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Konforme Partialwellen-Analyse höherer Ordnung in 4-dimensionaler Quantenfeldtheorie

Higher order conformal partial wave analysis in 4-dimensional quantum field theory

prepared by

Lena Marie Wallenhorst

from Bremen

at the Institut für Theoretische Physik

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First Referee: Prof. Dr. Karl-Henning Rehren

Second Referee: Prof. Dr. Laura Covi

Abstract

Positivity is an important property of the state spaces in quantum mechanical systems. Since this property is nonlinear, it is often difficult to investigate. In order to study quantum field theories and in particular Hilbert space positivity of higher correlation functions in globally conformal invariant theories, the partial wave expansion is subject to current investigations. Due to its complexity, an explicit expression of the expansion is only known in a few exceptional cases. Using orthogonality relations between partial waves, it is possible to express n-point functions by lower correlation functions. We have found operators performing the reduction in the case of correlation functions involving only scalar fields of the same scaling dimension. In a more general case, we have obtained some intermediate results, but further work is necessary to complete the analysis. Furthermore, implications of conservation laws resulting from the twist-2 contribution to the operator product expansion in a correlation function are investigated. These laws can also be used as a test of positivity.

Zusammenfassung

Positivität ist ein wichtiges Merkmal der Zustandsräume in quantenmechanischen Systemen. Da diese Eigenschaft nichtlinear ist, treten bei der Untersuchung der Positivität oft Probleme auf. Um Quantenfeldtheorien und insbesondere Hilbertraumpositivität höherer Korrelationsfunktionen in global konform invarianten Theorien zu überprüfen, steht die Partialwellenentwicklung im Fokus aktueller Studien. Aufgrund der komplexen Struktur ist die explizite Berechnung der Entwicklung nur in wenigen Ausnahmefällen bekannt. Mithilfe von Orthogonalitätsrelationen zwischen den Partialwellen können n-Punkt-Funktionen durch niedrigere Korrelationsfunktionen beschrieben werden. Für Korrelationsfunktionen von skalaren Feldern gleicher Skalendimension wurden Operatoren gefunden, die diese Reduktion durchführen. In allgemeineren Fällen konnten wir einige Zwischenresultate erzielen, eine vollständige Betrachtung erfordert jedoch weitere Arbeiten. Des Weiteren wurden Folgerungen von Erhaltungssätzen untersucht, die von dem Twist-2 Beitrag zur Operatorproduktentwicklung in einer Korrelationsfunktion herrühren. Diese können ebenso als Positivitätstest genutzt werden.

Contents

Notation

We give a short list of notations and abbreviations used throughout this thesis.

Symbol or Abbreviation	**Meaning**
OPE	operator product expansion
PDE	partial differential equation
GCI	global conformal invariance
x_{ij}	$x_i - x_j - i\varepsilon e_0$
ϱ_{ij}	squared distance $(x_i - x_j - i\varepsilon e_0)^2$
$\mathcal{P}_+^\uparrow$	proper orthochronous Poincaré group
$\mathcal{H}$	Hilbert space
Ω	vacuum state
$\mathbb{M}$	four dimensional Minkowski space
$Conf(\mathbb{M})$	conformal group acting on Minkowski space
$conf(\mathbb{M})$	conformal algebra acting on Minkowski space
∂_i	partial derivative w. r. t. x_i
∂_{ij}	partial derivative w. r. t. ϱ_{ij}
∇_i	derivative w. r. t. ∂_i
$\mathcal{W}_n$	n-point function
P_μ	generators of translations
D	generator of dilations
$M_{\mu\nu}$	generators of the connected Lorentz group
K_μ	generators of special conformal transformations
$\mathcal{S}(\mathbb{R}^4)$	Schwartz space

1 Introduction

1.1 Motivation

One of the most remarkable facts of nature is that microscopic fields and particles are described by a quantum theory. Quantum field theories have been studied intensively for many decades. Most physicists have followed and still follow an approach where classical fields are quantized, leading e. g. to the standard model in particle physics which is to date a very successful theory. Unfortunately, this kind of theories may yield problems concerning the mathematical structure. Starting in the 1950s, an alternative way of formulating a QFT has been established. The idea is to start from a certain set of minimal properties which should be satisfied by any physical theory and translate them into the language of mathematics. Next, concrete physical models should be constructed.

Despite the success of rigourous constructions of quantum field theories in two spacetime dimensions, there are no physical models present in four dimensions but free field models so far. Many different axiomatic formulations provide us with mathematically sound frameworks but the universality of these frameworks, e. g. a pure Wightman theory [21], is problematical: Namely, it is difficult to construct specific models. As a result, a typical strategy is to narrow the wide range of possible models by extending the symmetry group.

An often studied class of theories are conformal theories. In such a framework, all angle preserving transformations are required to be symmetry transformations. Although our world is not conformally covariant, some physical systems might exhibit at least an approximate conformal symmetry, e. g. in high energy physics [7] or in statistical mechanics [19]. The postulation of conformal covariance implies many helpful consequences. One effect is that the operator product expansion and the partial wave expansion (PWE) of correlation functions (i. e. the main objects of interest in a Wightman-theory) are determined in some cases [15]. The PWE turns out to be very useful in order to investigate a theory and is already well-studied in the case of four-point functions [4, 5]. In order to restrict the theories even more, we can demand global conformal invariance (GCI) where we postulate the existence of a true representation of the conformal group on our Hilbert space [18]. This highly restrictive symmetry leads e. g. to rationality of all correlation functions and commutativity of covariant fields not only at spacelike but also at timelike distances [18]. Under these conditions, candidates for contributions to 6-point functions which cannot arise in a free theory have been found [16]. These candidates automatically fulfil almost all Wightman axioms, except for positivity. Therefore, they have to be tested if they also satisfy this last condition and thereby would yield a physical theory. PWE has proven to be a helpful tool to study in particular positivity. However, the partial wave expansion is not known for 6-point functions or higher.

The purpose of current studies is a further developement of this tool in four but also in two dimensional conformal quantum field theories.

In this thesis, we restrict our analyses to four spacetime dimensions. An alternative to the actual decomposition is provided by using orthogonality relations between partial waves [13]. The aim is to find operators which project separately on different contributions of the irreducible positive energy representations of the conformal group. Hence, they express a n-point function by (n-1)-point partial waves. Successive reduction would permit studying Hilbert space positivity using PWE without knowing the explicit expansion of the n-point functions.

1.2 Organization

This thesis is structured as follows: In a first part, we review some general aspects of the framework used throughout the thesis, in particular we focus on the notion of conformal covariance and global confomal invariance. Next, consequences of these additional symmetries within a Wightman theory on a Minkowski space are discussed and we work out the central question. We then present the partial wave expansion, a tool used very often in the investigation of conformal theories and starting point for the analyses performed in this thesis. Properties of the partial waves are used in order to express n-point functions by expansions of lower correlation functions for the purpose of avoiding an explicit partial wave expansion. To conclude, these results are then applied to the investigation of conservation laws occuring in a GCI theory.

2 The setting

This first chapter is dedicated to the general framework used in the following. Every physical quantum theory should satisfy a couple of fundamental principles such as locality, covariance under the action of the Poincaré group, stability (existence of an energy ground state) and unitarity (existence of a probability interpretation since we are interested in a quantum theory). These principles are fixed among others in the Wightman-formalism in which the objects of interest are the correlation functions since they carry all information about the fields [21]. In particular, it is important that the axioms require Hilbert space positivity. Throughout the whole thesis, only the case of four spacetime dimensions is considered even though many results can be generalized to higher dimensions.

We first review the general physical input. In the second part, we focus on the notion of conformal covariance and global conformal invariance and conclude with some definitions used in the following.

2.1 Wightman theory

There are two ways of working in a Wightman formalism. The first possibility is to formulate axioms for the fields themselves, but it is often more useful to consider their correlation functions. The reconstruction theorem [21] permits to recover the fields from their correlation functions, so that both formulations are equivalent [9, 19, 21]. The following axioms are valid for bosonic fields.

2.1.1 Axioms for the fields

Relativistic quantum theory

Let $\mathcal{H}$ be a separable Hilbert space and $U(g)$ a unitary representation of $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ on $\mathcal{H}$. A state is a unit ray in $\mathcal{H}$ and there exists a unique state Ω (vacuum), which is invariant under the action of $\mathcal{P}_+^\uparrow$. Since the representation is unitary, a pure translation can be written as $U(x) = e^{iP_\mu x^\mu}$, where the spectrum of P_μ lies in the closure of the forward cone $\{p_\mu | p_\mu p^\mu \geq 0, p_0 \geq 0\}$.

The fields

There exists a multiplet of operator valued distributions on $\mathcal{S}(\mathbb{R}^4)$ defined (together with their adjoints) on a domain D dense in $\mathcal{H}$. This domain contains the vacuum and is invariant under the action of the field algebra and $\mathcal{P}_+^\uparrow$. The adjoint $\Phi^*(f)$ of a field $\Phi(f)$ is defined by

$$\langle \Psi_2 | \Phi^*(f) | \Psi_1 \rangle = \overline{\langle \Psi_1 | \Phi(f) | \Psi_2 \rangle}, \tag{2.1.1}$$

where $\Psi_1,\ \Psi_2\ \in\ D$.

Locality

If f and g are test functions with spacelike separated support, then

$$[\Phi_j(f), \Phi_k(g)] = 0. \tag{2.1.2}$$

In other words, fields should commute for spacelike distances.

Transformation law

Under the action of $\mathcal{P}_+^\uparrow$, the fields transform according to

$$U(a, A)\Phi_j(f)U(a, A)^* = \sum_k S_{jk}(A^{-1})\Phi_k(f(A^{-1}(x - a))). \tag{2.1.3}$$

Here $S(A)$ is a finite dimensional representation matrix of the Lorentz group specifying the tensor or spinor character of the field.

Completeness

The vacuum Ω is cyclic for the smeared fields:

$$\mathcal{H} = \overline{span\{\Phi_1(f_1)\Phi_2(f_2)\ldots\Omega\}}. \tag{2.1.4}$$

2.1.2 Axioms for the correlation functions

The field axioms yield the following set of axioms for their correlation functions. These Wightman distributions are defined as

$$\mathcal{W}_{1\ldots n}(x_1, x_2, \ldots, x_n) = (\Omega, \Phi_1(x_1), \ldots, \Phi_n(x_n)\Omega). \tag{2.1.5}$$

Hermiticity condition

The Wightman distributions satisfy

$$(\Omega, \Phi_1(x_1)\ldots\Phi_n(x_n)\Omega) = \overline{(\Omega, \Phi_1^*(x_1)\ldots\Phi_n^*(x_n)\Omega)} \tag{2.1.6}$$

Locality

For Bose fields, the following equality holds if $(x_i - x_{i+1})^2 > 0$.

$$\mathcal{W}_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n). \tag{2.1.7}$$

Covariance

Wightman distributions are translation and Lorentz invariant. It follows that they only depend on the relative coordinates $x_{ij} = x_i - x_j$ (reduced Wightman distribution).

Spectral conditions

The Fourier transforms of the reduced Wightman distributions are tempered distributions and have their support in the product of the future light cones.

Positivity

For any sequence $\{f_i\}$ of test functions, the Wightman distributions of a single field Φ have to satisfy

$$\sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \overline{f}_j(x_1 \ldots x_j) \mathcal{W}_{jk}(x_j, x_{j-1}, \ldots, x_1, y_1, \ldots y_k) f_k(y_1, \ldots, y_k) \geq 0, \tag{2.1.8}$$

where $\mathcal{W}_{jk} = (\Omega, \Phi^*(x_j) \dots \Phi^*(x_1)\Phi(y_1) \dots \Phi(y_k)\Omega)$. For several fields, appropriate generalizations are satisfied.

