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De-Sitter-Vakua in Supergravitationsmodellen

deSitter vacua in supergravity models

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Abstract

We study large-volume limits of heterotic string compactifications on Calabi-Yau threefolds. By demanding the existence of a metastable deSitter vacuum in the low-energy supergravity approximations of such theories we find constraints on the Calabi-Yau intersection numbers. It turns out that in most nontrivial cases these constraints are usually not global on the moduli space parameterizing the Calabi-Yau Kähler structure, i.e. they depend on non-topological properties of the compactification manifold. We formulate the constraints in terms of a non-linear eigenvalue problem and use this formulation to identify one example of a global invariant, the hyperdeterminant of the Calabi-Yau intersection tensor. We also perform a complete analysis of the metastability condition in some special cases. Finally, the sGoldstino mass is computed and shown to be unbounded on moduli spaces of metastable deSitter theories.

Zusammenfassung

Wir betrachten Large-Volume-Approximationen von auf Calabi-Yau-Mannigfaltigkeiten kompaktifizierter heterotischer Stringtheorie. Die Bedingung der Existenz von metastabilen deSitter-Vakua in der zugehörigen Niedrigenergie-Supergravitationsnäherung ergibt Einschränkungen an die Calabi-Yau-Schnittzahlen. Diese Einschränkungen sind im Allgemeinen nicht global in Bezug auf den Kähler-Moduliraum, hängen also von nicht-topologischen Eigenschaften der Calabi-Yau-Mannigfaltigkeit ab. Wir geben eine Formulierung dieser Einschränkungen in Form eines nichtlinearen Eigenwertproblems an und benutzen diese Formulierung, um eine globale Invariante zu identifizieren, die Hyperdeterminante der Calabi-Yau-Schnittzahlen. Außerdem analysieren wir wichtige Spezialfälle von heterotischen Kompaktifizierungen im Detail. Wir schließen mit einer Berechnung der sGoldstino-Masse und zeigen, dass diese in Moduliräumen von metastabilen deSitter-Theorien unbeschränkt ist.

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1. Introduction

1.1. Introduction

Our understanding of nature on its most fundamental level is essentially based on two theories: The general theory of relativity, which describes gravitational interactions for sufficiently low energies, and the standard model of particle physics, which is used to describe all non-gravitational interactions known to us. Both theories have passed many highly nontrivial experimental tests since their initial formulation (see [1] for a review on tests of general relativity and [2] for a collection of precision results including a comparison to theoretical predictions in particle physics) and up to now, no significant deviation of experimental results from theoretical predictions has been discovered.

However, a number of conceptual as well as practical shortcomings illustrate the necessity of an even more sophisticated description of nature, which is not known to us yet. The most obvious problem of the status of fundamental theoretical physics is the incompatibility between general relativity and quantum field theory, the theoretical concept underlying the standard model of particle physics. This incompatibility manifests itself immediately in naive approaches to a reconcilement of the two theories by the observation that the resulting theory of quantum gravity lacks the property of renormalizability, i.e. there is no satisfactory way to deal with divergent quantities in computations and thus the predictivity of the theory is spoiled (see [3] and [4] for very accessible reviews on the topic and [5] for a more advanced overview). On the other hand, there is no reasonable doubt that gravity has to be, at least partially, described as a quantum theory at some more fundamental level. A quantum theory of gravity is for example believed to be needed to describe the evolution of the universe shortly after the big bang.

The standard model of particle physics has, even beyond the absence of gravitational interactions, additional shortcomings. These do usually not spoil its predictive power or internal consistency (which is believed to last at least up to the scale of grand unification $m_{GUT} \sim 10^{16}$ GeV if the recent clues for a Higgs mass of $m_H \approx 125$ GeV are correct [6]) but are more of a conceptual and aesthetic nature. A well-known example for such problems is the Higgs hierarchy problem: in its simplest formulation this is the observation of a huge and seemingly unnatural hierarchy between the GUT scale m_{GUT} and the scale of electroweak symmetry breaking m_H. As the Higgs is a scalar particle, its propagator and therefore its mass potentially receives corrections from all other particles present in the theory, driving its mass up to very large scales if not a rather miraculous cancellation occurs.

Another puzzling aspect of the interplay of general relativity and quantum field theory is the cosmological constant, i.e. the energy density of the vacuum of space. The existence of a cosmological constant is compatible with the concepts of general relativity and since the discovery of the accelerating expansion of the universe (see [7] [8] for some of the experimental measurements) it is the most widely accepted explanation for the hypothesized dark energy driving this acceleration. Combining astrophysical observations with measurements of the cosmic microwave background it can be shown (see e.g. [9]) that the cosmological constant has to be of order

$$\Lambda \sim 10^{-122} M_{Pl}^4 \tag{1.1}$$

if expressed in Planck units $M_{Pl} \approx 2.44 \cdot 10^{18}$ GeV. The smallness of this value poses severe problems if quantum field theory is used to explain dark energy, as typical estimates from QFT are of the order M_{Pl}^4 (see [10] and [11] for more details).

Several strategies have been proposed to deal with the aforementioned (and other) shortcomings on different levels. The two paradigms arguably dominating research in fundamental theoretical physics during the last decades are supersymmetry and string theory. The first assumes the existence of a symmetry relating bosonic and fermionic particles. In addition to the mathematical appeal of supersymmetric theories, this idea has the potential to cure some conceptual problems of the standard model of particle physics without leaving the framework of relativistic quantum field theory. The hierarchy problem, for example, can be mitigated drastically by the supersymmetric cancellation of fermionic with bosonic contributions to the quantum corrections to the Higgs mass. We will discuss supersymmetry in more detail in the next section.

String theory on the other hand is conceptually quite different from quantum field theory, though it is still a Lorentz invariant quantum theory. It promises to provide a consistent theory containing the standard model of particle physics as well as general relativity as different limits. However, due to the limited understanding of string theory it still lacks predictivity. We give more details in sections 1.3 and 1.4.

1.2. Supersymmetry and Supergravity

There are good reasons to believe that fundamental physics is Lorentz invariant (see for example [12] for astrophysical observations of gamma ray bursts and [13] for a broader review article on the topic). Lorentz invariance in (quantum) field theories is implemented by requiring that the Poincaré Lie algebra is part of the symmetry Lie algebra of the field theory. Under some basic assumptions the Coleman-Mandula theorem states that the symmetry group of any Lorentz invariant quantum field theory has (locally) to be of the form

$$G_{sym.} = \mathbb{R}^4_{translations} \rtimes SO(1,3) \times G_{int} \tag{1.2}$$

where G_{int} is a (in most interesting cases compact) group of internal symmetries [14]. Thus, allowing only Lie algebras for the generators of symmetries, a nontrivial commutator between internal symmetries and spacetime symmetries is forbidden. However, the assumptions can be violated by extending the Lie algebra to a so-called super Lie algebra, i.e. an algebra containing two sets of elements: even (bosonic) and odd (fermionic) ones, where the latter satisfy anticommutator relations between each other. This results in more interesting possibilities for the symmetry (super) Lie algebra. Theories invariant under such extended symmetries are called *supersymmetric*.

In the simplest variant, $\mathcal{N} = 1$ supersymmetry, one family of fermionic operators is introduced, usually labeled Q_α, where α is a spinor index. All (anti-)commutators of the symmetry algebra can then be fixed using the Haag-Lopuszanski-Sohnius theorem [15]:

$$[P_\mu, Q_\alpha] = 0 = [P_\mu, Q^\dagger_{\dot\alpha}] \tag{1.3}$$

$$[M_{\mu\nu}, Q_\alpha] = \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \tag{1.4}$$

$$[M_{\mu\nu}, Q^{\dagger\dot\alpha}] = \mathrm{i}(\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} Q^{\dagger\dot\beta} \tag{1.5}$$

$$\{Q_\alpha, Q^\dagger_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \tag{1.6}$$

$$\{Q_\alpha, Q_\beta\} = 0 = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\}. \tag{1.7}$$

Here, we adopted the convention in [15] in which the index of $Q^\dagger$ carries an additional dot. σ^μ are the Pauli matrices and $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\sigma^\mu, \sigma^\nu]$ denote the commutators of Pauli matrices. The generators of translations are denoted P_μ and the Lorentz generators $M_{\mu\nu}$.

Additional fermionic symmetries can be introduced, thus extending $\mathcal{N} = 1$ supersymmetry to e.g. $\mathcal{N} = 2$ supersymmetry. In $3+1$ spacetime dimensions, only $\mathcal{N} = 1$ supersymmetry has direct phenomenological relevance, because more supersymmetry would prohibit chiral fermions, a necessary ingredient of the standard model of particle physics [16].

One may wonder if global supersymmetry can be promoted to a local symmetry, a process referred to as 'gauging' by particle physicists. From Eq. (1.6) one immediately realizes that the gauging of supersymmetry seems to imply a gauging of spacetime translations as well. An invariance under local translations should make a theory invariant under diffeomorphisms, thus implying the presence of (some variant of) Einstein gravity in the field theory. This argument can be made precise and turns out to be correct (see again [15] for a detailed analysis) and the resulting field theory is usually called *supergravity* as it contains a supersymmetric version of Einstein gravity. We will give more details, in particular about the scalar sector of supergravity theories, in section 2.3.

1.3. String Theory and its compactifications

The most promising known candidate for a consistent theory of quantum gravity is string theory (a nice introduction on undergraduate level is [17], a more advanced introduction is provided by [18] and a useful freely available introduction is provided by

[19]). String theory is a relativistic quantum theory of interacting one-dimensional objects called strings. Gauge bosons can be identified with oscillation modes on strings and a detailed study of the string spectrum reveals the presence of a spin-2 particle, the graviton. A careful analysis of consistency relations for the embedding of strings in a curved spacetime can be shown to imply the Einstein equations for the background geometry, thus incorporating the theory of general relativity into the string theoretic framework (see [18] for an accessible account of this calculation).

As only bosonic particles can be identified in the most naive approach to string theory (which is therefore called bosonic string theory), the framework cannot give realistic descriptions of nature. The only consistent way known to incorporate fermions into the theory is to supersymmetrize bosonic string theory, giving superstring theory. Superstring theory has other important advantages over bosonic string theory, even if realistic model building is not of immediate concern: bosonic string theories seem to necessarily require a state of negative mass-squared, a tachyon, thus rendering the theory unstable. This tachyon state can be eliminated in superstring theories.

A peculiar feature of string theory is that the dimensionality of spacetime is not arbitrary but strongly constrained by the requirement of Lorentz invariance (and the requirement of anomaly freedom). Without additional, rather contrived mechanisms, the spacetime dimensionality of bosonic string theory is fixed to 26 and the spacetime dimensionality of superstring theory to 10. It turns out that there are 5 different classes of superstring theories, usually denoted type I, type IIa, type IIb, and two heterotic theories. Type I string theory involves both open and closed strings, type IIa and type IIb have only closed strings and have non-chiral or chiral fermions respectively. A heterotic theory has only closed strings and half of the string spectrum is given by a 26-dimensional bosonic string theory. The additional 16 dimensions show up as a gauge group which is either $SO(32)$ or $E_8 \times E_8$. All five string theories are conjectured to be different limits of a single 11-dimensional theory, called M-theory. This claim is supported by the discovery of dualities between some string theories.

The typical strategy to reconcile the presence of 10 spacetime dimensions in superstring theory with the observation of only 4 spacetime dimensions at low energies is to assume that the whole spacetime (at least locally) consists of a flat 4-dimensional part (the local Minkowski spacetime we are living in) and a compact six-dimensional part whose size is small enough to have evaded detection in experiments so far. Schematically, we have

$$M_{10} = \mathbb{R}^4 \times Y_6, \tag{1.8}$$

where Y_6 is some (a priori arbitrary) compact six-dimensional manifold.

The choice of the compact manifold can be restricted by additional assumptions. A well-motivated requirement is the preservation of $\mathcal{N} = 1$-supersymmetry in the flat part $\mathbb{R}^4$. Besides technical and aesthetic reasons this is a necessary requirement if one wants to construct an MSSM-like model as a low-energy limit from string theory. In [20] it was shown that the only class of compact manifolds satisfying this requirement are so-called *Calabi-Yau manifolds*. A Calabi-Yau manifold is in particular a *Kähler manifold*. A Kähler manifold is a complex manifold (with local coordinates z^i and $\bar{z}^i$) with a positive definite metric $g_{i\bar{j}}$ called Kähler metric which is locally given by

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}} K(z^i, \bar{z}^{\bar{i}}), \tag{1.9}$$

where K is a real function called Kähler potential. See section 2.2 for a more detailed discussion of Kähler manifolds.

A compact Kähler manifold is a Calabi-Yau manifold if and only if its first Chern class vanishes. An equivalent condition is that the Kähler metric is Ricci-flat.

If one chooses a Calabi-Yau manifold as the compact part of spacetime and then performs a low-energy limit (including neglecting all higher Kaluza-Klein modes from the compact part) one obtains a 4-dimensional field theory which turns out to be locally supersymmetric, i.e. $\mathcal{N} = 1$-supergravity. The geometrical information about the compact manifold Y_6 is not completely gone from the low-energy theory. In fact, new scalar fields appear – called *moduli fields* – which have the property that their vacuum expectation values parameterize the Calabi-Yau manifold Y_6 (for example its Kähler potential K). Moduli often have flat directions in field space, yielding a huge number of inequivalent vacua of the theory. This corresponds to the large number of possible distinct Calabi-Yau manifolds.

The choice of string theory and compactification manifold fixes the low-energy theory uniquely. However, in practice it is often very difficult or even impossible to determine the Lagrangian of the low-energy field theory from the high-energy input. On the other hand, the geometric part of the supergravity field theory, which is again given by a Kähler manifold (see section 2.3 for details), is usually much simpler and the Kähler potential is known in many important cases; we will give examples later. This raises the question if it is possible to study low-energy phenomenology in such a way that constraints on the Kähler potential can be deduced which are independent from the precise form of the full Lagrangian. These constraints then have a chance to give important restrictions on the high-energy input. As we will see in the next chapter, there are indeed very simple and basic requirements on the low-energy field theory resulting in non-trivial constraints on the low-energy Kähler geometry. The main part of this thesis is concerned with the problem of extracting useful information about the Calabi-Yau manifold Y_6 out of these constraints.

1.4. deSitter vacua in string theory

The most convenient way to obtain a deSitter universe, i.e. a universe with a positive cosmological constant, in string theory would be a positively curved spacetime at the classical level. On first sight, it may seem natural to aim for a Minkowski vacuum at tree level and then involve higher order corrections to lift this to a deSitter vacuum. However, this idea comes with many problems, one of them being the so called Dine-Seiberg problem. The problem is sometimes summarized by the saying "When corrections can be computed, they are not important, and when they are important, they cannot be computed" [21]. The idea is the following: Assume we have a parameter ρ (e.g. a modulus) such that the limit $\rho \to \infty$ corresponds to the weak coupling limit. Let $V(\rho)$ denote the vacuum expectation value of the potential (which is equivalent to the cosmological constant as we will see in section 2.3). By assumption, in the limit $\rho \to \infty$ the tree level prediction is correct and we have $\lim_{\rho\to\infty} V(\rho) = 0$. Now, if for some large ρ we have $V(\rho) > 0$, we have a runaway toward $\rho = \infty$ (i.e. the classical solution) and if for some large ρ we have $V(\rho) < 0$ the scalar is pulled toward the strong coupling regime (see fig. 1.1 for an illustration). There can only be a local minimum if higher order corrections are included and are sizable, but then the weak coupling limit is spoiled. A more precise form of this argument can be found in [22].

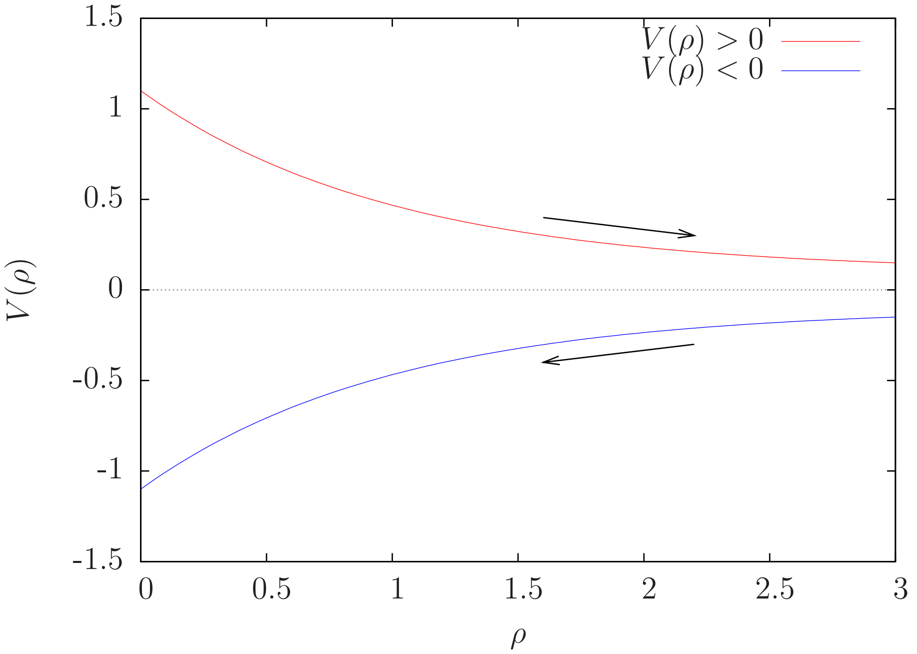

Figure 1.1.: Two possible behaviors of the tree level effective potential. The arrows indicate the direction the field ρ is pulled.

One may conclude from this observation that the string theory vacuum is probably strongly coupled. Alternatively, one may seek for tree level deSitter vacua. Perhaps a bit surprisingly, this turns out to be quite difficult to achieve. In fact, there are several 'no-go theorems' on the market, illustrating the difficulty of constructing a metastable deSitter vacuum in string theory, in particular if additional cosmological constraints such as slow-roll inflation are taken into account; see [23], [24] and [25] for several examples.

These no-go theorems are usually proven by enumerating all possible contributions to the 4-dimensional scalar potential, determining their scaling behavior with respect to the moduli parameterizing the string coupling (i.e. the dilaton) and the volume of the compact manifold and then showing that either the vacuum conditions or slow-roll inflation is incompatible with a positive cosmological constant. Though most available no-go theorems are applicable directly only to type II string theories, the existence of dualities between the five string theories can be seen as evidence that the existence of metastable deSitter vacua is a subtle issue for all string inspired models. However, during the last years, additional techniques have been developed, in particular the idea of 'flux compactifications' (see [26] for a review), which can be used to partially circumvent the known no-go theorems at the cost of making the theory more complicated. This provides additional motivation for the strategy briefly outlined at the end of the last section, with which one may hope to be able to make nontrivial phenomenological statements without full knowledge of the complicated low-energy Lagrangian.

1.5. Thesis outline

This thesis is organized as follows: We review the basic properties of (the scalar sector of) supergravity theories in chapter 2. The basic phenomenological requirements of metastability and a positive cosmological constant we are going to impose are explained in sections 2.4 and 2.5. We then briefly repeat the arguments in [27] to obtain a simple encoding of these requirements into the sign of a single function σ which only depends on the internal geometry of the scalar sector. We go on to review no-scale supergravity models in section 2.6 and in particular the no-scale models obtained from low-energy heterotic string theory in chapter 3. In these more specific situations some simplifications are possible and the problem of metastability can be reformulated in terms of a second function ω. We state the precise form of σ and ω in this case.

Chapter 4 contains the main part of this thesis. It is concerned with the maximization of ω as the sign of its global maximum has to be determined for the metastability analysis. In section 4.1 we calculate ω and study its dependence on complex phases of its argument, the Goldstino direction G^i. We conclude that these phases can be discarded at least in the two- and three-dimensional cases. The situation for two-dimensional moduli spaces has already been studied in [27]. We review and simplify the analysis in this situation in section 4.2 and find a natural formulation of the problem in terms of a tensorial eigenvalue problem in section 4.2.2. We then study the three-dimensional case in section 4.3. After performing part of the extremization problem we show in section 4.3.1 that the remaining part can be expressed by the same tensorial eigenvalue problem used in the two-dimensional case. This observation is used to find the generalization of the two-dimensional formula Eq. (4.26) in Eq. (4.108). We study important special cases of three-dimensional moduli spaces in section 4.4. We then further generalize the results in section 4.3 to arbitrary-dimensional moduli spaces in section 4.5. We close the chapter with some remarks about the inclusion of matter fields into the analysis in section 4.7.

Chapter 5 studies the implications of the analysis on the sGoldstino mass. We show in section 5.1 that the maximization of ω and the maximization of σ are equivalent problems in the case of heterotic compactifications and generalize the explicit formula for the sGoldstino mass obtained for two-dimensional moduli spaces in [28] to the arbitrary-dimensional case. We then show in section 5.2 that metastable heterotic models necessarily have to have arbitrarily massive sGoldstinos in their moduli spaces with divergences occurring at certain singularities.

We conclude and summarize in chapter 6.

2. Stability in Supergravity

In this chapter we review the scalar sector of locally supersymmetric field theories and explain the phenomenological constraints we are going to impose on low-energy limits of string theory.

2.1. Units and conventions

During this thesis we always use (reduced) Planck units, i.e.

$$\hbar = c = 1, \quad M_{Pl} \equiv \frac{1}{\sqrt{8\pi G_N}} = 1, \tag{2.1}$$

where G_N is the Newton constant.

If not explicitly stated otherwise, repeated indices are always summed.

2.2. Kähler geometry

Kähler geometry will play an important role in the following. In this section we give a brief introduction to Kähler manifolds and state the most important formulas.

A Kähler manifold is a complex analytic manifold (with local coordinates z^i and $\bar{z}^{\bar{i}}$), i.e. a smooth manifold with analytic transition maps, with a positive definite Hermitian form (the Kähler metric) g, obtained locally by

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}} K(z^i, \bar{z}^{\bar{i}}). \tag{2.2}$$

The real function K is called the *Kähler potential.* The components g_{ij} and $g_{\bar{i}\bar{j}}$ vanish by hermiticity.

The Christoffel symbols are defined by the same formula as in Riemannian geometry:

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right), \tag{2.3}$$

where μ, ν, ρ, σ denote both holomorphic and antiholomorphic indices. Using Eq. (2.2) and the hermiticity of g it follows that the only non-vanishing Christoffel symbols are

$$\Gamma_{ij}{}^{k} = \left(\partial_i g_{j\bar{l}}\right) g^{\bar{l}k}, \qquad \Gamma_{\bar{i}\bar{j}}{}^{\bar{k}} = \overline{\left(\Gamma_{ij}{}^{k}\right)}. \tag{2.4}$$

Defining the Christoffel symbols by Eq. (2.3) makes sure that the covariant derivatives ∇_i and $\nabla_{\bar{i}}$ defined in the usual way, e.g. for a vector V_i

$$\nabla_i V_j := \partial_i V_j - \Gamma^k_{ij} V_k \tag{2.5}$$

$$\nabla_{\bar{i}} V_j := \partial_{\bar{i}} V_j, \tag{2.6}$$

are in fact covariant and the Kähler metric g is covariantly constant:

$$\nabla_k g_{i\bar{j}} = 0 = \nabla_{\bar{k}} g_{i\bar{j}}. \tag{2.7}$$

As in Riemannian geometry, the Riemann tensor is defined by the commutator of two covariant derivatives:

$$\left[\nabla_i, \nabla_{\bar{j}}\right] V_k =: R_{i\bar{j}k}{}^{l} V_l =: R_{i\bar{j}k\bar{l}} V^{\bar{l}}. \tag{2.8}$$

The Riemann tensor can be computed directly from the Christoffel symbols. The non-vanishing components are given by

$$R_{i\bar{j}k\bar{l}} = -R_{\bar{j}ik\bar{l}} = -g_{m\bar{l}} \partial_{\bar{j}} \Gamma_{ik}{}^{m}, \qquad R_{i\bar{j}\bar{l}k} = -R_{\bar{j}i\bar{l}k} = g_{k\bar{m}} \partial_i \Gamma_{\bar{j}\bar{l}}{}^{\bar{m}}. \tag{2.9}$$

The Ricci tensor can now be defined by

$$R_{i\bar{j}} := -g^{k\bar{l}} R_{k\bar{l}i\bar{j}}. \tag{2.10}$$

2.3. The supergravity action

The most general supergravity Lagrangian involving only derivatives up to second order is uniquely fixed by a single real function G of the superfields Φ^i (or equivalently of the scalar fields ϕ^i) and their conjugates $\overline{\Phi}^{\bar{i}}$. G can be decomposed in terms of the real Kähler potential K and the holomorphic superpotential W as (see for example [15] for a derivation)

$$G(\Phi, \overline{\Phi}) = K(\Phi, \overline{\Phi}) + \log |W(\Phi)|^2. \tag{2.11}$$

Note that this decomposition is only unique up to Kähler transformations:

$$K \to K + f + \bar{f} \tag{2.12}$$

$$W \to W e^{-f} \tag{2.13}$$

for an arbitrary holomorphic function f.

