazing collisions, regardless of the magnitude of the relative tangential velocity of the contact point (see Fig. 1). For realistic materials one would expect some residual friction due to surface roughness. This effect can be modeled crudely by a minimal roughness β_{\min} such that $\beta(\gamma) \geq \beta_{\min}$. In addition, Coulomb's law, $|\mathbf{F}_{\text{fric}}| = \mu|\hat{\mathbf{n}} \cdot \mathbf{F}_{\text{load}}|$, which we have used only holds for sufficiently large normal loads. When the normal load gets very small, as it happens in the late stages of cooling

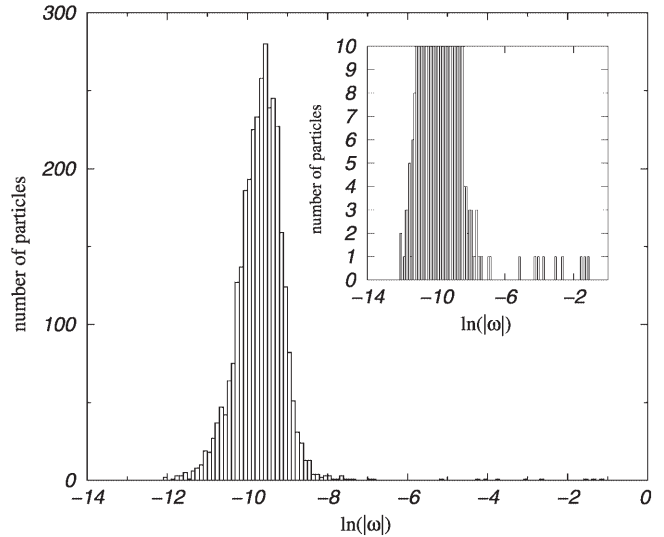


Fig. 9. Histogram at $\tau = 10^6$ of $\ln(|\omega|)$ for cellulose acetate ($\epsilon = 0.87$, $\mu = 0.25$, $\beta_0 = 0.43$)

and in particular for small ϵ , cohesion begins to play a role so that there will always be friction even for zero-load impacts as discussed by Johnson, Kendall, and Roberts [13]. They modify Coulomb's law according to:

$$|\mathbf{F}_{\text{fric}}| = \mu \left(|\hat{\mathbf{n}} \cdot \mathbf{F}_{\text{load}}| + F_0 + \sqrt{2|\hat{\mathbf{n}} \cdot \mathbf{F}_{\text{load}}| + F_0^2} \right) \quad (28)$$

where $F_0 = \frac{3}{2}\pi a E_s > 0$ and E_s denotes the material-specific surface energy. Even simpler is the following version discussed as early as 1934 [14]

$$|\mathbf{F}_{\text{fric}}| = \mu \left(|\hat{\mathbf{n}} \cdot \mathbf{F}_{\text{load}}| + F_0 \right) \quad (29)$$

with a small positive quantity (F_0) due to cohesion. To estimate the effects of cohesion we integrate Eq. (29) over the duration of a collision $\Delta t \sim ac^{-4/5}v^{-1/5}$ as given by the theory of Hertz [15] (c denotes the velocity of sound). We then obtain a modified Coulomb law

$$m|\hat{\mathbf{n}} \times \Delta \mathbf{v}| = \mu(m|\hat{\mathbf{n}} \cdot \Delta \mathbf{v}| + F_0 \Delta t) \quad (30)$$

which corresponds to an impact angle dependent coefficient of tangential restitution for sliding contacts

$$\beta^*(\gamma) = -1 + \frac{1+q}{q} \mu \left\{ (1+\epsilon)|\cot \gamma| + \frac{2F_0 \Delta t}{m|\mathbf{g} \times \hat{\mathbf{n}}|} \right\}. \quad (31)$$

and generalizes Eq. (5) to include cohesion forces for low impact collisions. The most important feature is a finite coefficient of tangential restitution for impact angles $\gamma = \pi/2$.

The question arises, whether the time persistence of the rotational energy survives if finite $\beta(\frac{\pi}{2})$ is taken into account as predicted by surface roughness as well as by cohesion forces. In Fig. 10 we show results of simulations with an angle dependent coefficient of restitution, as given in Eq. (7), however, with a lower bound β_{\min} . The rotational energy shows a plateau for intermediate times and decays asymptotically like t^{-2} for long times. The length of the plateau and the onset of the decay depend on the value of β_{\min} as expected: The plateau is longer and the decay sets in at later times the closer β_{\min} is to -1 .