Cluster decomposition property

For a spacelike vector a we have:

$$\mathcal{W}(x_1, \dots, x_j, x_{j+1} + \lambda a, x_{j+2} + \lambda a, \dots, x_n + \lambda a) \to \mathcal{W}(x_1, \dots x_j)\mathcal{W}(x_{j+1}, \dots, x_n) \tag{2.1.9}$$

as $\lambda \to \infty$.

2.2 The conformal group and global conformal invariance

We would like to enlarge the symmetry group. In order to obtain a physical theory, the external symmetry transformations have to preserve at least the causal structure. The largest group satisfying this demand is the conformal group [6]. This subsection refers mainly to information from [9, 11, 19].

Definition 2.2.1. (Conformal group)
The conformal group $Conf(\mathbb{M})$ *is the group of all angle preserving transformations:*

$$x_\mu \mapsto \tilde{x}_\mu, \qquad \mathrm{d}\tilde{x}_\mu \mathrm{d}\tilde{x}^\mu = w(x)^2 \mathrm{d}x_\mu \mathrm{d}x^\mu, \tag{2.2.1}$$

with a real, smooth function $w(x)$ *(scale factor).*

These include

- Poincaré transformations,
- dilations: $x_\mu \mapsto \lambda x_\mu$,
- and special conformal transformations:

$$x_\mu \mapsto \frac{x_\mu - b_\mu}{1 - b \cdot x + b^2 x^2}. \tag{2.2.2}$$

Note that the special conformal transformations cannot give a global symmetry since they admit singularities where the denominator vanishes. Hence, working on the Minkowski space itself, we encounter problems because a special conformal transformation may map points from Minkowski space to infinity. Thus, we can find a unitary conformal transformation which violates causality by mapping spacelike separated fields on timelike separated fields [18], unless they commute also at timelike distances. This problem can be avoided by working with a projective representation of the universal covering group of $Conf(\mathbb{M})$ on the covering space of the Minkowski space [10] and pass to the *conformally compactified Minkowski space*:

Definition 2.2.2. (Conformally compactified Minkowski space)
The compactified Minkowski space is defined as

$$\bar{\mathbb{M}} = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z = \frac{\bar{z}}{\bar{z}^2},\ z^2 = \sum_{ß=1}^{4} z_i^2 = \vec{z}^2 + z_4^2\}. \tag{2.2.3}$$

In the following, we would like to require the existence of a well-defined unitary representation on the compactified Minkowski space itself, which leads to *global conformal invariance* (GCI). This strong constraint will have an important impact on the field content itself and on correlation functions.

In [18], global conformal invariance is defined in the following way:

Definition 2.2.3. (Global conformal invariance)
A quantum field theory defined on Minkowski space and satisfying Wightman axioms is called globally conformal invariant if the Wigthman distributions are invariant under the action of the conformal group.

We will see that any QFT satisfying GCI defined on Minkowski space admits an extension to the compactified Minkowski space.

In the following, we are mostly interested in theories generated by scalar fields Φ which transform under dilations according to

$$x \mapsto \lambda x, \qquad \Phi(x) \mapsto \lambda^d \Phi(\lambda x), \tag{2.2.4}$$

where d is the *scaling dimension*.

It can be shown [11] that all irreducible unitary positive energy representations are lowest weight representations. This demand already restricts the possible weights $\alpha = (d, -j_1, -j_2)$ where d is the scaling dimension and $j_1,\ j_2\ \in\ SL(2,\mathbb{C})$ spins. The admissible representations can be classified [11]:

1) $d = j_1 = j_2 = 0$ (trivial 1-dimensional representation)

2) $j_1, j_2 \neq 0,\ d \geq j_1 + j_2 + 2$

3) $j_1 \cdot j_2 = 0,\ d \geq j_1 + j_2 + 1.$

The corresponding Lie algebra $conf(\mathbb{M})$ is generated by the generators P_μ of translations, the generators $M_{\mu\nu}$ of the connected Lorentz group, the dilations D and the special conformal transformations K_μ. A positive energy representation is characterized by $P_0 > 0$, which is equivalent to positivity of the conformal Hamiltonian $H_0 = \frac{1}{2}(P_0 + K_0)$ [11]. The reason for this is that there exists a unitary conformal transformation mapping K_0 to P_0. With a scalar field $\Phi(x)$, the generators have the following commutation relations:

$$\begin{aligned} i\,[P_\mu, \Phi] &= \partial_\mu \Phi \\ i\,[D, \Phi] &= (x \cdot \partial + d)\Phi \\ i\,[M_{\mu\nu}, \Phi] &= (x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi \\ i\,[K_\mu, \Phi] &= (2x_\mu(x \cdot \partial) - x^2 \partial_\mu + 2d x_\mu)\Phi. \end{aligned} \tag{2.2.5}$$

It is sometimes useful to define the *twist*:

Definition 2.2.4. (Twist)
The twist 2κ is the quantity

$$2\kappa = d - L \tag{2.2.6}$$

for a rank L operator of scaling dimension d.

Furthermore, we will often use *cross ratios*:

Definition 2.2.5. (Cross ratios)
The cross ratios are defined as

$$R^{ij}_{kl} = \frac{\varrho_{ij}\varrho_{kl}}{\varrho_{ik}\varrho_{jl}}$$

with $\varrho_{ij} = x_{ij}^2 = (x_i - x_j - i\varepsilon e_0)^2$. In the case of four points in spacetime, we can choose two algebraically independent cross ratios:

$$s = \frac{\varrho_{12}\varrho_{34}}{\varrho_{13}\varrho_{24}}, \qquad t = \frac{\varrho_{14}\varrho_{23}}{\varrho_{13}\varrho_{24}}. \tag{2.2.7}$$

All other cross ratios can be expressed by products or ratios of s and t.

Additional postulation of GCI leads to very constrained correlation functions and therefore, it simplifies the calculations a lot. This permits a better understanding of the structural properties of possible theories. Among others, these consequences are discussed in the following chapter.

3 Consequences of conformal covariance and GCI

Conformal covariance and in particular global conformal invariance have far reaching consequences. In conformally covariant theories, the operator product expansion of products of scalar fields $\Phi_1\Phi_2$ is achievable, since its coefficients are determined by conformal symmetry up to a (model dependent) normalization. In addition, postulation of GCI strongly restricts the admissible correlation functions and the field content. These implications simplify calculations a lot and may even lead to an example for a nontrivial theory.

In a theory satisfying GCI, the twist-2 contribution to the OPE plays an important role since it gives rise to infinitely many conserved tensor currents. They are the starting point to derive a concrete condition for a (twist-2 contribution to a) n-point function to be nontrivial and in addition, it is possible to compute the whole twist-2 contribution of a n-point function from its leading part.

In this chapter, we first summarize the consequences of GCI on the field content and on the correlation functions. We continue with an overview about OPE and point out the importance of the twist-2 contribution to the OPE in a GCI QFT. Last we focus on the notion of nontriviality and how it manifests.

3.1 Field content and structure of correlation functions

Even in a conformally covariant theory, the 1-, 2- and 3-point functions of scalar fields are fixed up to a constant [7]:

$$\begin{aligned}\mathcal{W}_1(x) &= 0\\ \mathcal{W}_2(x_1,x_2) &= \delta_{d_1,d_2}\frac{C}{\varrho_{12}^{d_1}}\\ \mathcal{W}_3(x_1,x_2,x_3) &= \frac{C'}{\varrho_{12}^{(d_1+d_2-d_3)/2}\varrho_{13}^{(d_1+d_3-d_2)/2}\varrho_{12}^{(d_2+d_3-d_1)/2}}.\end{aligned} \tag{3.1.1}$$

The demand of global conformal invariance (combined with the Wightman axioms) in Minkowski space strongly restricts the field content and the structure of correlation functions [15, 18]. Nikolov et al. have studied the consequences of GCI in detail [18]. The most important implications for scalar fields are [15, 18]:

1) The correlation functions of fields $\Phi(x)$ are invariant under transformations

$$\Phi(x) \mapsto \det\left(\frac{\partial g}{\partial x}\right)^{d/4}\Phi(g(x)), \tag{3.1.2}$$

where $g \in Conf(\mathbb{M})$ and $\frac{\partial g}{\partial x}$ is the Jacobi matrix. Furthermore, it follows that $d \in \mathbb{N}$.

2) Fields satisfy Huygens locality: for sufficiently large $N \in \mathbb{N}$,

$$\varrho_{12}^{N}\left[\Phi_1(x_1),\Phi_2(x_2)\right] = 0, \tag{3.1.3}$$

i. e. fields commute if $(x_1-x_2)^2 \neq 0$.

3) Correlation functions are rational:

$$\langle\Phi_1\ldots\Phi_n\rangle = \sum_{\{\mu_{jk}\}} C_{\{\mu_{jk}\}}\prod_{j<k}(\varrho_{jk})^{\mu_{jk}}, \tag{3.1.4}$$

with $\mu_{jk} = \mu_{kj} \in \mathbb{Z}$ and, as a consequence of (3.1.2),

$$\sum_{j,\,j\neq k}\mu_{jk} = -d_k. \tag{3.1.5}$$

Here, d_k is the scaling dimension of the field Φ_k. Furthermore, the exponents μ_{jk} satisfy the pole bounds (unitarity bounds)

$$\mu_{jk} \geq -\lfloor \frac{d_j + d_k + \delta_{d_j d_k} - 1}{2} \rfloor, \tag{3.1.6}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. It follows that the sum in (3.1.4) is finite.

These results admit, among others, an extension of any GCI QFT to the compactified Minkowski space.

3.2 Operator product expansion and twist-2 contribution

The operator product expansion is a very powerful tool to study conformal theories. Combined with the demand of conformal invariance, it reveals candidates for non-trivial theories. The main concern is whether these candidates satisfy Hilbert space positivity.

In order to find a candidate for a contribution to a n-point function, it is useful to define

$$U(x_1, x_2) = (\varrho_{12})^{d-1} \cdot (\Phi_1(x_1)\Phi_2(x_2) - \langle 0|\Phi_1\Phi_2|0\rangle)\,, \tag{3.2.1}$$

where Φ_1 and Φ_2 are two scalar fields of the same scaling dimension d [15]. The subtracted term corresponds to the (most singular and trivial) twist-0 contribution (vacuum) and the factor $(\varrho_{12})^{d-1}$ ensures regularity of the correlation functions involving U in ϱ_{12} because the pole bounds (3.1.6) exclude poles of higher order than $d-1$. U is Huygens bilocal but not conformally covariant. The Taylor expansion in x_{12}

$$U(x_1, x_2) = \sum_{n=0}^{\infty} \sum_{\mu_1,\ldots,\mu_n=0}^{3} x_{12}^{\mu_1} \cdots x_{12}^{\mu_n} \cdot X^n_{\mu\ldots}(x_2) \tag{3.2.2}$$

introduces the OPE of $\Phi_1\Phi_2$. It involves fields X which do not transform irreducibly under conformal transformations. They can be replaced by introducing quasiprimary fields $O^L_{\mu_1\ldots\mu_n}$ of rank L (i. e. fields which transform irreducibly under the action of the conformal group) by subtracting derivatives of lower dimensional fields. These fields $O^L_{\mu_1\ldots\mu_n}$ are traceless tensors. The OPE can be reorganized according to the twist $2\kappa = d - L$:

$$U(x_1, x_2) = \sum V_\kappa(x_1, x_2) \cdot \varrho_{12}^{\kappa-1}, \tag{3.2.3}$$

with

$$V_\kappa(x_1, x_2) = \sum_{L=0}^{\infty} K_\kappa^{\mu_1\ldots\mu_L}(x_{12}, \partial_{x_2}) O^{L+2\kappa}_{\mu_1\ldots\mu_n}(x_2). \tag{3.2.4}$$

Due to the universality of the 2- and 3-point functions of scalar fields in a conformal theory, these $K_\kappa^{\mu_1\ldots\mu_L}(x_{12}, \partial_{x_2})$ can, up to a normalization, be determined globally for any (!) conformal QFT. Hence, the whole operator product expansion can be performed explicitly in a conformal theory.