The bosonic part of the action (without non-gravitational gauge fields) can be written as

$$S = \int \sqrt{-h} \left[\frac{1}{2} R[h] - g_{i\bar{j}} \partial\phi^i \cdot \partial\overline{\phi}^{\bar{j}} - V(\phi, \overline{\phi}) \right]. \tag{2.14}$$

The first term involves the space-time metric h and is simply the Einstein-Hilbert action. This part will not play an important role in the following. The second term is the kinetic part, which is non-standard if $g_{i\bar{j}} \neq \delta_{i\bar{j}}$[1]. This metric is a Kähler metric given by $g_{i\bar{j}} = K_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \frac{\partial^2 K}{\partial\phi^i \partial\bar{\phi}^{\bar{j}}}$ and can be interpreted as the metric on a Kähler manifold whose coordinates are the scalar fields. For the kinetic energy of the scalar fields to be positive, g is assumed to be positive definite. Note that the standard kinetic term can be recovered as a special case by choosing the Kähler potential to be

$$K = \sum_i |\phi^i|^2. \tag{2.15}$$

[1] Note however that the dot product between the gradients of the scalar fields is still given by the spacetime metric h.

The last term is the scalar potential, given by

$$V = e^{G}\left(G^{i}G_{i} - 3\right), \tag{2.16}$$

where $G_i = \partial_i G$.

To study phenomenology, a vacuum has to be fixed. Possible vacua are configurations in field space satisfying

$$\partial_i V = 0, \tag{2.17}$$

i.e. critical points of the potential. Using Eq. (2.16), this equation is equivalent to

$$e^{G}\left(G_i + G^k \nabla_i G_k\right) = 0, \tag{2.18}$$

where ∇_i denotes the Kähler-covariant derivative defined in Eq. (2.5).

Every chiral multiplet contains additional scalar auxiliary fields which are non-dynamical and therefore are fixed by their (algebraic) equations of motions, in this case to $F^i = e^{G/2}G^i$. Spontaneous F-term supersymmetry breaking requires that the vacuum expectation value of F^i does not vanish: $\langle F^i \rangle \neq 0$. We will in the following omit the $\langle \dots \rangle$-brackets for quantities evaluated at the vacuum, because this should be clear from context.

The vector G^i defines the direction of the Goldstino in the space of chiral fermions, i.e. the Goldstino is given by $\psi = G^i \chi_i$, where χ_i are chiral fermions. The Goldstino itself gets absorbed by the gravitino via the so-called super Higgs effect (see [29] for details), thus giving the gravitino a mass $m_{3/2} = e^{G/2}$. The scalar partner of the Goldstino is called sGoldstino.

2.4. Masses and stability

The mass squared matrix is given by the Hessian of the potential, evaluated at the vacuum:

$$M^2 = \begin{pmatrix} V_{i\bar{j}} & V_{ij} \\ V_{\bar{i}\bar{j}} & V_{\bar{i}j} \end{pmatrix}. \tag{2.19}$$

Using Eq. (2.16) and Eq. (2.18), the entries can be worked out and are given by (see [27])

$$V_{i\bar{j}} = e^G \left(G_{i\bar{j}} + \nabla_i G_k \nabla_{\bar{j}} G^k - R_{i\bar{j}m\bar{n}} G^m G^{\bar{n}}\right) + \left(G_{i\bar{j}} - G_i G_{\bar{j}}\right) V \tag{2.20}$$

$$V_{ij} = e^G \left(2\nabla_i G_j + G^k \nabla_i \nabla_j G_k\right) + \left(\nabla_i G_j - G_i G_j\right) V, \tag{2.21}$$

where $R_{i\bar{j}m\bar{n}}$ is the Riemann tensor of the complex Kähler geometry.

Metastability of the vacuum is equivalent to the positive definiteness of this $2n \times 2n$-matrix. Because a quadratic matrix is positive definite if and only if all its principal submatrices are positive definite, we can obtain a weaker (necessary) condition for metastability by the requirement that the $n \times n$-matrix $V_{i\bar{j}}$ is positive definite.

We now think of the superpotential W as arbitrary and the Kähler potential K as fixed. By tuning the superpotential (i.e. by adding large supersymmetric mass terms) most of the eigenvalues of M^2 can be made positive. There is however one exception: The Goldstino direction G^i vanishes at supersymmetric vacua and the Goldstino multiplet therefore cannot receive supersymmetric mass contributions. Thus, we can expect that the projection of $V_{i\bar{j}}$ along G^i does not depend on the superpotential. To be more specific, define the quantity

$$\lambda = e^{-G} V_{i\bar{j}} G^i G^{\bar{j}}. \tag{2.22}$$

This is a positive combination of eigenvalues of $V_{i\bar{j}}$ and should therefore be positive if M^2 is a positive definite matrix. Using Eq. (2.20) and Eq. (2.18) to calculate λ more explicitly, one finds (as first derived in [30])

$$\lambda = e^{-G} V_{i\bar{j}} G^i G^{\bar{j}} = 2 g_{i\bar{j}} G^i G^{\bar{j}} - R_{i\bar{j}m\overline{n}} G^i G^{\bar{j}} G^m G^{\overline{n}}. \tag{2.23}$$

The vacuum expectation value of G^i does depend on the superpotential and can be varied by varying W. The coefficients in λ however depend only on the Kähler geometry and the condition

$$\max_{G^i} \{\lambda\} = \max_{G^i} \left\{2 g_{i\bar{j}} G^i G^{\bar{j}} - R_{i\bar{j}m\overline{n}} G^i G^{\bar{j}} G^m G^{\overline{n}}\right\} \overset{!}{>} 0 \tag{2.24}$$

for the existence of a supersymmetry breaking metastable vacuum therefore does only depend on the Kähler potential K and gives a constraint on possible Kähler potentials in viable theories.

2.5. The cosmological constant

From Eq. (2.14) one sees that the cosmological constant in supergravity theories is given by the vacuum expectation value of the potential V. In units of the gravitino mass $m_{3/2}$, the dimensionless quantity

$$\gamma = \frac{V}{3M_{Pl}^2 m_{3/2}^2} \tag{2.25}$$

can be taken to parameterize the cosmological constant.

As recent measurements show that the cosmological constant is small and positive, a viable model needs a small and positive γ. [2]

To take the constraints coming from a non-negative cosmological constant into account, it is useful to rewrite λ as (see again [27])

$$\lambda = -\frac{2}{3} e^{-G} V \left(e^{-G} V + 3\right) + \sigma \tag{2.26}$$

$$\text{with} \qquad \sigma = \left[\frac{2}{3} g_{i\bar{j}} g_{m\bar{n}} - R_{i\bar{j}m\bar{n}}\right] G^i G^{\bar{j}} G^m G^{\bar{n}}. \tag{2.27}$$

Then the sign of σ does only depend on the orientation of G^i, not on its length. Assume there is a vector G^i such that $\sigma(G^i) > 0$. By a scaling $G^i \to rG^i$ we can always achieve $V(rG^i) = 0$ (i.e. a Minkowski vacuum) and because $\sigma(rG^i)$ is still positive we have $\lambda(rG^i) > 0$. By increasing r a bit further, we get $V(rG^i) > 0$ and – if the increase in r is small enough – still have $\lambda(rG^i) > 0$. Conversely, if $\sigma < 0$ for all directions of G^i, λ can never be made positive as long as we demand $V(G^i) > 0$. In summary, we have

$$V \geq 0 \text{ and } \lambda > 0 \text{ is possible} \Leftrightarrow \sigma > 0 \tag{2.28}$$

and the necessary condition on the existence of metastable deSitter vacua is reduced to the analysis of the sign of σ.

[2] Note that the gravitino mass scale coincides with the scale of supersymmetry breaking, which should at least be 1TeV according to current experimental observations. The cosmological constant is roughly of the order $(10^{-12}\text{GeV})^4$.

2.6. No-scale models

To proceed, more assumptions on the model have to be made. We are particularly interested in supergravity models arising in string theory compactifications and therefore it is natural to look for typical properties of these classes of models. One prominent feature (at least approximately) satisfied by most low energy string theory models is the no-scale property

$$K_i K^i = 3, \tag{2.29}$$

i.e. a normalization of the gradient of the Kähler potential.

As Eq. (2.29) is valid at any point in field space, one can take derivatives of this relation and deduce restrictions on the Riemann tensor of the Kähler geometry. Taking one derivative, one finds

$$K_i + K^k \nabla_i K_k = 0. \tag{2.30}$$

Taking two derivatives, one can deduce

$$g_{i\bar{j}} + \nabla_i K_k \nabla_{\bar{j}} K^k - R_{i\bar{j}m\overline{n}} K^m K^{\overline{n}} = 0 \tag{2.31}$$

$$2\nabla_i K_j + K^k \nabla_i \nabla_j K_k = 0. \tag{2.32}$$

Taking contractions with $K^i K^{\bar{j}}$ and $K^{\bar{j}}$, one finds

$$R_{i\bar{j}m\overline{n}} K^j K^{\bar{j}} K^m K^{\overline{n}} = 6 \tag{2.33}$$

$$R_{i\bar{j}m\overline{n}} K^{\bar{j}} K^m K^{\overline{n}} = 2K_i. \tag{2.34}$$

As the direction K^i is somewhat special, it is natural to consider a decomposition of the Goldstino direction of the form

$$G^i = \alpha K^i + N^i, \tag{2.35}$$

where $N^i K_i = 0$, i.e. N^i lies in the orthogonal complement of K^i. The projector onto this orthogonal complement is simply given by

$$P_i{}^j = \delta_i{}^j - \frac{1}{3} K_i K^j \tag{2.36}$$

if the no-scale property Eq. (2.29) is satisfied.

Plugging Eq. (2.35) into Eq. (2.27) one finds

$$\sigma = 4|\alpha|^2 \left(g_{i\bar{j}} - R_{i\bar{j}m\bar{n}} K^m K^{\bar{n}}\right) N^i N^{\bar{j}} - \left(\overline{\alpha}^2 R_{i\bar{j}m\bar{n}} K^i K^m N^{\bar{j}} N^{\bar{n}} + \text{c. c.}\right) \\ - 2\left(\overline{\alpha} R_{m\bar{n}i\bar{j}} K^m N^{\bar{n}} N^i N^{\bar{j}} + \text{c. c.}\right) + \left[\frac{2}{3} g_{i\bar{j}} g_{m\bar{n}} - R_{m\bar{n}i\bar{j}}\right] N^i N^{\bar{j}} N^m N^{\bar{n}}. \quad (2.37)$$

Obviously, we have $\sigma = 0$ for $N^i = 0$, i.e. for Goldstinos in K^i-direction.

2.7. Real homogeneous no-scale models

No-scale models arising from string theory compactifications usually have an even stronger property: It turns out that they possess n independent shift symmetries in the superfields: $\delta_i \Phi^j = \mathrm{i}\varepsilon \delta_i{}^j$ with ε real. This implies the existence of a coordinate system in which e^{-K} is a homogeneous polynomial of degree 3 in the variables $\Phi^i + \overline{\Phi}^i$. In particular, the no-scale property $K^i K_i = 3$ can be shown to follow from this [27].

As only the real part of the scalar fields appear in the theory, all derivatives can be thought of as derivatives with respect to real quantities and by the substitutions $G_i \to G_i$, $G_{\bar{i}} \to \overline{G}_i$, $G^i \to \overline{G}^i$, $G^{\bar{i}} \to G^i$ we can therefore drop the bars on top of indices.

In this class of models, σ simplifies to (see again [27])

$$\sigma = -2\left(\alpha\overline{N}^i + \overline{\alpha}N^i\right)\left(\alpha\overline{N}_i + \overline{\alpha}N_i\right) - 2K_{imn}\left(\alpha\overline{N}^i + \overline{\alpha}N^i\right) N^m \overline{N}^n \\ + \left[\frac{2}{3} g_{ij} g_{mn} - R_{ijmn}\right] N^i \overline{N}^j N^m \overline{N}^n. \quad (2.38)$$

By completing the squares in $\alpha\overline{N}^i + \overline{\alpha}N^i$, σ can be rewritten as

$$\sigma = -2 s_i s^i + \omega \quad (2.39)$$

with

$$s^i = \alpha\overline{N}^i + \overline{\alpha}N^i + \frac{1}{2} e^K P^{ij} K_{jmn} N^m \overline{N}^n \quad (2.40)$$

$$\omega = \left[\frac{2}{3} g_{ij} g_{mn} - R_{ijmn} + \frac{1}{2} K_{ijp} P^{pq} K_{qjn}\right] N^i \overline{N}^j N^m \overline{N}^n. \quad (2.41)$$

The first term $-2s_is^i$ in σ is negative-semidefinite. Therefore, a necessary condition for the positivity of σ and therefore for the existence of metastable deSitter vacua is

$$\max_{N^i \perp K^i} \omega(N^i) > 0. \tag{2.42}$$

The next task is to compute ω in specific models, determine its global maximum as a function of N^i and decide whether (and when) this is positive. We will perform such an analysis in the case of compactifications of heterotic string theory in the next two chapters.

3. Compactifications of heterotic string theory

The main part of this thesis is concerned with the study of supergravity models arising from compactifications of heterotic string theory on Calabi-Yau manifolds. In this chapter, we briefly review the known results about the Kähler and complex structure moduli sectors of such theories and prepare the detailed analysis contained in the next chapter.

3.1. Calabi-Yau moduli space

Part of the moduli space of heterotic compactifications consists of the deformations of the Calabi-Yau manifold Y_6. These in turn are divided into deformations of the complex structure and deformations of the Kähler form. The whole moduli space $\mathcal{M}$ of the theory includes an additional one-dimensional space spanned by the dilaton which parameterizes the string coupling. Locally, the moduli space factorizes as

$$\mathcal{M} = \mathcal{M}^{ks} \times \mathcal{M}^{cs} \times \frac{SU(1,\,1)}{U(1)}, \tag{3.1}$$

where the factors are the Kähler structure deformations, the complex structure deformations and the dilaton. Interestingly, it turns out that $\mathcal{M}^{ks}$ and $\mathcal{M}^{cs}$ are itself Kähler manifolds [31].

We will only consider the moduli fields in the metastability analysis. This is valid under the assumption that non-moduli fields (i.e. matter fields) do not participate in supersymmetry breaking and therefore do not contribute to the Goldstino. Note that this assumption is not well motivated from a physics point of view and the analysis has to be generalized eventually. However, we will later give an argument based on [32] that an independent analysis of the moduli space is possible, even if matter fields do contribute significantly to supersymmetry breaking (see section 4.7).

We additionally assume that the dilaton can be neglected in the metastability analysis. Because complex structure moduli and Kähler moduli are interchanged by mirror-symmetry (see for example [33]), we can concentrate on the latter. Let $\mathcal{V}$ denote the classical volume of the Calabi-Yau manifold. It holds (see [31])

$$\mathcal{V} = \frac{4}{3} \int_{Y_6} J \wedge J \wedge J, \tag{3.2}$$

where J is the Kähler $(1,1)$-form. J is a harmonic form and can therefore be written as $J = v^i w_i$, where w_i, $i = 1, \ldots, h^{1,1}$ is a basis of the $H^{1,1}$-cohomology group. In string theory compactifications, an additional geometric structure arises, a real two-form $B = b^i w_i$, which is connected to the metric by supersymmetry. As argued in [31], natural local coordinates on the moduli space $\mathcal{M}^{ks}$ are given by $T^i = v^i + \mathrm{i} b^i$ and the classical volume can be written as

$$\mathcal{V} = \frac{1}{6} d_{ijk} \left(T^i + \overline{T}^i\right) \left(T^j + \overline{T}^j\right) \left(T^k + \overline{T}^k\right). \tag{3.3}$$

The symmetric rank-3 tensor d_{ijk} is defined as

$$d_{ijk} = \int_{Y_6} w_i \wedge w_j \wedge w_k, \tag{3.4}$$

and consists of the Calabi-Yau intersection numbers.

In the large-volume limit, i.e. if the volume of the Calabi-Yau is large compared to the string scale (but still small compared to e.g. LHC scales), the Kähler potential of $\mathcal{M}^{ks}$ is simply given by

$$K = -\log \mathcal{V}. \tag{3.5}$$

Note that in particular the dimension of the moduli space $\mathcal{M}^{ks}$ is given by the $(1,1)$-Betti number of the Calabi-Yau manifold Y_6:

$$p := \dim \mathcal{M}^{ks} = \dim H^{1,1}(Y_6). \tag{3.6}$$

We will usually denote this dimension by p. It coincides with the number of moduli fields we consider.

It is now obvious from Eq. (3.3) that these models are real-homogeneous, as the Kähler potential depends only on the real parts of the fields T^i. The computation of the Kähler metric and the Riemann tensor is straightforward and one finds (see [27])

$$g_{ij} = -\frac{\mathcal{V}_{ij}}{\mathcal{V}} + \frac{\mathcal{V}_i \mathcal{V}_j}{\mathcal{V}^2} = e^K d_{ijk} K^k + K_i K_j \tag{3.7}$$

$$R_{ijmn} = g_{ij} g_{mn} + g_{in} g_{mj} - e^{2K} d_{imp} g^{pq} d_{qjn} \tag{3.8}$$

with

$$K^i = -\left(T^i + \overline{T}^i\right) \tag{3.9}$$

$$K_i = -\frac{1}{2} e^K d_{ijk} \left(T^j + \overline{T}^j\right)\left(T^k + \overline{T}^k\right). \tag{3.10}$$

The quantities ω and s^i from Eq. (2.40) and Eq. (2.41) can now be computed as

$$s^i = \alpha \overline{N}^i + \overline{\alpha} N^i - \frac{1}{2} e^K P^{ij} d_{jmn} N^m \overline{N}^n \tag{3.11}$$

$$\omega = \left(-\frac{4}{3} g_{ij} g_{mn} + \frac{1}{3} g_{im} g_{jn} + \frac{1}{2} e^{2K} d_{ijp} P^{pq} d_{qmn} + e^{2K} d_{imp} P^{pq} d_{qjn}\right) N^i \overline{N}^j N^m \overline{N}^n. \tag{3.12}$$

Using Eq. (3.7) and the no-scale property $K_i K^i = 3$, a couple of useful equations can be derived:

$$K_i = -\frac{1}{2} e^K d_{ijk} K^j K^k \tag{3.13}$$

$$N_i = e^K d_{ijk} N^j K^k \tag{3.14}$$

$$-6 = e^K d_{ijk} K^i K^j K^k \tag{3.15}$$

$$0 = e^K d_{ijk} N^i K^j K^k \tag{3.16}$$

$$1 = e^K d_{ijk} N^i \overline{N}^j K^k, \tag{3.17}$$

where N^i is a unit vector orthogonal to K^i.

3.2. One-moduli Models

The simplest situation is a one-dimensional moduli space. The no-scale property alone implies by Eq. (2.27) and Eq. (2.33) that in these situations

$$\sigma \equiv 0, \tag{3.18}$$

simply by the fact that $G^i \propto K^i$.

In fact, for 1-dimensional moduli spaces a completely general result can be obtained, derived for example in [30]. If the Kähler potential is of the form

$$K = K(\phi^1, \overline{\phi}^{\bar{1}}) \tag{3.19}$$

it holds (as can be easily checked with Eq. (2.9))

$$R_{1\bar{1}1\bar{1}} = g_{1\bar{1}} R = K_{1\bar{1}} R \tag{3.20}$$

with the scalar curvature

$$R = \frac{K_{11\bar{1}\bar{1}}}{K_{1\bar{1}}^2} - \frac{K_{11\bar{1}} K_{1\bar{1}\bar{1}}}{K_{1\bar{1}}^3}. \tag{3.21}$$

It therefore follows with Eq. (2.27) that

$$\sigma = \frac{2}{3} - R. \tag{3.22}$$

This quantity vanishes for the Kähler potential

$$K = -\log\left[d_{111}\phi^1\phi^1\phi^1\right] \tag{3.23}$$

with a real field $\phi^i := T^i + \overline{T}^i$, confirming Eq. (3.18).

3.3. Factorizable models

Particularly simple situations arise if the Kähler geometry factorizes. Locally, this is equivalent to the existence of a coordinate system in which the volume can be written as

$$\mathcal{V} = \frac{1}{6} d_{1ab} \left(T^1 + \overline{T}^1\right) \left(T^a + \overline{T}^a\right) \left(T^b + \overline{T}^b\right), \tag{3.24}$$

where a, b run from 2 to $h^{1,1}$. Examples in which such Kähler potentials arise are given in [34].

It has been shown (see for example [27]) that σ satisfies $\sigma \leq 0$ for every possible Goldstino direction G^i and has its maximum $\sigma = 0$ at $G^i \propto K^i$. As we will mainly study the function ω in the following, we will now attempt to calculate ω explicitly in this case.