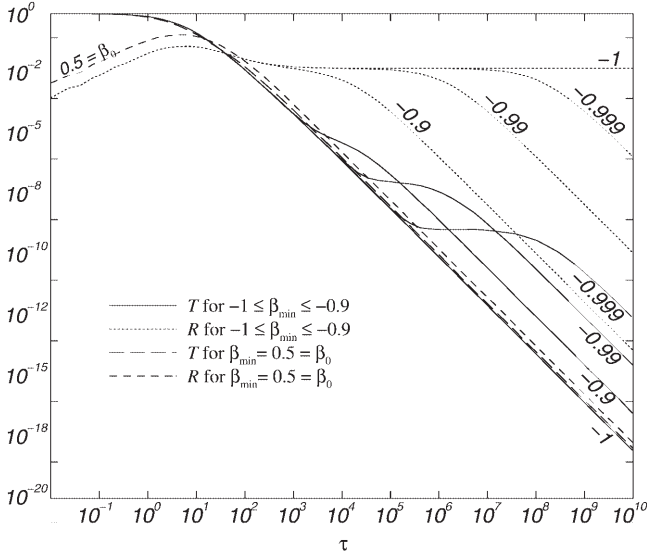


Fig. 10. Simulations for $\epsilon = 0.3$, $\mu = 0.2$, and $\beta_0 = 0.5$ for different β_{\min} . The values of β_{\min} are shown at the corresponding curves. Any $-1 < \beta_{\min} \leq \beta_0$ causes the rotational energy to decay. As β_{\min} approaches -1 the plateau of the rotational energy persists for longer times. $\beta_{\min} = \beta_0$ cancels all μ -dependence of β and thus reveals a constant β

5 Summary and Outlook

We have investigated the effects of friction on the cooling properties of granular particles. We observe three distinct phases which differ qualitatively in their late stage of cooling. In the *rough* phase cooling is characterized by a constant ratio of translational and rotational energies whereas the *smooth* phase is characterized by a time persistent rotational energy even for the latest times. These two regimes are separated by an intermediate regime in which the late stage of cooling can be either smooth or rough depending on the initial conditions. The computed phase boundaries are such that one should be able to observe both phases in experiments, too. Soda lime glass ($\epsilon = 0.97$, $\mu = 0.092$, $\beta_0 = 0.44$) definitely lies in the rough regime while acetate spheres ($\epsilon = 0.87$, $\mu = 0.25$, $\beta_0 = 0.43$) fall in the intermediate regime [1].

Both regimes are also observed in the simulations. In fact, approximate analytical theory and simulation agree well within both phases. Close to the intermediate regime we find a strongly non-Gaussian angular velocity distribution which causes the analytical theory to fail. Deviations between theory and simulation may also be due to finite size effects which are expected to also play a role in experiments on granular media, in contrast to experiments on conventional systems of statistical mechanics, where finite size effects are usually negligible.

Any small friction for grazing collisions as generated for example by surface roughness causes the rotational energy to decay. However, a plateau survives for intermediate time scales, such that the decay of the rotational energy sets in at much later times than the decay of the translational temperature. The length of the plateau of the rotational energy, and hence the time of decay diverges as the roughness goes to zero.

A possible extension of our work are driven systems. A particular way of driving – adding random velocity vectors – has been investigated recently with simulations and approximate analytical theory by Cafiero et al. [16]. They find an asymptotic state in which both kinetic energies take on constant values due to driving.