For twist-2, the 2-point functions are conserved and so are the fields, since a vanishing norm square of a vector implies the vanishing of the vector itself and the Reeh-Schlieder theorem now implies that the fields themselves have to vanish. In a GCI theory, this yields that the twist-2 contribution V_1 is biharmonic $\Box_1 V_1 = 0 = \Box_2 V_1$ (the argument uses the explicit expressions of the $K_\kappa^{\mu_1\ldots\mu_L}(x_{12}, \partial_{x_2})$) [17].

In general, we can use the following property [2]:

Lemma 3.2.1. *(Harmonic completion)*
For every power series p in $z \in \mathbb{C}^n$ there exists a unique $h = p + z^2 \cdot q$, q another power series, such that h is harmonic.

Therefore, we can find the unique harmonic decomposition of U:

$$U(x_1, x_2) = V_1(x_1, x_2) + \varrho_{12} \tilde{U}(x_1, x_2).$$

The fact that V_1 is harmonic w. r. t. x_1 and also x_2 gives a condition on possible candidates of twist-2 contributions, since the two a priori different harmonic completions have to coincide. This condition yields a differential equation, which is solved by candidates u_0 for the leading part of the twist-2 part [15]:

$$(E_1 D_2 - E_2 D_1) u_0 = 0, \tag{3.2.5}$$

with

$$D_1 = \sum_{3 \leq j < k \leq n} \varrho_{jk} \partial_{1j} \partial_{1k}, \qquad E_1 = \sum_{3 \leq i} \varrho_{2i} \partial_{2i},$$

where $\partial_{jk} = \frac{\partial}{\partial_{\varrho_{jk}}}$. (3.2.5) ensures that these leading parts can be completed to the full twist-2 contribution. Issues regarding convergences of the decompositions are studied in [15].

3.3 Pole structure and nontriviality

The question is whether there exist harmonic bilocal fields which are nontrivial, i. e. which are not a biharmonic combinations of canonical free fields, e. g.

$$
\begin{aligned}
&: \varphi(x_1)\varphi(x_2) :, \\
&: \bar{\psi}(x_1)(x_1 - x_2)_\mu \gamma^\mu \psi(x_2) :, \\
&: F_{\mu\nu}(x_1)(2x_{12}^\nu x_{12}^\kappa - \eta^{\nu\kappa} x_{12}^2) F_\kappa^\mu(x_2) :,
\end{aligned}
$$

where $: \cdot :$ denotes normal ordering and $F_{\mu\nu}$ is the Maxwell field, φ a scalar field and ψ a Dirac field. Investigation of the pole structure of possible candidates yields a useful classification.

It can be shown that the PDE (3.2.5) combined with rationality of the correlation functions also leads to constraints on the pole structure of the candidates for u_0 [16]. The most involved structure we can find is of the form

$$
\frac{1}{\varrho_{12}^{d-1}} \frac{\text{polynomial}}{\varrho_{1i}^p \varrho_{1j}^q \varrho_{2i}^r \varrho_{2j}^s} \cdot \text{some factors.}
$$

If both p and q or r and s are positive, the structure has a *double pole*, otherwise it is called a *single pole*. It can be shown [15], that the following statements are equivalent:

1) The twist-2 bifield $V_1(x_1, x_2)$ converges to a Huygens bilocal field (i. e. spacelike and timelike commutativity is satisfied w. r. t. both variables).

2) The correlation functions $\langle \cdot V_1 \cdot \rangle$ are rational.

3) $\langle \cdot V_1 \cdot \rangle$ admit only single poles.

Furthermore, a trivial V_1 has these properties: for trivial fields, the correlation functions are sums of products of two-point functions according to Wick's theorem. Those in turn are fixed by conformal invariance and rational.

One good example for a typical solution to the PDE (3.2.5) with double pole structure is the following six-point structure [16]:

$$F_0 = \frac{(\varrho_{15}\varrho_{26}\varrho_{34} - 2\varrho_{15}\varrho_{23}\varrho_{46} - 2\varrho_{15}\varrho_{24}\varrho_{36})_{[1,2][5,6]}}{\varrho_{12}^{d-1} \cdot \varrho_{14}\varrho_{23}\varrho_{13}\varrho_{24}\varrho_{35}\varrho_{36}\varrho_{45}\varrho_{46}\varrho_{56}^{d-1}}, \tag{3.3.1}$$

where $(\cdot)_{[i,j]}$ denotes the antisymmetrization in the variables x_i, x_j. To date, positivity is neither proved nor rejected. In order to study Hilbert space positivity of this candidate (and similar ones), we would like to use partial wave expansion.

4 Partial Wave Expansion

The partial wave expansion (PWE) is similar to the OPE and provides a very useful tool in order to study a quantum field theory, in particular to investigate Hilbert space positivity. In the following, we first explain the main idea. In a conformal field theory, the partial waves can be determined explicitly for correlation functions of (up to four) scalar fields. In the second part, we show a possible way of computing them. In addition, this is then applied to the well-studied case of four-point functions.

4.1 General idea

The idea is to expand a correlation function by projecting onto the irreducible positive energy representations of the conformal group (a similar expansion is known from quantum mechanics, where wave functions are expressed by spherical harmonics which are eigenfunctions of the Casimir operators of the angular momentum algebra). The partial waves depend only on the (conformal) algebra and are therefore universal for any conformal field theory. Thus, the physical content of a specific theory is only characterized by the coefficients. In particular the PWE of four-point functions has been studied successfully, but so far we do not know much about the PWE of higher correlation functions.

By inserting projections $\mathbf{1} = \sum_{\alpha_i} \Pi_{\alpha_i}$ into a general correlation function, we obtain the expansion:

$$\begin{aligned}\langle \Phi_1(x_1)\dots\Phi_n(x_n)\rangle &= \sum_{\alpha} \langle \Phi_1\Phi_2\Pi_{\alpha_2}\Phi_3\Pi_{\alpha_3}\dots\Pi_{\alpha_{n-2}}\Phi_{n-1}\Phi_n\rangle \\ &= \sum_{\alpha} B_\alpha \beta_\alpha(x_1,\dots,x_n),\end{aligned}$$

where $\alpha = (\alpha_2, \dots \alpha_{n-2})$, β_α are *partial waves* and $\alpha_i = (d_i, j_{1_i}, j_{2_i})$ representations of the conformal group.

For up to four scalar fields, the partial waves are determined by GCI (again because of the explicit knowledge about the 2- and 3-point functions). Furthermore, they are universal (i. e. they do not depend on the fields Φ_i). This implies that if a specific β is known to exist in any free theory, the contribution is neccessarily positive and reduces therefore the study of positivity to investigation of the coefficients. In the case of four-point functions of free massless fields, each partial wave in the expansion appears with a positive coefficient and, as a consequence, also the universal partial waves are positive. For other fields or more than four scalar fields, the three-point functions are linear combinations of a finite basis of structures. These structures are universal but unfortunately, they are not known to date.

4.2 Determination of the partial waves

In order to find the partial waves, Casimir operators are useful objects. In four dimensions, the conformal group has rank 3 and therefore, the corresponding algebra has 3 Casimir operators. A Casimir operator is composed of the generators of the algebra which have known commutation relations with the fields. Thus, the commutation relations of the Casimir operators with the fields are also known. By Schur's Lemma, every irreducible representation space is an eigenspace of the Casimir operators at the same time. In $D = 4$ spacetime dimensions, the quadratic Casimir operator has the form:

$$C = \frac{1}{2}\left(P_\mu K^\mu + K_\mu P^\mu\right) - D^2 + \frac{1}{2} M_{\mu\nu} M^{\mu\nu} \tag{4.2.1}$$

with the eigenvalues:

$$c_\alpha = d(d-4) + L(L+2).$$

A state $\Phi\Omega$ is irreducible, but not $\Phi_1\Phi_2\Omega$. Consider $C\Phi_1\Phi_2\Omega$. By using commutation relations and $X\Omega = 0$ (for $X =$ any generator of the conformal algebra), we obtain the equation

$$C\Phi_1\Phi_2\Omega = D_{12}\Phi_1\Phi_2\Omega,$$

where D is a differential operator. We insert operators projecting on irreducible states:

$$\Phi_1\Phi_2\Omega = \sum_\alpha \Pi_\alpha \Phi_1\Phi_2\Omega.$$

The Casimir operator commutes with the projection operators. It leads to the equalities

$$\begin{aligned} C\Pi_\alpha\Phi_1\Phi_2\Omega &= c_\alpha\Pi_\alpha\Phi_1\Phi_2\Omega \\ &= \Pi_\alpha C\Phi_1\Phi_2\Omega = D_{12}\Pi_\alpha\Phi_1\Phi_2\Omega. \end{aligned}$$

In total, it yields the following differential equation:

$$(\Omega, \Phi\cdots\Phi\Pi_\alpha D_{12}\Phi_1\Phi_2\Omega) = c_\alpha \underbrace{(\Omega, \Phi\cdots\Phi\Pi_\alpha\Phi_1\Phi_2\Omega)}_{=\beta_\alpha}$$

$$\implies D_{12}\beta_\alpha = c_\alpha\beta_\alpha.$$

In the same way, projections have to be inserted at all other places. Thus, Casimir operators can, in principle, be used to determine the explicite form of the partial waves (unfortunately, the differential operators are in general pretty involved). In two spacetime dimensions, this method is succesful [13] but it is not practical in four dimensions.

4.3 Special case: four scalar fields

Dolan et al. computed the partial waves of correlation functions of four scalar fields [5]. In the following, this is shown for the special case of equal scaling dimensions and in four spacetime dimensions:

Commutation of $C = \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu) - D^2 + \frac{1}{2}M_{\mu\nu}M^{\mu\nu}$ past the fields $\Phi_3\Phi_4$ in $\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle$ using the commutation relations (2.2.5) yields

$$C\beta = D_{34}\beta \quad \text{with}$$

$$D_{34} = 2d(2d-4) + 2d\, x_{34}\cdot(\partial_3 - \partial_4) + x_{34}^2(\partial_3\cdot\partial_4) - 2x_{34}\otimes x_{34}\cdot\partial_3\otimes\partial_4. \quad (4.3.1)$$

In general, GCI implies rational correlation functions of the form

$$\langle \Phi_1(x_1)\cdots\Phi_n(x_n)\rangle = f\,(\text{cross ratios})\cdot \prod_{i<j} \varrho_{ij}^{\mu_{ij}}$$

with μ_{ij} satisfying the constraints (3.1.5) and (3.1.6). For $n = 4$, we only have the two cross ratios

$$s = \frac{\varrho_{12}\varrho_{34}}{\varrho_{13}\varrho_{24}}, \qquad t = \frac{\varrho_{14}\varrho_{23}}{\varrho_{13}\varrho_{24}}$$

and therefore

$$\langle \Phi(x_1)\cdots\Phi(x_4)\rangle = \frac{1}{\varrho_{12}^d\varrho_{34}^d}\cdot f(s,t).$$

Substitution leads to:

$$2D_{s,t}f(s,t) = c_\alpha \cdot f(s,t)$$

with

$$\begin{aligned} D_{s,t} = {} & s(t-s-3)\partial_s + (1-2t-st-s+t^2)\partial_t \\ & + 2st(t-s-1)\partial_s\partial_t \\ & + s^2(1-s+t)\partial_s^2 + t(1-2t-st-s+t^2)\partial_t^2. \end{aligned} \tag{4.3.2}$$