As in [27], it follows from Eq. (3.24) that g_{ij} is block diagonal ($g_{1a} = 0$) and that

$$K^1 K_1 = 1, \quad K^a K_a = 2. \tag{3.25}$$

According to Eq. (3.7) we have

$$d_{1ab} = e^{-K} K_1 \left(g_{ab} - K_a K_b\right) \tag{3.26}$$

giving

$$e^{2K} d_{1ac} g^{ec} d_{e1b} = g_{11} g_{ab}, \quad e^{2K} d_{ab1} g^{11} d_{1ce} = \left(g_{ab} - K_a K_b\right) \left(g_{ce} - K_c K_e\right). \tag{3.27}$$

Using Eq. (2.36) and Eq. (3.13) to Eq. (3.17), we can compute

$$\begin{aligned}
\omega &= \left(-\frac{4}{3} g_{ij} g_{mn} + \frac{1}{3} g_{im} g_{jn} + \frac{1}{2} e^{2K} d_{ijp} P^{pq} d_{qmn} + e^{2K} d_{imp} P^{pq} d_{qjn}\right) N^i \overline{N}^j N^m \overline{N}^n \\
&= -\frac{4}{3} \left(N_i \overline{N}^i\right)^2 + \frac{1}{3} \left|N_i N^i\right|^2 + \left[\frac{1}{2} \left(g^{pq} - \frac{1}{3} K^p K^q\right) d_{ijp} d_{qmn} \right. \\
&\qquad\qquad \left. + \left(g^{pq} - \frac{1}{3} K^p K^q\right) d_{imp} d_{qjn}\right] N^i \overline{N}^j N^m \overline{N}^n \\
&= -\frac{4}{3} \left(N_i \overline{N}^i\right)^2 + \frac{1}{3} \left|N_i N^i\right|^2 - \frac{1}{6} \underbrace{\left(e^K d_{ijp} N^i \overline{N}^j K^k\right)^2}_{\left(N_i \overline{N}^i\right)^2} - \frac{1}{3} \underbrace{\left|e^K d_{ijp} N^i N^j K^k\right|^2}_{\left|N_i N^i\right|^2} \\
&\qquad + \left(\frac{1}{2} e^{2K} d_{ijp} g^{pq} d_{qmn} + e^{2K} d_{imp} g^{pq} d_{qjn}\right) N^i \overline{N}^j N^m \overline{N}^n \\
&= -\frac{3}{2} \left(N_i \overline{N}^i\right)^2 + \left(\frac{1}{2} e^{2K} d_{ijp} g^{pq} d_{qmn} + e^{2K} d_{imp} g^{pq} d_{qjn}\right) N^i \overline{N}^j N^m \overline{N}^n.
\end{aligned} \tag{3.28}$$

The second term can be evaluated using Eq. (3.27):

$$
\begin{aligned}
&e^{2K} d_{ijp} g^{pq} d_{qmn} N^i \overline{N}^j N^m \overline{N}^n \\
&\quad = e^{2K} \left[d_{ab1} g^{11} d_{1cd} N^a \overline{N}^b N^c \overline{N}^d + d_{ije} g^{ef} d_{fmn} N^i \overline{N}^j N^m \overline{N}^n \right] \\
&\quad = \Big(\left(g_{ab} - K_a K_b \right) N^a \overline{N}^b \Big)^2 \\
&\qquad + e^{2K} \Big[d_{1ae} g^{ef} d_{f1b} N^1 \overline{N}^a N^1 \overline{N}^b + d_{1ae} g^{ef} d_{fb1} N^1 \overline{N}^a N^b \overline{N}^1 \\
&\qquad + d_{a1e} g^{ef} d_{f1b} N^1 \overline{N}^1 N^a \overline{N}^b + d_{a1e} g^{ef} d_{fb1} N^a \overline{N}^1 N^b \overline{N}^1 \Big] \\
&\quad = \Big(N_a \overline{N}^a - K_a N^a K_b \overline{N}^b \Big)^2 + \left(N_1 N^1 \overline{N}^a \overline{N}_a + \text{c. c.} \right) + 2 N_1 \overline{N}^1 N_a \overline{N}^a \\
&\quad = \Big(N_a \overline{N}^a - K_1 N^1 K_1 \overline{N}^1 \Big)^2 + \left(N_1 N^1 \overline{N}^a \overline{N}_a + \text{c. c.} \right) + 2 N_1 \overline{N}^1 N_a \overline{N}^a \\
&\quad = \Big(N_a \overline{N}^a + N_1 \overline{N}^1 \Big)^2 + \left(N_1 N^1 \overline{N}^a \overline{N}_a + \text{c. c.} \right) - 2 N_1 \overline{N}^1 N_a \overline{N}^a \\
&\quad = \Big(N_i \overline{N}^i \Big)^2 + \left(N_1 N^1 \overline{N}^a \overline{N}_a + \text{c. c.} \right) - 2 N_1 \overline{N}^1 N_a \overline{N}^a,
\end{aligned}
\tag{3.29}
$$

where we used

$$
0 = K_1 N^1 + K_a N^a = K_1 \overline{N}^1 + K_a \overline{N}^a \tag{3.30}
$$

and $K_1 K_1 = g_{11}$. Analogously, we find

$$
\begin{aligned}
&e^{2K} d_{imp} g^{pq} d_{qjn} N^i \overline{N}^j N^m \overline{N}^n \\
&\quad = e^{2K} \left[d_{ab1} g^{11} d_{1cd} N^a \overline{N}^c N^b \overline{N}^d + d_{ije} g^{ef} d_{fmn} N^i \overline{N}^m N^j \overline{N}^n \right] \\
&\quad = \left| \left(g_{ab} - K_a K_b \right) N^a N^b \right|^2 + 4 N_1 \overline{N}^1 N_a \overline{N}^a \\
&\quad = \left| N_i N^i \right|^2 - 2 \left(N_1 N^1 \overline{N}^a \overline{N}_a + \text{c. c.} \right) + 4 N_1 \overline{N}^1 N_a \overline{N}^a.
\end{aligned}
\tag{3.31}
$$

Thus, ω is given by

$$
\omega = - \Big(N_i \overline{N}^i \Big)^2 + \left| N_i N^i \right|^2 - \frac{3}{2} \Big(N_a N^a \overline{N}_1 \overline{N}^1 + \text{c. c.} \Big) + 3 N_a \overline{N}^a N_1 \overline{N}^1. \tag{3.32}
$$

Obviously, we have $\omega \equiv 0$ if N^i is real. Calculating ω more explicitly for complex N^i turns out to be surprisingly difficult. One can directly check that $\omega \equiv 0$ for all N^i for $p = 2$ and $p = 3$. If we choose a coordinate system in which

$$
K^a = 0 \quad \text{for} \quad a > 2, \tag{3.33}
$$

which is always possible, we obtain

$$\omega = -2\frac{\det g}{K_1^2 K_2^2}\left(\Im\left[N^3\overline{N}^4\right]\right)^2 \tag{3.34}$$

in the $p = 4$-dimensional case. If we additionally assume that the intersection numbers d_{12a} vanish for $a > 2$, the metric components g_{2a} vanish for $a > 2$ and ω can be computed to be

$$\omega = -\sum_{a,b=3}^{p}\left(N_a\overline{N}^a N_b\overline{N}^b - N_a N^a \overline{N}_b\overline{N}^b\right). \tag{3.35}$$

So in this case, it always holds that $\omega \leq 0$ and in addition that $\omega = 0$ if and only if N^i is (up to a global phase) real. We have not been able to find a simple formula for ω if the last assumption gets dropped, but there is no apparent reason to believe that the conclusion will change. We will prove in chapter 4 that this is indeed correct, see Lemma 4.1.1.

4. Metastability analysis of heterotic string models

In this chapter we analyze the behavior of ω and σ in the case of compactifications of heterotic string theory. We find evidence that the task of determining the global maximum of ω is equivalent to the study of a nonlinear eigenvalue problem. We then use the mathematical theory of such problems to derive a global constraint on the critical points of ω, valid on the whole moduli space. We close with a few remarks about the inclusion of matter fields in the analysis.

4.1. General form of ω

The first step in the analysis is to calculate ω more explicitly in the general case of a p-dimensional Kähler moduli space for compactifications of heterotic string theory. For this, we choose a real orthonormal basis (consisting of the $p-1$ vectors n^i_α) of the orthogonal complement of K^i, i.e.

$$K_i n^i_\alpha = 0, \quad n_{\alpha i} n^i_\beta = \delta_{\alpha\beta}, \quad \bar{n}^i_\alpha = n^i_\alpha \quad \text{for } \alpha,\, \beta = 1, \ldots, p-1. \tag{4.1}$$

In terms of these basis vectors, the projector P^{ij} onto the orthogonal complement of K^i can simply be written as

$$P^{ij} = \sum_{\alpha=1}^{p-1} n^i_\alpha n^j_\alpha. \tag{4.2}$$

A general unit vector N^i orthogonal to K^i can be parameterized as

$$N^i = \sum_{\alpha=1}^{p-1} e^{\mathrm{i}\varphi_\alpha} c_\alpha n^i_\alpha \tag{4.3}$$

with real phases φ_α and real c_α satisfying

$$\sum_\alpha c_\alpha^2 = 1. \tag{4.4}$$

We also introduce an abbreviation for the contraction of the basis vectors n_α^i and the intersection numbers d_{ijk}:

$$D_{\alpha\beta\gamma} := e^K d_{ijk} n_\alpha^i n_\beta^j n_\gamma^k. \tag{4.5}$$

We call the $D_{\alpha\beta\gamma}$ *transverse intersection numbers*. They are completely symmetric under interchange of indices.

With Eq. (4.2) and Eq. (4.3) ω (see Eq. (3.12)) can be written as

$$\begin{aligned}
\omega &= -\frac{4}{3} + \frac{1}{3}\left|\sum_\alpha c_\alpha^2 e^{2\mathrm{i}\varphi_\alpha}\right|^2 + \frac{1}{2}\sum_\alpha\left(\sum_{\beta\gamma} c_\beta c_\gamma D_{\alpha\beta\gamma} e^{\mathrm{i}(\varphi_\beta-\varphi_\gamma)}\right)^2 \\
&\qquad\qquad + \sum_\alpha\left|\sum_{\beta\gamma} c_\beta c_\gamma D_{\alpha\beta\gamma} e^{\mathrm{i}(\varphi_\beta+\varphi_\gamma)}\right|^2 \\
&= -\frac{4}{3} + \frac{1}{3}\sum_{\alpha\beta} c_\alpha^2 c_\beta^2 \cos(2\varphi_\alpha - 2\varphi_\beta) \\
&\quad + \sum_{\alpha\beta\gamma\delta\eta} c_\beta c_\gamma c_\delta c_\eta D_{\alpha\beta\gamma} D_{\alpha\delta\eta}\left[\frac{1}{2}\cos(\varphi_\beta - \varphi_\gamma - \varphi_\delta + \varphi_\eta)\right. \\
&\qquad\qquad\qquad \left. + \cos(\varphi_\beta + \varphi_\gamma - \varphi_\delta - \varphi_\eta)\right] \\
&= -\frac{3}{2} + \sum_{\beta\gamma\delta\eta} c_\beta c_\gamma c_\delta c_\eta\, C_{\beta\gamma\delta\eta}\left[\frac{1}{2}\cos(\varphi_{\beta\delta} - \varphi_{\gamma\eta}) + \cos(\varphi_{\beta\delta} + \varphi_{\gamma\eta})\right]
\end{aligned} \tag{4.6}$$

where we defined the rank 4 tensor

$$C_{\beta\gamma\delta\eta} := \sum_\alpha D_{\alpha\beta\gamma} D_{\alpha\delta\eta} + \frac{1}{3}\delta_{\beta\gamma}\delta_{\delta\eta} \tag{4.7}$$

and used the abbreviation $\varphi_{\beta\delta} := \varphi_\beta - \varphi_\delta$.

It is obvious from Eq. (4.6) that $\varphi_\alpha = 0$ for all $\alpha = 1, \ldots, p-1$ will always be a critical point of ω (as a function of φ_α). It is however not clear that this critical point will correspond to the global maximum of ω as a function of c_α and φ_α. By explicitly calculating the derivatives $\frac{\partial}{\partial \varphi_\alpha}\omega$ one sees that the critical points of ω with respect to the phases φ_α are given by the solutions of a system of quartic trigonometric equations. Thus, a direct analysis is probably impossible.

However, at least for very low p, the influence of the complex phases can be analyzed directly. The case $p = 2$ has only a single global phase which drops out in ω completely. For $p = 3$, ω is given by

$$\begin{aligned}\omega = &-\frac{3}{2} + \frac{3}{2}c_1^4 C_{1111} + \frac{3}{2}c_2^4 C_{2222} + 6\left(c_1^3 c_2 C_{1112} + c_1 c_2^3 C_{1222}\right)\cos(\varphi_1 - \varphi_2) \\ &+ c_1^2 c_2^2 \left[C_{1122} + 5C_{1212} + (2C_{1122} + C_{1212})\cos(2\varphi_1 - 2\varphi_2)\right]. \end{aligned} \tag{4.8}$$

By swapping the sign of c_1, the term $c_1^3 c_2 C_{1112} + c_1 c_2^3 C_{1222}$ can always be made a positive contribution to ω which is maximal if $\varphi_1 - \varphi_2 = 0$. The term proportional to $2C_{1122} + C_{1212}$ can potentially give a negative contribution which can be reduced (or even turn to a positive one) if the complex phases do not vanish. Therefore, we find that the global maximum of ω in the three-dimensional case $p = 3$ can only have a non-vanishing (non-global) phase $\varphi_1 - \varphi_2$ if

$$2C_{1122} + C_{1212} = \frac{2}{3} + D_{112}^2 + D_{122}^2 + 2D_{111}D_{122} + 2D_{112}D_{222} < 0. \tag{4.9}$$

ω has to be (up to a redefinition of the c_α and the φ_α) independent of the choice of basis vectors n_α^i. In fact, the projector in Eq. (4.2) does not change at all if we rotate two basis vectors n_α^i and n_β^i into each other by

$$\begin{aligned} n_\alpha'^i &= \cos\vartheta\, n_\alpha^i + \sin\vartheta\, n_\beta^i \\ n_\beta'^i &= -\sin\vartheta\, n_\alpha^i + \cos\vartheta\, n_\beta^i \end{aligned} \tag{4.10}$$

and in N^i (Eq. (4.3)) only the coefficients have to be redefined.

The transverse intersection number $D_{\alpha\beta\beta}$ (no sum over β implied) changes its sign if we perform this rotation with $\vartheta = \pi$. This implies that there is a coordinate change given by a rotation with an angle $\vartheta = \vartheta_0$ such that in the new basis it holds $D_{\alpha\beta\beta} = 0$. We have $p - 2$ independent such rotations of the $p - 1$ basis vectors n_α^i, meaning that we can always assume without loss of generality that the $p - 2$ quantities $D_{1\beta\beta}$ for $\beta = 2, \ldots, p - 1$ vanish. For $p = 3$ this assumption is simply

$D_{122} = 0$ and Eq. (4.9) simplifies to

$$\frac{2}{3} + D_{112}^2 + 2D_{112}D_{222} < 0. \tag{4.11}$$

This is only fulfilled if

$$-D_{222} - \sqrt{D_{222}^2 - \frac{2}{3}} < D_{112} < -D_{222} + \sqrt{D_{222}^2 - \frac{2}{3}}. \tag{4.12}$$

In particular, it has to hold that

$$D_{222}^2 > \frac{2}{3}, \tag{4.13}$$

implying that $\omega(c_1 = 0,\, c_2 = 1, \varphi_\alpha = 0) > 0$.

In conclusion, we found that in the $p = 3$-dimensional case it is always safe to neglect the complex phase φ_α if one is only interested in the sign of ω at its global maximum: Even if the true global maximum is attained for $\varphi_\alpha \neq 0$, there will be a (smaller) ω at a point with $\varphi_\alpha = 0$, which is positive. For later reference, we record this result in the following Lemma:

Lemma 4.1.1. *Let $p = 3$ and let N^i_{max} denote the global maximizer of ω. If N^i_{max} has a non-vanishing and non-global complex phase, then there is a real vector N^i_0 such that*

$$\omega(N^i_0) > 0. \tag{4.14}$$

This completes the argument in section 3.3 showing that in the case of factorizable models it always holds $\omega \leq 0$. We have $\omega \equiv 0$ for all real N^i. Lemma 4.1.1 then implies that the global maximum cannot be attained for a complex N^i, thus ω is bounded by 0 from above.

The analysis of the complex phases becomes very complicated for larger p, at least in the parameterization in Eq. (4.3). However, some progress can be made by using the parameterization

$$N^i = \cos\vartheta\, n^i_1 + \sin\vartheta\, e^{i\varphi} n^i_2 \tag{4.15}$$

where the real orthonormal vectors n^i_1 and n^i_2 now depend on additional $p-2$ and

$p-3$ angles respectively.

That such a parameterization is always possible can be seen as follows: Every p-dimensional vector N^i can be written using its real and imaginary parts N_1^i and N_2^i as

$$\begin{aligned} N^i = N_1^i + \mathrm{i} N_2^i &= \left(1 + \mathrm{i}\frac{N_1 \cdot N_2}{N_1 \cdot N_1}\right) N_1^i + \mathrm{i}\left(N_2^i - \frac{N_1 \cdot N_2}{N_1 \cdot N_1} N_1^i\right) \\ &= \sqrt{N_1 \cdot N_1}\left(1 + \mathrm{i}\frac{N_1 \cdot N_2}{N_1 \cdot N_1}\right) n_1^i + \mathrm{i}\sqrt{N_2 \cdot N_2 - \frac{(N_1 \cdot N_2)^2}{N_1 \cdot N_1}}\, n_2^i, \end{aligned} \tag{4.16}$$

thus giving a decomposition into two real orthonormal vectors n_1 and n_2. n_1 is parameterized by $p-2$ angles, n_2 by $p-3$ angles. The three remaining free parameters are given by $N_1 \cdot N_1$, $N_1 \cdot N_2$ and $N_2 \cdot N_2$. By a redefinition of the parameters, N^i can be written as

$$N^i = \sqrt{N \cdot N}\left(\cos\vartheta\, n_1^i + \sin\vartheta\, e^{i\varphi} n_2^i\right), \tag{4.17}$$

if an irrelevant global phase is discarded. Eliminating the length by demanding N^i to be a unit vector gives the parameterization in Eq. (4.15). However, N^i now depends on $(p-2)+(p-3)+1$ angles and one phase, i.e. on $2(p-2)+1$ parameters. Because a global phase of N^i drops out immediately in ω, one parameter is spurious.

We complete n_1^i and n_2^i to a basis n_α^i, $\alpha = 1, \ldots, p-1$. Note that all basis vectors depend on the $2p-5$ angles we did not write down as arguments of n_1^i and n_2^i explicitly. Equation (4.15) has precisely the form of Eq. (4.3) for $p = 3$, but the projector P^{ij} in Eq. (4.2) still involves the full set of $p-1$ basis vectors. Therefore, the form of ω is now identical to the form in Eq. (4.6) for $p = 3$, but the $C_{\beta\gamma\delta\eta}$ involve a sum from 1 to $p-1$. This observation implies that, if the global maximum should have a non-vanishing complex phase φ, we need again

$$2C_{1122} + C_{1212} = \frac{2}{3} + \sum_{\alpha=1}^{p-1}\left(D_{12\alpha}^2 + 2D_{11\alpha}D_{22\alpha}\right) < 0. \tag{4.18}$$

This inequality is satisfied if and only if

$$-D_{222} - \Delta < D_{112} < -D_{222} + \Delta \tag{4.19}$$

with

$$\Delta := \sqrt{D_{222}^2 - \frac{2}{3} - \sum_{\alpha=2}^{p-1} D_{12\alpha}^2 - 2 \sum_{\alpha=1,\,\alpha\neq 2}^{p-1} D_{11\alpha} D_{22\alpha}}. \tag{4.20}$$

Therefore

$$D_{222}^2 > \frac{2}{3} + \sum_{\alpha=2}^{p-1} D_{12\alpha}^2 + 2 \sum_{\alpha=1,\,\alpha\neq 2}^{p-1} D_{11\alpha} D_{22\alpha}. \tag{4.21}$$

However, in the new parameterization Eq. (4.15) we cannot set all terms in the second sum to zero, because the first two basis vectors n_1^i and n_2^i have to be treated differently than the remaining $p-3$ vectors. We are therefore missing an argument why

$$\sum_{\alpha=2}^{p-1} D_{12\alpha}^2 + 2 \sum_{\alpha=1,\,\alpha\neq 2}^{p-1} D_{11\alpha} D_{22\alpha} \geq 0 \tag{4.22}$$

should hold to draw the same conclusion as in the $p = 3$-case.

Nevertheless, in the following we will assume that the complex phases can be discarded. This simplifies the problem significantly. ω is now given by

$$\omega = -1 + \frac{3}{2} \sum_{\alpha\beta\gamma\delta\eta} c_\beta c_\gamma c_\delta c_\eta D_{\alpha\beta\gamma} D_{\alpha\delta\eta} = -1 + \frac{3}{2} \sum_{\alpha=1}^{p-1} D_{\alpha NN}^2, \tag{4.23}$$

with the abbreviation

$$D_{\alpha NN} := e^K d_{ijk} n_\alpha^i N^j N^k. \tag{4.24}$$

Eventually, the influence of the complex phases on the analysis should of course be checked.

4.2. $p = 2$-dimensional moduli spaces

The two-dimensional case $p = 2$ has been analyzed in [27] already. If $p = 2$, the subspace orthogonal to K^i is 1-dimensional and the (up to orientation) only real unit vector orthogonal to K^i is

$$N^i = \frac{1}{\sqrt{3 \det g}} \begin{pmatrix} K_2 \\ -K_1 \end{pmatrix}. \tag{4.25}$$

Because N^i contains no free parameters, no extremization needs to be performed. ω can be obtained directly by a somewhat tedious calculation. The result is

$$\omega = \frac{9}{8} e^{4K} \frac{\mathrm{Det}\,[d_{ijk}]}{\det g^3}, \tag{4.26}$$

where $\mathrm{Det}\,[d_{ijk}]$ is a homogeneous polynomial in the intersection numbers given by the formula

$$\mathrm{Det}\,[d_{ijk}] = d_{111}^2 d_{222}^2 + 4 d_{111} d_{122}^3 + 4 d_{222} d_{112}^3 - 3 d_{112}^2 d_{122}^2 - 6 d_{111} d_{222} d_{112} d_{122}. \tag{4.27}$$

Up to a numerical factor of -27, this is precisely the discriminant of the homogeneous polynomial $f(x^1, x^2) = d_{ijk} x^i x^j x^k$.

The quantities $e^K = \mathcal{V}^{-1}$ and $\det g$ both have to be positive in physical regions of the moduli space, therefore ω is positive if and only if the homogeneous polynomial $\mathrm{Det}\,[d_{ijk}]$ is positive.

The discriminant $-27\,\mathrm{Det}\,[d_{ijk}]$ also shows up at another place: The volume $\mathcal{V} = -6^{-1} d_{ijk} K^i K^j K^k$ can be factorized into three factors

$$-6^{-1} d_{ijk} K^i K^j K^k = -6^{-1} \mathcal{V}_0 \prod_{i=1}^{3} \left(K^1 - \alpha_i K^2\right), \tag{4.28}$$

where α_i are the roots of $\left(K^2\right)^{-3} \mathcal{V}$ in K^1/K^2 and $\mathcal{V}_0 = d_{111} K^1 K^1 K^1$ is the normalization, if and only if the discriminant vanishes: $\mathrm{Det}\,[d_{ijk}] = 0$. In this case, the model is factorizable as defined in section 3.3. Therefore, for $p = 2$ in some sense factorizable geometries form the boundary between models with positive and models with negative ω.

4.2.1. Tensorial eigenvalue problems

We now prepare an alternative and a bit simpler derivation of the formula (4.26) which allows, in some sense, a generalization of this result to larger p. The argument is based on the observation that the discriminant can be understood as a generalization of the determinant of matrices. The determinant $\det A_{ij}$ of some matrix A_{ij} can be (up to a normalization) uniquely defined by its property that it is the minimal-degree homogeneous polynomial in the matrix entries that satisfies

$$\det A_{ij} = 0 \Leftrightarrow \exists v^i \neq 0 : \; A_{ij} v^j = 0, \tag{4.29}$$

i.e. it vanishes if and only if a null-eigenvector exists. Discriminants of homogeneous polynomials have an analogue property:

$$\mathrm{Det}\,[d_{ijk}] = 0 \Leftrightarrow \exists v^i \neq 0 : \; d_{ijk} v^j v^k = 0. \tag{4.30}$$

This motivates the following definition of a tensorial (or nonlinear) eigenvalue problem for (rank 3) tensors:

$$d_{ijk} v^j v^k = \lambda E_{ijk} v^j v^k, \tag{4.31}$$

where $E_{ijk} = \delta_{ij}\delta_{jk}$ is the unit tensor (j is not summed here). v^i is called eigenvector and λ is the corresponding eigenvalue. Equation (4.30) implies that all eigenvalues are roots of the characteristic polynomial

$$\mathrm{Det}\,[d_{ijk} - \lambda E_{ijk}] = 0. \tag{4.32}$$

The definition of the discriminant in Eq. (4.30) does not only make sense for 2-dimensional tensors but in fact defines a homogeneous polynomial in the tensor components d_{ijk} for any dimension (and even for higher-rank tensors). The normalization of this polynomial is fixed by requiring that

$$\mathrm{Det}\,[E_{ijk}] = 1. \tag{4.33}$$

Equation (4.32) is valid for arbitrary tensors. Based on this property, the polynomial $\mathrm{Det}\,[d_{ijk}]$ defined by Eq. (4.30) and Eq. (4.33) is called the *hyperdeterminant* of the tensor d_{ijk}. [35] gives an easily readable account for the most important properties of the hyperdeterminant. The full mathematical theory can be found in [36] in a quite sophisticated treatment.

The hyperdeterminant of a rank m tensor in p dimensions is a homogeneous polynomial in the tensor components of degree $p(m-1)^{p-1}$, i.e. in particular

$$\deg \mathrm{Det}\,[d_{ijk}] = p \cdot 2^{p-1}. \tag{4.34}$$

The hyperdeterminant is an invariant of d_{ijk}, meaning that under the transformation $d_{ijk} \to d'_{ijk} = d_{lmn} U^l{}_i U^m{}_j U^n{}_k$ with some $p \times p$-matrix U the hyperdeterminant transforms as

$$\mathrm{Det}\,[d_{ijk}] \to \mathrm{Det}\left[d'_{ijk}\right] = (\det U)^{\frac{3}{p} \deg \mathrm{Det}\left[d_{ijk}\right]} \mathrm{Det}\,[d_{ijk}]\,. \tag{4.35}$$

A last important (and by Eq. (4.32) and Eq. (4.33) obvious) property is that the hyperdeterminant is equal to the product of all tensorial eigenvalues:

$$\prod_{n=1}^{p \cdot 2^{p-1}} \lambda_n = \mathrm{Det}\,[d_{ijk}]\,, \tag{4.36}$$

where $\lambda = \lambda_n$ are the solutions of Eq. (4.31).