Appendix

We show here in detail the derivation of Eqs. (18). The calculations are almost identical for translational and rotational energies, so we will restrict ourselves to $T_{\text{tr}}(t)$, whose time derivative is given by:

$$\begin{aligned} \langle i\mathcal{L}_+ E_{\text{tr}} \rangle_t &= \left\langle \frac{1}{2} \sum_{k \neq l} iT_+^{kl} E_{\text{tr}} \right\rangle_t \\ &= \frac{1}{2N} \sum_{k=1}^N \sum_{\substack{l=1 \\ l \neq k}}^N \frac{m}{2} \int \prod_{j=1}^{3N} d\mathbf{r}^j d\mathbf{v}^j d\boldsymbol{\omega}^j \rho(\mathbf{r}, \mathbf{v}, \boldsymbol{\omega}) \\ &\quad \times iT_+^{kl} (|\mathbf{v}_k|^2 + |\mathbf{v}_l|^2) \end{aligned} \quad (32)$$

We have used that the binary collision operator iT_+^{kl} yields zero acting on any variable other than the ones of the two spheres involved in the collision. Defining

$$\begin{aligned} d\Gamma &:= \prod_{j=1}^{3N} d\mathbf{r}^j d\mathbf{v}^j d\boldsymbol{\omega}^j \prod_{j \neq l} \Theta(|\mathbf{r}_{jl}| - a) \\ &\quad \times \exp \left(- \sum_{k=1}^N \frac{m}{2T_{\text{tr}}(t)} |\mathbf{v}_k|^2 - \sum_{k=1}^N \frac{I}{2T_{\text{rot}}(t)} |\boldsymbol{\omega}_k|^2 \right) \end{aligned} \quad (33)$$

and using the definition of iT_+^{12} we can write

$$\begin{aligned} \langle i\mathcal{L}_+ E_{\text{tr}} \rangle_t &= - \frac{N-1}{2} \int d\Gamma \hat{\mathbf{r}}_{12} \cdot \mathbf{v}_{12} \delta(|\mathbf{r}_{12}| - a) \\ &\quad \Theta(-\hat{\mathbf{r}}_{12} \cdot \hat{\mathbf{v}}_{12}) (b_+^{12} - 1) \frac{m}{2} (|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \end{aligned}$$

The change of energy $\Delta E_{\text{tr}} := \frac{m}{2} (b_+^{12} - 1) (|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)$ that results from a collision of sphere 1 and 2 depends only on the phase space variables of sphere 1 and 2. Since we assume spatial homogeneity this change of energy can only depend on the *relative* distance vector as well as the *relative* translational and rotational velocities $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ and $\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Further, we assume instantaneous collisions. Therefore the change of energy can only depend on the *direction* of the distance vector $\hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$ relative to \mathbf{v}_{12} and relative to $\boldsymbol{\omega}_{12}$. We will see this later explicitly when we calculate ΔE_{tr} in Eq. (34). We summarize that $\Delta E_{\text{tr}} = \Delta E_{\text{tr}}(\hat{\mathbf{r}}_{12}, \mathbf{v}_{12}, \boldsymbol{\omega}_{12})$ is a function of *no more* variables than the specified ones. Now we can perform the integrations over $d^3 r_3 \cdots d^3 r_N$, $d^3 v_3 \cdots d^3 v_N$ and $d^3 \omega_3 \cdots d^3 \omega_N$. The integration over $d^3 v_3 \cdots d^3 v_N$ and $d^3 \omega_3 \cdots d^3 \omega_N$ are simple Gaussian integrals. To integrate over $d^3 r_3 \cdots d^3 r_N$ we introduce two (six) more integrals over δ functions $\int d^3 R_1 d^3 R_2 \delta^3(\mathbf{R}_1 - \mathbf{r}_1) \delta^3(\mathbf{R}_2 - \mathbf{r}_2)$. Using $\mathbf{R}_{12} := \mathbf{R}_1 - \mathbf{R}_2$ and the definition of the pair correlation function

$$\frac{g(|\mathbf{R}_1 - \mathbf{R}_2|)}{V^2} : \\ = \frac{\int \prod_{j=1}^{3N} dr^j \prod_{j \neq l} \Theta(|\mathbf{r}_{jl}| - a) \delta(\mathbf{R}_1 - \mathbf{r}_1) \delta(\mathbf{R}_2 - \mathbf{r}_2)}{\int \prod_{k=1}^{3N} dr^k \prod_{j \neq l} \Theta(|\mathbf{r}_{jl}| - a)}$$