This differential equation is still pretty complicated and therefore, we use a second helpful substitution to obtain a differential equation which is invariant under the exchange of the two new variables u and v:

$$s = uv,\;\; t = (1-u)(1-v).$$

The Jacobi matrix for this transformation is:

$$\begin{pmatrix} \partial_s u & \partial_t \\ \partial_s v & \partial_t v \end{pmatrix} = \begin{pmatrix} \partial_u s & \partial_v s \\ \partial_u t & \partial_v t \end{pmatrix}^{-1} \tag{4.3.3}$$

$$= \frac{1}{u-v}\begin{pmatrix} u-1 & -u \\ 1-v & v \end{pmatrix}. \tag{4.3.4}$$

Hence, we find the new differential equation

$$2D_{uv}f(u,v) = c_\alpha f(u,v)$$

with

$$\begin{aligned} D_{uv} &= u^2(1-u)\partial_u^2 + v^2(1-v)\partial_v^2 \\ &+ \left(-u^2 + 2\frac{uv}{u-v}(1-u)\right)\partial_u \\ &+ \left(-v^2 - 2\frac{uv}{u-v}(1-v)\right)\partial_v. \end{aligned}$$

Dolan and Osborn [5] studied the solutions in detail and in a more general case (arbitrary spacetime dimension and different scaling dimensions). For the scaling dimensions $d_1 = d_4$, $d_2 = d_3$ they could express the solutions in terms of hypergeometric functions. For four-point functions, it is shown that all occuring partial waves are positive and therefore, only the positivity of the coefficients has to be checked. In this case, a recurrence formula for the coefficients is known [14].

5 Reduction of n-point functions

Due to its complicated structure, the explicit PWE of higher ($n > 4$) correlation functions or correlation functions of nonscalar fields in four spacetime dimensions is still unknown today. By reducing the n-point functions, we can try to avoid the resulting problems since we do not perform the partial wave expansion explicitly. In order to express a n-point function by (n-1)-point functions, we can use that different partial waves are orthogonal w. r. t. a suitable mesure $\mathrm{d}\mu(x_1, x_2)$ and project on the contribution of every representation to the expansion separatly.

In the case of the twist-2 part of a correlation function of scalar fields, local differential operators have been found that perform the projection. The second part of this chapter addresses the reduction of arbitrary twist contributions of scalar fields carrying the same scaling dimension and also lays the ground for a generalization of the analysis to symmetric traceless tensor fields and/or different scaling dimensions. If not mentioned otherwise, we consider scalar fields of equal scaling dimensions in this chapter.

5.1 Reduction of twist-2 correlation functions

In general there exists a differential operator D projecting a state $\Phi_1\Phi_2\Omega$ on an irreducible state $\Phi_\alpha\Omega$ for every representation α occuring in the OPE of $\Phi_1\Phi_2$ (local linear maps):

$$\Phi_\alpha(x)\Omega = \iota_x \circ D\Phi_1(x_1)\Phi_2(x_1)\Omega, \tag{5.1.1}$$

where ι_x denotes evaluation at $x_1 = x_2 = x$. To simplify the calculation, we project in one direction by contracting the tensor fields with polarization vectors: $T(v) = T^{\mu_1 \dots \mu_n} v_{\mu_1} \dots v_{\mu_n}$. The original tensor field is recovered by taking the derivative with respect to the vectors. Eq. (5.1.1) implies that applied to the twist-2 contribution $V_1(x_1, x_2)$, we obtain

$$\iota_x \circ D^{\mu_1 \dots \mu_n} V_1(x_1, x_2) = T^{\mu_1 \dots \mu_n}(x)$$

in case $T^{\mu_1 \cdots \mu_n}$ occurs in the OPE of $\Phi_1\Phi_2$. Hence, $\iota_x \circ D(v)$ projects the twist-2 contribution to a correlation function $\langle \ldots V_1 \rangle$ onto the contributions $\langle \ldots T(v) \rangle$. In the twist-2 case, the image of these operators D has very specific properties, so that we can use them in order to find those differential operators.

It is known that only representations $\alpha = (d, j, j)$ contribute to the OPE of two scalar fields, which corresponds to symmetric and traceless tensors [12]. Therefore, it is sufficient to contract the tensors with only one vector v. Thus, possible contributions to the operators are only contractions of the vector v and ∂_i, $i = 1, 2$.

In addition all correlation functions involving the twist-2 part of the product $\Phi_1\Phi_2$ are biharmonic ($\partial_i^2 V_1 = 0$) [17]. Furthermore, we know that correlation functions involving this part are conserved. Since we have specified the twist to be $2\kappa = 2$, the representations differ only in the rank $L = 2j$.

To summarize, the image of D has to be:

- symmetric: all contractions with the same vector v ($v_{\mu_1} \ldots v_{\mu_n} T^{\mu_1 \cdots \mu_n}$),
- of rank L: $(v \cdot \partial_v) D V_1 = L\, D V_1$ (gives the homogenity in v),
- traceless: $\partial_v^2 D V_1 = 0$,
- conserved: $(\partial_v \cdot \partial_1 + \partial_v \cdot \partial_2) D V_1 = 0$.

The rank L contribution is isolated by applying D and setting $x_1 = x_2$. In this way, we reduce an n-point function to (n-1)-point functions.

The following examples for $L = 0, 1, 2, 3, 4$ are calculated by hand (for $L = 0, 1, 2$ cf. [14]) and are used to check more general results:

- $D^0 = id$
- $D^1 = v \cdot \partial_1 - v \cdot \partial_2$
- $D^2 = (v \cdot \partial_1)^2 + (v \cdot \partial_2)^2 - 4(v \cdot \partial_1)(v \cdot \partial_2) + v^2(\partial_1 \cdot \partial_2)$
- $D^3 = (v \cdot \partial_1)^3 - (v \cdot \partial_2)^3 + 9(v \cdot \partial_1)(v \cdot \partial_2)^2$
  $-9(v \cdot \partial_2)(v \cdot \partial_1)^2 + 3v^2(v \cdot \partial_1)(\partial_1 \cdot \partial_2) - 3v^2(v \cdot \partial_2)(\partial_1 \cdot \partial_2)$
- $D^4 = (v \cdot \partial_1)^4 + (v \cdot \partial_2)^4 + 36(v \cdot \partial_1)^2(v \cdot \partial_2)^2$
  $-16(v \cdot \partial_1)(v \cdot \partial_2)^3 - 16(v \cdot \partial_2)(v \cdot \partial_1)^3 + 6v^2(v \cdot \partial_1)^2(\partial_1 \cdot \partial_2)$
  $+6v^2(v \cdot \partial_2)^2(\partial_1 \cdot \partial_2) + \frac{3}{2}v^4(\partial_1 \cdot \partial_2)^2 - 18v^2(v \cdot \partial_1)(v \cdot \partial_2)(\partial_1 \cdot \partial_2)$.

In order to find a general formula for the differential operators, we can start with the ansatz

$$D^L = \sum_{m+n=L} \sum_{c \le L/2} A^c_{mn} (v \cdot \partial_1)^{m-c} (v \cdot \partial_2)^{n-c} (v^2)^c (\partial_1 \cdot \partial_2)^c.$$

The property that the image is traceless and conserved yields two recursive equations which are solved by

$$\begin{aligned} A^c_{mn} &= \frac{-(m-c+1)(n-c+1)}{2c(c+1)+2c(m+n-2c)} A^{c-1}_{mn} \\ \Longrightarrow \quad A^c_{mn} &= (-1)^c \frac{m!n!(m+n-c)!}{2^c c!(m+n)!(m-c)!(n-c)!} A^0_{mn} \end{aligned}$$

with

$$A^0_{mn} = (-1)^n \frac{(m+n)!^2}{m!^2 n!^2}.$$

We conclude with the following proposition.

Proposition 5.1.1. *The operator*

$$D^L = \sum_{m+n=L} \sum_{c \le L/2} \binom{L}{c} \binom{L-c}{m} \binom{L-c}{n} \left(\frac{v^2}{2}\right)^c (\partial_1 \cdot \partial_2)^c (v \cdot \partial_1)^{m-c} (-v \cdot \partial_2)^{n-c} \tag{5.1.2}$$

returns a traceless, symmetric and conserved rank L tensor if applied to a biharmonic function.

Thus, using orthogonality of the partial waves, we can reduce a twist-2 contribution to a correlation function of n scalar fields to (n-1)-point functions of (n-2) scalar and one tensor field.

5.2 Reduction of arbitrary correlation functions

In the following, we discuss two possible ways of reducing contributions to n-point functions of arbitrary twist. First, we make use of conformal covariance, which implies that the 3- and 2-point functions of scalar fields are determined and we therefore know, how the differential operators should act. We have applied this procedure in order to compute examples of operators corresponding to some specific representations $\alpha = (\kappa, L)$, but we have not found a systematic way of determining the operators corresponding to an arbitrary representation.

Second, we use the fact that the local differential operators should commute with the action of the algebra $conf(\mathbb{M})$ in order to obtain conditions which completely determine them. So far, the second method seems to be more promising. For two scalar fields of equal scaling dimension coupling to a symmetric traceless tensor field, we could determine the operators that project on an arbitrary representation.

5.2.1 Reduction of n-point functions using the explicit formulas of 3- and 2-point functions

In order to find differential operators which reduce an arbitrary contribution to a correlation function, we try to use the fact that the 3- and 2-point functions are explicitly known. Furthermore, we know that our Hilbert space $\mathcal{H}$ is given by the smeared states $\Phi_\alpha(x)|0\rangle$ and therefore, an arbitrary state $|\psi\rangle$ can be written as a linear combination of these basis vectors. In particular, it is known that 3-point functions $\langle\Phi\Phi\Phi_\alpha\rangle$ of a tensor field coupled to two scalar fields of equal dimensions are only nonzero, if Φ_α is a symmetric and traceless tensor, $\alpha = (\kappa, L)$, $L = 2j = j_1 + j_2$ and $d = 2\kappa + L$ [12].

Hence, we can write a 3-point function as:

$$\langle 0|\Phi_1\Phi_2|\psi\rangle = \sum_{\kappa',L'} \int \mathrm{d}x_3 f_{\kappa',L'}(x_3)\langle\Phi_1\Phi_2 T_{\kappa',L'}(x_3)\rangle \tag{5.2.1}$$

with the functions $f_{\kappa',L'}(x_3)$ and symmetric traceless tensor fields $T_{\kappa',L'}$.