4.2.2. Alternative derivation of the $p = 2$-formula

It turns out that the result in Eq. (4.26) follows naturally if one considers the tensorial eigenvalue problem

$$e^K d_{ijk} v^i v^j = \lambda I_{ijk} v^i v^j \tag{4.37}$$

with the non-standard right-hand side

$$I_{ijk} = \frac{1}{3}\left(K_i g_{jk} + K_j g_{ki} + K_k g_{ij}\right). \tag{4.38}$$

It is plausible that the unit tensor E_{ijk} in Eq. (4.31) has to be altered as we work in a non-euclidean geometry. The motivation for the precise form of the right-hand side will be given later (see section 4.5.1). It is based on the fact that I_{ijk} encodes the constraint $K_i N^i = 0$ in a 'cubic homogeneous' way:

$$I_{ijk} v^i v^j v^k = 0 \Leftrightarrow K_i v^i = 0. \tag{4.39}$$

Note that the new right-hand side changes the normalization of the characteristic polynomial by an additional factor $\mathrm{Det}\,[I_{ijk}]$ in Eq. (4.36).

According to Eq. (4.34) we expect 4 eigenvalues. To find these, we write a general ansatz for the eigenvector in the form

$$v^i = \alpha K^i + \beta N^i. \tag{4.40}$$

Plugging this into Eq. (4.37) gives (after multiplying the resulting equation with K^i and N^i respectively) the following set of equations:

$$-6\alpha^2 + \beta^2 = \lambda I_{ijk} v^i v^j K^k = \lambda(9\alpha^2 + \beta^2) \tag{4.41}$$

$$\beta^2 D_{111} + 2\alpha\beta = \lambda I_{ijk} v^i v^j N^k = 2\lambda\alpha\beta, \tag{4.42}$$

To derive these, we used the identities Eq. (3.13) to Eq. (3.17).

The first equation (Eq. (4.41)) leads to

$$\beta^2 = 3\alpha^2 \frac{3\lambda + 2}{1 - \lambda}. \tag{4.43}$$

For $\beta = 0$ we find the first eigenvalue to be $\lambda_1 = -2/3$ with an eigenvector in K^i direction. This can easily be checked directly. If $\beta \neq 0$ it follows from the second equation (Eq. (4.42)) that

$$D_{111}^2 = \frac{4}{3} \frac{(1-\lambda)^3}{3\lambda + 2}. \tag{4.44}$$

This cubic equation has three solutions, giving the remaining three eigenvalues λ_2, λ_3 and λ_4. By Vieta's formula we can compute their product as

$$\prod_{i=2}^{4} \lambda_i = 1 - \frac{3}{2} D_{111}^2 = -\omega. \tag{4.45}$$

We then know from Eq. (4.36) that

$$-\frac{2}{3}(-\omega) = \prod_{i=1}^{4} \lambda_i = \frac{\mathrm{Det}\left[e^K d_{ijk}\right]}{\mathrm{Det}\left[I_{ijk}\right]}. \tag{4.46}$$

As the hyperdeterminant in two dimensions is just the discriminant Eq. (4.27), the denominator on the right-hand side can be calculated directly:

Lemma 4.2.1. *Let $p = 2$ and $c_{ijk} = w_{(i} A_{jk)}$ with a vector w_i and an (w.l.o.g.*

symmetric) invertible matrix A_{jk}. Then

$$\mathrm{Det}\,[c_{ijk}] = \frac{4}{27}(A^{ij}w_i w_j)^2 \det A^3, \tag{4.47}$$

where $A^{ij} = A^{-1}_{ij}$.

Proof. The claim can be verified by a straight forward calculation. □

With $w_i = K_i$ and $A_{jk} = g_{jk}$ this gives

$$\mathrm{Det}\,[I_{ijk}] = \frac{4}{3}\det g^3. \tag{4.48}$$

Combining this result with Eq. (4.46) gives

$$\omega = \frac{9}{8}e^{4K}\frac{\mathrm{Det}\,[d_{ijk}]}{\det g^3}. \tag{4.49}$$

This is precisely the result in Eq. (4.26).

4.3. Maximization for $p = 3$-dimensional moduli spaces

We now perform a careful analysis of the three-dimensional case. This allows us to gain enough experience to tackle the general p-dimensional case afterwards. However, the three-dimensional case allows more explicit calculations and therefore the analysis presented in this section will be more complete than the analog analysis for $p > 3$ in section 4.5.

Let us parameterize N^i as

$$N^i = \cos\vartheta\, n_1^i + \sin\vartheta\, n_2^i \tag{4.50}$$

with real orthonormal vectors n_1^i and n_2^i as in Eq. (4.1). The (almost unique) real unit vector orthogonal to K^i and N^i is then given by

$$M^i = -\sin\vartheta\, n_1^i + \cos\vartheta\, n_2^i. \tag{4.51}$$

In ω, this gives as in Eq. (4.23)

$$\begin{aligned}\omega &= -1 + \frac{3}{2}\left(e^K d_{ijk} n_1^i N^j N^k\right)^2 + \frac{3}{2}\left(e^K d_{ijk} n_2^i N^j N^k\right)^2 && (4.52)\\ &= -1 + \frac{3}{2}\left(D_{1NN}^2 + D_{2NN}^2\right), && (4.53)\end{aligned}$$

where again

$$D_{\alpha NN} := e^K d_{ijk} n_\alpha^i N^j N^k. \tag{4.54}$$

As already noted, ω is independent of the choice of basis n_1^i and n_2^i used in the projector P^{ij}. In particular, we can replace

$$\begin{aligned}n_1^i &\to N^i\\ n_2^i &\to M^i,\end{aligned} \tag{4.55}$$

which gives

$$\omega = -1 + \frac{3}{2}\left(D_{NNN}^2 + D_{NNM}^2\right). \tag{4.56}$$

This has the advantage that the differential operator $\partial/\partial\vartheta$ acts in a simple way on the transverse intersection numbers:

$$\frac{\partial}{\partial\vartheta}\begin{pmatrix} D_{NNN}\\ D_{NNM}\\ D_{NMM}\\ D_{MMM}\end{pmatrix} = \begin{pmatrix} 3D_{NNM}\\ 2D_{NMM} - D_{NNN}\\ D_{MMM} - 2D_{NNM}\\ -3D_{NMM}\end{pmatrix}. \tag{4.57}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\omega &= \frac{3}{2}\left(6D_{NNN}D_{NNM} + 2D_{NNM}(2D_{NMM} - D_{NNN})\right)\\ &= 6D_{NNM}(D_{NNN} + D_{NMM})\end{aligned} \tag{4.58}$$

and a local extremum of ω has to satisfy

$$D_{NNM}(D_{NNN} + D_{NMM}) = 0. \tag{4.59}$$

This leaves the two possibilities

$$D_{NNM} = 0 \tag{4.60}$$

and

$$D_{NNN} + D_{NMM} = 0. \tag{4.61}$$

It can be shown that the first condition always gives maxima greater than all critical points given by the second condition and the latter can therefore be discarded in the following analysis. To see this, we write all transverse intersection numbers in terms of $\cos\vartheta = c_\vartheta$ and $\sin\vartheta = s_\vartheta$. This gives

$$D_{NNN} = c_\vartheta^3 D_{111} + 3c_\vartheta^2 s_\vartheta D_{112} + 3c_\vartheta s_\vartheta^2 D_{122} + s_\vartheta^3 D_{222} \tag{4.62}$$

$$D_{NNM} = -c_\vartheta^2 s_\vartheta D_{111} + \left(c_\vartheta^3 - 2c_\vartheta s_\vartheta^2\right) D_{112} + \left(2c_\vartheta^2 s_\vartheta - s_\vartheta^3\right) D_{122} + c_\vartheta s_\vartheta^2 D_{222} \tag{4.63}$$

$$D_{NMM} = c_\vartheta s_\vartheta^2 D_{111} + \left(s_\vartheta^3 - 2c_\vartheta^2 s_\vartheta\right) D_{112} + \left(c_\vartheta^3 - 2c_\vartheta s_\vartheta^2\right) D_{122} + c_\vartheta^2 s_\vartheta D_{222} \tag{4.64}$$

$$D_{MMM} = -s_\vartheta^3 D_{111} + 3c_\vartheta s_\vartheta^2 D_{112} - 3c_\vartheta^2 s_\vartheta D_{122} + c_\vartheta^3 D_{222}. \tag{4.65}$$

The equation $D_{NNN} + D_{NMM} = 0$ simplifies significantly and is equivalent to

$$\tan\vartheta = -\frac{D_{111} + D_{122}}{D_{112} + D_{222}}. \tag{4.66}$$

Note that this simplification was actually expected. Using D_{nNN} and D_{mNN} in ω gives an equation of only fourth order in ϑ while writing ω in terms of D_{NNN} and D_{MNN} is a priori of order six.

As the following expressions are somewhat unhandy, we simplify them by choosing a basis such that $D_{122} = 0$. At the critical point of ω given by Eq. (4.66) we then find

$$\begin{aligned}D_{NNN}^2 + D_{NNM}^2 &= \frac{D_{122}^2\left(D_{111} + D_{122}\right)^2 + D_{112}^2\left(D_{112} + D_{222}\right)^2 + D_{111}^2 D_{222}^2}{\left(D_{111} + D_{122}\right)^2 + \left(D_{112} + D_{222}\right)^2} \\ &\quad - \frac{D_{112}^2 D_{122}^2 + 2D_{112}D_{122}\left(D_{111}D_{112} + D_{122}D_{222} + 2D_{111}D_{222}\right)}{\left(D_{111} + D_{122}\right)^2 + \left(D_{112} + D_{222}\right)^2} \\ &\overset{D_{122}=0}{=} \frac{D_{112}^2\left(D_{112} + D_{222}\right)^2 + D_{111}^2 D_{222}^2}{D_{111}^2 + \left(D_{112} + D_{222}\right)^2},\end{aligned} \tag{4.67}$$

which follows after some straightforward algebra.

Proposition 1. *Let $D_{122} = 0$. Then*

$$\max_{\vartheta\in[0,2\pi]} D^2_{NNN}(\vartheta) \geq \frac{D^2_{112}(D_{112}+D_{222})^2 + D^2_{111}D^2_{222}}{D^2_{111} + (D_{112}+D_{222})^2}. \tag{4.68}$$

Proof. The proof can be found in appendix A.1. □

4.3.1. Tensorial eigenvalue approach

We saw that the tensorial eigenvalue problem Eq. (4.37) can be used to derive a simple formula for ω in the two-dimensional case. We now consider the same eigenvalue problem for the $p = 3$-case to find out how much of the analysis can be generalized.

We have shown that for $p = 3$ moduli the problem of finding the global maximum of ω is equivalent to solving the problem

$$D_{NNM}(\vartheta) = 0, \tag{4.69}$$

i.e. determining the critical points of $D_{NNN}(\vartheta)$ (cf. Eq. (4.57)). This problem will turn out to be equivalent to solving the tensorial eigenvalue problem

$$e^K d_{ijk} v^j v^k = \lambda I_{ijk} v^j v^k \tag{4.70}$$

with

$$I_{ijk} = \frac{1}{3}\left(K_i g_{jk} + K_j g_{ki} + K_k g_{ij}\right). \tag{4.71}$$

A general proof of this claim can be found in section 4.5.1. For now, we study the eigenvalue problem directly. Using the ansatz

$$v^i = \alpha K^i + \beta n^i_1 + \gamma n^i_2 \tag{4.72}$$

we obtain the following set of equations by multiplying Eq. (4.70) with K^i, n^i_1 and

n_2^i respectively:

$$-6\alpha^2 + \beta^2 + \gamma^2 = \lambda I_{ijk} v^i v^j K^k = \lambda(9\alpha^2 + \beta^2 + \gamma^2) \tag{4.73}$$

$$\beta^2 D_{111} + \gamma^2 D_{122} + 2\alpha\beta + 2\beta\gamma D_{112} = \lambda I_{ijk} v^i v^j n_1^k = 2\lambda\alpha\beta \tag{4.74}$$

$$\beta^2 D_{112} + \gamma^2 D_{222} + 2\alpha\gamma + 2\beta\gamma D_{122} = \lambda I_{ijk} v^i v^j n_2^k = 2\lambda\alpha\gamma. \tag{4.75}$$

Multiplying the second equation with γ, the third equation with β and subtracting them from each other, we find

$$-\beta^2\gamma D_{111} + (\beta^3 - 2\beta\gamma^2) D_{112} + (2\beta^2\gamma - \gamma^3) D_{122} + \beta\gamma^2 D_{222} = 0. \tag{4.76}$$

Identifying $\beta = \rho\cos\vartheta$ and $\gamma = \rho\sin\vartheta$ we see that Eq. (4.76) is equivalent to $D_{NNM} = 0$ and that the corresponding N^i is given by the part of the eigenvector v^i orthogonal to K^i (see Eq. (4.63)), assuming that $\rho \neq 0$. In addition, the equation $e^K d_{ijk} v^i v^j v^k = \lambda I_{ijk} v^i v^j v^k$ reads

$$-6\alpha^3 + 3\alpha(\beta^2 + \gamma^2) + 3\beta\gamma^2 D_{122} + 3\beta^2\gamma D_{112} + \beta^3 D_{111} + \gamma^3 D_{222} = \lambda I_{ijk} v^i v^j v^k. \tag{4.77}$$

From this we obtain (see Eq. (4.62))

$$\rho^3 D_{NNN} = \lambda I_{ijk} v^i v^j v^k + 6\alpha^3 - 3\alpha(\beta^2 + \gamma^2), \tag{4.78}$$

where D_{NNN} is evaluated at ϑ, i.e. at one of its three critical points.

We use Eq. (4.73) to calculate ρ and find

$$\rho^2 = \beta^2 + \gamma^2 = 3\alpha^2 \frac{3\lambda + 2}{1 - \lambda}. \tag{4.79}$$

If $\rho = 0$, we obtain $\beta = \gamma = 0$ and

$$\lambda_1 = -2/3 \tag{4.80}$$

as in the $p = 2$-case. There is one special case in which we get two additional eigenvalues. $\rho = 0$ is also solved by

$$\beta = \pm \mathrm{i}\gamma. \tag{4.81}$$

Equation (4.76) then gives a constraint on the transverse intersection numbers:

$$D_{111} = 3D_{122} \tag{4.82}$$

$$D_{222} = 3D_{112}. \tag{4.83}$$

Equation (4.63) shows that $D_{NNM}(\vartheta) = 0$ degenerates to a linear problem in this case. From Eq. (4.74) and (4.75) we see that $\alpha = 0$ is a contradiction, assuming not all transverse intersection numbers vanish. Then Eq. (4.73) implies the existence of two additional eigenvalues $\lambda_2 = \lambda_3 = -2/3$. We proceed by ignoring this special case for the time being.

For $\rho \neq 0$, we find

$$\begin{aligned} \rho^3 D_{NNN} &= \lambda I_{ijk} v^i v^j v^k + 6\alpha^3 - 3\alpha(\beta^2 + \gamma^2) \\ &= 3\alpha\lambda(3\alpha^2 + \beta^2 + \gamma^2) + 6\alpha^3 - 3\alpha(\beta^2 + \gamma^2) && (4.84) \\ &= 3\alpha(\lambda - 1)(\beta^2 + \gamma^2) + (6 + 9\lambda)\alpha^3 && (4.85) \\ &= -6\alpha^3(3\lambda + 2). && (4.86) \end{aligned}$$

Squaring this equation, we find with Eq. (4.79)

$$D^2_{NNN} = \frac{4}{3}\frac{(1-\lambda)^3}{3\lambda + 2}, \tag{4.87}$$

where D_{NNN} is evaluated at one of its critical points.

It is not difficult to see that this equation has exactly one real and two complex solutions for λ if the left-hand side is positive, i.e. that each real critical point $\vartheta = \vartheta_j$ of D_{NNN} gives exactly one real and two complex eigenvalues. The product of the three eigenvalues from the j-th critical point $\lambda_{3j+1}, \lambda_{3j+2}, \lambda_{3j+3}$ by Vieta's formula satisfies

$$\prod_{i=1}^{3} \lambda_{3j+i} = 1 - \frac{3}{2} D_{NNN}(\vartheta_j)^2 = -\omega(\vartheta_j) \quad \text{for} \quad j = 1, 2\,, 3. \tag{4.88}$$

The three critical points of D_{NNN} give 9 eigenvalues via Eq. (4.87) and together with the one in Eq. (4.80) we found 10 eigenvalues. The characteristic polynomial appears to have degree 12 (see Eq. (4.34)), so we would be missing two eigenvalues.

In fact, the right-hand side I_{ijk} as defined here is degenerate:

$$\mathrm{Det}\,[I_{ijk}] = 0, \tag{4.89}$$

as we will see later. The characteristic polynomial is only of degree 10, because I_{ijk} has two zero eigenvalues. This may be interpreted as the existence of eigenvalues 'at infinity': $\lambda_2 = \lambda_3 = \infty$ with corresponding eigenvectors $\alpha = 0$, $\beta = \pm \mathrm{i}\gamma$.

To proceed, we have to regularize the ill-posed eigenvalue problem by a substitution

$$I_{ijk} \rightarrow I^{\varepsilon}_{ijk} = I_{ijk} + \varepsilon \delta I_{ijk} \tag{4.90}$$

such that for $\varepsilon > 0$

$$\mathrm{Det}\left[I^{\varepsilon}_{ijk}\right] \neq 0. \tag{4.91}$$

One obvious possibility is

$$\delta I_{ijk} = E_{ijk} = \delta_{ij}\delta_{jk}. \tag{4.92}$$

This substitution will deform the eigenvalues found above only by terms of order ε but gives two additional eigenvectors. These can be found by expanding in ε.

The new eigenvectors and eigenvalues have to be of the form

$$\alpha = \varepsilon\alpha_1, \quad \beta = \pm\mathrm{i}\gamma + \varepsilon\beta_1, \quad \lambda = \frac{\mu}{\varepsilon}. \tag{4.93}$$

Plugging this into the Eq. (4.74) and Eq. (4.75) with the right-hand side modified by $\varepsilon\delta I_{ijk}$ gives

$$-D_{111} + D_{122} \pm 2\mathrm{i}D_{112} = \mu(\pm 2\mathrm{i}\alpha_1 + C_1) + \mathcal{O}(\varepsilon) \tag{4.94}$$

$$-D_{112} + D_{222} \pm 2\mathrm{i}D_{122} = \mu(2\alpha_1 + C_2) + \mathcal{O}(\varepsilon), \tag{4.95}$$

where

$$C_\eta = E_{ijk}v^i v^j n^k_\eta = (v^1)^2 n^1_\eta + (v^2)^2 n^2_\eta + (v^3)^2 n^3_\eta, \quad \eta = 1, 2 \tag{4.96}$$

$$v^i = \pm\mathrm{i}n^i_1 + n^i_2. \tag{4.97}$$

Solving for μ, we find

$$\mu = \frac{D_{111} \mp 3\mathrm{i}D_{112} - 3D_{122} \pm \mathrm{i}D_{222}}{C_1 \mp \mathrm{i}C_2}, \tag{4.98}$$

i.e. for the product of the two missing eigenvalues

$$\lambda_2\lambda_3 = |\lambda_2|^2 = \frac{(D_{111} - 3D_{122})^2 + (D_{222} - 3D_{112})^2}{\varepsilon^2|\mathrm{i}C_2 - C_1|^2}. \tag{4.99}$$

Note that this result again shows the specialty of the degenerated case first noted in Eq. (4.82). The appearance of $|\mathrm{i}C_2 - C_1|^2 = \left|\sum_{i=1}^3 (\mathrm{i}n_1^i + n_2^i)^3\right|^2$ is an artifact of the chosen regularization method and will cancel with part of the normalization as we will see.

We now see that

$$\begin{aligned}\prod_{j=1}^3 \omega(\vartheta_j) &= -\frac{\prod_{i=4}^{12}\lambda_i}{\lambda_1\lambda_2\lambda_3} \\ &= \frac{3}{2}e^{12K}\,\mathrm{Det}\,[d_{ijk}]\,\frac{|\mathrm{i}C_2 - C_1|^2}{(D_{111} - 3D_{122})^2 + (D_{222} - 3D_{112})^2}\left[\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{\mathrm{Det}\left[I^\varepsilon_{ijk}\right]}\right].\end{aligned} \tag{4.100}$$

It remains to compute the limit $\lim_{\varepsilon\to 0}\left[\varepsilon^{-2}\,\mathrm{Det}\left[I^\varepsilon_{ijk}\right]\right]$. This is done by the following Lemma.

Lemma 4.3.1. *Let w_i denote a 3-dimensional vector and A_{ij} an arbitrary positive definite 3×3 matrix. Then it holds for $c_{ijk} = w_{(i}A_{jk)} + \varepsilon E_{ijk}$:*

$$\mathrm{Det}\,[c_{ijk}] = \varepsilon^2\frac{4^3}{3^9}|\mathrm{i}C_2 - C_1|^2(A^{ij}w_iw_j)^5\det A^6 + \mathcal{O}(\varepsilon^3), \tag{4.101}$$

where again

$$C_\eta = E_{ijk}v^iv^jn_\eta^k = (v^1)^2n_\eta^1 + (v^2)^2n_\eta^2 + (v^3)^2n_\eta^3, \quad \eta = 1,\,2 \tag{4.102}$$

$$v^i = \pm\mathrm{i}n_1^i + n_2^i \tag{4.103}$$

and n_1^i and n_2^i are (arbitrary) vectors satisfying

$$w_in_1^i = w_in_2^i = 0 \quad \textit{and} \quad n_\eta^jA_{ij}n_\kappa^i = \delta_{\eta\kappa}. \tag{4.104}$$

In particular, it follows that

$$\operatorname{Det}\left[I^{\varepsilon}_{ijk}\right] = \varepsilon^2 \frac{4^3}{3^4} |\mathrm{i}C_2 - C_1|^2 \det g^6 + \mathcal{O}(\varepsilon^3). \tag{4.105}$$

Proof. The proof can be found in appendix A.1. □

For later reference, we collect all results of this section in a single theorem:

Theorem 4.3.2. *Let λ_i, $i = 1, \ldots, 12$ denote the eigenvalues of $\mathrm{e}^K d_{ijk}$ with respect to the right-hand side*

$$I_{ijk} = \frac{1}{3}\left(K_i g_{jk} + K_j g_{ki} + K_k g_{ij}\right). \tag{4.106}$$

Then $\lambda_1 = -\frac{2}{3}$ is an eigenvalue with eigenvector K^i.

Let ϑ_j, $j = 1, 2, 3$ denote the three critical points of D_{NNN}. Then the solutions of

$$D^2_{NNN}(\vartheta_j) = \frac{4}{3} \frac{(1-\lambda)^3}{3\lambda + 2} \tag{4.107}$$

are eigenvalues of $\mathrm{e}^K d_{ijk}$. If D_{NNN} is real, then exactly two of them are complex and one is real. The real one also satisfies $-2/3 < \lambda_{3j+1} \leq 1$. It follows that $\omega(\vartheta_j)$ is positive if and only if the corresponding real eigenvalue is negative. The extremizer N^i is (up to a normalization) given by the projection of the eigenvector corresponding to the real eigenvalue onto the orthogonal complement of K^i.

In addition, the formula

$$\prod_{j=1}^{3} \omega(\vartheta_j) = \frac{243}{128} e^{12K} \frac{\operatorname{Det}\left[d_{ijk}\right]}{\det g^6\left[(D_{111} - 3D_{122})^2 + (D_{222} - 3D_{112})^2\right]} \tag{4.108}$$

holds. Using Eq. (4.57) it is easy to check that $(D_{111} - 3D_{122})^2 + (D_{222} - 3D_{112})^2$ does not depend on the choice of basis vectors n^i_1 and n^i_2. In fact, it can be shown that

$$(D_{111} - 3D_{122})^2 + (D_{222} - 3D_{112})^2 = e^{2K} d_{pij} g^{pq} d_{qmn}(4g^{im}g^{jn} - 3g^{ij}g^{mn}) - \frac{40}{3}. \tag{4.109}$$

An important Corollary is

$$\mathrm{Det}\,[d_{ijk}] > 0 \Rightarrow \omega = -1 + \frac{3}{2}(D^2_{NNN} + D^2_{NNM}) > 0 \tag{4.110}$$

at at least one critical point of ω in every physical region (these are regions where $g > 0$ and $\mathcal{V} > 0$) of the moduli space. Note however that the converse does not necessarily hold: for negative $\mathrm{Det}\,[d_{ijk}]$ either one or all critical points of ω on the left-hand side of Eq. (4.108) can be negative and the number of negative critical points may even vary on the moduli space.

