AdS/CFT Holography of the $O(N)$-symmetric ϕ^4 Vector Model

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Chapter 1

Introduction

1.1 The AdS/CFT Correspondence

The AdS/CFT correspondence which has been conjectured by Maldacena in its original form [68, 2] relates two seemingly very different theories. It states that there exists an equivalence between a type IIB superstring theory [80] on $\mathrm{AdS}_5 \times S^5$ on one hand, and supersymmetric $\mathcal{N} = 4$ Yang-Mills-theory with gauge group $U(N)$ on a compactified four-dimensional Minkowski space in the limit of large N on the other. Anti-de-Sitter space is a maximally symmetric solution of Einstein's equations for a constant negative cosmological constant; the radius of the 5-sphere S^5 has to equal the curvature radius of AdS_5 in this context.

A necessary condition for the two theories to be equivalent is that both theories carry the same symmetry group. Since the symmetry group of AdS_5 is $SO_o(2,4)$ (the identity component of $SO(2,4)$), we must have the same set of symmetries on the Yang-Mills side, a fact easily ascertained by recognising that $\mathcal{N} = 4$ SYM theory is a conformal theory and the conformal group in four dimensions is $SO_o(2,4)$. There are more symmetries to be matched: The rotational symmetry of the 5-sphere S^5 corresponds to the so-called R-symmetry on the SYM side (an internal symmetry due to supersymmetry), and the $U(N)$ gauge symmetry corresponds to a similar "gauge-type" symmetry in the IIB string theory, in connection with Chan-Paton factors.

The original formulation of the conjecture set out from a specific physical situation. Maldacena examined how a string theory would behave in the vicinity of N D-branes embedded into ten-dimensional spacetime. It turned out that there are two different ways such a system can be described: Since the branes are infinitely massive extended objects, strings attached to the brane cannot move very far away by gravitational attraction: Their centre of mass always stays at a finite distance from the brane. One has a choice of computing the classical solutions of the Einstein equations surrounding the massive branes, and likewise the solutions for the equations governing the form flux fields around the branes (since the branes carry a "non-commutative" charge).

In this setting, a physical description is obtained by formulating a closed IIB string theory on the curved background spacetime near the branes. Alternatively, one may consider open strings on a flat background coupled to the branes (such a system is effectively described by the SYM theory). The background fields are not computed explicitly in this picture; rather, one assumes that the "part" of the string worldsheet which couples directly to the branes models the fields surrounding the branes. In [51], it has been demonstrated for the first time that in this way, the correspondence can be used to compute gauge theory correlators from string theory.

Subsequently, it became clear that this is not the only geometry which lends itself to such a dual description: Other brane geometries can be constructed which admit a similar correspondence, but all these approaches were based on the assumption that string theory is consistent and that the correspondence is one of its implications (and a full description goes hand in hand with an understanding of string theory). So far, there has been no example of a string theory which in some manner contradicts the correspondence.

A rather striking feature of the correspondence is that it relates a theory with gravity to another without. A quantum theory incorporating gravity should be expected to have a dynamical causal structure. On the SYM field theory side, the causal structure is fixed. In practical approaches, one usually goes to the limit where only small perturbations of the metric around the AdS metric are expected and uses the supergravity approximation. In this limit, the closed string theory can be treated like a (quantum) field theory with a specific set of interactions. By linearising the (classical) supergravity field equations in the small deviations from the $\mathrm{AdS}_5 \times S^5$ background solution, one finds a set of free modes and an excitation spectrum on the AdS side [54], which can be compared to the spectrum of the boundary theory, with good agreement in many cases. By this simplification, the dynamical aspect of the causal structure is lost, however; in order to appreciate the richness of gravity, it should be treated by its fully non-linear equations, reckoning on the possibility of large deviations from the AdS geometry.

However, since the advent of algebraic holography (Rehren duality) [83, 82, 84], it has become evident that a correspondence between a theory on curved AdS spacetime and a conformal field theory on its conformal boundary does not necessarily have to be based on string theoretical notions. In the algebraic framework, the correspondence is based on an identification of the underlying nets of observable algebras on the AdS bulk and its conformal boundary respectively, and the bulk theory is an ordinary quantum field theory. The proof of this statement is simple and universal, and it makes some direct structural statements on the observables of the boundary theory.

How can these two seemingly conflicting approaches be reconciled? There are only few works which seriously try to argue this matter. Arnsdorf and Smolin [6] examine the assumption that algebraic holography does in fact reproduce the correspondence which is "meant" by the original AdS/CFT proposal (and not something entirely different). Without retracing their arguments in detail, let us just mention that the "programmatic" character of the Maldacena conjecture places Rehren duality in a completely new light: If we believe both approaches to be correct side by side, then we are given a recipe how to *interpret* the bulk theory on AdS_5 which is obtained by Rehren duality from the boundary SYM theory. Namely, it should be an alternative description of the closed string theory on $\mathrm{AdS}_5 \times S^5$. If this were true, the implications would be, to say the least, very puzzling.

Coupling Constants. A crude impression of the correspondence is obtained by examination of the mapping of parameters on both sides. The string coupling and the Yang-Mills coupling are conjectured to be equal, $g_s = g_{\mathrm{YM}}^2$. If L is the curvature radius of AdS and also the radius of the 5-sphere S^5, then we have $L^4 = 4\pi g_s N \alpha'^2$, with α' the Regge slope (it is related to the string tension by $T = (2\pi\alpha')^{-1}$). A characteristic length scale for strings is given by $\ell_s = \alpha'^{1/2}$.

In the broadest form, the correspondence holds for all values of N and g_s. There are several limiting cases which are still interesting but easier to examine. We can fix the *t'Hooft coupling* $\lambda = g_s N$ and let $N \to \infty$; this implies that the string coupling goes to zero as N increases. Consequently, in the limit string loops involving a higher number of coupling constants g_s (ie string worldsheets with higher genus) are suppressed. A "classical" or "tree-level" string theory containing only worldsheets of minimal genus should be a good approximation to this limit, as long as the string coupling in this theory is small.

Conversely, we might consider letting $\lambda \to \infty$; this implies (with the AdS radius L held fixed) that the Regge slope $\alpha' \to 0$, or equivalently the string tension $T \to \infty$. Therefore, we expect that the strings become more and more pointlike objects and in the limit, we obtain the IIB supergravity approximation to string theory [1]. Of course, both limits may be combined, with the requirement that N grows faster than λ (so that $g_s = \lambda/N \to 0$): The result is weakly coupled (classical) IIB supergravity.

On the gauge theory side, things look different: If we perform a large-N expansion of the gauge theory, then only the planar diagrams survive the limit of large N; this is in coincidence with the string theory. However, the complexity of the cross-linking (or "webbing") of the graphs with a given "gauge genus" (the expansion parameter in the large-N expansion) is proportional to the t'Hooft coupling λ in the large-N expansion; so it would rather be the limit $\lambda \to 0$ which is tractable on the gauge theory side. In the limit of both large N and small λ, we obtain the free gauge theory.

1.1.1 Holography of the ϕ^4-model

In this thesis, we shall examine a much simpler model of a conformal field theory and investigate the holographic theory matching its correlations. We will follow the line of Klebanov and Polyakov [55] who suggested that starting from a simple conformal

[1] Since the Regge slope decreases, higher string modes need an increasing amount of energy to be excited, and therefore only the lowest massless modes need to be considered.

theory, the UV- or IR- scaling limit of the model containing a vector field ϕ^a with N entries and the $O(N)$-symmetric interaction $(\phi^a\phi^a)^2$ in three dimensions, one might in the limit of large N obtain a holographic description in terms of higher-spin (HS) gauge fields on AdS_4. Since the theory contains a coupling constant with a mass dimension, we expect the coupling constant to drop out in the UV and IR scaling limits; the UV limit is a free theory and the IR limit is strongly coupled. Although these limits are very different, the conjecture applies to both, and we will see that there exists a relation between their holographic duals.

Fields of higher spin have been studied for a long time, started off by the work of Fronsdal [40] (see also de Wit and Freedman [24] for a very agreeable systematic exposition); it was soon realised that in flat space, there is no way of implementing a consistent interaction preserving unitarity. In more recent times, it has been shown by Vasiliev that on background spacetimes with nonzero constant curvature, an interaction can be constructed by using the inverse cosmological constant in the definition of the coupling constants [98, 97, 99]. In the limit of a flat space, the cosmological constant vanishes and the couplings are diverging. Nowadays, there are algorithms present for obtaining the complete field equations on a constant curvature background with the interactions to all orders [89, 18]. Note the by symmetry breaking of the metric field, it might be possible to reobtain a flat spacetime dynamically.

Sezgin and Sundell have examined whether the HS gauge theory on AdS can be understood as a truncation of IIB superstring theory, at least in certain limits [90]. Note that the suggestion of holography via HS fields can be immediately understood immediately in the context of Rehren duality, without resorting to string theory: We have an ordinary field theory in the bulk, which is taken to be defined perturbatively. It contains an infinite tower of fields with arbitrary spin, and these fields are constructed on an AdS_4-background. On the assumption that the local net of observables can be constructed from gauge invariant combinations of the associated local field operators, there is no reason why we should not work on the healthy assumption that this net will finally turn out to be the Rehren net. Note however that the applicability of the Rehren duality demands that the boundary ϕ^4-theory can be defined rigorously in the framework of algebraic quantum field theory.