The next step is to find operators $E_{\kappa,L}$, which select a specific representation in the expansion and reduce the 3-point function to a 2-point function. Here we use the expression [12]

$$\langle\Phi_1(x_1)\Phi_2(x_2)T_{\kappa,L}(v,x_3)\rangle = \varrho_{12}^{-d}(X^2)^\kappa(v\cdot X)_0^L \tag{5.2.2}$$

for the 3-point function, where $(\cdot)_0$ denotes the harmonic part w. r. t. v and Φ_1, Φ_2 are of equal scaling dimensions. Moreover, we used these simplified expressions:

$$X_\mu = \frac{x_{23\mu}}{\varrho_{23}} - \frac{x_{13\mu}}{\varrho_{13}}, \quad X^2 = \frac{\varrho_{12}}{\varrho_{13}\varrho_{23}}.$$

The 2-point function is given by

$$\langle T_{\kappa,L}(v,x_1)T_{\kappa,L}(w,x_2)\rangle = \varrho_{12}^{-(2\kappa+L)} \cdot I(v,w;x_{12})^L_{00} \tag{5.2.3}$$

with

$$I_{\mu\nu} = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \tag{5.2.4}$$

$I(v,w) = I_{\mu\nu}v^\mu w^\nu$ and $(\cdot)_{00}$ denotes the harmonic part w. r. t. v and w. Thus, we have

$$\iota_{x_0} \circ E_{\kappa,L}\, \varrho_{12}^d \langle \Phi_1\Phi_2 T_{\kappa',L'}(x_3)\rangle = \delta_{\kappa\kappa'}\delta_{LL'}\langle T_{\kappa,L}(x_0)T_{\kappa',L'}(x_3)\rangle, \tag{5.2.5}$$

where ι_{x_0} refers to evaluation at $x_1 = x_2 = x_0$. The idea is to use eq. (5.2.5) in order to determine $E_{\kappa,L}$. Once they are known, we can apply them to eq. (5.2.1), leading to

$$\iota_{x_0} \circ E_{\kappa,L}\, \varrho_{12}^d \langle \Phi_1\Phi_2|\psi\rangle = \int \mathrm{d}x_3 f_{\kappa,L}(x_3)\langle T_{\kappa,L}(x_0)T_{\kappa,L}(x_3)\rangle.$$

Choosing the state $|\psi\rangle$ to be $|\psi\rangle = \Phi_3 \ldots \Phi_n\Omega$, the operator $E_{\kappa,L}$ isolates a specific (κ, L) in

$$\langle \Phi_1 \ldots \Phi_n\rangle = \sum_{\kappa',L'} \langle \Phi_1\Phi_2\Pi_{\kappa',L'}\Phi_3 \ldots \Phi_n\rangle.$$

In principle, further application of these local differential operators permits a successive reduction of an arbitrary n-point function, but 3-point functions involving the resulting additional tensor fields have to be reduced, too.

Using the method just presented, these operators have been found for some specific values of (κ, L):

$$\begin{aligned}
E_{10} &= (\partial_1\partial_2)\\
E_{11} &= (\partial_1\partial_2)\left[(v\partial_1) - (v\partial_2)\right] - \left[\partial_1^2(v\partial_2) - \partial_2^2(v\partial_1)\right]\\
E_{12} &= (\partial_1\partial_2)\left[(v\partial_1)^2 + (v\partial_2)^2\right] - 4(\partial_1\partial_2)(v\partial_1)(v\partial_2) + 3\left[\partial_1^2(v\partial_2)^2 + \partial_2^2(v\partial_1)\right]\\
&\quad - 3(\partial_1^2 + \partial_2^2)(v\partial_1)\\
E_{20} &= (\partial_1\partial_2)^2 + (\partial_1\partial_2)\left[(v\partial_1)^2 + (v\partial_2)^2\right] + \frac{1}{2}\partial_1^2\partial_2^2\\
E_{0L} &= \sum_{s+t=L}(-v\partial_1)^s(v\partial_2)^t\frac{1}{s!(s-1)!t!(t-1)!}
\end{aligned} \tag{5.2.6}$$

Note that for the last operators, corresponding fields do not exist. $\kappa = 0$ and $L > 0$ violates unitarity and $\kappa = L = 0$ denotes the vacuum contribution, which is not taken into account, because we apply the operators only to the modified correlation functions. Those are regular in ϱ_{12} and do not contain the vacuum contribution, cf. eq. (3.2.1). Hence, any correlation function has to be annihilated by E_{0L} in a positive theory. In the following, the operators (5.2.6) will be used to check results.

5.2.2 Reduction of n-point functions using intertwining differential operators

The previous way of determining operators which reduce n-point functions seems to date too complicated to be generalized to arbitrary representations α. Furthermore, an explicit formula for the three-point functions is not known for general fields. We now focus on a possible way of determining them by using the so called *intertwining property* in order to find partial differential equations whose solutions are the operators E_α [13].

We first consider two scalar fields Φ_1 and Φ_2 of the same scaling dimension d. As before, we use that their 3-point functions can only be nonzero if they couple to a symmetric and traceless tensor field. Therefore, we only investigate this case. In the following, the fields are always thought of as being involved in a correlation function. The aim is to find local differential operators E'_α projecting $\Phi_1(x_1)\Phi_2(x_2)$ on a field Φ_α which transforms under the representation $\alpha = (\kappa, L)$:

$$\Phi_\alpha(x) = \iota_x \circ E'_\alpha \Phi_1(x_1)\Phi_2(x_1) \tag{5.2.7}$$

where ι_x refers again to evaluation at $x_1 = x_2 = x$.

It follows by conformal covariance that E'_α annihilates 3-point functions involving a field carrying a representation $\mu \neq \alpha$:

$$\iota_x \circ E'_\alpha \langle \Phi_\mu(y)\Phi_1(x_1)\Phi_2(x_1)\rangle = \delta_{\mu\alpha}\langle \Phi_\mu(y)\Phi_\alpha(x)\rangle.$$

Applied to the OPE of $\Phi_1\Phi_2$, the operators E'_α precisely isolate the contribution of the field Φ_α. Therefore, they reduce the particular n-point partial wave to a (n-1)-point partial wave in a correlation function.

In order to determine the operators E'_α, we use that they are supposed to satisfy the *intertwining property* [8], so that they leave the action of the conformal algebra invariant:

Definition 5.2.1. (Intertwining property)
Let (E_1, α_1) and (E_2, α_2) be representations of an algebra $\mathcal{A}$. We say that a linear map $T : E_1 \to E_2$ intertwines α_1 and α_2 if

$$\forall\, a \in \mathcal{A} \quad \alpha_2(a) \circ T = T \circ \alpha_1(a). \tag{5.2.8}$$

In this case T is called an intertwining operator for α_1 and α_2.

Thus, we demand

$$\begin{aligned} E'_\alpha \langle [A, \Phi_1 \Phi_2] \Phi_\alpha \rangle &= \langle [A, \Phi_\alpha] \Phi_\alpha \rangle \\ &= \langle [A, E'_\alpha \Phi_1 \Phi_2] \Phi_\alpha \rangle \end{aligned} \tag{5.2.9}$$

for any generator A of the conformal algebra. Using the commutation relations with a covariant field $\Phi(x)$ (scalar or tensor) transforming under the representation λ,

$$i[A, \Phi(x)] = a^\lambda \Phi(x)$$

gives rise to differential operators $a_1^d + a_2^d$ and $a^{(\kappa L)}$, respectively. For scalar fields a^d are given by the commutation relations (2.2.5) and for the symmetric traceless tensor fields (contracted with a polarization vector v) $a^{(\kappa L)}$ are specified by

$$\begin{aligned} i[P_\mu, \Phi(x)] &= \partial_\mu \Phi(x), \\ i[D, \Phi(x)] &= (x \cdot \partial + (2\kappa + L))\Phi(x), \\ i[M_{\mu\nu}, \Phi(x)] &= (x_\mu \partial_\nu - x_\nu \partial_\mu + v_\mu \partial^v_\nu - v_\nu \partial^v_\mu)\Phi(x) \\ &= (x \wedge \partial + v \wedge \partial^v)\Phi(x), \text{and} \\ i[K_\mu, \Phi(x)] &= (2x_\mu(x \cdot \partial) - x^2 \partial + 2(2\kappa + L)x_\mu + 2v_\mu(x\partial^v) - 2(x \cdot v)\partial^v)\Phi(x) \\ &= (2x_\mu(x \cdot \partial) - x^2 \partial + 2(2\kappa + L)x_\mu - 2x \cdot (v \wedge \partial^v))\Phi(x). \end{aligned} \tag{5.2.10}$$

Now equation (5.2.9) reads

$$\iota_x \circ E'_\alpha \circ (a_1^d + a_2^d) \langle \Phi_1 \Phi_2 \Phi_\alpha \rangle = a^{(\kappa L)} \circ \iota_x \circ E'_\alpha \langle \Phi_1 \Phi_2 \Phi_\alpha \rangle \tag{5.2.11}$$

We start with the ansatz $E'_\alpha = E_\alpha(x_1 + x_2, v, \partial_1, \partial_2) \circ (\varrho_{12})^d$. A dependence on $x_1 - x_2$

does not need to be taken into account since E'_α is always followed by ι_x. v is again the polarisation vector. Note that by conformal invariance, the pole bounds ensure a singularity in ϱ_{12} of at most ϱ_{12}^{-d} and therefore, the evaluation map is applied to a regular function if E_α is regular. We tighten the condition and demand equality also on the operator level:

$$\iota_x \circ E_\alpha \circ (\varrho_{12})^d \circ (a_1^d + a_2^d) = a^{2\kappa+L} \circ \iota_x \circ E_\alpha \circ (\varrho_{12})^d. \tag{5.2.12}$$

Commuting the factor $(\varrho_{12})^d$ past the operators $(a_1^d + a_2^d)$ on the left-hand side systematically removes the d dependencies for every generator of the conformal algebra. Hence, we are left with the condition

$$\iota_x \circ E_\alpha \circ (a_1 + a_2) = a^{2\kappa+L} \circ \iota_x \circ E_\alpha \tag{5.2.13}$$

for two scalar fields of the same scaling dimension.

In the following calculations, we often need to commute $x_i \circ E_\alpha$. Since E_α depends on ∂_i, we can use the identity

$$E_\alpha \circ x_i = x_i \circ E_\alpha + \nabla_i E_\alpha, \tag{5.2.14}$$

where ∇_i denotes the derivative w. r. t. ∂_i. We already know that they have to be homogeneous of degree L and harmonic in v. Evaluating the intertwining conditions (5.2.13) with the generators

1) $A = P_\mu$,

2) $A = D$, and

3) $A = M_{\mu\nu}$

gives information about the general shape of E_α, namely:

1) The generators of translations yield

$$(\partial_1 + \partial_2)E_\alpha = 0, \tag{5.2.15}$$

i. e. E_α does not depend on $x_1 + x_2$.

2) The generator of dilations gives rise to

$$(\partial_1\nabla_1 + \partial_2\nabla_2)E_\alpha = (2\kappa + L)E_\alpha, \tag{5.2.16}$$

where ∇_i denotes again the derivative w. r. t. ∂_i. This means that E_α is homogeneous in ∂_1 and ∂_2 of degree $2\kappa + L$.

3) Finally, the generators of Lorentz transformations lead to

$$(\partial_1 \wedge \nabla_1 + \partial_2 \wedge \nabla_2 + v \wedge \partial^v)E_\alpha = 0, \tag{5.2.17}$$

i. e. E_α is a Lorentz scalar depending only on the variables v^2, $(v\partial_i)$ and $(\partial_i\partial_j)$, $i, j = 1, 2$.

It follows from the homogeneities that E_α can only be a polynomial in the variables y_1 and v if κ is an integer which is ensured by global conformal invariance.

The generators of the special conformal transformations K_μ give detailed information about the operators. Evaluation of the intertwining condition (5.2.13) and using the previous results (5.2.16) and (5.2.17) yields a second order partial differential equation for E_α:

$$[2(y_1\nabla_1)\nabla_1 - y_1\nabla_1^2 + 2(y_2\nabla_2)\nabla_2 - y_2\nabla_2^2]E_\alpha(v, y_1, y_2) = 0, \tag{5.2.18}$$

where $y_i = \partial_i$.