4.3.2. Some classical invariant theory for $p = 3$

We have seen that in the analysis of $p = 3$-dimensional moduli spaces an invariant of the polynomial $\mathcal{V} = -6^{-1} d_{ijk} K^i K^j K^k$ appears, namely the hyperdeterminant of d_{ijk}. It therefore appears useful to review the classical results from the study of such invariants.

A simple closed formula for the hyperdeterminant with $p > 3$ is not known [36]. However, for $p = 3$ there is extensive classical work (see a paper by Aronhold [37]). We give a brief review of his most important results. Some of this discussion can also be found in [38] in a modern exposition.

Aronhold found two invariants of the polynomial

$$d_{ijk} x^i x^j x^k, \tag{4.111}$$

where $(x^i) = (x^1,\, x^2,\, x^3)$. These so-called Aronhold invariants are denoted S and T and are homogeneous polynomials of degree 4 and 6 respectively in the coefficients d_{ijk}. Their defining property as invariants of Eq. (4.111) is their transformation behavior under $d_{ijk} \to d'_{ijk} = d_{lmn} U^l{}_i U^m_j U^n_k$:

$$S \to S' = (\det U)^4\, S \tag{4.112}$$

$$T \to T' = (\det U)^6\, T. \tag{4.113}$$

It was shown that the hyperdeterminant can be expressed in terms of S and T as

$$\mathrm{Det}\,[d_{ijk}] = T^2 - S^3. \tag{4.114}$$

Note that in the original work this quantity is called R.

S and T can be calculated directly from the coefficients d_{ijk}. For this, from given 3×3 matrices a and b define the new matrices

$$(ab)_{ij} = a_{i+1,j+1} b_{i+2,j+2} - a_{i+1,j+2} b_{i+2,j+1} + (a \leftrightarrow b)\,, \tag{4.115}$$

where the indices are understood as modulo 3.

Denote by d_i the matrices

$$(d_i)_{jk} = d_{ijk} \tag{4.116}$$

obtained by fixing the first index of the cubic coefficients d_{ijk}.

The first Aronhold invariant S can be computed as

$$S = \sum_{i,j=1}^{3} (d_k d_k)_{ij} (d_i d_j)_{kk}, \tag{4.117}$$

which is independent of $k = 1, 2, 3$.

To compute T, one needs the coefficients e_{ijk} of the cubic polynomial

$$6^{-2} \det \left[d_{ijk} x^k \right] = e_{ijk} x^i x^j x^k. \tag{4.118}$$

Interestingly, this is up to a factor of $-18e^{3K}$ equal to $\det g$, the determinant of the Kähler metric.

The second Aronhold invariant T can then be computed as

$$T = \frac{1}{2} \sum_{i,j,l=1}^{2} (d_k e_l)_{ij} (d_i e_j)_{kl}, \tag{4.119}$$

which is again independent of $k = 1, 2, 3$. A Maple implementation for the computation of S and T can be found in appendix A.3.2.

Another important result of the classical analysis is that the polynomial $d_{ijk} x^i x^j x^k$ is factorizable (thus giving a factorizable model as defined in section 3.3) if and only if

$$d_{ijk}x^i x^j x^k \equiv C \det \left[d_{ijk}x^k\right] \tag{4.120}$$

for some constant C.

A last interesting fact is that $S = 0$ holds if and only if $d_{ijk}x^i x^j x^k$ can be written as a sum of three cubes. In the following section 4.4.1 we will see that in this case ω is positive for all Goldstino directions.

4.4. Explicit examples of $p = 3$-dimensional moduli spaces

We now treat two special cases in more detail, which allow a more detailed and explicit analysis.

4.4.1. Diagonal intersection numbers

The first case we are going to study is if the intersection numbers are diagonal, i.e. $d_{iii} \neq 0$ and all other d_{ijk} vanish. A Maple implementation of the analysis is in this section can be found in appendix A.3.3.

It holds

$$\mathcal{V}_{ij} = \begin{pmatrix} d_{111}K^1 & & \\ & d_{222}K^2 & \\ & & d_{333}K^3 \end{pmatrix} \tag{4.121}$$

and in particular

$$\begin{aligned} \det g^3 &= -\frac{e^{9K}}{8}(d_{111}d_{222}d_{333}K^1K^2K^3)^3 \\ &= \frac{e^{8K}}{8}(d_{111}d_{222}K^1K^2)^3 d_{333}^2 \left[e^K d_{111}(K^1)^3 + e^K d_{222}(K^2)^3 + 6\right], \end{aligned} \tag{4.122}$$

where the last equality is simply the no-scale property $K_iK^i = 3$.

We choose orthonormal basis vectors orthogonal to K by

$$n_1^i = \frac{1}{\sqrt{C_{12}}}(K_2, -K_1, 0) \tag{4.123}$$

and

$$n_2^i = \sqrt{\frac{\det g}{3}}\varepsilon^{ijk} K_j n_{1k}, \tag{4.124}$$

where C_{12} is a normalization constant. It turns out that

$$C_{12} = \frac{1}{4} e^{3K} d_{111} d_{222} K^1 K^2 \left(d_{111}(K^1)^3 + d_{222}(K^2)^3\right). \tag{4.125}$$

With these choices we have $D_{122} = 0$. The other transverse intersection numbers are given by reasonable, but not very handy expressions and will not be stated explicitly here. The important point is that if D_{122} vanishes, the equation

$$\frac{\partial}{\partial \vartheta} D_{NNN} = 0 \tag{4.126}$$

has $\vartheta = \pi/2$ as a solution, i.e. ω has a critical point at $N^i = n_2^i$. It can be checked by a direct calculation that choosing n_1^i different from Eq. (4.123) as

$$\frac{1}{\sqrt{C_{23}}}\begin{pmatrix} 0 \\ K_3 \\ -K_2 \end{pmatrix} \quad \text{or} \quad \frac{1}{\sqrt{C_{13}}}\begin{pmatrix} K_3 \\ 0 \\ -K_1 \end{pmatrix} \tag{4.127}$$

also results in $D_{122} = 0$ and therefore gives the other two critical points of ω. In total, we find

$$\begin{aligned}
\omega(\vartheta_1) &= \frac{9e^{-K}}{[d_{111}(K^1)^3 + d_{222}(K^2)^3]\,[e^K d_{111}(K^1)^3 + e^K d_{222}(K^2)^3 + 6]} \\
&= \frac{9e^{-K}}{(K^3)^3 d_{333}\,[e^K d_{333}(K^3)^3 + 6]} = \frac{9}{32} e^{10K} d_{333}^2 \frac{(d_{111}d_{222}K^1K^2)^4}{C_{12}\det g^3} > 0 && (4.128) \\
\omega(\vartheta_2) &= \frac{9e^{-K}}{(K^2)^3 d_{222}\,[e^K d_{222}(K^2)^3 + 6]} = \frac{9}{32} e^{10K} d_{222}^2 \frac{(d_{111}d_{333}K^1K^3)^4}{C_{13}\det g^3} > 0 && (4.129) \\
\omega(\vartheta_3) &= \frac{9e^{-K}}{(K^1)^3 d_{111}\,[e^K d_{111}(K^1)^3 + 6]} = \frac{9}{32} e^{10K} d_{111}^2 \frac{(d_{222}d_{333}K^2K^3)^4}{C_{23}\det g^3} > 0. && (4.130)
\end{aligned}$$

At all three critical points ω is always positive (at least in physical regions, i.e. if

$g > 0$ and $\mathcal{V} > 0$).

Calculating the product of ω at ϑ_1, ϑ_2 and ϑ_3 we find

$$\prod_{j=1}^{3} \omega(\vartheta_j) = e^{3K} \frac{9^3}{8} \frac{\sqrt{\mathrm{Det}\,[d_{ijk}]}}{\det g^3} \left[\left(d_{111}(K^1)^3 + d_{222}(K^2)^3 \right) (2 \leftrightarrow 3)\, (1 \leftrightarrow 3) \right]^{-1}, \tag{4.131}$$

where we used $\mathrm{Det}\,[d_{ijk}] = d_{111}^4 d_{222}^4 d_{333}^4$ for diagonal tensors. Using computer algebra, it can be checked that

$$\begin{aligned} &\left[\left(d_{111}(K^1)^3 + d_{222}(K^2)^3 \right) (2 \leftrightarrow 3)\, (1 \leftrightarrow 3) \right] \\ &\qquad = \frac{48 e^{-9K}}{d_{111}^2 d_{222}^2 d_{333}^2} \det g^3 \left[(3D_{112} - D_{222})^2 + D_{111}^2 \right], \end{aligned} \tag{4.132}$$

which gives

$$\prod_{j=1}^{3} \omega(\vartheta_j) = e^{12K} \frac{243}{128} \frac{\mathrm{Det}\,[d_{ijk}]}{\det g^6 \left[(3D_{112} - D_{222})^2 + D_{111}^2 \right]}. \tag{4.133}$$

This result confirms Eq. (4.108).

4.4.2. Almost factorizing models

A second case we can treat in more detail occurs if the volume factorizes:

$$\mathcal{V} = -\frac{1}{6} d_i K^i d_{jk} K^j K^k, \tag{4.134}$$

where d_i is some vector and d_{jk} is a symmetric matrix. Note that this is in general not a factorizing model in the sense of section 3.3. From Lemma 4.3.1 (with $c_{ijk} = d_{(i} d_{jk)}$) we know that this is a situation where d_{ijk} has two zero eigenvalues. From Theorem 4.3.2 we can deduce that ω vanishes at two of its three critical points:

$$\omega(\vartheta_1) = \omega(\vartheta_2) = 0. \tag{4.135}$$

We can choose coordinates such that

$$d_i = \begin{pmatrix} d_1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.136}$$

Then the corresponding zero eigenvectors are easily computed to be

$$v_\pm = C_\pm \begin{pmatrix} 0 \\ -d_{23} \pm \sqrt{d_{23}^2 - d_{22}d_{33}} \\ d_{22} \end{pmatrix}, \tag{4.137}$$

where $C_\pm$ are some normalization constants.

We now derive an expression for $\omega(\vartheta_3)$ which depends only on the scalar product of v_+ and v_-. For this, choose $C_\pm$ such that $v_\pm$ satisfies $g_{ij}v_\pm^i v_\pm^j = 1$. Note that this means that $v_\pm$ are no unit vectors if they are not real, because their norm is given by $g_{ij}v_\pm^i \overline{v}_\pm^j$. Then it holds (due to $v_\pm^i$ being zero eigenvectors of d_{ijk}) that

$$1 = g_{ij}v_\pm^i v_\pm^j = e^K d_{ijk} v_\pm^i v_\pm^j K^k + (v_\pm^i K_i)^2 = (v_\pm^i K_i)^2. \tag{4.138}$$

This implies (possibly after changing the orientation) that

$$v_\pm^i = \frac{1}{3}K^i + \sqrt{\frac{2}{3}} u_\pm^i \tag{4.139}$$

where $u_\pm$ are normalized vectors orthogonal to K^i. By Theorem 4.3.2 $u_\pm$ are the extremizers of ω corresponding to the critical points in Eq. (4.135). Now we make the ansatz

$$v^i = \alpha K^i + \beta v_+^i + \gamma v_-^i \tag{4.140}$$

for the eigenvalue problem Eq. (4.169). Plugging v^i into the eigenvector equations gives

$$-2\alpha^2 K_i + 2\alpha e^K d_{ijk} K^j \left(\beta v_+^k + \gamma v_-^k\right) + 2\beta\gamma e^K d_{ijk} v_+^j v_-^k$$

$$= \lambda \left(3\alpha^2 K_i + 2\alpha\beta \left(\frac{2}{3} K_i + v_{+i} \right) + 2\alpha\gamma \left(\frac{2}{3} K_i + v_{-i} \right) + \beta^2 \left(\frac{2}{3} v_{+i} + \frac{1}{3} K_i \right) \right.$$
$$\left. + \gamma^2 \left(\frac{2}{3} v_{-i} + \frac{1}{3} K_i \right) + \frac{2}{3} \beta\gamma \left(\eta K_i + v_{+i} + v_{-i} \right) \right), \tag{4.141}$$

where we defined

$$\eta := g_{ij} v_+^i v_-^j . \tag{4.142}$$

By multiplying with K^i, v_+^i and v_-^i we obtain the complicated system of equations

$$\begin{aligned} & -6\alpha^2 - 4\alpha(\beta+\gamma) + 2\beta\gamma(\eta - 1) \\ & \quad = \lambda \left(9\alpha^2 + 6\alpha(\beta+\gamma) + \frac{5}{3}\left(\beta^2+\gamma^2\right) + 2\beta\gamma\left(\eta + \frac{2}{3}\right) \right) && (4.143) \\ & -2\alpha^2 + 2\alpha\gamma(\eta - 1) \\ & \quad = \lambda \left(3\alpha^2 + \frac{10}{3}\alpha\beta + \frac{2}{3}\alpha\gamma(2+3\eta) + \beta^2 + \frac{1}{3}\gamma(\gamma+2\beta)(2\eta+1) \right) && (4.144) \\ & -2\alpha^2 + 2\alpha\beta(\eta - 1) \\ & \quad = \lambda \left(3\alpha^2 + \frac{10}{3}\alpha\gamma + \frac{2}{3}\alpha\beta(2+3\eta) + \gamma^2 + \frac{1}{3}\beta(\beta+2\gamma)(2\eta+1) \right). && (4.145) \end{aligned}$$

Using a computer algebra system (see appendix A.3.4 for a Maple implementation of this calculation), one can show that this system has a solution with a (potentially) real λ, where $\beta = \gamma$ and α and λ are given by relatively complicated expressions. Fortunately, the ω corresponding to this eigenvalue simplifies drastically and can be shown to be

$$\omega(\vartheta_3) = -27\eta \frac{(\eta - 1)^2}{(1 + 3\eta)^3}. \tag{4.146}$$

We have $-1 < \eta < 1$ if v_+^i and v_-^i are linearly independent real vectors. If $v_\pm^i$ are complex vectors, they have to be complex conjugates of each other. The Cauchy-Schwarz inequality then implies

$$\eta = g_{ij} v_+^i v_-^j = g_{ij} v_+^i \overline{v}_+^j \geq |g_{ij} v_+^i v_+^j| = 1, \tag{4.147}$$

with a strict inequality if $v_\pm^i$ is not real up to a global phase.

Because $v^i_\pm K_i = 1$ it always holds that $\eta > -1/3$ and therefore

$$\omega(\vartheta_3) > 0 \quad \Leftrightarrow \quad \eta < 0. \tag{4.148}$$

Note that there is an apparent contradiction: We showed in section 4.1 (see Lemma 4.1.1) that if the global maximum of ω is attained at a complex N^i, then there is always another, real N^i such that $\omega > 0$ for this orthogonal direction. If $v^i_\pm$ are complex and therefore $\eta > 1$, the global maximum of ω restricted to real N^i is given by Eq. (4.146), which is negative. But there are complex N^i such that $\omega = 0$, see Eq. (4.135) (recall that N^i is given by the projection of the corresponding tensorial eigenvector onto the subspace orthogonal to K^i, see Theorem 4.3.2). Case studies suggest that the solution to this apparent paradox is that the metric g can never be positive definite for this class of models if a complex zero eigenvector exists. Then the argument that $\eta > 1$ for complex $v^i_\pm$ in Eq. (4.147) does not necessarily hold and the contradiction disappears. However, we have not been able to find a general proof for this conjecture yet.

We can now compute $\eta = g_{ij} v^i_+ v^j_-$ for $v_\pm$ defined in Eq. (4.137) and then use Eq. (4.146) to obtain

$$\omega(\vartheta_3) = \frac{1}{2} e^{7K} \left(d_{133} d_{122} - d_{123}^2\right)^2 \det d_{ij} \frac{\left(K^1\right)^3}{\det g^3}. \tag{4.149}$$

This calculation is straightforward but quite tedious and we do not state the details here. The result can however be easily checked in special cases and a Maple implementation for the general case can be found in appendix A.3.5.

The quantity $4(d_{133} d_{122} - d_{123}^2)^2$ turns out to be the first Aronhold invariant $S = S[d_{ijk}]$ (see Eq. (4.117)). If we perform arbitrary rotations to eliminate the restriction in Eq. (4.136) and use the invariance of S, $\det g$ and $\det d_{ij}$ under these rotations, we see that the general formula has to be

$$\omega(\vartheta_3) = \frac{1}{8} e^{7K} S \det d_{ij} \frac{\left(d_i K^i\right)^3}{\det g^3}. \tag{4.150}$$

As d_{ijk} has vanishing eigenvalues, the hyperdeterminant has to vanish and by

$$\mathrm{Det}\,[d_{ijk}] = T^2 - S^3 \tag{4.151}$$

the invariant S has to be non-negative. By

$$e^{-K} = \mathcal{V} = -\frac{1}{6} d_i K^i d_{jk} K^j K^k \tag{4.152}$$

it follows

$$\omega(\vartheta_3) = -\frac{3}{4} e^{6K} S \frac{\det d_{ij}}{d_{jk} K^j K^k} \frac{(d_i K^i)^2}{\det g^3}. \tag{4.153}$$

If d_{ij} is positive or negative definite, the factor $\frac{\det d_{ij}}{d_{jk} K^j K^k}$ will always be positive and ω is negative.

For indefinite d_{ij} the situation is more complicated. Independent of the sign of $\det d_{ij}$ there will always be K^i such that $\mathcal{V} > 0$ and $\omega > 0$ simultaneously. In more detail: If d_{ij} has one positive and two negative eigenvalues, $\det d_{ij} > 0$ and we need $d_i K^i > 0$ to get $\omega > 0$. Then K^i has to have a significant part inside one of the two negative eigenspaces to make $\mathcal{V}$ positive. Analogously if d_{ij} has two positive and one negative eigenvalue. The condition $g > 0$ may provide additional constraints. However, it can be checked that it only constrains K^i further and can never exclude the model completely.

4.5. Maximization for arbitrary-dimensional moduli spaces

We now extend the analysis for the $p = 3$-case to general p. The reasoning will parallel the one in section 4.3.

We first choose a parameterization for N^i in terms of $p-2$ variables ϑ_α (e.g. spherical coordinates) such that $N_i N^i = 1$. We then choose a set of $p-2$ orthonormal unit vectors N^i_β such that $\{N^i,\, N^i_\beta\}$ is an orthonormal basis of the orthogonal complement of K^i. ω can then be written as

$$\omega = -1 + \frac{3}{2}\left[D^2_{NNN} + \sum_{\beta=2}^{p-1} D^2_{NN\beta} \right], \tag{4.154}$$

where

$$D_{NN\beta} := e^K d_{ijk} N^i N^j N^k_\beta \tag{4.155}$$

and the vectors N^i_β now depend on the angles ϑ_α parameterizing N^i. We also define $N^i \equiv N^i_1$ to simplify the notation. For example, we now write

$$D_{11\alpha} \equiv D_{NN\alpha} \tag{4.156}$$

etc.

We first show that a solution of the set of equations

$$D_{NN\beta} = 0, \quad \beta = 2, \ldots, p-1 \tag{4.157}$$

is always a critical point of ω. To see this, observe that

$$0 = \frac{\partial}{\partial \vartheta_\alpha}\left(N_i N^i\right) = 2N_i \frac{\partial}{\partial \vartheta_\alpha} N^i \Rightarrow \frac{\partial}{\partial \vartheta_\alpha} N^i = \sum_{\gamma=2}^{p-1} a_{\alpha\gamma} N^i_\gamma \tag{4.158}$$

with some matrix of coefficients $a_{\alpha\gamma}$. Note that there is no contribution from K^i to $\partial_\alpha N^i$ because K^i does not depend on the ϑ_α and therefore $0 = \partial_\alpha\left(K_i N^i\right) = K_i \partial_\alpha N^i$.

Equation (4.158) implies

$$\begin{aligned} \frac{\partial}{\partial \vartheta_\alpha}\omega &= \frac{3}{2}\left[6D_{NNN}\sum_{\gamma=2}^{p-1} a_{\alpha\gamma} D_{NN\gamma} + 2\sum_{\beta=2}^{p-1} D_{NN\beta}\frac{\partial}{\partial \vartheta_\alpha} D_{NN\beta}\right] \\ &= \frac{3}{2}\sum_{\beta=2}^{p-1} D_{NN\beta}\left[6a_{\alpha\beta} D_{NNN} + 2\frac{\partial}{\partial \vartheta_\alpha} D_{NN\beta}\right] = 0 \end{aligned} \tag{4.159}$$

if Eq. (4.157) is satisfied.

There are additional critical points of ω. The system of equations Eq. (4.159) specifying these can be simplified further.

It follows from Eq. (4.158) that

$$N \cdot \partial_\alpha N_\beta = -\left(\partial_\alpha N\right) \cdot N_\beta = -\sum_{\delta=2}^{p-1} a_{\alpha\delta} N_\delta \cdot N_\beta = -\sum_{\delta=2}^{p-1} a_{\alpha\delta}\delta_{\delta\beta} = -a_{\alpha\beta} \tag{4.160}$$

and we can therefore write

$$\partial_\alpha N_\beta = -a_{\alpha\beta} N + \sum_{\gamma=2}^{p-1} a_{\alpha\beta\gamma} N_\gamma \tag{4.161}$$

with a tensor of coefficients $a_{\alpha\beta\gamma}$. By

$$N_\gamma \cdot \partial_\alpha N_\beta = -\left(\partial_\alpha N_\gamma\right) \cdot N_\beta \tag{4.162}$$

we have antisymmetry in the last two indices:

$$a_{\alpha\beta\gamma} = -a_{\alpha\gamma\beta}. \tag{4.163}$$

Using Eq. (4.158), Eq. (4.161) and Eq. (4.163), we can simplify Eq. (4.159):

$$\begin{aligned}
0 &= \sum_{\beta=2}^{p-1} D_{NN\beta} \left[3a_{\alpha\beta} D_{NNN} + \frac{\partial}{\partial \vartheta_\alpha} D_{NN\beta} \right] \\
&= \sum_{\beta=2}^{p-1} D_{NN\beta} \left[2a_{\alpha\beta} D_{NNN} + 2\sum_{\gamma=2}^{p-1} a_{\alpha\gamma} D_{N\gamma\beta} + \sum_{\gamma=2}^{p-1} a_{\alpha\beta\gamma} D_{NN\gamma} \right] \\
&= \sum_{\beta=2}^{p-1} D_{NN\beta} \left[2a_{\alpha\beta} D_{NNN} + 2\sum_{\gamma=2}^{p-1} a_{\alpha\gamma} D_{N\gamma\beta} \right] + \underbrace{\sum_{\beta=2}^{p-1} D_{NN\beta} \sum_{\gamma=2}^{p-1} a_{\alpha\beta\gamma} D_{NN\gamma}}_{=0} \\
&= \sum_{\beta=2}^{p-1} D_{NN\beta} \left[2a_{\alpha\beta} D_{NNN} + 2\sum_{\gamma=2}^{p-1} a_{\alpha\gamma} D_{N\gamma\beta} \right],
\end{aligned} \tag{4.164}$$

i.e. after a multiplication with a^{-1} (a should be invertible almost everywhere if the parameterization of N^i is exhausting)

$$\begin{aligned}
0 &= D_{NNN} D_{NN\alpha} + \sum_{\beta=2}^{p-1} D_{NN\beta} D_{N\alpha\beta} \\
&= \sum_{\beta=1}^{p-1} D_{NN\beta} D_{N\alpha\beta}, \quad \alpha = 2, \ldots, p-1.
\end{aligned} \tag{4.165}$$

This can also be written in terms of a $(p-2) \times (p-2)$-matrix:

$$\sum_{\beta=2}^{p-1} D_{NN\beta} \left[\delta_{\alpha\beta} D_{NNN} + D_{N\alpha\beta}\right] = 0 \tag{4.166}$$

and we see that all what remains from the simple factorization found in the $p = 3$-case (see Eq. (4.59)) is that either Eq. (4.157) holds or the determinant of the $(p-2) \times (p-2)$-matrix vanishes:

$$\det_{\alpha\beta} \left[\delta_{\alpha\beta} D_{NNN} + D_{N\alpha\beta}\right] = 0. \tag{4.167}$$

It is now possible to find further solutions of this equation. In particular, the analysis correctly reduces to the case $p = 3$ (see Eq. (4.61)). However, for general p this set of equations looks very complicated and in the following we just assume that the global maximum of ω is contained in the set of critical points given by Eq. (4.157), as it is the case for $p = 3$.