However, there are some aspects which seem to transcend this harmless interpretation. The gauge transformations implied by Vasiliev's HS gauge fields are "generalised coordinate transforms", ie for spin-1 fields they look like the usual vector Abelian gauge transformations for the Maxwell field, for spin 2 they look like coordinate transforms (diffeomorphisms), and for higher spins they are suitable generalisations. By construction, the local observables in the bulk constructed from gauge invariant combinations of field operators inherit the locality structure of these fields. It is an algebraic result that commutativity of local operators on AdS should be guaranteed on very mild assumptions if the localisation regions of these operators cannot be connected by a timelike geodesic [2]. The transformation properties of the

[2] This is a consequence of modular nuclearity; it is a somewhat stronger statement than what we are used to in flat space where a timelike curve should suffice. See eg [19].

HS fields suggest that they should be interpreted naïvely, ie the vector field is the Maxwell field, the symmetric 2-tensor is the metric tensor, the scalar is the dilaton field etc. This seems to clash with the specification of the causality structure by the mentioned perturbative construction. However, this is the usual puzzle faced by perturbative constructions of gravity throughout. The particular aspect which is interesting in this context is that we have the backup from Rehren duality, which is an *algebraic* and not a perturbative statement: The perturbative construction of the holographic HS gauge theory including gravity should coincide with the algebraic (dual) Rehren net! This looks like the perturbative construction of gravity leads after all in the right direction; on the other hand, Rehren duality has a chance of *containing* some description of quantum gravity in the bulk.

Another possibility which must be taken seriously is that ultimately, it may happen that the ϕ^4_3 $O(N)$-symmetric vector model does not exist in the strict axiomatic sense, but only perturbatively. This would forestall the application of Rehren duality; we could then conjecture that the difficulties faced in the perturbative construction of the boundary ϕ^4-theory are presumably of a similar type as the ones faced in the perturbative construction of the bulk HS theory.

For the ϕ^4_3 scalar theory, there are some rigorous constructive results available: Glimm and Jaffe have shown the positivity of the Hamiltonian [44]; Magnen and Seneor haven proven the Borel summability of the theory [67] and have studied the infrared behaviour of the theory [66]. Feldman and Osterwalder have proven the appearance of a mass gap in the weak coupling regime [38] and shown by Euclidean methods that the theory fulfills the Wightman axioms.

The $O(N)$ vector model in three dimensions is known to have a nontrivial conformal fixpoint from renormalisation group analysis [12, 102] (the IR fixpoint).

Euclidean AdS (EAdS). For simplicity, computations in this thesis are done in the Euclidean domain. While on general curved spaces, the concept of Wick rotation as yet has not been shown to make sense, Bros et al [13] have shown that on Anti-de-Sitter space, the concept of Wick rotation and the corresponding "Euclidean AdS" make sense. Its conformal boundary is Euclidean flat space, one-point compactified. One advantage of this treatment is that the representation theoretic treatment of Dobrev of the AdS/CFT correspondence [30] is formulated conveniently in the Euclidean setting. Also, Schwinger parametrisation as analytical tool is less problematic for Euclidean propagators.

1.1.2 Schwinger parametrisation and AdS-presentation.

In this thesis we will not concentrate on the full HS gauge theory; rather, we study the relation between the $O(N)$-symmetric ϕ^4 vector model on the boundary and its holographic dual directly, ie on the level of the path integral methods introduced by Witten and others [103, 51]. We will often resort to the technique of Schwinger parametrisation, in particular in the technical second part II. This wants an explanation.

The first point is a technical one. Actual computations of the correlation functions of in the bulk or correlation functions of the boundary CFT implied by the holographic bulk theory invariably are resorting to a Schwinger parametrised form of the propagators. However, the notion of Schwinger parametrisation in the AdS bulk is a vague one: What is termed a "Schwinger parametrisation" is often nothing more than some (seemingly arbitrary) integral representation of the propagators, introducing a new integration variable α running from 0 to infinity for each propagator, the so-called "Schwinger parameters". The integrations over the vertex coordinates or loop momenta of the AdS bulk graphs are then commuted with the integrations of the Schwinger parameters, and for an educated choice of integral representation, this makes the computation feasible after all. Although we feel that a sound physical theory may very well lead to analytic expressions for correlation functions and other quantities of interest which are difficult to integrate, and on the other hand a computational recipe which is simple to pull through is not necessarily an indication that the single steps of this computation have a physical meaning except being mathematically convenient, the question remains whether Schwinger parametrisation has an intrinsic meaning and how these integral representations may be generated systematically.

The second point concerns the recent programmatic approach of Gopakumar [45, 46, 47] who suggested that the Schwinger parametrised form of the correlation functions for dual boundary and bulk theories are related in a very specific manner. The AdS/CFT correspondence according to this suggestion may be seen as a two-step procedure: Starting from the correlation functions in the large N expansion of the conformal boundary theory given in Schwinger parametrised from, in a first step these correlations are "AdS-presented", ie the Feynman graphs of the boundary theory are expressed in a covariant manner as Feynman graphs whose domain is intrinsically AdS space, with vertices situated in AdS-space, "bulk-to-bulk" propagators between these vertices, and "bulk-to-boundary" propagators stretching all the way to conformal infinity where the sources are located. The boundary amplitude is obtained by integrating out the coordinates of the vertices all over the bulk. The AdS-presentation happens on a graph-by-graph level (or at least certain sums of graphs on the boundary correspond to certain sums of graphs in the bulk). There is a delicate relation between the topology of the graphs on the boundary and the corresponding graphs in the bulk, and there is some evidence that this relation could be understood efficiently by the method of Schwinger parametrisation. This is yet a purely mathematical reformulation of these amplitudes. We will examine in detail the AdS-presentation in later chapters, although not entirely from Gopakumar's perspective. In a second step, the AdS-presented amplitudes are re-interpreted in terms of a string theory on a highly curved AdS space in the limit of large N; this corresponds to the case where the string coupling g_s and consequently the Yang-Mills coupling g_{YM} on the boundary vanish and we are dealing with a free gauge theory on the boundary. The Schwinger parameters are in this case conjectured to be related to the moduli of the string worldsheet.

But the central technical problem, the AdS-presentation of boundary amplitudes, is essentially not clarified yet; an exhaustive treatment has been given only for the simple cases of three- and four point functions. The prescriptions for such a procedure are in a sense very arbitrary, and it is interesting to ask whether there is a precise sense in which such AdS-presentation can be performed, and what we can say about the structure of the resulting terms. While many of the arguments for the AdS/CFT correspondence are grounded in the perturbative approach, ultimately, we have to demand that both bulk and boundary theories obey the same physical requirements, notably unitarity (or, in the Euclidean domain, the corresponding axiom of reflection positivity in the Osterwalder-Schrader setting). These present strong restrictions, and it is an important question to clarify how the AdS/CFT correspondence accommodates itself with these requirements. The possibility of a correspondence between physical theories on different spacetimes is not only astonishing because of the equivalence of physical effects which should be observable, but also because there must exist an incorruptible equivalence of the basic universal notions like "locality", "causality", "probability". We think that the role of this underlying structural equivalence cannot be stressed enough.

1.2 Overview of this Thesis

This thesis has two parts. The first part is the main part and develops the AdS/CFT correspondence of the $O(N)$-symmetric ϕ^4 vector model.

We begin in chapter 2 with a discussion of the $O(N)$-symmetric vector model, its diagrammatics (in particular the $1/N$ expansion) and renormalisation group fixpoints. Two fixpoints, one in the UV and another one in the IR, will be of special interest; it will be argued by perturbative analysis that the UV fixpoint is the free $O(N)$-symmetric vector model and the IR fixpoint is an interacting conformal field theory. We will construct an astonishing relation between these fixpoint theories, the "UV/IR duality". We discuss the twist-2 quasi-primary bilinear tensor currents which are an important class of operators in the boundary theories (section 2.6) and use them for the construction of the “twist-2 conformal partial wave expansion” (twist-2 CPWE) in the free UV fixpoint theory, a variant of the usual CPWE which relies on twist-2 currents only, baring conformal partial waves of higher twist (section 2.7).

Chapter 3 is the main chapter. We discuss the geometry of Euclidean Anti-de-Sitter space (EAdS) in section 3.1 and introduce the functional integral perspective on the AdS/CFT correspondence which is central to this text (section 3.2). As a side result, we will make a proposition how to implement consistently the dual prescription for boundary source terms in the Dirichlet path integral over EAdS (section 3.2.2). We will use UV/IR duality to diagnose a relation between the holographic duals of both (conformal) fixpoint theories of the $O(N)$ vector model, contained in proposition 3.2 on page 65. This puts strong constraints on the expected form of a Lagrangian holographic bulk theory corresponding to the UV fixpoint theory.

Then, we briefly discuss representation theoretic issues focusing on a mini-review of

Dobrev's intertwiner realisation of the correspondence, and present how this is culminated in Rühl's "lifting programme", a protocol for the reconstruction of holographic bulk theories from their images on the boundary (section 3.3).