In order to solve this differential equation, one can make the ansatz

$$E_\alpha = (y_1y_2)^\kappa \left[(vy_1 + vy_2)^L \cdot e_\alpha(p, q, r)\right]_0, \tag{5.2.19}$$

with the new variables

$$p = \frac{y_1^2}{y_1y_2}, \qquad q = \frac{y_2^2}{y_1y_2}, \qquad r = \frac{vy_1 - vy_2}{vy_1 + vy_2}.$$

$[\cdot]_0$ denotes again the traceless part w. r. t. v. Note that E_α can only be a polynomial if $e_\alpha(p, q, r)$ is at most of degree L in r and of degree κ in p and q. The ansatz (5.2.19) has been chosen, because it respects the homogeneities and the new variables are three linearly independent combinations of the possible scalar combinations of the original variables. Since we take the traceless part w. r. t. v, we do not have to take into account terms $\propto v^2$. These terms are uniquely determined by the ansatz (5.2.19) and the harmonicity in v, cf. Lemma (3.2.1). Combined with the homogeneity conditions, only three linearly independent combinations exist.

With these variables, the specific operators (5.2.6) can be expressed as

$$\begin{aligned}
e_{10} &= 1, \\
e_{11} &= r + \frac{r}{2}(p+q) - \frac{p-q}{2}, \\
e_{12} &= 1 - 3r^2 - 3(p+q)r^2 + 3(p-q)r, \\
e_{20} &= 1 + p + q + \frac{pq}{2}, \text{ and} \\
e_{0L} &= \sum_{s+t=L} (r+1)^s (r-1)^t \frac{1}{s!(s-1)!t!(t-1)!}.
\end{aligned}$$

With

$$\begin{pmatrix} \nabla_1 p & \nabla_2 p \\ \nabla_1 q & \nabla_2 q \end{pmatrix} = \frac{1}{(y_1 y_2)} \begin{pmatrix} 2y_1 - py_2 & -py_1 \\ -qy_2 & 2y_2 - qy_1 \end{pmatrix} \tag{5.2.20}$$

and

$$\nabla_i r = \frac{v}{vy_1 + vy_2} \left((-1)^{i+1} - r \right) \tag{5.2.21}$$

and ignoring all terms $\propto v^2$, the substitution of (5.2.19) in eq. (5.2.18) gives rise to a system of three differential equations:

$$\left[L(L+1) + (1-r^2)\partial_r^2 + 2\kappa(L - r\partial_r) + 2(p\partial_p - q\partial_q)\partial_r \right] e_\alpha = 0 \tag{5.2.22}$$

$$\begin{aligned}
&[4(p\partial_p - 1)\partial_p - q(\kappa - p\partial_p - q\partial_q)(\kappa - 1 - \partial_p - q\partial_q) + \\
&\quad 2(\kappa - p\partial_p - q\partial_q)(\kappa - 1 - p\partial_p + q\partial_q + (r-1)\partial_r)] \, e_\alpha = 0,
\end{aligned} \tag{5.2.23}$$

$$\begin{aligned}
&[4(q\partial_q - 1)\partial_q - p(\kappa - p\partial_p - q\partial_q)(\kappa - 1 - \partial_p - q\partial_q) + \\
&\quad 2(\kappa - p\partial_p - q\partial_q)(\kappa - 1 + p\partial_p - q\partial_q + (r+1)\partial_r)] \, e_\alpha = 0.
\end{aligned} \tag{5.2.24}$$

Eq. (5.2.22) can now be solved in the following way: Since the equation is homogeneous in p and q, we can make the ansatz

$$e_\alpha(p,q,r) = \sum_{m,n \geq 0, m+n \leq \kappa} p^m q^n e_{mn}(r), \tag{5.2.25}$$

where every contribution $p^m q^n e_{mn}(r)$ has to solve eq. (5.2.22) separately. Using the known homogenities and therefore setting $p\partial_p = m$ and $q\partial_q = n$ (Euler operators), we find that (5.2.22) no longer involves ∂_p or ∂_q but only the difference $\delta = m - n$. The substitution $z = \frac{1-r}{2}$ finally yields Euler's hypergeometric differential equation:

$$\left[z(z-1)\partial_z^2 + (2\kappa z - (\kappa - \delta))\partial_z - L(2\kappa + L - 1)\right] e_{\alpha;m,n} = 0. \tag{5.2.26}$$

Its solutions are

$$\begin{aligned} e_{\alpha,mn}(r) = c_{\alpha;m,n} \cdot f_{\alpha,\delta}(r) &= c_{\alpha;m,n} \cdot (\kappa - \delta)_L \cdot {}_2F_1\left(-L, L + 2\kappa - 1, \kappa - \delta; \frac{1-r}{2}\right) \\ &= c_{\alpha;m,n}(-1)^L f_{\alpha,-\delta}(-r) \end{aligned} \tag{5.2.27}$$

with coefficients $c_{\alpha;m,n}$ to be determined.

The coefficients $c_{\alpha;m,n}$ are fixed by the other two differential equations (5.2.23) and (5.2.24). In order to solve for them we use

$${}_2F_1(\alpha, \beta, \gamma + 1, z) = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)} \left[(1-z)\partial_z - (\alpha + \beta - \gamma)\right] {}_2F_1(\alpha, \beta, \gamma, z) \tag{5.2.28}$$

and

$${}_2F_1(\alpha, \beta, \gamma - 1, z) = \frac{1}{\gamma - 1} \left[z\partial_z + (\gamma - 1)\right] {}_2F_1(\alpha, \beta, \gamma, z), \tag{5.2.29}$$

which follows from eq. (15.2.4) and (15.2.6) in [1]. It gives rise to operators

$$A^{\pm}_{\alpha,\delta} = \frac{(r \mp 1)\partial_r + \kappa - 1 \mp \delta}{L + \kappa - 1 \mp \delta} \tag{5.2.30}$$

which lower or raise the parameter δ in $f_{\alpha,\delta}$ by 1:

$$A^{\pm}_{\alpha,\delta} f_{\alpha,\delta} = f_{\alpha,\delta\pm1}. \tag{5.2.31}$$

Now we can use the operators (5.2.30) and the fact that $c_{\alpha;m,n} = 0$ for $m + n > \kappa$. The two equations (5.2.23) and (5.2.24) then give rise to recursive systems for the coefficients:

$$\begin{aligned} 4(m^2 - 1)c_{\alpha;m+1,n} + 2(\kappa - m - n)(L + \kappa - 1 - \delta)c_{\alpha;m,n} \\ -(\kappa - m - n)(\kappa - m - n + 1)c_{\alpha;m,n-1} = 0, \end{aligned} \tag{5.2.32}$$

$$\begin{aligned} 4(n^2 - 1)c_{\alpha;m,n+1} + 2(\kappa - m - n)(L + \kappa - 1 + \delta)c_{\alpha;m,n} \\ -(\kappa - m - n)(\kappa - m - n + 1)c_{\alpha;m-1,n} = 0. \end{aligned} \tag{5.2.33}$$

The following proposition summarizes our results.

Proposition 5.2.2. *The intertwining operators in eq. (5.2.13) which reduce a n-point function of two scalar fields of equal scaling dimension coupling to a symmetric traceless tensor field to a (n-1)-point function involving two tensor fields belonging to the same representation are given by*

$$E'_\alpha = (y_1 y_2)^\kappa \left[(vy_1 + vy_2)^L \sum_{m,n\geq 0, m+n\leq\kappa} c_{\alpha;m,n} p^m q^n (\kappa-\delta)_L \cdot {}_2F_1(\alpha,\beta,\gamma;z) \right]_0 \circ \varrho_{12}^d, \tag{5.2.34}$$

where $\alpha = -L$, $\beta = L + 2\kappa - 1$, $\gamma = \kappa - \delta$, $z = \frac{1-r}{2}$ *and the coefficients* $c_{\alpha;m,n}$ *solve the recurrences (5.2.32) and (5.2.33).*

In general, by applying the equations (15.2.2) and (15.4.4) from [1], we find that $f_{\alpha,0}$ are derivatives of Legendre polynomials P_m. We use

$${}_2F_1(\alpha+n,\beta+n,\gamma+n,z) = \frac{(\gamma)_n}{(\alpha)_n(\beta)_n}\partial_z^n {}_2F_1(\alpha,\beta,\gamma,z) \tag{5.2.35}$$

with $n = \kappa - 1$ and

$${}_2F_1(-m, m+1, 1, z) = P_m(1-2z) \tag{5.2.36}$$

with $m = L + \kappa - 1$. They result to

$$f_{\alpha,0} = \frac{2^{\kappa-1} L!}{(L+\kappa)_{\kappa-1}} \partial_r^{\kappa-1} P_{L+\kappa-1}(r). \tag{5.2.37}$$

Application of the operators (5.2.30) now permits to express all functions $f_{\alpha,\delta}$ by derivatives of Legendre polynomials. In the interesting case $\kappa = 1$ (twist-2), we find the expression:

$$e_{1L} = \left(1 + \frac{p}{2}(r-1)\partial_r + \frac{q}{2}(1+r)\partial_r\right) P_L(r). \tag{5.2.38}$$

So far, we have not investigated the recurrences (5.2.32) and (5.2.33) intensively and systematically. We have some first results in the case $L = 0$ and for small values of κ. It turns out that at least for $\kappa \leq 5$, most solution spaces are one-dimensional. In detail, we found (recall that $m + n \leq \kappa$ has to be satisfied)

- $\kappa = 1$: c_{00} is nonzero and can be chosen arbitrarily (corresponding to the choice of a normalization).
- $\kappa = 2$: Only c_{00}, c_{10}, c_{01} and c_{11} are nonzero and fixing one of them already

determines the other coefficients.

- $\kappa = 3$: There are two independent solutions: $c_{21} = 2c_{20}$ and $c_{12} = 2c_{02}$, otherwise we find $c_{m,n} = 0$.
- $\kappa = 4$: Only c_{22} is nonzero and can be chosen arbitrarily.
- $\kappa = 5$: $c_{23} = c_{32} = -\frac{2}{3}c_{22}$, otherwise $c_{m,n} = 0$.

Starting from these results, we can show (also for general $L \neq 0$) that in case $\kappa \geq 4$ the coefficients $c_{m,n}$ are zero for $m < 2$ or $n < 2$. Due to the factor $(n^2 - 1)$ or $(m^2 - 1)$, the coefficients $c_{m,2}$ and $c_{2,n}$ decouple from the equations for $n = 1$ and $m = 1$. Hence, the equations for $c_{m,n}$ are overdetermined for $n < 2$ or $m < 2$ and therefore, $c_{m,n} = 0$ in these cases. Using again the recurrences we find that for $m \leq 2$ and $n > 2$ (or $n \leq 2$ and $m > 2$), all $c_{m,n}$ are expressible by the former coefficients and therefore, $c_{m,n} = 0$ also in these cases. Hence, $c_{m,n}$ are zero for $m < 2$ or $n < 2$.

We have found closed formulas for c_{2,n_0} and $c_{m_0,2}$ (with $n_0 \leq \kappa - 2$ or $m_0 \leq \kappa - 2$, resp.):

$$c_{2,n_0} = \left(-\frac{1}{2}\right)^{n_0-2} \cdot \frac{(\kappa-4)!(L+\kappa-1)!}{(\kappa-n_0-2)!(L+\kappa-n_0+1)!} \cdot \prod_{a=1}^{n_0-2} \left((n_0-a)^2-1\right)^{-1} \cdot c_{22}$$

$$c_{m_0,2} = \left(-\frac{1}{2}\right)^{m_0-2} \cdot \frac{(\kappa-4)!(L+\kappa-1)!}{(\kappa-m_0-2)!(L+\kappa-m_0+1)!} \cdot \prod_{a=1}^{m_0-2} \left((m_0-a)^2-1\right)^{-1} \cdot c_{22}.$$

Here, we could use that $c_{1,n} = c_{m,1} = 0$ for all m, n and therefore the recurrences involved only two coefficients instead of three.