4.5.1. Tensorial eigenvalue approach

We now prove that the remaining part of the extremization of ω, given by Eq. (4.157), leads to the same tensorial eigenvalue problem we used to derive Eq. (4.26) in section 4.2, i.e. we show that

$$D_{NNN} = e^K d_{ijk} N^i N^j N^k \text{ extremal}, \quad N_i K^i = 0, \quad N_i N^i = 1 \tag{4.168}$$

is equivalent to the tensorial eigenvalue problem

$$e^K d_{ijk} v^j v^k = \lambda I_{ijk} v^j v^k, \tag{4.169}$$

where again

$$I_{ijk} = \frac{1}{3}\left(K_i g_{jk} + K_j g_{ki} + K_k g_{ij}\right). \tag{4.170}$$

To see this, we introduce a Lagrange multiplier μ to implement the constraint $N_i K^i = 0$ and consider the function

$$f(N^i,\, \mu) = \frac{D_{NNN} - \mu I_{ijk} N^i N^j N^k}{\|N^i\|^3}. \tag{4.171}$$

Maximizing this function is equivalent to maximizing D_{NNN}.

Differentiating with respect to N^i gives

$$\begin{aligned} 0 = \frac{\partial}{\partial N^l} f &= 3\frac{\|N^i\|^2 e^K d_{ljk} N^j N^k - N_l D_{NNN} - \|N^i\|^2 \mu I_{ljk} N^j N^k + \mu N_l I_{ijk} N^i N^j N^k}{\|N^i\|^5} \\ &= 3\left[e^K d_{ljk} N^j N^k - N_l D_{NNN} - \mu I_{ljk} N^j N^k\right], \end{aligned} \tag{4.172}$$

where in the second line we used both constraints from (4.168). We claim that if N^i solves Eq. (4.172), then

$$v^i = \alpha K^i + N^i \tag{4.173}$$

is a solution to the eigenvalue problem Eq. (4.169) with

$$\alpha^2 = \frac{1-\lambda}{6+9\lambda} \tag{4.174}$$

$$D^2_{NNN} = 4\frac{(1-\lambda)^3}{6+9\lambda}. \tag{4.175}$$

The first of these equations follows immediately by multiplication of Eq. (4.169) with K^i and the second by multiplication of Eq. (4.169) with N^i. Fixing the relative sign between D_{NNN} and α such that $\alpha D_{NNN} < 0$, these two equations imply

$$D_{NNN} = -2(1-\lambda)\alpha. \tag{4.176}$$

It remains to show that v^i really solves the eigenvalue problem. To see this, multiply Eq. (4.169) with an arbitrary vector n^i orthogonal to K^i. Using Eq. (4.172), (4.173) and (4.176) we find

$$\begin{aligned} e^K d_{ijk} n^i v^j v^k &\overset{(4.173)}{=} 2\alpha n^i N_i + e^K d_{ijk} n^i N^j N^k \\ &\overset{(4.172)}{=} n^i N_i \left(2\alpha + D_{NNN}\right) \overset{(4.176)}{=} n^i N_i 2\alpha\lambda \overset{(4.173)}{=} \lambda I_{ijk} n^i v^j v^k, \end{aligned} \tag{4.177}$$

which proves the claim.

The converse statement is proven by deducing Eq. (4.172) from Eq. (4.177).

4.5.2. The product formula for arbitrary p

We now discuss a possible generalization of the product formula Eq. (4.108) to arbitrary p. From our experience with $p = 3$ we know that not all eigenvalues of d_{ijk} are given by critical points of D_{NNN} via Eq. (4.175). d_{ijk} has (after regularization) $p \cdot 2^{p-1}$ eigenvalues, the degree of the hyperdeterminant. Let q_c denote the number of eigenvalues given by Eq. (4.175). Naively, this number is three times the number of solutions to the set of equations Eq. (4.157). However, Eq. (4.175) is invariant under $D_{NNN} \to -D_{NNN}$, so we have to count the number of different solutions to Eq. (4.157) modulo sign reversal.

Equation (4.157) is a system of cubic equations on the $p-2$-dimensional unit sphere. Introducing a Lagrange multiplier λ, such a system takes the generic form

$$\sum_{b,\,c=1}^{p-1} D_{abc} x^b x^c - \lambda x^a = 0 \tag{4.178}$$

$$\sum_{b=1}^{p-1} (x^b)^2 = 1. \tag{4.179}$$

One directly sees that for every solution $(x^a,\, \lambda)$ of this system we obtain another solution $(-x^a,\, -\lambda)$. As multiplication of Eq. (4.178) with x^a immediately gives $\lambda = D_{NNN}$, this is precisely the symmetry we have to mod out, which simply amounts to dividing the number of solutions of this system by 2.

We have not been able to find a general strategy to determine the number of solutions. A strong indication can however be obtained by considering only diagonal D_{abc}. Then the solution for x^a reads

$$x^a = 0 \quad \text{or} \quad x^a = \frac{\lambda}{D_{aaa}}. \tag{4.180}$$

There are 2^{p-1} combinations of solutions, though the solution $x^a = 0$ for all a has to be discarded, as it cannot solve the last equation Eq. (4.179).

The remaining equation is quadratic in λ and therefore gives

$$2(2^{p-1} - 1) \tag{4.181}$$

solutions for λ. As already discussed, the number of solutions has to be divided by 2, as we are interested in the number of distinct $D_{NNN}^2 = \lambda^2$. This gives

$$q_c = 2^{p-1} - 1. \tag{4.182}$$

In addition to this somewhat handwaving argument this formula correctly reproduces the results for $p = 2$ and $p = 3$. We also verified it for $p = 4$ numerically.

The eigenvalue problem Eq. (4.169) has always one solution with $v^i = K^i$ and corresponding eigenvalue $\lambda = -\frac{2}{3}$:

$$e^K d_{ijk} K^j K^k = -2K_i = -\frac{2}{3} I_{ijk} K^j K^k, \tag{4.183}$$

where we used Eq. (3.13) and the definition of I_{ijk} in Eq. (4.170).

In addition, there is an (as we will see) even number q_r of eigenvalues of order ε^{-1} introduced by the regularization Eq. (4.90). The numbers of eigenvalues given by each of the three families have to sum up to the degree of the hyperdeterminant, i.e.

$$1 + 3q_c + q_r = p \cdot 2^{p-1}. \tag{4.184}$$

Assuming Eq. (4.182) is correct, we find $q_r = (p - 3)2^{p-1} + 2$.

The next task would be to calculate Det $\left[I_{ijk}^{\varepsilon}\right]$ for arbitrary p. The direct method used in section 4.3.1 is probably not going to work and we have not been able to find a more powerful one yet. However, we expect the following final result:

$$\prod_{j=1}^{q_c} \omega(\vec{\vartheta}_j) = C_p \left[\frac{e^{2pK}}{\det g^3}\right]^{2^{p-2}} \mathrm{Det}\left[d_{ijk}\right], \tag{4.185}$$

where $\vec{\vartheta}_j$ are the critical points of ω given by Eq. (4.157) and $C_p > 0$ is a quantity depending only on the Kähler geometry. Note that if q_c is odd as argued above and if C_p is in fact positive, we have the same Corollary as in the 3-dimensional case:

$$\text{Det}\,[d_{ijk}] > 0 \quad \Rightarrow \quad \omega > 0 \tag{4.186}$$

at at least one critical point of ω in every physical region of the moduli space.

We now give an argument why q_r is always even and C_p is positive. For this, we have to find the eigenvalues of order ε^{-1} of the regularized version of the eigenvalue problem Eq. (4.169). These correspond to the (perturbed) zero eigenvalues of I_{ijk}. Making an ansatz for $I_{ijk}v^j v^k = 0$ of the form

$$v^i = \alpha K^i + \sum_{\delta=1}^{p-1} c_\delta n_\delta^i \tag{4.187}$$

we obtain the set of equations

$$9\alpha^2 + \sum_\delta c_\delta^2 = 0 \tag{4.188}$$

$$\alpha c_\delta = 0. \tag{4.189}$$

The general solution is given by the solutions of

$$\alpha = 0, \quad \sum_\delta c_\delta^2 = 0. \tag{4.190}$$

Clearly, there cannot be a real solution. To compute the next order in the ε-expansion, we define (see Eq. (4.93) for the analog definition in the $p = 3$-analysis)

$$\alpha = \varepsilon\alpha_1, \quad c'_\delta = c_\delta + \varepsilon c_\delta^1, \quad \lambda = \varepsilon^{-1}\mu. \tag{4.191}$$

Plugging this into

$$e^K d_{ijk} v^j v^k = \lambda \left(I_{ijk} + \varepsilon\delta I_{ijk}\right) v^j v^k \tag{4.192}$$

we find

$$e^K d_{ijk} \sum_{\delta\eta} c_\delta c_\eta n_\delta^j n_\eta^k = \mu \left[2\alpha_1 I_{ijk} K^j \sum_\delta c_\delta n_\delta^k + 2 I_{ijk} \sum_{\delta\eta} c_\delta c_\eta^1 n_\delta^j n_\eta^k + \delta I_{ijk} \sum_{\delta\eta} c_\delta c_\eta n_\delta^j n_\eta^k\right]. \tag{4.193}$$

Multiplying this equation with $\sum_\delta c_\delta n^i_\delta$ gives

$$\begin{aligned}\sum_{\gamma\delta\eta} c_\gamma c_\delta c_\eta D_{\gamma\delta\eta} &= \mu\left[\frac{2}{3}\alpha_1 \underbrace{\sum_\delta c_\gamma^2}_{=0} + 2\sum_{\gamma\delta\eta} c_\gamma c_\delta c^1_\eta \underbrace{I_{\gamma\delta\eta}}_{=0} + \sum_{\gamma\delta\eta} c_\gamma c_\delta c_\eta \delta I_{\gamma\delta\eta}\right] \\ &= \mu \sum_{\gamma\delta\eta} c_\gamma c_\delta c_\eta \delta I_{\gamma\delta\eta}, \end{aligned} \tag{4.194}$$

i.e.

$$\mu = \frac{\sum_{\gamma\delta\eta} c_\gamma c_\delta c_\eta D_{\gamma\delta\eta}}{\sum_{\gamma\delta\eta} c_\gamma c_\delta c_\eta \delta I_{\gamma\delta\eta}}. \tag{4.195}$$

This is the generalization of Eq. (4.98). It is clear that for every solution to Eq. (4.190), the complex conjugate is also a solution. This implies that the μ's given by Eq. (4.195) always come in pairs of complex conjugates. As the product of all such μ's enters C_p, this factor is positive. By a similar argument, part of the remaining factor of C_p, coming from Det $\left[I^\varepsilon_{ijk}\right]$, is also positive. That Det $\left[I^\varepsilon_{ijk}\right]$ is really positive is however not completely clear, as not all eigenvalues of I^ε_{ijk} are complex; see the proof of Lemma 4.3.1 in appendix A.1.

4.6. Characterization of models with vanishing ω

We now give a characterization of models satisfying $\omega \equiv 0$ for all real orthogonal directions N^i. According to [39], these are precisely the models with a homogeneous moduli space, i.e. they satisfy $\nabla_l R_{ijmn} \equiv 0$. The condition $\omega \equiv 0$ can be translated to a condition for the tensorial eigenvalues:

Proposition 2. *Let d_{ijk} have (the maximal number of) q_c zero eigenvalues. Then*

$$\omega \equiv 0 \tag{4.196}$$

and vice versa.

Proof. If there are q_c zero eigenvalues, by Eq. (4.175) it holds $D^2_{NNN} = \frac{2}{3}$ at all critical points of D_{NNN}, implying that

$$D^2_{NNN} \equiv \frac{2}{3}. \tag{4.197}$$

Parameterizing N^i in terms of $p-2$ angles ϑ_α and taking derivatives of this equation implies

$$e^k d_{ijk} N^i N^j \frac{\partial}{\partial \vartheta_\alpha} N^k \equiv 0. \tag{4.198}$$

If the ϑ_α are a parameterization of all possible N^i, the vectors $\frac{\partial}{\partial \vartheta_\alpha} N^k$ are linearly independent, implying that all $D_{NN\beta}$ in equation Eq. (4.159) vanish. Then by the same equation and Eq. (4.197), the first part of the claim follows. The converse is obvious because if $\omega \equiv 0$, all critical points of D_{NNN} have to satisfy Eq. (4.197). □

4.7. Possible extension to matter fields

In a more realistic physical situation not only moduli contribute to the Goldstino direction but also matter fields. Matter fields are scalar fields not corresponding to parameters of the high-energy theory. An example may be the scalar superpartner of the electron, the selectron. We denote scalar matter fields as ϕ^α, where $\alpha = 1, \dots, q$ enumerates the different fields. The precise form of the Kähler potential has been worked out in [40] and [41] and reads

$$K = -\log \mathcal{V}, \quad \mathcal{V} = \frac{1}{6} d_{ijk} J^i J^j J^k \tag{4.199}$$

with

$$J^i = T^i + \overline{T}^i - c^i_{\alpha\beta} \phi^\alpha \overline{\phi}^\beta, \tag{4.200}$$

where $c^i_{\alpha\beta}$ are Hermitian matrices in their lower entries.

The metric components g_{ij}, $g_{i\beta}$ and $g_{\alpha\beta}$, the Riemann tensor and finally σ and ω can now be calculated. The expressions are however relatively complicated, in particular because interference terms between moduli and matter fields (i.e. the components $g_{i\beta}$) do not vanish. By inspection of Eq. (4.200), it is obvious that

$$\partial_\alpha K = \frac{J^i}{\partial \phi^\alpha} \partial_i K = -\left(c^i_{\alpha\beta} \overline{\phi}^\beta + \overline{c}^i_{\alpha\beta} \phi^\beta \right) \partial_i K, \tag{4.201}$$

i.e. this quantity vanishes if the vacuum expectation values of the fields ϕ^i vanish. By the same argument, all metric components $g_{i\beta}$ vanish in such situations and the problem simplifies significantly. As matter fields usually participate in gauge interactions, a non-vanishing vacuum expectation value would spontaneously break the corresponding gauge symmetry, providing a physical motivation for the assumption that the vacuum expectation values of all matter fields are either zero or very small (e.g. of the order of electroweak symmetry breaking). Of course, this may not be true for new, yet unknown gauge interactions, broken at very high scales. Nevertheless, we will make the assumption of vanishing vacuum expectation values for the matter fields in the following.

If one restricts to a special coordinate system (called the *canonical frame* in the following), the following simplifications can be assumed at the vacuum (see [39] for a justification):

$$g_{ij} = \delta_{ij}, \quad g_{\alpha\beta} = \delta_{\alpha\beta}, \quad g_{i\beta} = 0, \quad K = 0, \quad T^i + \overline{T}^i = \sqrt{3}\delta_{i1}, \quad \phi^\alpha = 0. \quad (4.202)$$

In terms of intersection numbers and matter matrices these conditions imply

$$d_{111} = \frac{2}{\sqrt{3}}, \quad d_{11a} = 0, \quad d_{1ab} = -\frac{1}{\sqrt{3}}\delta_{ab}, \quad c^1_{\alpha\beta} = \frac{1}{\sqrt{3}}\delta_{\alpha\beta}, \quad (4.203)$$

where a and b run from 2 to p and all other d_{ijk} and $c^i_{\alpha\beta}$ are generic.

The coordinate change required to achieve this situation does of course depend on the initial values of T^i and ϕ^α and can usually not be calculated in an explicit form. This reduces the value of the canonical frame method if explicit results are needed. However, in [32] it has been proven that

$$\max\omega = \max\{0,\, a_{max},\, b_{max}\} \quad (4.204)$$

with

$$a_{max} = \max_{|y|=1} a_{abcd}\, y^a y^b y^c y^d \quad (4.205)$$

$$b_{max} = \max_{|y|=|z|=1} b^{ab}_{\alpha\beta}\, y^a y^b z^\alpha z^\beta, \quad (4.206)$$

where $a, b, \ldots$ run from 2 to p and α, β run from 2 to q. The coefficients are given by

$$a_{abcd} = \frac{1}{2}\left(d_{abr}d_{rdc} + d_{adr}d_{rbc} + d_{bdr}d_{rac}\right) - \frac{1}{3}\left(\delta_{ab}\delta_{dc} + \delta_{ad}\delta_{bc} + \delta_{bd}\delta_{ac}\right) \tag{4.207}$$

$$b^{ab}_{\alpha\beta} = \frac{1}{2}\left\{c^a, c^b\right\}_{\alpha\beta} - \frac{1}{3}\delta_{ab}\delta_{\alpha\beta} - \frac{1}{2}d_{abr}c^r_{\alpha\beta}. \tag{4.208}$$

a_{max} is identical to the global maximum of ω if one only considers moduli fields and works in the frame defined by Eq. (4.202). Therefore, one half of the metastability analysis can be performed without considering any matter fields. Thus our assumption that only moduli fields play a role are partially justified.

It is instructive to compare the results of the canonical frame method in the case of homogeneous moduli spaces with our results in section 4.6. See appendix A.2 for a few remarks on this.

5. The sGoldstino mass

Using the results from the last chapter, the sign of σ at its global maximum is shown to be equal to the sign of ω at its maximum and conclusions for the sGoldstino mass are drawn. In particular, we show that this mass can become arbitrarily large in a given model if it is positive at one point in the moduli space.

5.1. Computation of σ and the sGoldstino mass

The positivity of ω is a priori only a necessary condition for the positivity of σ (see section 2.7). We now show that in many cases, these two conditions are in fact equivalent. Part of the following discussion is based on [42].

We parameterize the Goldstino direction as

$$G^i = \rho K^i + N^i \tag{5.1}$$

with

$$N^i = \sum_{\alpha=1}^{p-1} c_\alpha n_\alpha^i, \tag{5.2}$$

where n_α^i is real. This time we do not split the c_α in pure phases and lengths explicitly, so the c_α are complex numbers. G^i is normalized, i.e.

$$1 = G_i \bar{G}^i = 3|\rho|^2 + \sum_{\alpha=1}^{p-1} |c_\alpha|^2. \tag{5.3}$$

It holds (see section 2.7 for the general formulas)

$$s^i = \sum_\alpha n^i_\alpha \left(\rho\bar{c}_\alpha + \bar{\rho} c_\alpha - \frac{1}{4}\sum_{\beta\gamma} D_{\alpha\beta\gamma}\left(c_\beta\bar{c}_\gamma + \bar{c}_\beta c_\gamma\right)\right) \tag{5.4}$$

$$\omega = -\frac{4}{3}\left(\sum_\alpha |c_\alpha|^2\right)^2 + \frac{1}{3}\left|\sum_\alpha c_\alpha^2\right|^2 + \sum_{\alpha\beta\gamma\delta\eta} D_{\alpha\beta\gamma}D_{\alpha\delta\eta}\left(\frac{1}{2}c_\alpha\bar{c}_\beta c_\gamma\bar{c}_\delta + c_\alpha c_\beta\bar{c}_\gamma\bar{c}_\delta\right) \tag{5.5}$$

and therefore

$$\begin{aligned} s_i s^i &= \sum_\alpha \left(\rho\bar{c}_\alpha + \bar{\rho}c_\alpha - \frac{1}{2}\sum_{\beta\gamma} D_{\alpha\beta\gamma}c_\beta\bar{c}_\gamma\right)^2 \\ &= \sum_\alpha \left(\rho\bar{c}_\alpha + \bar{\rho}c_\alpha\right)^2 + \frac{1}{4}\sum_{\alpha\beta\gamma\delta\eta} D_{\alpha\beta\gamma}D_{\alpha\delta\eta}c_\alpha\bar{c}_\beta c_\gamma\bar{c}_\delta - \sum_{\alpha\beta\gamma}\left(\rho\bar{c}_\alpha + \bar{\rho}c_\alpha\right) D_{\alpha\beta\gamma}c_\beta\bar{c}_\gamma. \end{aligned} \tag{5.6}$$

We then arrive at

$$\begin{aligned} \sigma &= -2s_i s^i + \omega = \\ &= -2\sum_\alpha \left(\rho\bar{c}_\alpha + \bar{\rho}c_\alpha\right)^2 + 2\sum_{\alpha\beta\gamma}\left(\rho\bar{c}_\alpha + \bar{\rho}c_\alpha\right) D_{\alpha\beta\gamma}c_\beta\bar{c}_\gamma \\ &\qquad - \frac{4}{3}\left(\sum_\alpha |c_\alpha|^2\right)^2 + \frac{1}{3}\left|\sum_\alpha c_\alpha^2\right|^2 + \sum_{\alpha\beta\gamma\delta\eta} D_{\alpha\beta\gamma}D_{\alpha\delta\eta}c_\alpha c_\beta\bar{c}_\gamma\bar{c}_\delta. \end{aligned} \tag{5.7}$$

We now assume that N^i is (up to a global phase) real. Then we can choose n^i_1 in the N^i direction, i.e. all c_α except for $c_1 = |c_1|e^{\mathrm{i}\gamma}$ vanish and it holds $N^i = c_1 n^i_1$. Then σ simplifies to

$$\sigma = |c_1|^2\left(-2\left(\rho e^{-\mathrm{i}\gamma} + \bar{\rho}e^{\mathrm{i}\gamma}\right)^2 + 2|c_1|\left(\rho e^{-\mathrm{i}\gamma} + \bar{\rho}e^{\mathrm{i}\gamma}\right)D_{111} - |c_1|^2 + |c_1|^2\sum_\alpha D^2_{\alpha 11}\right). \tag{5.8}$$

The sign of σ is determined by the sign of the quadratic polynomial $P(|c_1|)$ given by

$$P(x) = \left(-1 + \sum_\alpha D^2_{\alpha 11}\right) x^2 + 2D_{111}Ax - 2A^2 \tag{5.9}$$

with $A = \rho e^{-\mathrm{i}\gamma} + \bar{\rho}e^{\mathrm{i}\gamma}$. Because $P(0) < 0$, there have to be real roots of P for σ to become positive for some values of $|c_1|$. These only exist if

$$D^2_{111} + 2\sum_{\alpha=1}^{p-1} D^2_{\alpha 11} - 2 \geq 0 \quad \Leftrightarrow \quad \frac{3}{2}D^2_{111} + \sum_{\alpha=2}^{p-1} D^2_{\alpha 11} \geq 1. \tag{5.10}$$

This condition looks in fact stronger than the condition

$$\frac{3}{2}\sum_{\alpha=1}^{p-1} D^2_{11\alpha} \geq 1 \tag{5.11}$$

coming from $\omega > 0$. However, at least for $p = 3$ we have shown that the global maximum of ω occurs if $D_{11\alpha} = 0$ for $\alpha > 1$. So if $\omega > 0$ at its maximum, we automatically know that $\sigma > 0$ at its maximum, too. For general p, we are still missing a proof that the global maximum of ω is given for a point such that $D_{11\alpha} = 0$ for $\alpha > 1$. However, if this turns out to be correct, the same conclusion would hold for the sign of σ.

We now turn to the explicit maximization of σ. Although it is (under the above assumptions) true that a positive ω implies a positive σ (and vice versa), it is not obvious that the maximum occurs for the same orthogonal direction N^i. This however will turn out to be true.

In [28] it was shown that a (normalized) Goldstino direction G^i_0 which is a critical point of σ (as a function of G^i) automatically diagonalizes the mass matrix (as defined in Eq. (2.19)):

$$V_i{}^j G_{0j} = m_0^2 G_{0i}, \tag{5.12}$$

where

$$m_0^2 = \left[3(1+\gamma)\sigma(G^i_0) - 2\gamma\right] m^2_{3/2} \tag{5.13}$$

with the gravitino mass $m_{3/2} = e^{G/2}$. Recall from Eq. (2.25) that γ is the cosmological constant rescaled by the gravitino mass.