An important intermediary step for the lifting procedure is the construction of an *EAdS-presentation* of correlation functions on the boundary, ie an integral representation of the correlation functions relying on the use of covariant EAdS integrals. An EAdS-presentation of three-point functions of twist-2 bilinear tensor currents in the free UV fixpoint theory is attempted in section 3.4; the construction is almost finished, missing only a final technical computation. In section 3.5, this is generalised to the case of n-point functions, with the result that these are given in terms of the EAdS-presentation of the three-point functions.

Finally, in section 3.6 we analyse holographic bulk theories which are corresponding to the UV and IR fixpoint theories on the boundary. These holographic theories must harmonise with the EAdS-presentations of the boundary correlations; in proposition 3.2, we had found further restrictions on their possible structure, and these are now validated. Special attention is given to the question whether the bulk theories can be Lagrangian; under this asumption, we are able to derive unusual semi-classical path integrals for their generating functions in the bulk (theorem 3.9 on page 117). We also ask in section 3.6.5 whether they do make sense in the axiomatic setting of Osterwalder and Schrader, adapted to the case of EAdS.

Part I ends with the conclusions and an outlook. We have not included them after part II, because the second part does not contribute essentially to the main statements of this thesis.

The second part is largely technical and is devoted to a detailed analysis of the functional integral approach to a massive or massless scalar field on EAdS space; it is detached from the main line of this text. Particular attention is given to Schwinger parametrisation of the propagators, and the connection between the heat kernel and the propagator in case the path integral bears constraints. We obtain the principal results in an abstract Hilbert space setting relating to as few specific model assumptions as possible. Therefore, very different types of constraints are covered. We will find that constraints show up in the heat equation governing the Schwinger parametrisation consistently as absorption terms. We view this part as an "experimental lab" which allows to explicitely examine many statements from the general part and gather "hands-on" experience whenever necessary.

There are four appendices: In appendix A, we present some computational rules for conformal propagators. Appendix B contains a lengthy but important computation of a generic EAdS-integral. Appendix C gives some integrals of Bessel functions, and appendix D contains a very brief summary of the electronic publication [53] by the author.

Part I

AdS/CFT Correspondence

Chapter 2

The $O(N)$-symmetric ϕ^4-model

It is the aim of this text to study the AdS/CFT correspondence in the simple model of a real vector field ϕ^a with N entries transforming in the fundamental representation of $O(N)$. As a Lagrangian field theory, we can add various interaction terms to the Lagrangian. This will generate a multitude of different models situated "around" the free $O(N)$ vector model. These theories are not independent: by scaling the system (performing a renormalisation group transform), different relevant and irrelevant interaction terms are "switched off / on", and one can see that there exist interrelations between different interaction potentials. Holographic renormalisation methods point in the direction that there is a deeper link between AdS/CFT correspondence on one hand and the renormalisation group on the other hand. It will be therefore our strategy to consider the class of theories obtained from perturbations around the free $O(N)$ vector model. Our prominent example of an interaction is the $O(N)$-invariant quartic term $(\phi^a\phi^a)^2$, so our generic Lagrangian will be of the form [1]

$$S[\phi] = \int \mathrm{d}^{\mathrm{d}}x \left\{ \frac{1}{2}\partial_\mu\phi \cdot \partial_\mu\phi + \frac{m^2}{2}\phi \cdot \phi + \frac{\theta}{8N}(\phi \cdot \phi)^2 \right\}. \qquad (2.0\text{-}1)$$

The coupling θ furnished with a factor of $1/N$ is called the t'Hooft coupling. If we write the action in this form, we can perform an expansion in $1/N$ - the "large N expansion" (for a comprehensive review, see [71]). This is a systematic prescription for an expansion of correlation functions in a dimensionless expansion parameter; each term in the series is made up of infinitely many Feynman diagrams which can be summed up analytically.

We remark that this is a very popular model to study, both because the difficulty is on a manageable level and still, there is already a number of phenomena which are very characteristic for the AdS/CFT correspondence. The analysis splits naturally in two steps of increasing complexity: The *leading order* in $1/N$ in the ϕ^4 theory corresponds roughly to a classical field theory on AdS. There is already quite some amount of insight to be gained from this step; the path to be taken has been pegged out in a popular article by Klebanov and Polykov [55]. It must be stressed that

[1] The mass m includes a factor of $\hbar^{-1}$ and thus has unit L^{-1}.

although there is repeated mention of order $1/N$ corrections, their conclusions are generally (meant to be) valid only up to leading order in $1/N$. We will explicitly look at some of the conjectures and statements indicated there. However, it is the stated intention of this work to venture beyond the leading order and ask whether the clear structure emerging in the leading order can be extended seamlessly further.

2.1 Model Lagrangian

We are constructing the Lagrangian theory of a real vector field $\phi^n(x)$ in d-dimensional flat Euclidean space, where $2 < \mathrm{d} < 4$ and $n = 1 \dots N$ is a vector index. $1/N$ will feature as expansion parameter [21]. The interaction is $(\phi \cdot \phi)^2$, which we also abbreviate ϕ^4. The action is thus

$$S[\phi] = \int \mathrm{d}^{\mathrm{d}} x \left\{ \frac{1}{2} \partial_\mu \phi \cdot \partial_\mu \phi + \frac{m^2}{2} \phi \cdot \phi + \frac{\theta}{8N} (\phi \cdot \phi)^2 \right\} \geq 0. \tag{2.1-2}$$

Note the peculiar factor of N^{-1} multiplying the coupling θ. We have decided not to include Wick ordering in the action, as we will have to renormalise anyhow. We include a (scalar) source field $J(x)$, coupling to Wick squares $:\phi(x) \cdot \phi(x):$ of the vector field; these are defined as

$$:\phi(x) \cdot \phi(y) := \phi(x) \cdot \phi(y) - \langle \phi(x) \cdot \phi(y) \rangle, \tag{2.1-3}$$

where the expectation is taken in the vacuum state of the interacting theory. The partition function

$$\mathscr{Z}_{:\phi^2:}[J] = \int \mathscr{D}(\phi) \exp -\frac{1}{\hbar} \left\{ S[\phi] + \frac{i}{2} \langle J, :\phi \cdot \phi: \rangle \right\} \tag{2.1-4}$$

is a functional over J, and the correlation functions are generated by application of $i\partial_{J(x)}$. The free propagator is given by the integral kernel of $\hbar(m^2 - \triangle)^{-1} \delta_{nm}$, where n and m are the colour indices. The vertices are given by $-\frac{\theta}{N\hbar} \int \mathrm{d}^{\mathrm{d}} x \, \delta_{\text{colours}}$ (the symmetry factor $1/8$ has disappeared). There are still global symmetry factors, and we get a factor N for each closed colour index loop, due to the summation over n.

In order to perform the $1/N$-expansion, we perform a trick due to Coleman, Jackiw and Politzer [22]. We introduce a new scalar field $\sigma(x)$ to construct the Gaussian integral

$$\int \mathscr{D}(\sigma) \exp -\frac{1}{\hbar} \int \mathrm{d}^{\mathrm{d}} x \left\{ \frac{N}{2\theta} \left(\sigma + i \frac{\theta}{2N} \phi \cdot \phi \right)^2 \right\}. \tag{2.1-5}$$

By construction, this integral is independent of ϕ. Multiplying (2.1-4) with (2.1-5), we find that the ϕ^4-interaction term in the action cancels out, and we are left with the action

$$S[\phi, \sigma] = \int \mathrm{d}^{\mathrm{d}} x \left\{ \frac{1}{2} \partial_\mu \phi \cdot \partial_\mu \phi + \frac{m^2}{2} \phi \cdot \phi + \frac{N}{2\theta} \sigma^2 + \frac{i}{2} \sigma \, \phi \cdot \phi \right\}. \tag{2.1-6}$$

For later use, we will also include a source term K for correlations of σ which are generated by $-i\partial_{K(x)}$ (note the different sign)

$$\mathscr{Z}_{\phi^2,\sigma}[J,K] = \int \mathscr{D}(\phi)\mathscr{D}(\sigma) \exp -\frac{1}{\hbar}\left\{ S[\phi,\sigma] + \frac{i}{2}\left\langle J, {:}\phi^2{:}\right\rangle - i\left\langle K,\sigma\right\rangle \right\}. \qquad (2.1\text{-}7)$$

The Feynman rules of this modified theory are different: The ϕ-propagator G_ϕ is still given by the integral kernel of $\hbar(m^2 - \triangle)^{-1}\delta_{nm}$. However, instead of the ϕ^4-interaction, we have the σ-field with propagator $G_\sigma(x-y) = \frac{\theta\hbar}{N}\delta^{(\mathrm{d})}(x-y)$; and the $\sigma\phi^2$-vertex $-\frac{i}{\hbar}\int \mathrm{d}^{\mathrm{d}}x\, \delta_{mn}$. Odd correlations of the auxiliary field σ (eg three-point functions) will be purely imaginary, as one finds by complex conjugation and substituting $\sigma \to -\sigma$. This is due to the way we introduced this field. Alternatively, one might consider correlations of $i\sigma$; then, the two-point function is negative definite.

