Causality Problems for Fermi's Two-Atom System

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(Received 2 September 1993)

Let A and B be two atoms or, more generally, a "source" and a "detector" separated by some distance R. At $t = 0$ A is in an excited state, B in its ground state, and no photons are present. A theorem is proved that in contrast to Einstein causality and finite signal velocity the excitation probability of B is nonzero immediately after $t = 0$. Implications are discussed.

PACS numbers: 03.65.Bz

To study and check finite signal velocity, Fermi [1] considered two atoms A and B separated by a distance R. At time $t = 0$ atom A is assumed to be in an excited state $|e_A\rangle$ and B in its ground state $|g_B\rangle$, with no photons present. Atom A will decay to its ground state under the emission of a photon which may then be absorbed by atom B. Fermi asked when atom B will notice A and start to move out of its ground state. In accordance with Einstein causality, i.e., no propagation faster than the speed of light, he expected this to occur after a time $t = R/c$. This was indeed what Fermi found by his calculation.

More than thirty years later Shirokov [2] pointed out that Fermi's "causal" result was the artifact of an approximation. Indeed, Fermi had replaced an integral over positive frequencies by an integral ranging from $-\infty$ to ∞. Without this approximation his calculation would not have given the expected result.

Moreover, Fermi had calculated the probability for a transition to A nonexcited, B excited, and no photons, i.e., the transition probability from the state $|e_A\rangle|g_B\rangle|0_{\rm ph}\rangle$ to the state $|g_A\rangle|e_B\rangle|0_{\rm ph}\rangle$, which requires measurements on A, B, and photons simultaneously. Hence this "exchange" probability does not refer to finite signal velocity or Einstein causality but to "local" or "nonlocal" correlations. What is really needed for finite signal velocity is the probability of finding B excited, irrespective of the state of A and of possible photons. This will be called the excitation probability of B.

Fermi's problem was investigated by many authors in this or in a related form, e.g., by Heitler and Ma [3], Hamilton [4], Fierz [5], Ferretti [6], Milonni and Knight [7], Shirokov [2] and his review [8], Rubin [9], Biswas *et al.* [10], and Valentini [11]. The older papers confirmed Fermi's conclusion, while the results of the later papers depend on the model and the approximations used. At present there seems to be agreement that Fermi's local result is not correct, but that this nonlocality cannot be used for superluminal signal transmission since measurements on A and B as well as on photons are involved.

Usually previous authors have used "bare" states and a Hamiltonian of the form

$$H_{\rm bare} = H_A + H_B + H_F + H_{AF} + H_{BF} \ . \qquad (1)$$

where H_{AF} and H_{BF} represent the coupling of atoms A and B to the quantized radiation field. The Hilbert space is simply a tensor product,

$$\mathcal{H}_{\rm bare} = \mathcal{H}_A \times \mathcal{H}_B \times \mathcal{H}_F \ . \qquad (2)$$

The initial state is then

$$|\psi_0^{\rm bare}\rangle = |e_A\rangle|g_B\rangle|0_{\rm ph}\rangle \ . \qquad (3)$$

The probability of finding B in some excited state, irrespective of the state of A and photons, is a sum over all excited states $|e_B\rangle$ of B, over all states $|i_A\rangle$ of A and over all photon states $|\{\mathbf{n}\}\rangle$, i.e.,

$$\sum_{e_B}\sum_{i_A}\sum_{\{\mathbf{n}\}} |\langle\{\mathbf{n}\}|\langle e_B|\langle i_A|\psi_t^{\rm bare}\rangle|^2 = \langle\psi_t^{\rm bare}|\Bigg(\sum_{i_A,e_B,\{\mathbf{n}\}} |i_A\rangle|e_B\rangle|\{\mathbf{n}\}\rangle\langle\{\mathbf{n}\}|\langle e_B|\langle i_A|\Bigg)|\psi_t^{\rm bare}\rangle$$
$$= \langle\psi_t^{\rm bare}|\mathbf{1}_A \times \sum_{e_B}|e_B\rangle\langle e_B| \times \mathbf{1}_F|\psi_t^{\rm bare}\rangle \ , \qquad (4)$$

where the completeness relation for orthonormal bases has been used. The operator

$$\mathcal{O}_{e_B}^{\rm bare} \equiv \mathbf{1}_A \times \sum_{e_B}|e_B\rangle\langle e_B| \times \mathbf{1}_F \qquad (5)$$

represents the observable "B is in a bare excited state," and it is a projection operator. The expectation value of $\mathcal{O}_{e_B}^{\rm bare}$ gives the excitation probability of B.