5.2.3 Generalizations

The previous calculations are for two scalar fields of equal scaling dimension coupling to a symmetric and traceless tensor field. Next we generalize the considerations: The two fields will still be coupled to a symmetric and traceless tensor in all upcoming considerations, even though it is not neccessarily true that three-point functions involving the two fields are only non-zero if they couple to a symmetric traceless tensor. First we allow different scaling dimensions d_1 and d_2 for the scalar fields Φ_1 and Φ_2. This situation needs an ansatz

$$E'_\alpha = E_\alpha(v, \partial_1, \partial_2) \circ (\varrho_{12})^{\frac{d_1+d_2}{2}}, \tag{5.2.39}$$

where we have already used that the first condition ($A = P_\mu$) will exclude any dependence on $x_1 - x_2$. In a GCI theory, $d_1 + d_2$ has to be an even number because otherwise, the three-point function $\langle \Phi_1 \Phi_2 T_{\kappa,L} \rangle$ would not be rational. Again, we know that the differential operators have to be homogeneous of degree L and harmonic in v. Proceeding as before, we can commute the factor $(\varrho_{12})^{\frac{d_1+d_2}{2}}$ past $a_1^{d_1} + a_2^{d_2}$. For $A \neq K_\mu$ it cancels again the dependences of $a_i^{d_i}$ on the scaling dimensions. The intertwining property tells us again that the operators are Lorentz scalars and homogeneous of degree $2\kappa + L$ in ∂_i. In the case $A = K_\mu$, the dependences on the scaling dimensions do not cancel entirely but yield an additional term in the differential equation which determines our operators:

$$\left[2(y_1\nabla_1)\nabla_1 - y_1\nabla_1^2 + 2(y_2\nabla_2)\nabla_2 - y_2\nabla_2^2 + (d_1 - d_2)(\nabla_1 - \nabla_2)\right] E_\alpha = 0. \quad (5.2.40)$$

Next we can relax the assumption that Φ_1 and Φ_2 need to be scalar fields. They are now demanded to be symmetric traceless tensor fields transforming in the same representation $\beta = (\kappa', L')$ coupling to a symmetric traceless tensor field ($\alpha = (\kappa, L)$):

$$E'_\alpha \langle \Phi_1^\beta \Phi_2^\beta \Phi_\alpha \rangle.$$

Here, we make the ansatz

$$E'_\alpha = E_\alpha(v, y_1, y_2, \partial^{v_1}, \partial^{v_2}) \circ (\varrho_{12})^{2\kappa' + 2L'} \quad (5.2.41)$$

with v_1 and v_2 being two additional polarization vectors. In order to ensure regularity, the exponent of the factor ϱ_{12} has been chosen in such a way that it equals the pole bound. The operators have to be homogeneous of degree L and harmonic in v and ∂^{v_i}. Using $\partial^{v_i} E_\alpha = 0$, we find again from the intertwining property that E_α is a Lorentz scalar. Note that in this case, the Lorentz condition has changed to

$$(\partial_1 \wedge \nabla_1 + \partial_2 \wedge \nabla_2 + v \wedge \partial^v + \partial^{v_1} \wedge \nabla_{v_1} + \partial^{v_2} \wedge \nabla_{v_2}) E_\alpha = 0, \quad (5.2.42)$$

where ∇_{v_i} is the derivative w. r. t. ∂^{v_i}. Here, the differential operators are homogeneous in ∂_i of degree $2L' + 2\kappa + L$:

$$(\partial_1 \nabla_1 + \partial_2 \nabla_2) E_\alpha = (2L' + 2\kappa + L) E_\alpha. \quad (5.2.43)$$

Evaluating the interwining condition for $A = K_\mu$ and applying eq. (5.2.43) and (5.2.42) yields the differential equation which determines the differential operators:

$$\big[2(y_1\nabla_1)\nabla_1 - y_1\nabla_1^2 + 2\nabla_1(\partial^{v_1} \wedge \nabla_{v_1}) + 2(y_2\nabla_2)\nabla_2 - y_2\nabla_2^2 \\ +2\nabla_2(\partial^{v_2} \wedge \nabla_{v_2}) - 2L'(\nabla_1 + \nabla_2)\big]\, E_\alpha = 0, \qquad (5.2.44)$$

Combining the two previous cases, we now consider symmetric traceless tensors transforming in different representations $\alpha_1 = (\kappa_1, L_1)$, $\alpha_2 = (\kappa_2, L_2)$ and start with the ansatz

$$E'_\alpha = E_\alpha(v, y_1, y_2, \partial^{v_1}, \partial^{v_2}) \circ (\varrho_{12})^n.$$

The exponent n needs to be chosen in such a way that it ensures regularity. The operators E_α with $\alpha = (\kappa, L)$ are again homogeneous of degree L and harmonic in v and ∂^{v_i}. Furthermore, they are Lorentz scalars and homogeneous of degree $2\kappa + L + 2n - d_1 - d_2$ in ∂_i, where $d_i = 2\kappa_i + L_i$. Moreover, they have to satisfy the equation

$$\big[2(y_1\nabla_1)\nabla_1 - y_1\nabla_1^2 + 2\nabla_1(\partial^{v_1} \wedge \nabla_{v_1}) + 2(y_2\nabla_2)\nabla_2 - y_2\nabla_2^2 \\ +2\nabla_2(\partial^{v_2} \wedge \nabla_{v_2}) + 2d_1\nabla_1 + 2d_2\nabla_2 - 2n(\nabla_1 + \nabla_2)\big]\, E_\alpha = 0. \qquad (5.2.45)$$

Since this kind of partial differential equations are too complicated to be treated within the preparation of this master's thesis, we have so far been concentrating on the case of two scalar fields of equal scaling dimension coupling to one symmetric traceless tensor field in further considerations. Additionally, in the case of non-symmetric tensor fields, one would have to introduce more polarization vectors, leading to an even more complicated structure.

6 Conservation laws and applications

In the following, we would like to further investigate the twist $2\kappa = 2$ contribution to the OPE in a correlation function. We consider again a correlation function involving two scalar fields of equal scaling dimension coupling to a symmetric traceless tensor and use the operators found in the previous chapter in order to project on the twist-2 contribution. As we said in chapter 3.3, the twist-2 part of a correlation function has to be conserved. In this chapter, we start analyzing the resulting higher conservation laws. In particular, we discuss the strategy in Section 6.1 and perform the calculations for the cases $L = 1$ and $L > 1$ individually in Section 6.2. Last, we start applying our recent results in Section 6.3.

6.1 Strategy

Using the operators (5.2.38)

$$E_{1L} \propto (y_1 y_2)(vy_1 + vy_2)^L \left(1 + \frac{p}{2}(r-1)\partial_r + \frac{q}{2}(1+r)\partial_r\right) P_L(r)$$

found in the previous chapter, we can project a correlation function on the part which arises from the twist-2 contribution of the operator product expansion. Since this part is conserved in a positive theory, we can formulate conservation laws:

$$(\partial_x \partial_v) \circ \iota_x \circ E_{1L} \circ (\varrho_{12})U = \iota_x \circ (\partial_1 + \partial_2) \cdot \partial_v \circ E_{1L} \circ (\varrho_{12})U = 0. \tag{6.1.1}$$

These conservation laws can be regarded as positivity test.

We would like to apply the operators in (6.1.1) to regular correlation functions U without vacuum contribution, cf. expression (3.2.1). We therefore define $D_L = E_{1L} \circ (\varrho_{12})$. With the identity (5.2.14), we can commute the factor ϱ_{12} past the operator E_{1L} and make use of the application of the map ι_x afterwards. Terms involving a factor x_1 or x_2 systematically cancel and we are left with the expression

$$D_L = (\nabla_1 - \nabla_2)^2 E_{1L}. \tag{6.1.2}$$

Our conservation laws can now be written as

$$\left[\iota_x \circ (\partial_1 + \partial_2)\partial_v D_L\right]_0 U = 0 \tag{6.1.3}$$

and we define

$$\begin{aligned} G_L &:= (y_1 + y_2) \cdot \partial_v (\nabla_1 - \nabla_2)^2 E_{1L} \\ &= \left[\iota \circ (\partial_1 + \partial_2)\partial_v D_L\right]_0 . \end{aligned} \tag{6.1.4}$$

In order to take only the traceless part in the end, we have to consider terms $\hat{D}_L$ of D_L of order $(v^2)^0$ and also $(v^2)^1$, since we calculate the divergence $\partial_v D_L$. Making the ansatz $\hat{D}_L = D'_L + v^2 Q(v)$ with Q linear in v and calculating the trace, we find

$$\begin{aligned} 0 &= \partial_v \partial^v (D'_L + v^2 Q) \\ &= \partial_v \partial^v D'_L - 4v\partial_v Q - 8Q \\ &= \partial_v \partial^v D'_L - 4LQ \\ \implies \quad Q &= \frac{1}{4L} \partial_v \partial^v D'_L , \end{aligned}$$

where we used that Q is homogeneous in v of degree $L-2$ and that $v\partial_v$ is the Euler operator. In total, we find

$$\begin{aligned} G_L &= \left[(y_1 + y_2)\partial_v \left(D - \frac{1}{4L} v^2 \partial_v \partial^v D \right) \right]_0 \\ &= \left[\left((y_1 + y_2)\partial_v - \frac{1}{2L}(vy_1 + vy_2)\partial_v \partial^v \right) D \right]_0 . \end{aligned} \tag{6.1.5}$$

6.2 Calculations

In the case $L = 1$, the conservation law (6.1.3) is trivially fulfilled, because already $\left[\iota_x \circ (\partial_1 + \partial_2)\partial_v D_1\right]_0 = 0$. Instead, we can set

$$G_1 = y_1^2 - y_2^2, \tag{6.2.1}$$

because it results from the correct projection operator $D_1 = v \cdot (\partial_1 - \partial_2)$ on the conserved vector current by taking the divergence of D_1.

Below, we will often use the Legendre differential equation in order to avoid higher derivatives than ∂_r^2:

$$(1-r^2)\partial_r^2 P_L(r) - 2r\partial_r P_L(r) + L(L+1)P_L(r) = 0.$$

First, we calculate the operators D'_L which are the parts of D_L of order $(v^2)^0$. We find:

$$D'_L = 4(L^2 - L - 2)(vy_1 + vy_2)^L P_L(r). \tag{6.2.2}$$

In order to compare them with the operators (5.1.2), we use the representation

$$P_L(r) = \sum_{k=0}^{L} (-1)^k \binom{L}{k}^2 \left(\frac{1+r}{2}\right)^{L-k} \left(\frac{1-r}{2}\right)^k \tag{6.2.3}$$

of Legendre polynomials, which follows from Bonnet's recursion formula [1]

$$(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z).$$

Writing

$$\frac{1+r}{2} = \frac{vy_1}{vy_1+vy_2}, \qquad \frac{1-r}{2} = \frac{vy_2}{vy_1+vy_2}, \qquad \binom{L}{L-m} = \binom{L}{m} \tag{6.2.4}$$

and $m = L - k$ yields

$$D'_L \propto \sum_{m=0}^{L} (-1)^{L-m} \binom{L}{m}^2 (vy_1)^m (vy_2)^{L-m}. \tag{6.2.5}$$

Up to proportionality factors, these operators are equal to the $(v^2)^0$-part of the reduction operators we had found before, where $c = 0$ in (5.1.2).

Continuing with the calculation of the operators (6.2.6), we find the following proposition.

Proposition 6.2.1. *In a unitary GCI theory, the operators $\iota_x \circ G_L$ with*

$$G_L = \frac{2(L-1)(L+2)}{L} \left[(vy_+)^{L-1} \left(2y_+y_-\partial_r P_L + (2ry_+y_- - y_-^2 - y_+^2)\partial_r^2 P_L\right)\right]_0, \tag{6.2.6}$$

where $y_\pm = y_1 \pm y_2$, annihilate all (regular) correlation functions.