Remark. A short comment on our conventions for going over to wave number space. We insert between the vertices and the ends of each single propagator basis changes

$$\delta^{(\mathrm{d})}(x-y) = \int \mathrm{d}^{\mathrm{d}}k\, \frac{e^{ikx}}{(2\pi)^{\frac{\mathrm{d}}{2}}} \cdot \frac{e^{-iky}}{(2\pi)^{\frac{\mathrm{d}}{2}}}.$$

At each site, we push one exponential onto the propagator and the other exponential onto the vertex. Finally, the coordinate space integrations $\mathrm{d}^{\mathrm{d}}x$ are performed. This will leave us with distributions $(2\pi)^{\mathrm{d}}\delta^{(\mathrm{d})}(\sum_i k_i)$ at the vertices, and similarly $(2\pi)^{\mathrm{d}}\delta^{(\mathrm{d})}(k+k')$ on the propagators. We can perform some of the k-integrations, until all these δ-distributions have been cancelled by enforcing the respective momentum conservation. This protocol treats propagators and vertices on the same footing. The resulting factor is $(2\pi)^{\mathrm{d}-\frac{n\mathrm{d}}{2}}$ at an n-vertex; at the propagators ($n=2$), all factors cancel. In wave number space,

$$G_\phi(k) = \frac{\hbar}{m^2+k^2}, \qquad (2.1\text{-}8a)$$

$$G_\sigma(k) = \frac{\theta\hbar}{N}, \qquad (2.1\text{-}8b)$$

and at the $\sigma\phi^2$-vertices, we have couplings

$$-\frac{i}{(2\pi)^{\frac{\mathrm{d}}{2}}\hbar}\delta^{(\mathrm{d})}(k_1+k_2+k_3). \qquad (2.1\text{-}9)$$

Counting powers of N, we find a factor N for every ϕ-loop, and a factor N^{-1} for every σ-propagator. If we think of ϕ-loops ("bubbles") as effective vertices, then the leading power of N is given by the effective tree graphs; inserting additional σ-propagators, we explore the subleading orders of the $1/N$-expansion. In a first step, we will discuss the full propagators in the leading order of $1/N$ and discuss renormalisation of the model.

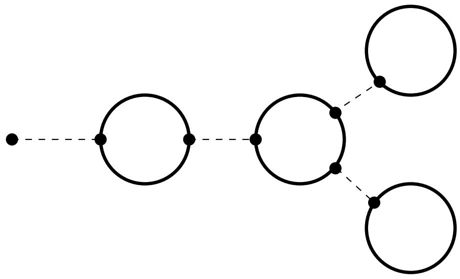

Figure 2.1: Typical example of a "bubble tree diagram". The thick lines are ϕ-propagators G_ϕ, the dashed lines represent σ-propagators G_σ, and the whole diagram is supposed to end on left at a $\sigma\phi^2$-vertex.

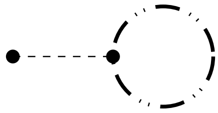

Figure 2.2: The diagram determining the mass correction due to a bubble tree diagram. The dashed-dotted line is the *full* ϕ propagator G'_ϕ (2.2-11).

2.2 Renormalisation to the Leading Order

The logical way to study the $1/N$-expansion begins with studying the renormalised mass of ϕ at leading order in $1/N$. This renormalised mass is generated by the so-called "bubble trees" [88], displayed in fig. 2.1. The bubble trees generate an additional mass for ϕ. In order to find out the mass correction, we have to calculate the loop integral which is depicted graphically in fig. 2.2. It reads

$$\begin{aligned} -\frac{m_{\text{tree}}^2}{\hbar} =& \frac{1}{2}\Big(-\frac{i}{(2\pi)^{\frac{\mathrm{d}}{2}}\hbar}\Big)^2 G_\sigma(0) N \int \mathrm{d}^{\mathrm{d}}k\, G'_\phi(k) \\ =& -\frac{\theta}{2(2\pi)^{\mathrm{d}}} \int \mathrm{d}^{\mathrm{d}}k\, \frac{1}{m'^2+k^2}. \end{aligned} \tag{2.2-10}$$

The factor $1/2$ is a symmetry factor. This integral does not converge in $\mathrm{d} \geq 2$ dimensions; however the malady can be cured by adding a mass counterterm δm^2 for ϕ which cancels the infinity. In fact, we do not perform the calculation, because it is obvious that by finite renormalisation, we can put the mass shift m_{tree}^2 to an arbitrary value (this does not say anything about the renormalisation flow, of course). The full ϕ-propagator

$$G'_\phi(p) = \frac{\hbar}{m^2 + m_{\text{tree}}^2 + k^2} \tag{2.2-11}$$

has a total, renormalised mass

$$m'^2 = m^2 + m_{\text{tree}}^2 \tag{2.2-12}$$

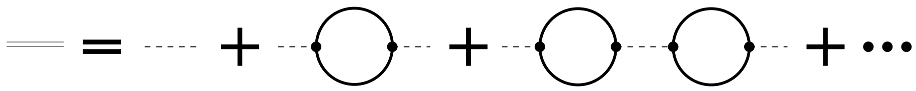

Figure 2.3: The full σ-propagator $G'_\sigma(k)$ (2.2-15) to first order in $1/N$.

which is some function of m^2 by solving with (2.2-10).

We now concentrate on the full σ-propagator (to leading order in $1/N$). Following closely Bjorken and Drell [11, Chpt. 19], it is given as a geometrical series over proper self-energy insertions, which are in fact ϕ-loops to first order in $1/N$ (see fig. 2.3). Such a loop (including the couplings) is simply given by

$$\begin{aligned}\Sigma_\sigma(k) =& \frac{1}{2}\left(-\frac{i}{(2\pi)^{\frac{\mathrm{d}}{2}}\hbar}\right)^2 N\int \mathrm{d}^{\mathrm{d}}q\, G'_\phi(q+k)G'_\phi(q)\\ =& -\frac{N}{2(2\pi)^{\mathrm{d}}}\int \mathrm{d}^{\mathrm{d}}q\,\frac{1}{(q^2+m'^2)((q+k)^2+m'^2)}.\end{aligned} \tag{2.2-13}$$

Again the factor $1/2$ is a symmetry factor of the loop. The integral does not converge in $4 \le \mathrm{d}$; we limit the examination to $2 < \mathrm{d} < 4$, so this does not matter for our purposes.

By standard textbook methods, the integral evaluates as

$$\Sigma_\sigma(k) = -\frac{N\,\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{2^{2\mathrm{d}-3}\,\pi^{\frac{\mathrm{d}}{2}}}\left(k^2+4m'^2\right)^{\frac{\mathrm{d}}{2}-2}\,{}_2\mathrm{F}_1\left(\frac{1}{2},2-\frac{\mathrm{d}}{2};\frac{3}{2};\frac{k^2}{k^2+4m'^2}\right). \tag{2.2-14}$$

The full σ-propagator is finally

$$G'_\sigma(k) = G_\sigma(k)\sum_{j=0}^{\infty}\left(\Sigma_\sigma(k)G_\sigma(k)\right)^j = \frac{\theta\hbar}{N}\frac{1}{1-\frac{\theta\hbar}{N}\Sigma_\sigma(k)}. \tag{2.2-15}$$

Indeed this result now strongly depends on the dimension d. For integer dimensions $\mathrm{d} = 1, 2, 3$, the result is expressible in standard functions:

- For $\mathrm{d} = 1$, have

$$\Sigma_\sigma^{(\mathrm{d}=1)}(k) = -\frac{N}{2m'\left(k^2+4m'^2\right)}. \tag{2.2-16}$$

  The full propagator is then

$$G'^{(\mathrm{d}=1)}_\sigma(k) = \frac{\theta\hbar}{N}\frac{k^2+4m'^2}{k^2+4m'^2+\frac{\theta\hbar}{2m'}}. \tag{2.2-17}$$

- For $\mathrm{d} = 2$, have

$$\Sigma_\sigma^{(\mathrm{d}=2)}(k) = -\frac{N\ln\left(\frac{\sqrt{k^2+4m'^2}+\sqrt{k^2}}{\sqrt{k^2+4m'^2}-\sqrt{k^2}}\right)}{4\pi\sqrt{k^2+4m'^2}\,\sqrt{k^2}}. \tag{2.2-18}$$

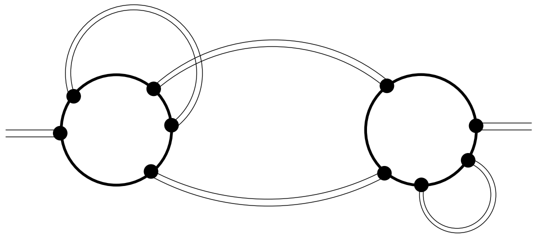

Figure 2.4: A typical (sub)diagram of 3rd-next-to-leading order in $1/N$. Thick lines: G'_ϕ. Double lines: G'_σ. This particular diagram contains a loop at the right bottom which needs to be renormalised *effectively*.