we obtain

$$\langle i\mathcal{L} + E_{\text{tr}} \rangle_t \\ = -\frac{N-1}{2V^2} \left(\frac{m}{2\pi T_{\text{tr}}(t)} \right)^3 \left(\frac{I}{2\pi T_{\text{rot}}(t)} \right)^3 \\ \times \int d^3 R_1 d^3 R_2 d^3 v_1 d^3 v_2 d^3 \omega_1 d^3 \omega_2 g(|\mathbf{R}_{12}|) \\ \times \exp \left(\frac{-\frac{m}{2}(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) - \frac{I}{2}(|\boldsymbol{\omega}_1|^2 + |\boldsymbol{\omega}_2|^2)}{T_{\text{tr}}(t)} \right) \\ \times \hat{\mathbf{R}}_{12} \cdot \mathbf{v}_{12} \delta(|\mathbf{R}_{12}| - a) \Theta(-\hat{\mathbf{R}}_{12} \cdot \hat{\mathbf{v}}_{12}) \Delta E_{\text{tr}} .$$

Since the change of energy ΔE_{tr} depends only on $\mathbf{R}_{12} = \mathbf{R}_1 - \mathbf{R}_2$, $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$, $\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ we introduce new variables

$$\mathbf{r} := \mathbf{R}_1 - \mathbf{R}_2, \quad \mathbf{v} := \frac{\mathbf{v}_1 - \mathbf{v}_2}{\sqrt{2}}, \quad \boldsymbol{\omega} := \frac{\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2}{\sqrt{2}} \\ \mathbf{R} := \mathbf{R}_1, \quad \mathbf{V} := \frac{\mathbf{v}_1 + \mathbf{v}_2}{\sqrt{2}}, \quad \boldsymbol{\Omega} := \frac{\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2}{\sqrt{2}}$$

The Jacobian of this transformation is 1. The new expression is independent of \mathbf{R} so that integration over $d^3 R$ yields the volume V . We write \mathbf{r} in spherical coordinates and integrate over dr . Integration over $d^3 V$ and $d^3 \Omega$ are plain Gaussians again. We obtain

$$\langle i\mathcal{L} + E_{\text{tr}} \rangle_t = -\sqrt{2} \frac{N-1}{2V} a^2 g(a) \left(\frac{m}{2\pi T_{\text{tr}}(t)} \right)^{\frac{3}{2}} \left(\frac{I}{2\pi T_{\text{rot}}(t)} \right)^{\frac{3}{2}} \\ \times \int d^2 \hat{r} d^3 v d^3 \omega \exp \left(\frac{-\frac{m}{2}|\mathbf{v}|^2}{T_{\text{tr}}(t)} + \frac{-\frac{I}{2}|\boldsymbol{\omega}|^2}{T_{\text{rot}}(t)} \right) \\ \times \hat{\mathbf{r}} \cdot \mathbf{v} \Theta(-\hat{\mathbf{r}} \cdot \hat{\mathbf{v}}) \Delta E_{\text{tr}} .$$

We choose the coordinate system for integration over $d^3 v$ and $d^3 \omega$ such that $\hat{\mathbf{r}}$ points in the z -direction. Since after integration over $d^3 v$ and $d^3 \omega$ the result does not depend on the special choice of $\hat{\mathbf{r}}$ integration over $d^2 \hat{r}$ yields a factor 4π . We now take a look at the change of energy ΔE_{tr} :

$$\frac{2}{m} \Delta E_{\text{tr}} := (b_+^2 - 1)(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \\ = (|\mathbf{v}'_1|^2 + |\mathbf{v}'_2|^2) - (|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \\ \stackrel{(3)}{=} 4\eta(\eta - 1)(|\mathbf{v}|^2 - (\hat{\mathbf{r}} \cdot \mathbf{v})^2) \\ - (1 - \epsilon^2)(\hat{\mathbf{r}} \cdot \mathbf{v})^2 + (a\eta)^2 |\hat{\mathbf{r}} \times \boldsymbol{\omega}|^2 \\ + 2a\eta(2\eta - 1)(\hat{\mathbf{r}} \times \boldsymbol{\omega}) \cdot \mathbf{v} \quad (34)$$

with $\eta = \frac{q}{1+q} \frac{1+\beta(\gamma)}{2}$ and $\beta(\gamma)$ given by Eq. (7). The change of energy does not depend on ω_z so that the integration over ω_z is a Gaussian integral which yields a factor of $\sqrt{(2\pi T_{\text{rot}}/I)}$. To perform the integration over \mathbf{v} and (ω_x, ω_y) we substitute

$$\mathbf{k} := (\hat{\mathbf{e}}_z \times \boldsymbol{\omega}) = (-\omega_y, \omega_x) \\ \mathbf{g} := \mathbf{v} + \frac{a}{2} \mathbf{k} \quad (35)$$