To obtain an upper bound on the sGoldstino mass m_0^2 we now maximize σ explicitly. For this, we parameterize n_1^i as in section 4.5 with $p-2$ angles ϑ_β and first differentiate σ w.r.t. to these angles. This gives

$$\frac{\partial}{\partial\vartheta_\beta}\sigma = |c_1|^4\left(6|c_1|^{-1}\left(\rho e^{-\mathrm{i}\gamma}+\bar{\rho}e^{\mathrm{i}\gamma}\right)\sum_{\alpha=2}^{p-1}a_{\beta\alpha}D_{11\beta}+2\sum_{\alpha=1}^{p-1}D_{11\alpha}\frac{\partial}{\partial\vartheta_\beta}D_{11\alpha}\right), \quad (5.14)$$

where we used Eq. (4.158) to calculate the derivative of n_1^i with respect to ϑ_β.

So as for ω, an orthogonal Goldstino component N^i such that $D_{11\beta}=0$ for $\beta=2,\ldots,p-1$ is a critical point. If $|c_1|^{-1}\left(\rho e^{-\mathrm{i}\gamma}+\bar{\rho}e^{\mathrm{i}\gamma}\right)D_{111}$ turns out to be very small and our assumption that the global maximum of ω is in fact determined by Eq. (4.157) is correct, this critical point indeed corresponds to the global maximum of σ. But even if $|c_1|^{-1}\left(\rho e^{-\mathrm{i}\gamma}+\bar{\rho}e^{\mathrm{i}\gamma}\right)D_{111}$ is not small, it can always be turned into a positive contribution by switching the sign of ρ, which does not change any other term in σ. This positive contribution in turn is maximal if D_{111} is maximal, which is precisely the case for the assumed maximum of ω (modulo an irrelevant sign swap to make D_{111} positive if needed). We are therefore left with the task of finding the global maximum of

$$\sigma = |c_1|^4\left(-2|c_1|^{-2}\left(\rho e^{-\mathrm{i}\gamma}+\bar{\rho}e^{\mathrm{i}\gamma}\right)^2+2|c_1|^{-1}\left(\rho e^{-\mathrm{i}\gamma}+\bar{\rho}e^{\mathrm{i}\gamma}\right)D_{111}-1+D_{111}^2\right) \quad (5.15)$$

as a function of ρ, γ and $|c_1|$, with the constraint $3|\rho|^2+|c_1|^2=1$ coming from the normalization of G^i.

Equation (5.15) however is completely independent of $D_{11\beta}$ for $\beta>1$ and identical in form for all p. The global maximum for $p=2$ has been determined in [28] and the result can be straightforwardly extrapolated to the general case. It reads

$$\sigma(G_0^i) = \frac{128}{3}\frac{a_{\mathcal{H}}+9\sqrt{(1+a_{\mathcal{H}})(1+a_{\mathcal{H}}/9)}-9}{\left(21+a_{\mathcal{H}}-3\sqrt{(1+a_{\mathcal{H}})(1+a_{\mathcal{H}}/9)}\right)^2}, \quad (5.16)$$

where

$$a_{\mathcal{H}} = -1+\frac{3}{2}D_{111}^2, \quad (5.17)$$

evaluated at the maximum of ω. Not that the ω entering σ via $\sigma = -2s_i s^i + \omega$ is given by $\omega = c_1^4 a_{\mathcal{H}}$, because ω is homogeneous of degree 4 in the orthogonal component of G^i.

In the next section we will show that in heterotic models which allow a positive σ, the quantity $a_{\mathcal{H}}$ can be made arbitrarily large while keeping the volume $\mathcal{V}$ constant, thus without leaving the realm of validity of the large-volume approximation. Equation (5.16) shows that σ grows asymptotically as $2/3a_{\mathcal{H}}$ and from Eq. (5.13) we therefore conclude that the sGoldstino mass can be made arbitrarily large, as long as the necessary condition for metastable deSitter vacua encoded in the sign of ω is fulfilled at one point in the moduli space.

5.2. Evolution of ω toward singularities

It was shown in section 4.5 that ω has critical points if

$$D_{NN\beta} = 0, \quad \beta = 2, \dots, p-1. \tag{5.18}$$

We assume again that N^i is normalized. Then this set of equations defines a family of unit vector fields on the moduli space, namely its solutions $N^i(K^i)$. We are now interested in the evolution of ω on an integral curve of one of these vectors fields, i.e. on a path $K^i(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} K^i(t) = N^i(K^i(t)). \tag{5.19}$$

Moving along this curve corresponds to a deformation of the vacuum expectation value of K^i along the special direction N^i defined by Eq. (5.18). We will find that along this curve ω necessarily diverges while the volume $\mathcal{V}$ stays constant. Mathematically, moving along the curve corresponds to a deformation of the Kähler structure of the Calabi-Yau manifold toward a point in the Kähler structure moduli space with a degenerate metric g, as will become clear during the analysis.

We first show that the volume $\mathcal{V} = e^{-K}$ does not change on the path defined by Eq. (5.19). This follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} e^K = -e^K K_i \dot{K}^i = -e^K K_i N^i = 0. \tag{5.20}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}\left[e^K d_{ijk} K^k + K_i K_j\right] = e^K d_{ijk} N^k + \dot{K}_i K_j + K_i \dot{K}_j. \tag{5.21}$$

From $N^i N_i \equiv 1$ we get

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[N^i N^j g_{ij}\right] = 2\dot{N}^i g_{ij} N^j + N^i N^j \frac{\mathrm{d}}{\mathrm{d}t} g_{ij} \\ &= 2\dot{N}^i g_{ij} N^j + N^i N^j \left[e^K d_{ijk} N^k + \dot{K}_i K_j + K_i \dot{K}_j\right] \\ &= 2\dot{N}^i g_{ij} N^j + e^K d_{ijk} N^i N^j N^k = 2\dot{N}^i g_{ij} N^j + D_{NNN} \end{aligned} \tag{5.22}$$

and from $K^i N_i \equiv 0$

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[K^i N^j g_{ij}\right] &= \underbrace{N^i N_i}_{=1} + K^i \dot{N}^j g_{ij} + \underbrace{e^k d_{ijk} K^i N^j N^k}_{=1} + \underbrace{3\dot{K}_j N^j}_{=-3} \\ = K^i \dot{N}^j g_{ij} - 1, \end{aligned} \tag{5.23}$$

because

$$\dot{K}_j = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} e^K d_{ijk} K^i K^k = -e^K d_{ijk} K^i N^k. \tag{5.24}$$

Equation (5.22) and Eq. (5.23) imply

$$\dot{N}^i = \frac{1}{3} K^i - \frac{1}{2} D_{NNN} N^i + \sum_{\alpha=2}^{p-1} \gamma_\alpha N^i_\alpha \tag{5.25}$$

where γ_α are some functions of t. The γ_α are determined by the requirement that Eq. (5.18) holds for every t but we do not need their explicit t-dependence in the following.

We can now determine a differential equation for D_{NNN} along the integral curve (as the terms proportional to the γ_α do not contribute by Eq. (5.18)):

$$\frac{\mathrm{d}}{\mathrm{d}t} D_{NNN} = 3 D_{NN\dot{N}} = 3\left[\frac{1}{3} D_{NNK} - \frac{1}{2} D^2_{NNN}\right] = 1 - \frac{3}{2} D^2_{NNN}. \tag{5.26}$$

This ODE can be solved and the form of the solution depends on the initial condition

$D_{NNN}(0)$. We find

$$D_{NNN}(t) = \begin{cases} \sqrt{\frac{2}{3}} \tanh\left[\sqrt{\frac{3}{2}}(t-t_0)\right] & \text{if } D^2_{NNN}(0) < \frac{2}{3} \\ \pm\sqrt{\frac{2}{3}} & \text{if } D^2_{NNN}(0) = \frac{2}{3} \\ \sqrt{\frac{2}{3}} \coth\left[\sqrt{\frac{3}{2}}(t-t_0)\right] & \text{if } D^2_{NNN}(0) > \frac{2}{3} \end{cases}, \tag{5.27}$$

where t_0 is an integration constant.

This gives

$$\omega(t) = \frac{3}{2} D^2_{NNN}(t) - 1 = \begin{cases} \frac{-1}{\cosh^2\left[\sqrt{\frac{3}{2}}(t-t_0)\right]} & \text{if } \omega(0) < 0 \\ 0 & \text{if } \omega(0) = 0 \\ \frac{1}{\sinh^2\left[\sqrt{\frac{3}{2}}(t-t_0)\right]} & \text{if } \omega(0) > 0 \end{cases}. \tag{5.28}$$

In particular, ω does not change its sign on an integral curve of N. The qualitative difference between integral curves with $\omega \leq 0$ and integral curves with $\omega > 0$ is that the latter are unbounded. This shows that the quantity $a_{\mathcal{H}}$ defined in Eq. (5.17) is unbounded in any heterotic model which satisfies the metastability condition at one point of its moduli space.

Because e^K stays constant on the integral curves considered here, the quantity $D_{NNN} = e^K d_{ijk} N^i N^j N^k$ (and therefore ω) can only diverge after the finite interval $[0, t_0]$ if the numerical values of N^i diverge, i.e. if the metric g gets degenerate and N^i aligns with a null eigenvector of g as $t \to t_0$. In particular, we need

$$\lim_{t\to t_0} \det g = -\frac{1}{2} e^{pK} \lim_{t\to t_0} \det\left[d_{ijk} K^k\right] = 0. \tag{5.29}$$

Here we used $g_{ij} = e^K d_{ijk} K^k + K_i K_j = e^K \left(\delta_i{}^l - \frac{1}{2} K_i K^l\right) d_{ljk} K^k$ and the formula

$$\det\left(\delta_i{}^l - v_i w^l\right) = 1 - v_i w^i, \tag{5.30}$$

which is easily proved by performing a rotation such that v_i lies parallel to the 1-axis.

In the $p = 2$-dimensional situation, this observation can be used for yet another derivation of Eq. (4.26). In this case, we have

$$\det\left[d_{ijk}K^k\right] = (d_{111}d_{222} - d_{112}d_{122})\,K^1K^2 + \left(d_{111}d_{122} - d_{112}^2\right)(K^1)^2 + \left(d_{112}d_{222} - d_{122}^2\right)(K^2)^2. \tag{5.31}$$

Denoting $z = K^1/K^2$ we see that $\det\left[d_{ijk}K^k\right] = 0$ is equivalent to

$$z^2(d_{111}d_{122} - d_{112}^2) + z(d_{111}d_{222} - d_{112}d_{122}) + d_{112}d_{222} - d_{122}^2 = 0. \tag{5.32}$$

The discriminant of this (quadratic) equation is given by

$$\begin{aligned}&(d_{111}d_{222} - d_{112}d_{122})^2 - 4(d_{111}d_{122} - d_{112}^2)(d_{112}d_{222} - d_{122}^2)\\&= d_{111}^2d_{222}^2 + 4d_{111}d_{122}^3 + 4d_{222}d_{112}^3 - 3d_{112}^2d_{122}^2 - 6d_{111}d_{222}d_{112}d_{122},\end{aligned} \tag{5.33}$$

i.e. precisely by the hyperdeterminant of d_{ijk} for $p = 2$. Therefore $\det g$ can only go to zero if $\mathrm{Det}\,[d_{ijk}] > 0$.

For arbitrary p, the situation is much more complicated. Except for very special moduli spaces, all models with $p > 3$ have field configurations at which $\det g = 0$ and $\mathcal{V} \neq 0$, thus giving a positive ω in the vicinity of these points (by aligning N^i to the null eigenvector of g). On the other hand, a good global understanding of the moduli space of a given model could probably be obtained by determining the algebraic varieties specified by the two different types of singularities $\det g = 0$ and $\mathcal{V} = 0$.

6. Summary and Outlook

6.1. Summary and conclusion

We studied the existence of metastable deSitter vacua in low-energy limits of heterotic string theory. We used the strategy proposed in [30] to encode a necessary condition for the existence of such vacua in the sign of a single function called ω. The maximum of this function has to be calculated to check the necessary condition. In this thesis, we made significant progress in this analysis for the situation in which only moduli fields contribute to supersymmetry breaking.

In section 4.1, section 4.3 and section 4.5 we were able to solve the maximization problem partially, though some simplifying assumptions had to be made (in the case of a $p = 3$-dimensional moduli space all results should be waterproof though). In particular, we devised a strategy to analyze the influence of complex phases in the vector G^i describing the Goldstino direction and succeeded in proving the irrelevance of such phases for the metastability analysis in the special case of $p = 3$ moduli fields. This directly benefits for example the analysis in [32], where the absence of additional phases is assumed in order to simplify the calculations.

The remaining part of the maximization problem has been put into the form of a tensorial eigenvalue problem in section 4.5.1 and this formulation allowed for the generalization of a known result in the special case of a two-dimensional moduli space (see section 4.2 for the two-dimensional case). This generalization consists of the formula Eq. (4.185) which constrains the product of a subset of the critical points of ω on the whole moduli space. Up to a (presumably) positive factor, the product of an odd number of such critical points is equal to an $SO(p)$ invariant polynomial in the intersection numbers defining the heterotic model, the hyperdeterminant (see Eq. (4.108) and Eq. (4.185). If the sign of this invariant is positive, one can immediately conclude that a metastable deSitter vacua is possible for all (physical) points on the Kähler moduli space (if additional tuning of the superpotential is allowed such that the necessary condition also becomes sufficient as was briefly explained in section 2.4). This seems to be a quite non-trivial result and raises the interesting question if the sign of the hyperdeterminant of the intersection numbers corresponds to a more intuitive geometric property of the Calabi-Yau manifold.

The tools we developed enabled us to study two non-trivial types of examples of heterotic moduli spaces in detail in section 4.4. While the first one, consisting of models with only diagonal intersection numbers, always allows metastable deSitter vacua, the second class of examples, models with a factorizing volume, satisfies the metastability condition only on one half of its moduli space. This class of models is particularly interesting as it constitutes a natural generalization of the class of models briefly discussed in section 3.3, which always at least marginally violate the metastability condition but are often studied in the literature due to their simplicity (see for example [34]).

In view of the no-go theorems concerning the construction of classical metastable deSitter vacua in string theory (see section 1.4), the setting studied here appears to be very promising. At least roughly one half of all possible Calabi-Yau manifolds – the ones with a positive hyperdeterminant – pose no immediate obstructions to the construction of viable deSitter models, while the other half requires a more detailed analysis to identify the allowed regions in their moduli spaces, see the remark after Theorem 4.3.2.

Chapter 5 was concerned with the sGoldstino mass which is essentially given by σ if the Goldstino direction is a critical point of σ. We showed that the orthogonal part of the Goldstino is (up to its length) the same for the maximum of ω and for a maximum of σ and that the size of the preferred Goldstino direction parallel to K^i can be calculated explicitly. In particular, it was shown that a positive maximum of ω implies a positive maximum of σ and because the converse is obvious, this establishes an equivalence between the two functions as discriminators of the existence of metastable deSitter vacua. As ω is usually simpler to study, this can be used to simplify the metastability analysis at least for models from heterotic string theory. Another important result following from the analysis in chapter 5 was a proof that the sGoldstino mass can become arbitrarily large by just moving the vacuum expectation values of the moduli if the sGoldstino mass is positive at one point in the moduli space. The sGoldstino mass diverges (for bounded volume $\mathcal{V}$) if and only if the moduli space metric becomes degenerate. This observation may be of interest for attempts to connect singularities on moduli spaces to spacetime topology changes. See for example [43] for a discussion of such a proposal.

6.2. Outlook

This work can be naturally extended in several directions. We propose a few possibilities in the following. In section 4.4 we identified two important special cases: models with diagonal intersection numbers and models with a factorizing volume. We studied these examples for a three-dimensional moduli space but it seems reasonable that these examples can be understood for arbitrary-dimensional moduli spaces and that the results are not significantly more complicated.

Another logical extension would be to include matter fields in the analysis. We briefly commented on this possibility in section 4.7, mainly citing results obtained with a canonical frame method in [32]. It seems plausible that a general-frame analysis as performed in this thesis for Kähler moduli only would lead to an equivalent formulation for the condition that b_{max} (see Eq. (4.206)) is positive similar in spirit to the formulation of the condition on $a_{max} = \max \omega$ in terms of a tensorial eigenvalue problem.

Eventually, all assumptions made in this thesis should be checked carefully. First, the analysis of the influence of complex phases in G^i should be completed for arbitrary-dimensional moduli spaces (see the end of section 4.1 for such an analysis for $p = 3$). Next, the claim that the solutions to Eq. (4.157) include the global maximum of ω should be verified for $p > 3$. The last missing part is a careful derivation of the results used in section 4.5.2 to generalize the product formula Eq. (4.108) to the p-dimensional case in Eq. (4.185).

We concluded from the examples in section 4.4 that the behavior of ω on the Kähler moduli space is in general rather complicated if $p > 2$. It would certainly be useful to understand the global situation better. We found in section 5.2 that singularities of $\det g$ and of the volume $\mathcal{V}$ play an important role. A critical point of ω can only change its sign while moving the vacuum expectation value of the moduli if one of these quantities vanishes (because if both $\det g$ and $\mathcal{V}$ stay finite, an additional zero tensorial eigenvalue would appear at the transition point, but the number of zero eigenvalues of $e^K d_{ijk}$ cannot change). On the other hand, ω diverges if one moves toward a point at which $\det g = 0$ and $\mathcal{V} \neq 0$ if the orthogonal direction of the Goldstino is aligned to the would-be zero eigenvector of g. If and how this observation can be used to characterize the positivity of ω in a useful way is however unclear at this point.

There may be additional global constraints on ω involving other invariants of d_{ijk}. The classical results reviewed in section 4.3.2 reveal the existence of two independent invariants (the Aronhold invariants S and T) in the $p = 3$-dimensional case, but we only found one global constraint. The example studied in section 4.4.1 of a model with only diagonal intersection numbers is characterized by the vanishing of the first Aronhold invariant:

$$S = 0, \tag{6.1}$$

as briefly explained at the end of section 4.3.2. In this situation, ω is positive at all of its critical points given by the tensorial eigenvalue problem, so S may have something to do with negative contributions to ω. The invariant S also appeared in the second class of examples discussed in section 4.4.2. If either of these observations bears deeper meaning, remains to be seen.

Another aspect not studied in this thesis due to time constraints is the extension of the analysis to models from compactifications of type IIb string theory (using orientifolds to overcome no-go theorems for type IIb string theory concerning the construction of deSitter vacua). These models are closely connected to the heterotic compactifications studied here by the substitutions $e^K \to e^{-K}$, $\det g \to (\det g)^{-1}$ and $\omega \to -\omega$. In particular, a positive ω in a heterotic model gets mapped onto a negative ω in an orientifold model and vice versa (see the appendix of [44] for details). This should be checked more explicitly and in particular it should be checked if the correspondence is manifestly realized in the tensorial eigenvalue framework.

A. Appendix

A.1. Proofs from chapter 4

Proposition 3. *Let $D_{122} = 0$. Then*

$$\max_{\vartheta\in[0,2\pi]} D^2_{NNN}(\vartheta) \geq \frac{D^2_{112}\left(D_{112}+D_{222}\right)^2 + D^2_{111}D^2_{222}}{D^2_{111}+\left(D_{112}+D_{222}\right)^2}. \tag{A.1}$$

Proof. First note that if $D^2_{222} \geq D^2_{112}$, then Eq. (4.62) gives for D^2_{NNN} at $\vartheta = \pi/2$

$$D^2_{NNN}(\vartheta=\pi/2) = D^2_{222} \geq \frac{D^2_{112}\left(D_{112}+D_{222}\right)^2 + D^2_{111}D^2_{222}}{D^2_{111}+\left(D_{112}+D_{222}\right)^2} \tag{A.2}$$

$$\Leftrightarrow D^2_{222}\left(D_{112}+D_{222}\right)^2 \geq D^2_{112}\left(D_{112}+D_{222}\right)^2, \tag{A.3}$$

which is true by assumption. If on the other hand $D^2_{222} < D^2_{112}$, then we have

$$\frac{D^2_{112}\left(D_{112}+D_{222}\right)^2 + D^2_{111}D^2_{222}}{D^2_{111}+\left(D_{112}+D_{222}\right)^2} = D^2_{112} + \frac{D^2_{111}\left(D^2_{222}-D^2_{112}\right)}{D^2_{111}+\left(D_{112}+D_{222}\right)^2} < D^2_{112}. \tag{A.4}$$

Let us denote $x = \tan\vartheta$, $\lambda = D_{111}/D_{112}$ and $\mu = D_{222}/D_{112}$. This gives

$$D^2_{NNN} = D^2_{112}\frac{1}{\left(1+x^2\right)^3}\left(\lambda+3x+\mu x^3\right). \tag{A.5}$$

The claim now follows if there is an x such that

$$\frac{1}{\left(1+x^2\right)^3}\left(\lambda+3x+\mu x^3\right)^2 \geq 1. \tag{A.6}$$

The case $|\mu| \geq 1$ is already done and the case $|\lambda| \geq 1$ is easily solved by $x = 0$ (i.e. $\cos(\vartheta) = \pm 1$). So let us assume that $|\lambda|, |\mu| < 1$.

An extremum in x of the left-hand side of Eq. (A.6) must satisfy

$$\lambda + 3x_0 + \mu x_0^3 = (1 + x_0^2)\left(\frac{1}{x_0} + \mu x_0\right). \tag{A.7}$$

This equation is in fact only quadratic in x_0 and is solved by

$$x_0 = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\mu + 8}}{2(2-\mu)}. \tag{A.8}$$

Taking the + solution if $\lambda \geq 0$ and the - solution if $\lambda < 0$ we see that (by concavity of the square root)

$$x_0^2 \leq \frac{1}{2-\mu}, \tag{A.9}$$

so

$$\frac{1}{1+x_0^2} \geq \frac{2-\mu}{3-\mu}. \tag{A.10}$$

The next step is to show that $\left(\frac{1}{x_0} + \mu x_0\right)$ grows monotonically for $0 \leq \lambda \leq 1$. This can be seen from

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{x_0} + \mu x_0\right) = \left(-\frac{1}{x_0^2} + \mu\right)\frac{\partial}{\partial\lambda}x_0 \geq 0, \tag{A.11}$$

because

$$\frac{\partial}{\partial\lambda}x_0 = \frac{1}{2(2-\mu)}\left(\frac{\lambda}{\sqrt{\lambda^2+\varepsilon^2}} - 1\right) < 0 \tag{A.12}$$

with $\varepsilon^2 = 8 - 4\mu > 0$ and

$$-\frac{1}{x_0^2} + \mu \leq -\frac{1}{2-\mu} + \mu = -\frac{(1-\mu)^2}{2-\mu} \leq 0. \tag{A.13}$$

This finally gives for $0 \le \lambda \le 1$

$$\frac{1}{(1+x_0^2)^3}\left(\lambda + 3x_0 + \mu x_0^3\right)^2 \overset{(A.7)}{=} \frac{1}{1+x_0^2}\left(\frac{1}{x_0} + \mu x_0\right)^2 \tag{A.14}$$

$$\overset{(A.10)}{\geq} \frac{2-\mu}{3-\mu}\left(\frac{1}{x_0} + \mu x_0\right)^2 \tag{A.15}$$

$$\overset{(A.11)}{\geq} \frac{2-\mu}{3-\mu}\left(2-\mu+2\mu+\frac{\mu^2}{2-\mu}\right) \tag{A.16}$$

$$= \frac{4}{3-\mu} > 1. \tag{A.17}$$

The calculation for $\lambda < 0$ is analogous. Alternatively, the claim follows by letting $x \to -x$ in Eq. (A.6). □

Lemma A.1.1. *Let w_i denote a 3-dimensional vector and A_{ij} an arbitrary positive definite 3×3 matrix. Then it holds for $c_{ijk} = w_{(i}A_{jk)} + \varepsilon E_{ijk}$:*

$$\mathrm{Det}\,[c_{ijk}] = \varepsilon^2 \frac{4^3}{3^9} |\mathrm{i}C_2 - C_1|^2 (A^{ij} w_i w_j)^5 \det A^6 + \mathcal{O}(\varepsilon^3), \tag{A.18}$$

where again

$$C_\eta = E_{ijk} v^i v^j n_\eta^k = (v^1)^2 n_\eta^1 + (v^2)^2 n_\eta^2 + (v^3)^2 n_\eta^3, \quad \eta = 1,\, 2 \tag{A.19}$$

$$v^i = \pm \mathrm{i} n_1^i + n_2^i \tag{A.20}$$

and n_1^i and n_2^i are (arbitrary) vectors satisfying

$$w_i n_1^i = w_i n_2^i = 0 \quad \text{and} \quad n_\eta^j A_{ij} n_\kappa^i = \delta_{\eta\kappa}. \tag{A.21}$$

In particular, it follows that

$$\mathrm{Det}\left[I_{ijk}^\varepsilon\right] = \varepsilon^2 \frac{4^3}{3^4} |\mathrm{i}C_2 - C_1|^2 \det g^6 + \mathcal{O}(\varepsilon^3). \tag{A.22}$$

Proof. The proof makes use of computer algebra (a Maple implementation of these parts of the proof can be found in appendix A.3.1) and works as follows:
First exploit the SGl(3) invariance of the hyperdeterminant:

$$\mathrm{Det}\,[c_{ijk}] = \mathrm{Det}\,[U_{il} U_{jm} U_{kn} c_{lmn}] \quad \text{if } \det U = 1. \tag{A.23}$$

Choosing first an orthogonal matrix V such that $V_{ij}w_j$ lies in the 1-direction and then another orthogonal matrix W, consisting only of a rotation around the 1-axis, such that for $U = WV$ it holds $U_{il}U_{jm}A_{lm}$ has vanishing 2-3-component, we can assume that after a coordinate change

$$w_i = \begin{pmatrix} w_1 \\ 0 \\ 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & 0 \\ g_{31} & 0 & g_{33} \end{pmatrix}. \tag{A.24}$$

The leads (in zeroth order in ε) to the vanishing of the components c_{222}, c_{223}, c_{233}, c_{333} and c_{123} in the new coordinates. Writing down the eigenvalue equations $c_{ijk}v^j v^k = \lambda\delta_{ij}\delta_{ik}v^j v^k$ and denoting $v = (x, y, z)$ we find for the $i = 2$, 3 equations

$$c_{112}x^2 + 2c_{122}xy = \lambda y^2 \tag{A.25}$$

$$c_{113}x^2 + 2c_{133}xz = \lambda z^2. \tag{A.26}$$

These equations are quadratic in y and z respectively and decouple from each other. The solutions of this equations can be plugged into the $i = 1$ equation. Using computer algebra (or a very tedious but completely straightforward manual calculation) the resulting equation can be brought into polynomial form. One finds a polynomial of degree 10 in λ. Using Vieta's formula, we find

$$\prod_{i=1}^{10} \lambda_i = 16c_{122}^2 c_{133}^2 (c_{122}^3 + c_{133}^3)(4c_{111}c_{122}c_{133} - 3c_{122}c_{113}^2 - 3c_{112}^2 c_{133}) \tag{A.27}$$

for the product of its roots.