- For d = 3, have

$$\Sigma_\sigma^{(\mathrm{d}=3)}(k) = -\frac{N}{8\pi\sqrt{k^2+4m'^2}} \cdot \frac{\arcsin\sqrt{\frac{k^2}{k^2+4m'^2}}}{\sqrt{\frac{k^2}{k^2+4m'^2}}}. \tag{2.2-19}$$

2.3 The $1/N$-expansion: An Effective Weak-coupling Expansion

Let us summarise what we have found so far: For the leading order of the $1/N$-expansion, we have to sum up two classes of diagrams: The "bubble trees" of figure 2.1 on page 18 (which can be done by a recursive method, yielding an algebraic equation for the mass - which however can be given an arbitrary value by adjusting the mass counterterm properly); and the full σ-propagator of figure 2.3 on page 19 (which requires the integration of the σ-self-energy to one-loop order).

We can now use these "building blocks" to assemble larger structures. These structures consist of two different elements: The remaining ϕ-propagators G'_ϕ still can form loops. These loops in turn are connected by the full σ-propagators G'_σ. An example for such a structure is given in fig. 2.4. Note that by construction, each ϕ-loop has at least 3 external σ-propagators attached to it: ϕ-loops with only 2 external σ's have been taken care of already in G'_σ (with the exception of the loop coupling two sources J); loops with only one external σ are part of the bubble trees.

We are now in a position to understand the nature of the $1/N$-expansion: Because $G'_\sigma \equiv \mathcal{O}(1/N)$, and each ϕ-loop contributes a colour factor of N, we can see that each loop containing at least one effective σ-propagator in the *effective* theory is punished by a factor $\hbar/N$. One should compare this to the "usual" Feynman expansion in the coupling constants, which amounts to punishing each loop with another factor $\hbar$. The analogy is clear: The effective theory is established with an effective Planck constant

$$\hbar_e = \frac{\hbar}{N}. \tag{2.3-20}$$

This effective Planck constant will decrease as N increases, and in the limit of $N \to \infty$ we expect that the connected correlation functions are dominated by effective diagrams with an *effective* tree structure - the only loops which remain are the ϕ-loops which are taking the role of *effective vertices*. Perturbative expansions which only take into account the tree level diagrams lead to classical theories; so it is often said that by letting the Planck constant vanish, we obtain the classical limit of the underlying quantum theory. However, we strongly oppose this denomination, as it is misleading: The nature of such theories (with small Planck constant) is rather that of a very *weakly coupled* system. For, to stay with the effective theory, consider the full four-point function $\langle :\phi^2::\phi^2::\phi^2::\phi^2:\rangle$. The connected contribution is proportional to N whereas there are three disconnected contributions factorising into two-point functions, proportional to N^2. For large N (small $\hbar_e$), therefore, the disconnected contributions are dominant! If the Planck constant diminishes, we are rapidly approaching a *free* theory.

If we compute the two point function of the field σ in the leading order of $1/N$, we find that it is suppressed by a factor N^{-1}. We therefore should couple the source terms K and the field σ with a factor $\sqrt{N}$. Similarly, the two-point function of $:\phi^2:$ is of order N, and to make it finite, we have to include a factor $\sqrt{N}^{-1}$ when the sources are coupled to the fields. Alternatively, by field strength renormalisation $\phi \to N^{1/4}\phi$ and $\sigma \to \sqrt{N}^{-1}\sigma$, we arrive at the action

$$S'[\phi,\sigma] = \int \mathrm{d}^{\mathrm{d}}x \left\{ \sqrt{N}\left(\frac{1}{2}\partial_\mu\phi\cdot\partial_\mu\phi + \frac{m^2}{2}\phi\cdot\phi\right) + \frac{1}{2\theta}\sigma^2 + \frac{i}{2}\sigma\,\phi\cdot\phi \right\}.$$

We can see that the correlation functions of more than two operators (either of $:\phi^2:$ and σ) are suppressed by increasing powers of $\sqrt{N}^{-1}$, since a ϕ-loop with k external legs carries an effective coupling constant $N^{1-k/2}$ (remember that the summation over colours gives a factor N to the ϕ-loop). Only the two-point functions remain finite. The theory becomes free in the large-N limit. The $1/N$-expansion should rather be termed a $1/\sqrt{N}$-expansion in consequence.

2.4 UV and IR Fixed Points

Renormalisation of the $O(N)$ symmetric ϕ^4-theory has been carried out by Wilson and Kogut in $4-\varepsilon$ dimensions through the ε-expansion [102]. We will reobtain their results in a slightly different way, separating completely the procedure of infinite renormalisation (by introducing counterterms) and finite renormalisation (analysing the effect of scale changes on the system). To begin, we assume that the already renormalised, massive theory is given on a certain arbitrary scale. It is important to stress that we assume that the counterterms are determined ab initio on the scale indicated, and that all UV and IR divergences are taken care of before. We then determine the action of the renormalisation group [63] by scaling the system. Although we will only consider the lowest order corrections (self energy of the σ field to one-loop order), we obtain complete agreement with the literature (see eg Petkou [77] for an approach combining diagrammatic and OPE methods).

The system is scaled by the dimensionless scaling parameter $\alpha > 0$, ie we perform a substitution $x \mapsto \alpha \cdot x$, $k \mapsto \alpha^{-1} \cdot k$ on the external and internal coordinates and wave numbers (we must not forget to substitute $\mathrm{d}^{\mathrm{d}}x \to \alpha^{\mathrm{d}}\,\mathrm{d}^{\mathrm{d}}x$ and $\mathrm{d}^{\mathrm{d}}k \to \alpha^{-\mathrm{d}}\,\mathrm{d}^{\mathrm{d}}k$ in the integrals over internal vertex coordinates resp. loop momenta). The infrared regime lies in the direction $\alpha \to \infty$ (points moving away from each other), and the ultraviolet (short distance) regime lies in the direction $\alpha \to 0$. We determine the "weak scaling limit", ie the scaling limit of the correlation functions under this action.

In both cases, it will in addition generally be necessary to rescale field operators for their correlations to stay finite [2]. For example, let the two-point function of a local field operator $O(x)$ behave asymptotically as $\langle O(\alpha x)O(\alpha y)\rangle \sim \frac{1}{\alpha^{2\Delta}}$. We have to scale the operator $O(x) \to \alpha^{\Delta}O(\alpha x)$ in order to obtain a finite limit. The parameter Δ is called the scaling dimension of the operator O. It need not be the same in the infrared and ultraviolet regime (renormalisation of scaling dimension). By Fourier transform, we get a factor of $\alpha^{-\mathrm{d}}$ from the Jacobian; so in the wave number domain, the corresponding scaling law is $O(k) \to \alpha^{\Delta-\mathrm{d}}O(\alpha^{-1}k)$.

We will demand that the system approaches asymptotically a fixed point (where all "reasonable" correlation functions are nonsingular, but some correlations may vanish completely). Once the fixed point is reached, the resulting theory is conformal, and the dependence on α drops out altogether. In this limit, we expect that $\hbar$ does not feature any more as loop counting parameter, because it has dimension mass times length, but there is no intrinsic mass present in the conformal limiting theory. The mass m' of the finite scale theory, measured in multiples of $\hbar$, has unit inverse length, and drops out similarly. However, it is by no means clear that the system will always approach the *same* IR or UV fixed point: Depending on the initial parameters, we may approach via the renormalisation flow different asymptotic fixed points. It is characteristic of the renormalisation group that the parameter regions making up the attractive neighbourhood of different fixed points possess different (co-)dimensionality.

Masslessness of ϕ-propagator. If we substitute the scaling behaviour for k, then the massive propagator for the field ϕ has the standard form

$$G'_\phi(\alpha^{-1}k) = \frac{\hbar}{m'^2 + \alpha^{-1}k^2}.$$

For $\alpha \to 0$, we approach the UV fixed point. In this limit, the mass term in the denominator will be suppressed by the wave number term; we conclude that the ϕ

[2] The normalisation of a field operator is no intrinsically defined quantity. We make the tacit and very reasonable assumption that this procedure is convergent. To be strict, we would have to construct the scaling algebras corresponding to the local field algebras following Buchholz and Verch [20]. However, we will see that our simple ansatz is fully sufficient.

will become massless in the leading order. The IR fixed point ($\alpha \to \infty$) is more intricate: If $m' \neq 0$, the derivative terms in the propagator are suppressed and the theory falls apart into a "theory of points" (the field ϕ becomes conformal with scaling dimension $\frac{d}{2}$). This is only natural; the mass m' supplies a natural inverse cutoff length. In order to obtain a nontrivial fixed point, we are forced to start with a *massless* theory with $m' \approx 0$. If we assume that the theory is massless on *all* scales, then the IR and the UV fixed point are lying on the same renormalisation trajectory, only at different ends. We will consequently adopt this view.

2.4.1 Divergences and Counterterms

We have to take into account the fact that in there will appear the usual divergences of perturbative quantum field theory. Divergences might occur in the initial finite scale theory; and they might arise once we approach the fixed points. We have to invent, and argue, some kind of regularisation for the divergent integrals. In both case, the regularisation has to be introduced on the level of the unscaled theory, *before* we take the approach to the UV or IR: We should not modify the theory by having to introduce new regularisations as we scale along.