The Jacobian of this transformation is 1. In terms of these new variables ΔE_{tr} reads

$$\frac{2}{m} \Delta E_{\text{tr}} = 4\eta(\eta - 1)(|\mathbf{g}|^2 - (\hat{\mathbf{e}}_z \cdot \mathbf{g})^2) \\ - (1 - \epsilon^2)(\hat{\mathbf{e}}_z \cdot \mathbf{g})^2 + 2a\eta \mathbf{g} \cdot \mathbf{k} \quad (36)$$

and, with $n_0 := \frac{N-1}{V} \approx \frac{N}{V}$

$$\langle i\mathcal{L} + E_{\text{tr}} \rangle_t \\ = -\frac{2\sqrt{2\pi}}{\pi^2} a^2 n_0 g(a) \left(\frac{m}{2T_{\text{tr}}(t)} \right)^{\frac{3}{2}} \left(\frac{I}{2T_{\text{rot}}(t)} \right) \\ \times \int d^3 g d^2 k \hat{\mathbf{e}}_z \cdot \mathbf{g} \Theta(-\hat{\mathbf{e}}_z \cdot \hat{\mathbf{g}}) \Delta E_{\text{tr}} \\ \times \exp \left(-\frac{m}{2T_{\text{tr}}(t)} (|\mathbf{g}|^2 - a\mathbf{g} \cdot \mathbf{k} + \frac{a^2}{4} |\mathbf{k}|^2) - \frac{I|\mathbf{k}|^2}{2T_{\text{rot}}(t)} \right) \quad (37)$$

where $d^2 k := dk^x dk^y$. We express \mathbf{g} in spherical coordinates (g, α, γ) and \mathbf{k} in polar coordinates (p, ψ) . Expression (36) becomes

$$\frac{2}{m} \Delta E_{\text{tr}} = 4\eta(\eta - 1)g^2 \sin^2 \gamma - (1 - \epsilon^2)g^2 \cos^2 \gamma \\ + 2\eta a g p \sin \gamma \cos(\alpha - \psi) \quad (38)$$

(please note that $\eta = \eta(|\cot \gamma|)$ in the Coulomb friction case) and

$$\langle i\mathcal{L} + E_{\text{tr}} \rangle_t \\ = -\frac{2\sqrt{2\pi}}{\pi^2} a^2 n_0 g(a) \left(\frac{m}{2T_{\text{tr}}(t)} \right)^{\frac{3}{2}} \left(\frac{I}{2T_{\text{rot}}(t)} \right) \\ \times \int_0^\infty dg \int_{-1}^0 d(\cos \gamma) \int_0^{2\pi} d\alpha \int_0^\infty dp \int_0^{2\pi} d\psi p g^3 \cos \gamma \\ \times \exp \left(-\frac{m}{2T_{\text{tr}}(t)} \left[g^2 - a g p \sin \gamma \cos(\alpha - \psi) \right. \right. \\ \left. \left. + \frac{a^2}{4} p^2 \right] - \frac{I}{2T_{\text{rot}}(t)} p^2 \right) \Delta E_{\text{tr}} .$$

The expression we are integrating depends on $\phi := \alpha - \psi$ only, so that the integration over α can be performed. Changing back from polar to Cartesian coordinates $(p, \phi) \rightarrow (x, y)$ we obtain Gaussian type integrals which can be calculated easily. Next the integration over g can be performed which, again, is a Gaussian type integral. Using $q \equiv \frac{4I}{ma^2}$ and $\alpha := \frac{T_{\text{rot}}(t)}{qT_{\text{tr}}(t)}$ we obtain

$$\langle i\mathcal{L} + E_{\text{tr}} \rangle_t = -16 \sqrt{\frac{\pi}{m}} a^2 n_0 g(a) T_{\text{tr}}^{\frac{3}{2}}(t) (1 + \alpha)^2 \\ \times \int_{\frac{\pi}{2}}^\pi d\gamma \frac{\sin \gamma \cos \gamma}{(1 + \alpha \cos^2 \gamma)^3} \\ \times \left(2\eta \left[\eta - \frac{1}{1 + \alpha} \right] \sin^2 \gamma - \frac{1 - \epsilon^2}{2} \cos^2 \gamma \right)$$