Note the prefactor $(L-1)$ in equation (6.2.6), which confirms that the case $L=1$ is specific and cannot be treated in the way presented above. Moreover, the operators involve only terms proportional to Laplacians and therefore, they automatically annihilate all twist-2 contributions.

By construction, the $\iota \circ G_L$ annihilate all functions containing a factor ϱ_{12}, which corresponds to $\kappa > 1$. The new information of these conservation laws is that $\iota \circ G_L$ also annihilates all leading twist $2\kappa = 2$ parts in the OPE.

Nikolov, Rehren and Todorov have studied previously [15] the case that a function U admits a biharmonic completion and found a third order partial differential equation as condition. The structure of this PDE is different from the structure of G_L. It would be interesting to study to what extent these different conditions are equivalent.

6.3 Applications

In order to start applying our recent results, we reduce the six-point structure (3.3.1) to the four-point structure corresponding to the $(2\kappa, L) = (2,1)$-contribution both in the OPE of $\Phi_1\Phi_2$ and of $\Phi_5\Phi_6$. The six-point structure F_0 is given by

$$F_0 = \frac{\left(\varrho_{15}\varrho_{26}\varrho_{34} - 2\varrho_{15}\varrho_{23}\varrho_{46} - 2\varrho_{15}\varrho_{24}\varrho_{36}\right)_{[1,2][5,6]}}{\varrho_{12}^{d-1} \cdot \varrho_{14}\varrho_{23}\varrho_{13}\varrho_{24}\varrho_{35}\varrho_{36}\varrho_{45}\varrho_{46} \cdot \varrho_{56}^{d-1}}. \tag{6.3.1}$$

Recall that it is the leading part of a biharmonic function satisfying all properties of a Wightman distribution except possibly positivity.

We use the operators introduced in Section 6.2. In this special case, they read

$$\begin{aligned} D_1^{12} &= (\nabla_1 - \nabla_2)^2 E_{11} \\ &= v \cdot (\partial_1 - \partial_2) \end{aligned} \tag{6.3.2}$$

and

$$D_1^{56} = w \cdot (\partial_6 - \partial_5). \tag{6.3.3}$$

In order to apply D_1^{12}, the following reorganisation of the terms of F_0 is useful:

$$F_0 = \frac{2}{\varrho_{12}^{d-1} \cdot \varrho_{14}\varrho_{23}\varrho_{13}\varrho_{24}\varrho_{35}\varrho_{36}\varrho_{45}\varrho_{46} \cdot \varrho_{56}^{d-1}} \big((\varrho_{15}\varrho_{26}\varrho_{34})_{[1,2]} - (\varrho_{15}\varrho_{23}\varrho_{46})_{[1,2]} - (\varrho_{15}\varrho_{24}\varrho_{36})_{[1,2]} - (\varrho_{26}\varrho_{13}\varrho_{45})_{[1,2]} - (\varrho_{26}\varrho_{14}\varrho_{35})_{[1,2]} \big) . \qquad (6.3.4)$$

Applying the operator $\iota_{x_0} D_1^{12}$ to one of the terms in the numerator yields

$$\iota_{x_0} v \cdot (\partial_1 - \partial_2) \ (\varrho_{1k}\varrho_{2j})_{[1,2]} = 4(v \cdot x_{0k}\varrho_{0j})_{[k,j]} . \qquad (6.3.5)$$

Moreover, the anti-symmetry in x_1, x_2 of the numerator of F_0 implies that any term multiplied by the numerator itself equals zero after application of ι_{x_0}. Using (6.3.5) and reorganizing the terms, $\iota_{x_0} D_1^{12}$ reduces the six-point structure to

$$\iota_{x_0} D_1^{12} \varrho_{12}^{d-1} \varrho_{56}^{d-1} F_0 = \frac{8}{\varrho_{03}^2 \varrho_{04}^2 \varrho_{35}\varrho_{36}\varrho_{45}\varrho_{46}} \big(-v \cdot x_{03}(\varrho_{45}\varrho_{06})_{[5,6]} - v \cdot x_{04}(\varrho_{35}\varrho_{06})_{[5,6]} + \varrho_{34}(v \cdot x_{05}\varrho_{06})_{[5,6]} - \varrho_{03}(v \cdot x_{05}\varrho_{46})_{[5,6]} - \varrho_{04}(v \cdot x_{05}\varrho_{36})_{[5,6]} \big) . \qquad (6.3.6)$$

Next, $\iota_{x_7} D_1^{56}$ is applied. We find

$$\iota_{x_7} w \cdot (\partial_6 - \partial_5)(v \cdot x_{k5}\varrho_{j6})_{[5,6]} = 2v \cdot w \varrho_{j7} - 4v \cdot x_{k7} w \cdot x_{j7} \qquad (6.3.7)$$

$$\iota_{x_7} w \cdot (\partial_6 - \partial_5)(\varrho_{k5}\varrho_{j6})_{[5,6]} = 4(x_{k7}\varrho_{j7})_{[j,k]} . \qquad (6.3.8)$$

Combining the results, we obtain the following four-point structure:

$$\iota_{x_7} D_1^{56} \iota_{x_0} D_1^{12} \varrho_{12}^{d-1} \varrho_{56}^{d-1} F_0 = \frac{16}{\varrho_{04}^2 \varrho_{03}^2 \varrho_{37}^2 \varrho_{47}^2} \cdot \big(v \cdot w(\varrho_{34}\varrho_{07} - \varrho_{03}\varrho_{47} - \varrho_{04}\varrho_{37}) - 2v \cdot x_{03}(w \cdot x_{47}\varrho_{07})_{[0,4]} - 2v \cdot x_{04}(w \cdot x_{37}\varrho_{07})_{[0,3]} + 2v \cdot x_{07}(\varrho_{03} w \cdot x_{47} + \varrho_{04} w \cdot x_{37}) - \varrho_{34} w \cdot x_{07} \big) . \qquad (6.3.9)$$

The result corresponds to a contribution to a correlation function of the form $\langle J_0 \Phi_3 \Phi_4 J_7 \rangle$, where J_0 and J_7 are vector currents.

For comparison, another six-point structure is presented in [13]:

$$
\begin{aligned}
B(x_1,\dots,x_6) &= \frac{1}{\varrho_{12}^{d-1}}\cdot\left(\frac{1}{\varrho_{14}\varrho_{23}}\right)_{[1,2]}\cdot\frac{1}{\varrho_{34}}\cdot\left(\frac{1}{\varrho_{36}\varrho_{45}}\right)_{[5,6]}\cdot\frac{1}{\varrho_{56}^{d-1}}\\
&= \frac{1}{\varrho_{12}^{d-1}}\cdot\left(\frac{1}{\varrho_{14}\varrho_{23}}-\frac{1}{\varrho_{24}\varrho_{13}}\right)\cdot\frac{1}{\varrho_{34}}\cdot\left(\frac{1}{\varrho_{36}\varrho_{45}}-\frac{1}{\varrho_{46}\varrho_{35}}\right)\cdot\frac{1}{\varrho_{56}^{d-1}}.
\end{aligned}
\tag{6.3.10}
$$

This structure exhibits the same symmetries as F_0, but it does not have any double poles (i. e. it arises in a free theory) and is separately biharmonic both in x_1, x_2 and x_5, x_6. It contributes to the six-point function of cubic Wick products of a massless complex scalar field.

Application of $\iota_{x_0}D_1^{12}$ and $\iota_{x_7}D_1^{56}$ yields

$$
\iota_{x_7}D_1^{56}\iota_{x_0}D_1^{12}\varrho_{12}^{d-1}\varrho_{56}^{d-1}B(x_1,\dots,x_6) = \frac{-16}{\varrho_{34}}\cdot\frac{(v\cdot x_{03}\varrho_{04})_{[3,4]}}{\varrho_{03}^2\varrho_{04}^2}\cdot\frac{(v\cdot x_{47}\varrho_{37})_{[3,4]}}{\varrho_{37}^2\varrho_{47}^2}. \tag{6.3.11}
$$

These calculations represent only the very beginning of using the methods presented to study positivity of the structure F_0. We would need to apply all operators E_α, and find operators which further reduce the resulting four-point structures of the form $\langle J_0\Phi_3\Phi_4J_7\rangle$. In Section 5.2.3, we have started the determination of these operators. The latter completes the analysis by reducing the structure to the final form of a two-point function.

7 Concluding remarks

We have shown that postulation of global conformal invariance in a Wightman theory permits far reaching investigations of structural aspects in axiomatic quantum field theories. Here, we have restricted our analysis to four spacetime dimensions.

In a theory satisfying GCI, the field content as well as the Wightman distributions are very constrained. In particular, we can make use of the fact that the twist-2 part of the OPE of two fields in a correlation function is conserved. Using other constraints imposed by GCI on the correlation functions, it is possible to formulate a condition for the leading part of the twist-2 contribution to be nontrivial, i. e. that it cannot arise from a combination of canonical free fields. These candidates have to be tested if they are positive and thereby would yield a physical theory.

A useful tool in order to investigate a conformal quantum field theory is the partial wave expansion. The PWE expresses a correlation function in terms of eigenfunctions of the Casimir operators. At the same time, the eigenspaces of the Casimir operators are irreducible representation spaces. The eigenfunctions or partial waves are universal for any conformal QFT and therefore, the physical properties of a particular theory is encoded in the coefficients. In principle, the PWE is determined in any QFT exhibiting a conformal symmetry, but the actual computation of it is very complicated. In the special case of four-point functions of scalar fields, the PWE is well studied and the partial waves as well as the coefficiants are explicitly known. The complexity of the problem for higher correlation functions has prevented us to date from performing an explicit PWE in a more general case.

Instead of trying a direct computation of the PWE, we have pursued a different strategy. Using orthogonality relation between partial waves, it is possible to reduce n-point functions. In the case of the twist-2 contribution to a correlation function of scalar fields, a closed formula for local differential operators which reduce the contribution to a n-point function to (n-1)-point functions is found. Here, we could use the special properties of the image of the operators in order to determine them.

For arbitrary twist contributions, we can use the intertwining property of the differential operators, i. e. they should commute with the group action. Applying commutation relations for the fields with the generators of the conformal group, it is possible to fix the structure of the operators and find differential equations which they have to fulfil. In the case of correlation functions involving only scalar fields of the same scaling dimension, we could solve the differential equation. In principle, the solution is a sum of polynomial hypergeometric functions with coefficients which are determined by recursive equations. To date, we have not found a closed formula for these coefficients.

Starting to relax some conditions, we have studied the intertwining condition in more general cases. Assuming different scaling dimensions and/or correlation functions involving tensor fields, we have determined the general shape of the differential operators and also the determining differential equations. Solving the equations is still an open problem.

Last, the conserved twist-2 contribution was again in the focus. We have studied implications of the conservation laws by using the differential operators which reduce n-point functions. By construction, the operators projecting on a twist-2 contribution annihilate all other twist contributions. Using the operators projecting a n-point function of scalar fields of the same scaling dimension on the twist-2 part, we have calculated the operators which annihilate all contributions to a correlation functions, provided the theory satisfies positivity. Investigating the structure of these conservation laws and a comparision with previous results [15] are still open tasks.

One major reason for the analyses in this thesis was the question, whether the 6-point structure (3.3.1) satisfies Hilbert space positivity. It would therefore be interesting to see if our recent results give already new information when applied to this structure which might be part of a nontrivial theory.

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Erklärung nach §18(8) der Prüfungsordnung für den Bachelor-Studiengang Physik und den Master-Studiengang Physik an der Universität Göttingen:

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(Lena Marie Wallenhorst)