We have to battle two types of divergences: UV divergences arise from an inadequately modelled short-distance behaviour; as indicated before, we introduce some kind of counterterm to match them. IR divergences are due to the existence of massless fields in the theory exhibiting long-range correlations; they could be understood as a "volume resonance effect". As a basis for their discussion, imagine we want to compute a scattering amplitude of some particles, coupled to a massless field. If we model the coupling of this process to the infinitely many low-energy background modes which cannot be detected because their energy is too small for any kind of detector available, we not only get the usual IR divergences from loop integrations containing massless propagators, but yet another kind which might be called "bremsstrahlung divergence" originating in this inclusion of the absorption and emission of non-detected long-wavelength background modes. It can be shown that the infrared and bremsstrahlung divergences cancel precisely [76]. The crude picture behind is that due to the universal limitation of detectors, low-energy excitations below a detector-dependent threshold inevitably escape our notice. If we build a better detector with a deeper low-energy resolution and enhance the search horizon for massless low-energy quanta produced (and absorbed [3]) in the scattering process, then we *will* measure an increasing overall amplitude for the processes. In the idealised limit of perfect detector sensitivity in the low energy range and perfect energy resolution of the detector, the amplitude indeed must diverge. So the *definition* of a sensible concept of a scattering amplitude or correlation function already has to contain a lower limit of detector sensitivity.

We conclude that these IR divergences appearing in computations of loop integrals have physical reality. As a side effect, we find that if the theory is unitary (as

[3] To "measure" the absorption of light quanta, we can eg use the energy-momentum balance.

any decent quantum field theory should be), then the IR divergences are limited by the phase space density of the corresponding low energy modes coupled to the process. Note that the "technical" limitation comes not only from the necessary size of the detector; also, detection times increase as the energy horizon is lowered. This is deeply related to the principle that we cannot measure the precise energies of the scattered particles themselves: They are, by definition, accompanied by the low-energy cloud carrying a certain, unobservable amount of energy.

One standard method to cope with the infrared problem is to introduce a cutoff wave number and acknowledge that the bremsstrahlung below the cutoff wave number cancels precisely the with those contributions from the loop integrals where the loop wave number is below the cutoff. One simply ignores the coupling to the low-energy background modes and introduces a regularisation for the massless propagators, eg by giving a very small mass to the massless particles. This has the simple effect of an effective cutoff for wave numbers smaller than this mass. It seems that introducing this mass regularisation has a huge influence on the amplitudes which we compute - after all, the infrared divergences are rendered finite. The point is that as we introduce this mass, we have to acknowledge that we do not detect any particle whose energy is around or below the regulator mass, and in this way the *definition* of the amplitude concept depends on the regulator mass as well.

There is a subtlety when we begin to scale the system. On a first glance, it seems that we should keep the physical cutoff mass fixed, as the definition of the scaling limit is that the physics should be scaled without modification. We meet two different problems, depending on the direction of the limit: Going to the IR fixpoint (blowing up the experimental device and operators), the dimensions of the system itself will at one point become so large that the typical wave number characterising the system will be smaller than the lower cutoff wavenumber. The cutoff - which was introduced as a mere technical tool - will act like a real physical effect then. In the UV, the problem is different: If the experiment is shrunk further and further, its characteristic wave number increases. However, if a Feynman diagram contains a massless loop, we may safely assume that all the modes with wave numbers (energies) lying below the characteristic wave number (energy) of the experiment and above the lower cutoff mass (energy) are contributing to the amplitude of the loop integration, which therefore ever increases. On the detection side, there are no changes to match the increasing number of field modes which are thus included explicitly. The consequence is that in the UV limit, the cutoff mass loses its abrasive power - we reobtain the same IR divergences that we tried to battle in the beginning.

The solution must then be to scale the cutoff mass $m' \mapsto \alpha^{-1}m'$ as we scale the system. Other scaling prescriptions for the mass are conceivable; however this is the simplest one, and it leads to finite results. Any *massive* field becoming massless during the scaling must be endowed with the cutoff mass m' as well.

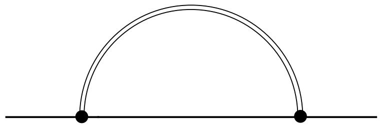

Figure 2.5: $\mathcal{O}(N)$ self-energy contribution to the propagator of ϕ. Double line: σ-propagator G'_σ.

2.4.2 Ultraviolet Fixpoint

As momenta $k \mapsto \alpha^{-1}k$ and cutoff mass $m' \mapsto \alpha^{-1}m'$ scale identically, the scaling behaviour of the effective building blocks is extremely simple under this regularisation: The scaled ϕ-propagator becomes

$$G'_\phi(k) \xrightarrow{\alpha} \alpha^2 G'_\phi(k) = \alpha^2 \frac{\hbar}{m'^2 + k^2}.$$

For the σ-self energy, we compute with (2.2-14)

$$\begin{aligned}\Sigma_\sigma(k) \xrightarrow{\alpha} & \alpha^{4-\mathrm{d}}\Sigma_\sigma(k) \\ = & -\alpha^{4-\mathrm{d}} \frac{N\,\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{2^{2\mathrm{d}-3}\,\pi^{\frac{\mathrm{d}}{2}}} \left(k^2+4m'^2\right)^{\frac{\mathrm{d}}{2}-2} {}_2\mathrm{F}_1\left(\frac{1}{2}, 2-\frac{\mathrm{d}}{2}; \frac{3}{2}; \frac{k^2}{k^2+4m'^2}\right) \\ = & -\alpha^{4-\mathrm{d}} N \left(k^2+4m'^2\right)^{\frac{\mathrm{d}}{2}-2} f_\Sigma\left(\frac{k^2}{k^2+4m'^2}\right),\end{aligned}$$

where the function $f_\Sigma(z)$ is bounded analytic on the closed unit disk $|z| \leq 1$. The α-dependence comes solely from the factor $(k^2+4m'^2)^{\frac{\mathrm{d}}{2}-2}$. In dimensions $2 < \mathrm{d} < 4$, $\Sigma_\sigma(k)$ scales with a positive power of α. It has a negative sign throughout for real momenta.

Counterterms for UV divergences. We use counterterms to cancel UV divergences, but we will not be very explicit in the precise construction and trust that the reader can imagine how the procedure works in the more complicated cases. The counterterms have to be presented in an integral representation, preferably over loop momenta, which are added in place so that the regularised amplitudes are absolutely convergent (as integrals) in the UV. Subsequently, we may perform the scaling limit (if it exists), using our knowledge about the scaling of the effective propagators.

To give an example, consider the $\mathcal{O}(1/N)$ contribution to the ϕ self-energy (fig. 2.5 on page 25) which is proportional to

$$\delta\Sigma'_\phi(k) \sim \int \mathrm{d}^\mathrm{d}q\, G'_\phi(k+q) G'_\sigma(q).$$

A possible subtraction scheme is

$$\delta^{\mathrm{reg}}\Sigma'_\phi(k) \sim \int \mathrm{d}^\mathrm{d}q\, \left[G'_\phi(k+q) - G^{\mathrm{reg}}_\phi(q)\right] G'_\sigma(q),$$

where G_ϕ^{reg} is a massive propagator for an arbitrary mass m^2_{reg} determined from the renormalisation conditions of the finite scale theory. When integrated, this second summand will be mass counterterm for ϕ. We have to make sure that this integral is rendered absolutely convergent by the subtraction; in special cases, we need to pick the symmetric part of the integral.

However, in the UV limit $\alpha \to 0$, the contribution of the unrenormalised loop scales as $\alpha^{2-\mathrm{d}}$, as a simple power counting argument shows. In $\mathrm{d} > 2$, the importance of the counterterm will gradually decrease, and finally, we will obtain again the usual UV divergence. So this is not a reasonable scaling limit. Of course, we could use the infrared regularised propagator G'_ϕ instead of G_ϕ^{reg}; this indicates that we change the renormalisation conditions as we scale. The meaning is that we do not stay on the same trajectory throughout. This implies that the UV fixpoint we are approaching cannot be reached from the finite scale theory by simple scaling. However, it is very convenient, because all the scaling arguments to follow below apply as well to the counterterms and we do not have to make special arguments.

In the IR ($\alpha \to \infty$), power counting reveals that the unrenormalised loop scales as α^{-2}, so a counterterm involving a fixed mass will always dominate the long-distance behaviour, with the consequence that in the deep IR, the amplitude will diverge (as the counterterm is divergent). Again, using G'_ϕ instead of G_ϕ^{reg} will change the renormalisation conditions as we scale and ultimately, we will reach a sensible fixpoint.

Ultraviolet Fixpoint. The UV fixed point lies in the direction $\alpha \to 0$. As a result, we have for the full σ-propagator

$$G'_\sigma(k) = \frac{\theta\hbar}{N}\frac{1}{1-\frac{\theta\hbar}{N}\Sigma_\sigma(k)} \overset{\alpha}{\to} \frac{\theta\hbar}{N}.$$

At the UV limit, we ultimately lose information about the large-distance behaviour of the σ-propagator of the finite-scale theory. Note that the same limiting behaviour is obtained under the assumption that $\theta\hbar$ is very small - so the UV limit point is the weak coupling limit of the finite scale theory.