Up to this point we have *not* specified, whether we are going to use constant coefficients of restitution or Coulomb friction. All this is hidden in η which is either a constant or a function of $\cot \gamma$. We are interested in the Coulomb

friction case so we use $\eta = \eta(\gamma)$. In the interval $\gamma \in [\frac{\pi}{2}, \pi]$, $\sin \gamma = |\sin \gamma|$ is monotonic decreasing while $(-\cos \gamma) = |\cos \gamma|$ is monotonic increasing. Therefore, $|\cot \gamma|$ is monotonic increasing for $\gamma \in [\frac{\pi}{2}, \pi]$. We introduce the notation $\eta = \frac{1+\epsilon}{2}\mu \min\{|\cot \gamma_0|, |\cot \gamma|\} \equiv \min\{\eta_0, \frac{1+\epsilon}{2}\mu|\cot \gamma|\}$ with $\eta_0 := \frac{q}{1+q} \frac{1+\beta_0}{2}$. We obtain

$$\begin{aligned} \langle i\mathcal{L}_+ E_{\text{tr}} \rangle_t &= -16 \sqrt{\frac{\pi}{m}} a^2 n_0 g(a) T_{\text{tr}}^{\frac{3}{2}}(t) (1+\alpha)^2 \\ &\times \left\{ \int_{\frac{\pi}{2}}^{\gamma_0} d\gamma \frac{\sin \gamma \cos \gamma}{(1+\alpha \cos^2 \gamma)^3} \times \left(\mu \frac{1+\epsilon}{1+\alpha} \sin \gamma \cos \gamma \right. \right. \\ &+ \left. \left(\frac{(1+\epsilon)^2}{2} \mu^2 - \frac{1-\epsilon^2}{2} \right) \cos^2 \gamma \right) \\ &+ \int_{\gamma_0}^{\pi} d\gamma \frac{\sin \gamma \cos \gamma}{(1+\alpha \cos^2 \gamma)^3} \\ &\times \left. \left(2\eta_0 \left[\eta_0 - \frac{1}{1+\alpha} \right] \sin^2 \gamma - \frac{1-\epsilon^2}{2} \cos^2 \gamma \right) \right\} \quad (39) \end{aligned}$$

After integration, using the fact that $\frac{1+\epsilon}{2}\mu|\cot \gamma_0| = \eta_0$ we obtain

$$\begin{aligned} \langle i\mathcal{L}_+ E_{\text{tr}} \rangle_t &= -16 \sqrt{\frac{\pi}{m}} a^2 n_0 g(a) T_{\text{tr}}^{\frac{3}{2}}(t) (1+\alpha)^2 \\ &\times \left\{ \left(2\eta_0 \frac{1}{8(1+\alpha)^2} \left(\frac{(1+\alpha) \cos^2 \gamma_0 - \sin^2 \gamma_0}{(1+\alpha \cos^2 \gamma_0)^2} \sin^2 \gamma_0 \right. \right. \right. \\ &+ \left. \left. \frac{\arctan(\sqrt{1+\alpha} \cot \gamma_0)}{\sqrt{1+\alpha} \cot \gamma_0} \right) \right. \\ &- \left. \left(2\eta_0^2 \tan^2 \gamma_0 - \frac{1-\epsilon^2}{2} \right) \frac{\cos^4 \gamma_0}{4(1+\alpha \cos^2 \gamma_0)^2} \right) \\ &- \left(2\eta_0 \left[\eta_0 - \frac{1}{1+\alpha} \right] \frac{1}{(1+\alpha \cos^2 \gamma_0)^2} \frac{\sin^4 \gamma_0}{4(1+\alpha)} \right. \\ &- \left. \frac{1-\epsilon^2}{2} \frac{1}{(1+\alpha \cos^2 \gamma_0)^2} \right. \\ &\times \left. \left. \left(\frac{\sin^4 \gamma_0}{4(1+\alpha)^2} + \frac{\sin^2 \gamma_0 \cos^2 \gamma_0}{2(1+\alpha)} \right) \right) \right\} \quad (40) \end{aligned}$$

which simplifies to Eq. (18) when we set $\eta_c := \eta_0 \sin \gamma_0$.

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