Plugging in $c_{ijk} = w_{(i}A_{jk)} + \mathcal{O}(\varepsilon)$, we obtain

$$\prod_{i=1}^{10} \lambda_i = \frac{4^3}{3^9}(A^{ij}w_i w_j)^5 \det A^6 \frac{A_{22}^3 + A_{33}^3}{A_{22}^3 A_{33}^3} \tag{A.28}$$

in lowest order in ε.

Next, we set[1]

$$n_1^i = \frac{1}{\sqrt{A_{22}}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad n_2^i = \frac{1}{\sqrt{A_{33}}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.29}$$

These vectors satisfy

$$w_i n_1^i = w_i n_2^i = 0 \quad \text{and} \quad n_\eta^j A_{ij} n_\kappa^i = \delta_{\eta\kappa}. \tag{A.30}$$

We have to compute two more eigenvalues, namely those two which vanish at order ε^0. The corresponding eigenvectors to lowest order are

$$v_0^i = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{A_{22}}} \\ \frac{\pm \mathrm{i}}{\sqrt{A_{33}}} \end{pmatrix} = n_1^i \pm \mathrm{i} n_2^i. \tag{A.31}$$

The next order in the ε expansion of λ_{11} and λ_{12} can be calculated by a perturbative ansatz for the eigenvectors of the type

$$v^i = v_0^i + \varepsilon \delta v^i. \tag{A.32}$$

Plugging this into the eigenvector equations and setting $\lambda_{11/12} = \varepsilon\lambda$ gives

$$2c_{ijk} v_0^j \delta v^k + e_{ijk} v_0^j v_0^k = \lambda E_{ijk} v_0^j v_0^k + \mathcal{O}(\varepsilon), \tag{A.33}$$

where

$$e_{ijk} = U_{il} U_{jm} U_{kn} E_{lmn} = \sum_{l=1}^{3} U_{il} U_{jl} U_{kl}. \tag{A.34}$$

The linear system of equations (A.33) for δv has a solution if and only if the constraint

$$e_{ijk} v_0^i v_0^j v_0^k = \lambda E_{ijk} v_0^i v_0^j v_0^k \tag{A.35}$$

[1] We have A_{22}, $A_{33} > 0$ if A is positive definite.

is satisfied, giving

$$\lambda = \frac{e_{ijk} v_0^i v_0^j v_0^k}{E_{ijk} v_0^i v_0^j v_0^k}. \tag{A.36}$$

The two eigenvalues λ_{11} and λ_{12} are therefore complex conjugates of each other with

$$\lambda_{11}\lambda_{12} = \varepsilon^2 \left| \frac{e_{ijk} v_0^i v_0^j v_0^k}{E_{ijk} v_0^i v_0^j v_0^k} \right|^2 . \tag{A.37}$$

One easily checks that

$$|E_{ijk} v_0^i v_0^j v_0^k|^2 = \frac{A_{22}^3 + A_{33}^3}{A_{22}^3 A_{33}^3}, \tag{A.38}$$

i.e. this factor cancels in Eq. (A.28). The remaining factor

$$|e_{ijk} v_0^i v_0^j v_0^k|^2 = |E_{ijk}(Uv_0)^i (Uv_0)^j (Uv_0)^k|^2 = |\mathrm{i}C_{1'} - C_{2'}|^2 \tag{A.39}$$

gives precisely what is missing to prove the claim after the coordinate change has been reverted. □

A.2. Comparison of the $\omega \equiv 0$-characterization with the literature

We perform a short comparison of our analysis with the results obtained via the canonical frame method in [32] for the case of homogeneous $p = 3$-dimensional moduli spaces. In [32] it is shown that the moduli space is homogeneous if and only if

$$a_{abcd} = 0 \tag{A.40}$$

for all transverse components (see Eq. (4.207) for the definition of a_{abcd}). On the other hand, Proposition 2 assures that this is precisely the case when all tensorial eigenvalues of $e^K d_{ijk}$ associated with critical points of ω vanish. These are clearly less conditions than suggested by Eq. (A.40). A generic totally-symmetric $p-1$-dimensional rank 4 tensor has a priori $\binom{2+p}{p-2}$ independent components. As a_{abcd} is completely specified by d_{abc}, which has only $\binom{p+1}{p-2}$ independent components, it is clear that the set of conditions $a_{abcd} = 0$ is redundant. In fact, at least for $p = 3$ it is easy to check that there is even more redundancy. Writing down Eq. (A.40) explicitly one finds

$$a_{2222} = \frac{3}{2}\left[d_{222}^2 + d_{223}^2\right] - 1 = 0 \tag{A.41}$$
$$a_{2223} = \frac{3}{2}\left[d_{222}d_{223} + d_{223}d_{233}\right] = 0 \tag{A.42}$$
$$a_{2233} = \frac{1}{2}\left[d_{222}d_{233} + d_{333}d_{223} + 2d_{223}^2 + 2d_{233}^2\right] - \frac{1}{3} = 0 \tag{A.43}$$
$$a_{2333} = \frac{3}{2}\left[d_{333}d_{233} + d_{233}d_{223}\right] = 0 \tag{A.44}$$
$$a_{3333} = \frac{3}{2}\left[d_{333}^2 + d_{233}^2\right] - 1 = 0. \tag{A.45}$$

Solving the second and fourth equation (and assuming that neither d_{223} or d_{233} vanishes) one finds

$$d_{222} = -d_{233}, \quad d_{333} = -d_{223}. \tag{A.46}$$

Plugging this into the remaing equations gives only one additional independent condition, namely

$$\frac{3}{2}d_{223}^2 + \frac{3}{2}d_{233}^2 - 1 = 0. \tag{A.47}$$

Equation (A.46) and Eq. (A.47) are also true if either d_{223} or d_{233} vanishes and in turn imply the Eq. (A.41) to (A.45). Thus the condition that the moduli space is homogeneous is completely encoded in three independent constraints, in agreement with $q_c = 3$ in the $p = 3$-dimensional case. In fact, using the explicit formula for the hyperdeterminant from section 4.3.2 it can be manually checked that in the coordinate system defined by Eq. (4.202) there is a triple zero eigenvalue if and only if Eq. (A.46) and Eq. (A.47) hold.

A.3. Maple codes

These are all Maple codes used in this thesis to perform calculations. All codes have only been tested with Maple 16; older versions may not work.

Listing A.3.1: Maple code used in the proof of Lemma 4.3.1 to obtain Eq. (A.27).

```
with(LinearAlgebra);
```

```
lhs1 := c111*x^2+c122*y^2+c133*z^2+2*c123*y*z+2*c112*x*y+2*c113*x*z;
lhs2 := 2*c122*x*y+2*c123*x*z+c222*y^2+c233*z^2+c112*x^2+2*c223*y*z;
lhs3 := 2*c123*x*y+2*c133*x*z+2*c233*y*z+c333*z^2+c113*x^2+c223*y^2;

c222 := 0; c223 := 0; c233 := 0; c333 := 0; c123 := 0;
y1 := solve(lhs2 = lambda*y^2, y)[1];
z1 := solve(lhs3 = lambda*z^2, z)[1];
tmpeq := subs(x = 1, simplify(subs(y = y1, z = z1, lhs1))) = lambda;
tmpeq2 := isolate(tmpeq, sqrt(c122^2+lambda*c112));
tmpeq3 := lhs(tmpeq2)^2 = simplify(rhs(tmpeq2)^2);
tmpeq4 := denom(rhs(tmpeq3))*lhs(tmpeq3) = expand(numer(rhs(tmpeq3)));
tmpeq5 := isolate(tmpeq4, sqrt(c133^2+lambda*c113));
tmpeq6 := (lhs(tmpeq5)*denom(rhs(tmpeq5)))^2 = numer(rhs(tmpeq5))^2;
lhsfinal := factor(rhs(tmpeq6)-lhs(tmpeq6))/lambda^2;
coeff(lhsfinal, lambda^10);
hypdet := factor(simplify(subs(lambda = 0, lhsfinal)));

A := Matrix(3, 3, symbol = alpha, shape = symmetric);
A[2, 3] := 0;

v := Vector(3);
v[1] := v1; v[2] := 0; v[3] := 0;

f := array(1 .. 3, 1 .. 3, 1 .. 3);
for a to 3 do
  for b to 3 do
    for c to 3 do
      f[a, b, c] := (1/3)*(v[a]*A[b, c]+v[b]*A[c, a]+v[c]*A[a, b])
    end do
  end do
end do;

B := MatrixInverse(A);

detI := subs(c111 = f[1, 1, 1], c112 = f[1, 1, 2], c113 = f[1, 1, 3], c122 = f[1,
    2, 2], c133 = f[1, 3, 3], c233 = f[2, 3, 3], c333 = f[3, 3, 3], c222 = f[2, 2,
    2], hypdet);

factor(detI/((v1^2*B[1, 1])^5*Determinant(A)^6));
```

Listing A.3.2: Maple code for the computation of Aronhold invariants (see section 4.3.2), used in the example in section 4.4.2 to obtain S.

```
with(LinearAlgebra);

modthree := proc (number)
  return (`mod`(number-1, 3))+1
end proc;

calcab := proc(mata,matb)
  local matab:=Matrix(3);
  local a,b;
```

```
  for a from 1 by 1 to 3 do
    for b from 1 by 1 to 3 do
      matab[a,b]:=mata[modthree(a+1),modthree(b+1)]*matb[modthree(a+2),modthree(b
          +2)]+mata[modthree(a+2),modthree(b+2)]*matb[modthree(a+1),modthree(b+1)]-
          mata[modthree(a+1),modthree(b+2)]*matb[modthree(a+2),modthree(b+1)]-mata[
          modthree(a+2),modthree(b+1)]*matb[modthree(a+1),modthree(b+2)];
    od;
  od;
  return(matab);
end;

fillMatrix := proc(l,d)
  local matrix:=Matrix(3);
  local a,b;

  for a from 1 by 1 to 3 do
    for b from 1 by 1 to 3 do
      matrix[a,b]:=d[l,a,b];
    od;
  od;
  return(matrix);
end;

calcDet := proc(d)
  local g:=Matrix(3,3);
  local a,b,c,detg;
  local e:=array(1..3,1..3,1..3):

  for a from 1 by 1 to 3 do
    for b from 1 by 1 to 3 do
      g[a,b]:=factor(simplify(add(d[a, b, i]*vK[i], i = 1 .. 3);
    od;
  od;
  detg:=Determinant(g);
  e[1,1,1]:=coeff(detg,vK[1]^(3)): e[2,2,2]:=coeff(detg,vK[2]^(3)):e[3,3,3]:=coeff(
      detg,vK[3]^(3)):e[1,1,2]:=(coeff(coeff(detg,vK[1]^(2)),vK[2]))/(3):e
      [1,2,2]:=(coeff(coeff(detg,vK[2]^(2)),vK[1]))/(3):e[1,1,3]:=(coeff(coeff(detg
      ,vK[1]^(2)),vK[3]))/(3):e[1,3,3]:=(coeff(coeff(detg,vK[3]^(2)),vK[1]))/(3):e
      [2,2,3]:=(coeff(coeff(detg,vK[2]^(2)),vK[3]))/(3):e[2,3,3]:=(coeff(coeff(detg
      ,vK[3]^(2)),vK[2]))/(3):e[1,2,3]:=(coeff(coeff(coeff(detg,vK[1]),vK[2]),vK
      [3]))/(6):
  for a from 1 by 1 to 3 do
    for b from 1 by 1 to 3 do
      for c from 1 by 1 to 3 do
        e[a,b,c]:=6*e[a,b,c];
        if (a>b) and (c>=b) and (c>=a) then
          e[a,b,c]:=e[b,a,c]
        fi; if (a>b) and (a>c) and (c>=b) then
          e[a,b,c]:=e[b,c,a]
        fi; if (a>b) and (a>c) and (b>=c) then
          e[a,b,c]:=e[c,b,a]
        fi; if (b>c) and (c>=a) and (b>=a) then
```

```
          e[a,b,c]:=e[a,c,b]
        fi; if (b>c) and (a>c) and (b>=a) then
          e[a,b,c]:=e[c,a,b]
        fi;
      od;
    od;
  od;
  return(e);
end;

calcS:= proc(d)
  local a,b,l,S;
  l:=1;
  S := add(add(calcab(fillMatrix(a,d),fillMatrix(b,d))[l,l]*calcab(fillMatrix(l,d),fillMatrix(l,d))[a,b],a=1..3),b=1..3);
  return(S);
end;

calcT:=proc(d)
  local e:=array(1..3,1..3,1..3);
  local a,b,l,k,T;
  k:=1;
  e:=calcDet(d);
  T := add(add(add(calcab(fillMatrix(k,d),fillMatrix(l,e))[a,b]*calcab(fillMatrix(a,d),fillMatrix(b,d))[k,l],a=1..3),b=1..3),l=1..3);
  return(T/(2));
end;

calcR := proc (d)
  (return )(calcT(d)^2-calcS(d)^3)
end proc;

d := array(1 .. 3, 1 .. 3, 1 .. 3);

for a to 3 do
  for b to 3 do
    for c to 3 do
      if b < a and b <= c and a <= c then
        d[a, b, c] := d[b, a, c] end if;
      if b < a and c < a and b <= c then
        d[a, b, c] := d[b, c, a] end if;
      if b < a and c < a and c <= b
        then d[a, b, c] := d[c, b, a] end if;
      if c < b and a <= c and a <= b
        then d[a, b, c] := d[a, c, b] end if;
      if c < b and c < a and a <= b then
        d[a, b, c] := d[c, a, b] end if;
    end do
  end do
end do;

S := calcS(d);
T := calcT(d);
```

```
factor(simplify(subs(d[2, 2, 2] = 0, d[2, 2, 3] = 0, d[2, 3, 3] = 0, d[3, 3, 3] =
    0, S)));
```

Listing A.3.3: Maple code for the example with diagonal intersection numbers in section 4.4.1.

```
with(LinearAlgebra);
with(VectorCalculus, CrossProduct);
with(VectorCalculus, DotProduct);

d := array(1 .. 3, 1 .. 3, 1 .. 3);

for a to 3 do
  for b to 3 do
    for c to 3 do
      if `or`(`or`(a <> b, a <> c), b <> c) then
        d[a, b, c] := 0
      end if
    end do
  end do
end do;

vN1 := Vector(3);
vN2 := Vector(3);

for a to 3 do
  vk[a] := -1/2*expK*add(add(d[a, i, j]*vK[i]*vK[j], i = 1 .. 3), j = 1 .. 3)
end do;

g := Matrix(3, 3);
for a to 3 do
  for b to 3 do
    g[a, b] := factor(simplify(expK*add(d[a, b, i]*vK[i], i = 1 .. 3)+vk[a]*vk[b]))
  end do
end do;

expK := factor(-6/add(add(add(d[a, b, c]*vK[a]*vK[b]*vK[c], a = 1 .. 3), b = 1 ..
    3), c = 1 .. 3));
detg := simplify(Determinant(g));
G := MatrixInverse(g);

calcOmega := proc (number)
  local tmpK, vn2;
  global vN1, vN2, C, normN2, D122, D222, omega;
  if number = 1 then
    vN1[1] := vk[2]; vN1[2] := -vk[1]; vN1[3] := 0
  end if;
  if number = 2 then
    vN1[1] := vk[3]; vN1[2] := 0; vN1[3] := -vk[1]
  end if;
  if number = 3 then
    vN1[1] := 0; vN1[2] := vk[3]; vN1[3] := -vk[2]
```

```
  end if;
  C := factor(simplify(DotProduct(vN1, Multiply(g, vN1))));
  vN1 := Multiply(1/sqrt(C), vN1);
  tmpK := Vector([vK[1], vK[2], vK[3]]);
  vn2 := Vector(3);
  vn2 := CrossProduct(tmpK, vN1);
  vN2 := simplify(Multiply(G, vn2));
  normN2 := factor(simplify(DotProduct(vN2, Multiply(g, vN2))));
  vN2 := Multiply(1/sqrt(normN2), vN2);
  D122 := factor(simplify(expK*add(add(add(d[i, j, k]*vN1[i]*vN2[j]*vN2[k], i = 1
      .. 3), j = 1 .. 3), k = 1 .. 3)));
  D222 := factor(simplify(expK*add(add(add(d[i, j, k]*vN2[i]*vN2[j]*vN2[k], i = 1
      .. 3), j = 1 .. 3), k = 1 .. 3)));
  omega := simplify(-1+3/2*D222^2)
end proc;

calcOmega(1);
omega1 := omega;
factor(simplify(omega*detg^3*C/expK^10));

calcOmega(2);
omega2 := omega;
factor(simplify(omega*detg^3*C/expK^10));

calcOmega(3);
omega3 := omega;
factor(simplify(omega*detg^3*C/expK^10));

D111 := factor(simplify(expK*add(add(add(d[i, j, k]*vN1[i]*vN1[j]*vN1[k], i = 1 ..
    3), j = 1 .. 3), k = 1 .. 3)));
D112 := factor(simplify(expK*add(add(add(d[i, j, k]*vN1[i]*vN1[j]*vN2[k], i = 1 ..
    3), j = 1 .. 3), k = 1 .. 3)));

factor(simplify(((3*D112-D222)^2+D111^2)*detg^3/expK^9));

factor(simplify(omega1*omega2*omega3*detg^6*((3*D112-D222)^2+D111^2)/expK^12));
```

Listing A.3.4: Maple code to solve the eigenvalue problem in section 4.4.2, leading to the result in Eq. (4.146).

```
eq1 := -6*alpha^2-4*alpha*(beta+delta)+2*beta*delta*(eta-1) = lambda*(9*alpha^2+(2*
    alpha*beta)*3+2*3*alpha*delta+5/3*beta^2+5/3*delta^2+2*beta*delta*(eta+2/3));
eq2 := -2*alpha^2+2*alpha*delta*(eta-1) = lambda*(3*alpha^2+1/3*2*5*alpha*beta
    +2*(2/3)*alpha*delta+2*alpha*delta*eta+beta^2+2/3*delta^2*eta+1/3*delta
    ^2+2*(2/3)*beta*delta*eta+2/3*beta*delta);
eq3 := -2*alpha^2+2*alpha*beta*(eta-1) = lambda*(3*alpha^2+2*(5/3)*alpha*delta
    +2*(2/3)*alpha*beta+2*alpha*beta*eta+delta^2+2/3*beta^2*eta+1/3*beta^2+2*(2/3)*
    beta*delta*eta+2/3*beta*delta);

solve({eq1, eq3, eq2}, {alpha, beta, delta, lambda});

sol := solve({subs(beta = delta, eq3), subs(beta = delta, eq1)}, [lambda, alpha]);
```

```
lambda1 := rhs(sol[1][1]);

omega := -1+(3/2)*(4*(1-lambda)^3/(6+9*lambda));
factor(simplify(subs(lambda = lambda1, omega)));
```

Listing A.3.5: Maple code to compute η and ω in 4.4.2, leading to the result in Eq. (4.149).

```
with(LinearAlgebra);
with(VectorCalculus, DotProduct);

d := array(1 .. 3, 1 .. 3, 1 .. 3);

d[2, 2, 2] := 0;
d[2, 2, 3] := 0;
d[2, 3, 3] := 0;
d[3, 3, 3] := 0;

for a to 3 do
  for b to 3 do
    for c to 3 do
      if b < a and b <= c and a <= c then
        d[a, b, c] := d[b, a, c] end if;
      if b < a and c < a and b <= c then
        d[a, b, c] := d[b, c, a] end if;
      if b < a and c < a and c <= b
        then d[a, b, c] := d[c, b, a] end if;
      if c < b and a <= c and a <= b
        then d[a, b, c] := d[a, c, b] end if;
      if c < b and c < a and a <= b then
        d[a, b, c] := d[c, a, b] end if;
    end do
  end do
end do;

for a to 3 do
  vk[a] := -1/2*expK*add(add(d[a, i, j]*vK[i]*vK[j], i = 1 .. 3), j = 1 .. 3)
end do;

g := Matrix(3, 3);
for a to 3 do
  for b to 3 do
    g[a, b] := factor(simplify(expK*add(d[a, b, i]*vK[i], i = 1 .. 3)+vk[a]*vk[b]))
  end do
end do;

expK := factor(-6/add(add(add(d[a, b, c]*vK[a]*vK[b]*vK[c], a = 1 .. 3), b = 1 ..
    3), c = 1 .. 3));

detg := factor(simplify(Determinant(g)));

vplus := Vector(3);
vplus[1] := 0;
```

```
vplus[2] := -d[1, 2, 3]+sqrt(-d[1, 2, 2]*d[1, 3, 3]+d[1, 2, 3]^2);
vplus[3] := d[1, 2, 2];

vminus := Vector(3);
vminus[1] := 0;
vminus[2] := -d[1, 2, 3]-sqrt(-d[1, 2, 2]*d[1, 3, 3]+d[1, 2, 3]^2);
vminus[3] := d[1, 2, 2];

vplusnorm := simplify(vk[1]*vplus[1]+vk[2]*vplus[2]+vk[3]*vplus[3]);
vminusnorm := simplify(vk[1]*vminus[1]+vk[2]*vminus[2]+vk[3]*vminus[3]);

simplify(vk[1]*vplus[1]+vk[2]*vplus[2]+vk[3]*vplus[3])^2/factor(simplify(vplusnorm2
    ));
eta := simplify(DotProduct(vminus, Multiply(g, vplus))/(vplusnorm*vminusnorm));

factor(simplify(factor((-27*eta*(1-eta)^2*detg^3/(1+3*eta)^3)*(1/expK^7))));
```

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Erklärung nach §18(8) der Prüfungsordnung für den Bachelor-Studiengang Physik und den Master-Studiengang Physik an der Universität Göttingen:

Hiermit erkläre ich, dass ich diese Abschlussarbeit selbständig verfasst habe, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe und alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen wurden, als solche kenntlich gemacht habe.
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Göttingen, den September 5, 2012

(Dirk Rathlev)