For bare states, however, there is a serious difficulty. Even with atom A absent and no photons present atom B will be immediately excited under simultaneous emission of photons. This well-known unphysical behavior is a consequence of the interaction term H_{BF} because then the bare ground state $|g_B\rangle|0_{\rm ph}\rangle$ is no longer an eigenstate of the bare Hamiltonian. Therefore, all results for bare states have to be considered with caution.

Valentini [11] and also Biswas *et al.* [10] have found

0031-9007/94/72(5)/596(4)$06.00

the following interesting result for bare states by using perturbation theory and cutoffs. They calculated that for $t \leq R/c$ the bare ground state of B behaves as if the excited atom A were not present. This result seems to indicate a causal behavior and suggests a similar result for a properly renormalized theory. This, however, will be shown not to be the case.

Fermi's problem of finite signal velocity will now be treated under very general assumptions without bare states. Although a renormalized theory has yet to be constructed, only two basic properties of such a supposedly existing theory are needed. The first is that the states of such a theory form a Hilbert space, denoted by $\mathcal{H}_{\rm ren}$. The other property needed is a renormalized self-adjoint Hamiltonian $H_{\rm ren}$ which is bounded from below, e.g., by 0. The assumption of positive energy is standard and physically well motivated.

In general $\mathcal{H}_{\rm ren}$ is no longer a tensor product,

$$\mathcal{H}_{\rm ren} \neq \mathcal{H}_A \times \mathcal{H}_B \times \mathcal{H}_F, \tag{6}$$

and the initial state, denoted by $|\psi_0\rangle$, will not be a simple product state,

$$|\psi_0\rangle \neq |e_A\rangle|g_B\rangle|\phi_{\rm ph}\rangle \ .$$

Similarly, if the observable "B is in an excited state" makes sense and is represented by an operator $\mathcal{O}_{e_B}$ then in general $\mathcal{O}_{e_B} \neq \mathcal{O}_{eB}^{\rm bare}$. However, $\mathcal{O}_{e_B}$ will still be a projection operator since its eigenvalues are 1 for "yes" and 0 for "no." The excitation probability of B at time t is then given by the expectation value

$$\langle\psi_t|\mathcal{O}_{e_B}|\psi_t\rangle \ .$$

Alternatively one may assume that the excitation probability of B is an expectation value of some positive operator, or one may measure the excitation through a positive observable which vanishes for the ground state, e.g., some operator related to the square of the dipole moment [12]. In all these cases one will run into difficulties with Einstein causality.

No pointlike localization of A and B is required. As a generalization of Fermi's setup A and B may be systems initially localized in two regions separated by a distance R as in Fig. 1, with no (real) photons present. The ground state of B may be degenerate.

We note that measurements of the excitation probability of B involve measurements on B only and that $P_B^e(t=0)=0$. One would expect, as Fermi, that

$$P_B^e(t) = 0 \ \text{ for } \ 0 \leq t \leq R/c \ . \tag{7}$$

However, in a slightly different context a theorem of the author [13] as well as prior [14] and later results [15–18] showed difficulties with causality in particle localization [19]. Although the theorem is not applicable here—it applies to free particles or to the center of mass of systems—it makes one wary. Indeed, as a complement

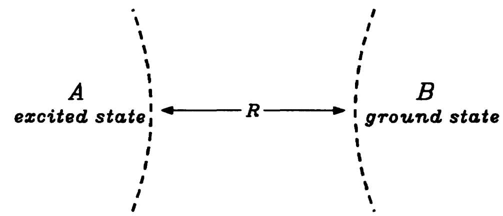

FIG. 1. Two systems located at time $t=0$ in separate regions a distance R apart. A excited, B in ground state, and no photons initially.

to this first theorem I will now show a second theorem which includes interactions.