To find out how the operator $:\phi^2:$ has to be scaled asymptotically, we compute the n-point function of $:\phi^2(k):$. This will give us the proper scaling dimensions. In the leading order of θ, the connected n-point function is given by a single diagram: the "free" diagram with a single ϕ-loop, illustrated in fig. 2.6 (left) on page 27. It will suffice to count powers of α. We display the full argument in this example: The "free" contribution is proportional to

$$G_{\mathrm{free}}(k_1,\dots,k_n) \sim \delta^{(\mathrm{d})}(k_1+\dots+k_n)\int \mathrm{d}^{\mathrm{d}}q\,(G'_\phi)^n.$$

We count n propagators $G'_\phi \sim \alpha^2$, one loop integral contributing $\alpha^{-\mathrm{d}}$ and the momentum conserving $\delta^{(\mathrm{d})}$-distribution giving a factor α^{d}; so altogether, the free diagram has weight α^{2n} in momentum space.

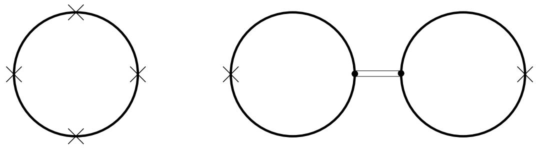

Figure 2.6: Free UV theory: (left) Connected n-point function of $:\phi^2:$. (right) Split of the two-point function of $:\phi^2:$ by a full σ-propagator G'_σ.

What happens if we add an internal σ-propagator to this loop (splitting the G'_ϕ propagators at each end and introducing an additional loop integral)? For the case of a two-point function, this is displayed in fig. 2.6 (right). Let us count the correction in terms of α,

$$\mathrm{d}^{\mathrm{d}}q\, G'_\sigma (G'_\phi)^2 \sim \alpha^{-\mathrm{d}}(\alpha^2)^2 = \alpha^{4-\mathrm{d}}.$$

This vanishes for $\alpha \to 0$ in $\mathrm{d} < 4$. It is easily checked that the introduction of additional ϕ-loops or every other possible modification makes things even worse. We conclude that all σ-propagators, and therefore also all $1/N$ corrections, are suppressed in the ultraviolet limit: Only the free diagram survives.

Proposition 2.1. *By perturbative renormalisation of the propagators including self-energy contributions to order $\mathcal{O}(N^0)$, the $O(N)$-symmetric ϕ^4 vector model in the ultraviolet approaches asymptotically the free (non-interacting) $O(N)$ vector model in $2 < \mathrm{d} < 4$.*

Its connected n-point functions of $:\phi^2:$-operators are solely generated by Wick contractions of the form displayed in figure 2.6 (left) on page 27.
The operator $:\phi^2(k):$ has to be joined by a factor α^{-2} for non-vanishing correlation functions, and therefore in coordinate space, its scaling dimension is $\Delta(:\phi^2:) = \mathrm{d}-2$. For a single ϕ, we have $\Delta(\phi) = \frac{\mathrm{d}-2}{2}$, this is the minimal scaling dimension allowed by unitarity [64]. By a similar analysis, the σ-field asymptotically becomes a free field with scaling dimension $\Delta(\sigma) = \frac{\mathrm{d}}{2}$; its (connected) $n \geq 3$-point functions all vanish by definition.

2.4.3 Infrared Fixpoint

This is the limit $\alpha \to \infty$. For the σ-self energy, we compute in the dimensional range $2 < \mathrm{d} < 4$ under investigation

$$\begin{aligned} G'_\sigma(k) =& \frac{\theta\hbar}{N}\frac{1}{1-\frac{\theta\hbar}{N}\Sigma_\sigma(k)} \\ =& \frac{\theta\hbar}{N}\frac{1}{1+\alpha^{4-\mathrm{d}}\,\theta\hbar\,(k^2+4m'^2)^{\frac{\mathrm{d}}{2}-2}\, f_\Sigma\left(\frac{k^2}{k^2+4m'^2}\right)} \xrightarrow{\alpha} -\frac{\alpha^{\mathrm{d}-4}}{\Sigma_\sigma(k)}. \end{aligned} \tag{2.4-21}$$

The same behaviour of G'_σ is obtained in the limit where $\theta\hbar$ is very large - so the IR fixpoint theory is the strong coupling limit of the finite scale theory.

At the IR limit, ultimately G'_σ behaves as $G'_\sigma(k) \sim \alpha^{\mathrm{d}-4}(k^2+4m'^2)^{2-\frac{\mathrm{d}}{2}}$. This is a very bad short-distance behaviour ($\sim x^{-4}$). A detailed analysis shows that for finite scaling, there is a UV regularisation efficient at scales

$$k \gg \alpha\,(\theta\hbar)^{\frac{1}{4-\mathrm{d}}} = \alpha\triangle k_{\mathrm{max}}, \qquad x \ll \alpha^{-1}\,(\theta\hbar)^{-\frac{1}{4-\mathrm{d}}} = \alpha^{-1}\triangle x_{\mathrm{min}}. \tag{2.4-22}$$

The reason for the bad behaviour at infinite scaling is that the massless field ϕ propagates over long distances (the massless propagator behaves as $\sim x^{2-\mathrm{d}}$); it is this conformal long-distance behaviour that seeps into the short-distance regime when we scale towards the IR fixpoint.

If we perform the scaling analysis of the global structure, we find a crucial difference: The addition of a G'_σ-bridge to any given graph results in a factor

$$\mathrm{d}^{\mathrm{d}}q\, G'_\sigma(G'_\phi)^2 \sim \alpha^{-\mathrm{d}}\alpha^{\mathrm{d}-4}(\alpha^2)^2 = \alpha^0.$$

That means: *Any* connected n-point function of $:\!\phi^2(k)\!:$ scales as α^{2n}; so *all* $1/N$ corrections persist in the IR limit:

Proposition 2.2. *By perturbative renormalisation of the propagators including self-energy contributions to order $\mathcal{O}(N^0)$, the $O(N)$-symmetric ϕ^4 vector model in the infrared is an interacting conformal field theory in $2 < \mathrm{d} < 4$.*

This is in agreement with the literature [102, 12]. In fact, in-depth analysis of G'_σ shows that it is equivalent to the nonlinear $O(N)$ "σ-model", a vector field with *fixed* modulus (see eg [79]). We will see that it is more natural in the IR fixpoint theory to study correlations of the field σ. This does not make a big difference: The correlations of σ are obtained from those of $:\!\phi^2\!:$ by glueing G'_σ on the sources corresponding to $:\!\phi^2\!:$; so we find that a tree diagram with n external σ-insertions scales as $(\alpha^{\mathrm{d}-2})^n$ in wave number space. This implies that the scaling dimension is $\Delta(\sigma) = 2$ in coordinate space.

2.4.4 Interpretation of Results

Finally we obtain some correlation functions which correspond to the scaling limit theories. They still do contain the infinitesimal mass m'; so it is a question how to interpret them. The trouble we have with the regulator mass m' is that we are interested in obtaining conformal fixpoint theories; and the regularisation parameter m' destroys the conformal nature of the fixpoint theories.

We have already indicated that m' is a natural cutoff and we should assume that our detectors are not reliable for wave numbers smaller than m'. In the free UV limit theory where the connected correlations contain only a single ϕ-loop, one can immediately see that when the external momenta are *non-exceptional*, ie no partial sum of the external momenta is zero [57], then there are no IR divergences (a typical exceptional point is when one external momentum is set to 0 in an arbitrary correlation function). In coordinate space, the correlations between local operators are automatically IR convergent by construction. So we may set $m' = 0$. The physical reason is that the massless background modes completely decouple in any free theory.

In the IR limit theory, the mass m' appears inside of logarithms in the combination $\ln(k^2/m'^2)$ or similar generated by the IR region of loop diagrams; and as an IR regulator of the G'_ϕ-propagator. Taking $m' \to 0$, we find that IR divergences appear at higher orders of $1/N$ in the ϕ-loops, eg when there is a self-energy contribution as in figure 2.5 on page 25. This happens when several ϕ-propagators G'_ϕ are "transporting" the same momentum in "serial connexion".

We will simply ignore these IR divergences, with the following argument: These divergences appear at higher orders of $1/N$. The holographic lifting which we examine is happening on the level of the $1/N$-expansion which we have performed so far, ie on the level of the $\mathcal{O}(N^0)$-renormalised propagators and ϕ-loops. We take the view that the renormalisation of IR-divergences happens on a higher level of complexity than the holographic correspondence we are examining; ie it occurs (in a different disguise) on *both* sides of the holographic correspondence. Ultimately, it has to be addressed, but only after the basic outline of AdS/CFT correspondence is under control in the IR fixpoint theory.