Theorem.—Let the Hamiltonian be positive or bounded from below and let the initial state at time $t=0$ be

$$|\psi_0\rangle = \begin{cases} A & \text{in an excited state,} \\ B & \text{in a ground state, no photons.} \end{cases}$$

Let $P_B^e(t)$ be the probability of finding B excited,

$$P_B^e(t) = \langle\psi_t|\mathcal{O}_{e_B}|\psi_t\rangle \ , \tag{8}$$

where $\mathcal{O}_{e_B}$ is a projection operator or, more generally, a positive operator.

Then either (i) the excitation probability of B is nonzero for almost all t, and the set of such t's is dense and open or (ii) the excitation probability of B is identically zero for all t.

Remarks.—Alternative (i) means that B starts to move out of the ground state immediately and is thus influenced by A instantaneously, in contrast to Einstein causality. Alternative (ii) is clearly unphysical since in this case B is never excited so that B is never influenced by A.

The proof is basically very simple and uses only the positivity of $H_{\rm ren}$, or rather its boundedness from below, and the fact that one deals with the expectation value of a positive self-adjoint operator.

Proof of theorem.—Since $|\psi_t\rangle$ is continuous in t, so is $P_B^e(t)$. Hence, if for some t_1 one has $P_B^e(t_1) > 0$ then this also holds in a small interval around t_1, and therefore the set is open. Now let us assume that the set of t's with $P_B^e(t) > 0$ is not dense. Then there is a small but finite interval I such that

$$P_B^e(t) = 0 \ \text{ for } \ t \in I \ . \tag{9}$$

It will now be shown that this implies that alternative (ii) holds. Equation (9) can be written as

$$\langle\psi_t|\mathcal{O}_{e_B}|\psi_t\rangle = 0 \ \text{ for } \ t \in I \ . \tag{10}$$

If $\mathcal{O}_{e_B}$ is a projection operator then $(\mathcal{O}_{e_B})^2 = \mathcal{O}_{e_B}$. Therefore Eq. (10) can be written as

$$\begin{aligned}\langle\psi_t|(\mathcal{O}_{e_B})^2|\psi_t\rangle &= \|\mathcal{O}_{e_B}|\psi_t\rangle\|^2 \\ &= 0 \ \text{ for } \ t \in I \ . \end{aligned} \tag{11}$$

This means that

$$\mathcal{O}_{e_B}|\psi_t\rangle = 0 \quad \text{for} \quad t \in I\ . \tag{12}$$

For $\mathcal{O}_{e_B}$ a positive operator the argument is similar [20]. Now let ϕ be any fixed vector and define the auxiliary function $F_\phi(t)$ by

$$F_\phi(t) = \langle\phi|\mathcal{O}_{e_B} e^{-iH_{\rm ren}t/\hbar}|\psi_0\rangle\ . \tag{13}$$

Then, by Eq. (12),

$$F_\phi(t) = 0 \quad \text{for} \quad t \in I\ .$$

Since $H_{\rm ren} \geq -$const, one has that the operator

$$e^{-iH_{\rm ren}(t+iy)/\hbar}$$

is well defined for $y \leq 0$. Putting $z = t + iy$ one sees that $F_\phi(z)$ can be defined as a continuous function for ${\rm Im}z \leq 0$, and, moreover, $F_\phi(z)$ is analytic for ${\rm Im}z < 0$ (cf. Fig. 2) [21]. However, such an analytic function cannot have boundary values vanishing on a real interval unless

$$F_\phi(z) \equiv 0$$

for ${\rm Im}z \neq 0$ [22]. But then, by continuity, one also has $F_\phi(t) = 0$ for all real t. Hence the right side of Eq. (13) vanishes for all t. Since ϕ was arbitrary, one has

$$\mathcal{O}_{e_B}|\psi_t\rangle \equiv 0 \quad \text{for all} \quad t$$

and this gives $P_B^e(t) \equiv 0$, i.e., case (ii).

This proves that $P_B^e(t)$ is either nonzero on a dense open set or that it vanishes identically. In a slightly more sophisticated way it will now be shown directly that $P_B^e(t)$ is either nonzero for almost all t or vanishes identically. Let the set of zeros of $P_B^e(t)$ be denoted by $\mathcal{N}_0$. The same argument as before shows that $F_\phi(t)$ vanishes there too. As a boundary value of a bounded analytic function $F_\phi(t)$ satisfies, unless it vanishes identically, the inequality [23]

$$\int_{-\infty}^{\infty} dt \ln|F_\phi(t)|/(1+t^2) > -\infty\ .$$

If $\mathcal{N}_0$ had positive measure the integral would be $-\infty$ and thus $F_\phi(t)$ would vanish identically in t, for each ϕ. This would again imply case (ii). Hence if case (ii) does not hold $P_B^e(t)$ can only vanish on a null set [24]. This completes the proof of the theorem.

A typical behavior of the excitation probability of B according to (i) is shown in Fig. 3. No estimate of the actual magnitude of $P_B^e(t)$ is provided by the above argument, except that it is nonzero for almost all t. It follows trivially for alternative (i) that the set of zeros of $P_B^e(t)$ is not only of measure 0 but also nowhere dense.

It should be noted that the above proof makes no use of any spatial separation of the two subsystems nor of its photon content. In fact, the theorem is a mathematically rigorous result which holds for any initial state $|\psi_0\rangle$, any positive Hamiltonian and expectation value of any positive operator [25]. Physics comes in only when one thinks of $|\psi_0\rangle$ as representing two spatially separated subsystems with no photons. Of course, if the systems are not spatially separated part (i) of the theorem comes as no surprise.

Extensions.—The derivation does not need that A and B are atoms. The result clearly extends to more general situations. (a) Larger systems: A may be some "source" of photons and B a "detector." (b) A and B may move. (c) Other particles and other interactions may be included.

Other positive observables can be considered. For example, for an excited localized atom (or system) with no real photons initially one obtains an acausal result for photons in regions not containing the atom. This is contrary to a result by Kikuchi [26] who, at the suggestion of Heisenberg, had studied this problem using the same approximation as Fermi [1]. The general case of a decaying particle or system can also be treated by the above approach.

If the effect implied by the theorem were real it could in principle be used for superluminal signals, with all the well-known consequences. However, the result may also be viewed as a difficulty for the formulation of the underlying theory. The theorem is of the "if-then" type. To avoid its physical consequences one has to check whether its conditions or any additional physical assumptions are fulfilled in a given situation. There are several possible ways out.

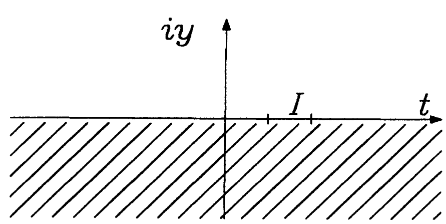

FIG. 2. The auxiliary function $F_\phi(z)$ of the proof is analytic for ${\rm Im}z < 0$ (shaded area) and vanishes in the interval I. Then it is identically zero.

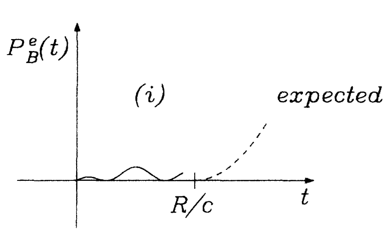

FIG. 3. Typical behavior of the excitation probability of B according to (i). Dashed curve: expected causal behavior.

(a) Systems localized in disjoint regions might not exist as a matter of principle, so that strictly speaking they always "overlap." Then an immediate excitation may evidently occur.

(b) Renormalization will introduce a sort of photon cloud around each system. This essentially implies an overlap of the systems, leading back to case (a).

(c) The notion of "ground state of B" in the presence of A may not make sense. Without A present one will expect a lowest energy state to exist for the system B plus radiation field, with no real photons. However, with A present, the lowest state of the complete system may change. Thus the ground state of B may not be preparable independently of A. Effectively this also leads back to case (a).

One may argue that any violation of Einstein causality would be so rare or so small as to be unobservable in practice and that it might hold only on the average. Decisive in physical applications of the theorem is the notion that for a certain time interval *absolutely no* excitation of B occurs. In addition to (a)–(c) other field theoretic mechanisms might be invoked to prevent this, mechanisms similar to those responsible for the nonpositivity of any energy density [27], although the overall integrated energy is strictly positive.

In conclusion, Fermi's original question on finite signal velocity has been generalized and analyzed in a model-independent way, without the use of any bare theory or any approximations. Only positivity of the energy has been used. It has been shown that this leads to violation of Einstein causality if one assumes that two subsystems, "source" and "detector," can be localized in disjoint regions at some initial time and that the detector is not immediately excited. The view has been taken that this difficulty is of a theoretical nature, and possible ways out have been discussed.

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