Another issue is the bad short-distance behaviour of the σ-propagator in the IR fixpoint theory. This will make diagrams containing loops with σ very prone to UV divergences, causing the need for yet more counterterms. As long as we do not reach the IR scaling limit, the σ-propagator will be regularised in the short-distance region; it has been established in (2.4-22) that the relevant distance is $\triangle x_{\min} \approx \alpha^{-1} (\theta\hbar)^{-\frac{1}{4-\mathrm{d}}}$. We can interpret this as the critical minimal length scale of physical effects (interactions). The massless modes lead to phenomena on all distance scales larger than $\triangle x_{\min}$. When we go towards the IR limit, we keep the correlation functions normalised so that for unit distance (in the *scaled* theory) all effects on shorter distances are integrated out. Going to short distances in the scaled theory, we have to subtract the effects on distances which are longer. It is this subtraction which leads to the short-distance infinities in the IR fixpoint theory. To put it succinctly: All those effects which used to be on such large distance scales that we could not see them on the observable distance scales of the unscaled theory (since they happened on huge scales) are now crowding the short distance scales in the IR limit theory.

In particular, the formula for $\triangle x_{\min}$ shows that the strong coupling limit $\theta \to \infty$ is equivalent to the IR limit theory. This is natural: From the parameters of the finite scale theory, we can construct the “typical” length scale $\triangle x_{\min}$ of the theory, and all others lengths have to be compared to this length.

A standard approach to generate the necessary counterterms is *dimensional regularisation* [9, 23], ie to assume that d is an arbitrary complex number. Most computations have a natural domain of convergence in d, and whenever necessary, one may analytically continue the results to any other dimension required. If a correlation function has a pole at some particular d, one may relatively easy regularise the correlation by subtracting the principal part at the pole, by the Mittag-Leffler theorem. After the subtraction, all correlations still are conformal. Dimensional renormalisation is a very short way of implementing exactly those counterterms which preserve the conformal character of the fixpoint theory, without having to state explicitly what these counterterms really are. We will take this approach in the sequel to control the UV divergences in the IR fixpoint model.

Related to this philosophy is the use of Schwinger parametrised conformal propagators, eg (A.1-2) or (A.1-3). As long as the Schwinger parameters are not integrated, no divergences appear in the formalism; when they are integrated, the dimension d again is conveniently taken to be a complex number.

An alternative to pass by these difficulties developed in the seventies of the last century is the method of bootstrap equations introduced by Migdal [69] and followed further in [25, 93]; see also [65]. It is based on the idea that once we have written down the amplitudes in terms of integrals in coordinate space, we may formulate differential equations ("bootstrap equations") for the proper vertex functions and propagators which can be solved to yield perfectly well-defined amplitudes. Naturally, the solutions will be available only as series; however, from the structure of the bootstrap equations, we can select an appropriate expansion parameter (eg some non-analytic function of the coupling) which will give a convergent series expansion. This approach is commonly adopted in the literature, see eg [26, 78]. With foresight to the AdS/CFT correspondence developed later on, it would maybe be a good idea to construct the AdS/CFT correspondence on the level of the bootstrap equations; however, we will not follow this programme.

Note that both in the IR and UV limiting theories, all factors of $\hbar$ have cancelled as anticipated - $\hbar$ is not a loop counting parameter in the asymptotic conformal theories. Also, the t'Hooft coupling θ has disappeared in *both* limit theories - in the free theory, there isn't any coupling at all; in the interacting $O(N)$ vector model, the coupling is independent of the finite scale coupling constant θ.

2.5 UV/IR Duality

There is an interesting connection between the UV fixpoint theory (free massless $O(N)$ vector model, see proposition 2.1 on page 27) and the IR fixpoint theory (the interacting massless $O(N)$ vector model, see proposition 2.2 on page 28). The partition function for the UV theory can be stated explicitly: we have

$$\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[J] = \mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[0] \exp N \left[\sum_{n=2}^{\infty} \frac{1}{2n} \frac{(-i)^n}{\hbar^n} \operatorname{Tr}(J G'_\phi)^n \right],$$

where J and G'_ϕ are interpreted as multiplication resp. convolution operators. The $n = 1$-term is missing because we have used Wick ordering for the operator $:\phi^2:$

multiplying the source J in the original partition function (2.1-4). The symmetry factor $(2n)^{-1}$ takes care of mirror symmetry and cyclic permutations.

The UV-IR duality is established by treating J as a dynamical field. Ignoring questions of renormalisability, let us see what happens if we perform a "functional Fourier transform" [32], ie a a path integral

$$\mathscr{F}\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[K] \equiv \mathrm{Reg}\int \mathscr{D}(J) e^{\frac{i}{\hbar}\langle J,K\rangle}\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[J]. \tag{2.5-23}$$

The new source term K is the "conjugate variable" to J. The integral is only defined after regularisation; see below. We evaluate it by splitting $\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[J]$ into a Gaussian part [4]

$$\exp -\frac{N}{4\hbar^2}\,\mathrm{Tr}\, JG'_\phi JG'_\phi$$

with an exponent quadratic in J which generates the propagator G_J, and the remaining summands with $n \geq 3$ which are treated as interaction polynomial for J. The wave number space kernel of this bilinear form corresponds to the amplitude of a single ϕ-loop Σ_σ as it has been computed in (2.2-13), pending the necessary couplings. The propagator G_J is the inverse of this bilinear form, it is equal to the full σ-propagator G'_σ (2.4-21) in the interacting conformal IR limit theory. Moreover, the "effective vertices" consisting of ϕ-loops with $n \geq 3$ external legs appear in the IR limit theory as well, as effective vertices for the field σ. By comparing carefully all the necessary terms involved, we find that the "field" J assumes the parallel role to the field σ in the IR fixpoint model. Because we have chosen all factors of i appropriately, we obtain the important

Proposition 2.3 (UV/IR duality). *The partition function $\mathscr{Z}^{\mathrm{IR}}_\sigma[K]$ of the interacting IR fixpoint theory of the $O(N)$-symmetric ϕ^4 vector model, with the sources coupled to the auxiliary scalar field σ, is identical to the (properly regularised) functional Fourier transform of the partition function $\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[J]$ of the corresponding free UV fixpoint theory, with the sources coupled to the field operators $:\phi^2:$,*

$$\left(\mathscr{F}\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}\right)[K] = \mathscr{Z}^{\mathrm{IR}}_\sigma[K]. \tag{2.5-24}$$

This gives us precise information on the kind of regularisation necessary to render $\mathscr{F}\mathscr{Z}^{\mathrm{UV}}_{:\phi^2:}[K]$ finite: We have to include exactly the same kind of counterterms which are necessary to regularise the effective IR limiting theory. There is a finite, small mass m'^2 for ϕ providing the infrared regulator for G'_ϕ; a (divergent) mass counterterm δm^2 for ϕ to regularise the self-energy contributions to ϕ at order $\mathcal{O}(1/N)$; if $\mathcal{O}(1/N)$-contributions to the self-energy of σ resp. J are taken into account, one should be able to control them by a field strength renormalisation $Z\langle J,K\rangle$ of the J/K-coupling (the original ϕ^4-coupling θ has dropped out completely, and therefore, there are only these two alternatives left). Practically, one will use dimensionally

[4] The prefactor i for the source J in (2.1-4) has been inserted in order to get the necessary minus sign in the exponent.

regularised functional Fourier transform to include the latter. By (2.1-4), we can see that the counterterms can be introduced by substituting

$$J \mapsto J - i(m'^2 + \delta m^2)$$

in the original action. So the regularised version of the functional Fourier transform is

$$\begin{aligned}\mathscr{F}\mathscr{Z}^{\mathrm{UV}}_{\phi^2}[K] &\equiv \int \mathscr{D}(J) e^{\frac{i}{\hbar}Z\langle J,K\rangle}\,\mathscr{Z}^{\mathrm{UV}}_{\phi^2}[J - i(m'^2 + \delta m^2)] \\ &= \int \mathscr{D}(J) e^{\frac{i}{\hbar}Z\langle J + i(m'^2+\delta m^2),K\rangle}\,\mathscr{Z}^{\mathrm{UV}}_{\phi^2}[J].\end{aligned}$$

The effects are simply a shift and rescaling of the source term K.

Is IR-UV duality a general phenomenon, ie are there general conditions under which UV- and IR-fixpoints are related by a functional Fourier transform? It does not seem easy to state general conditions which would enable IR-UV duality starting from arbitrary finite scale "seed" theories. On the level of Feynman diagrams, it is clear that any IR and UV fixpoint theories share similarities, as they are conformal and are generated by scaling of the same set of Feynman diagrams of the seed theory. In the UV and IR, different classes of diagrams survive this scaling; it is to be expected that a suitable UV fixpoint theory (possibly asymptotically free) provides "effective vertices" for building the (interacting?) IR fixpoint theory. However, there is a host of open questions: What if there are several IR or UV fixpoints? IR-UV duality (if it exists as a generic mechanism) implies that fixpoints appear in pairs. What is the connection between them?

Proposition 2.3 will be the decisive ingredient when we begin to construct the holographic pendants to the conformal fixpoint theories of the $O(N)$ vector model in section 3.2 and will eventually lead to the structural proposition 3.2 on page 65 underlying the AdS/CFT correspondence of these models.

2.6 Quasi-Primary Tensor Currents

In the $O(N)$ vector model there exists a set of traceless, totally symmetric tensor currents $\mathfrak{J}^s$ constructed from the bilinear field operators of the underlying theory. In this section, we introduce the currents and list some of the basic properties; however, no important results are presented, and since the relevant formulas of this section will be referred to by number whenever necessary, readers who are familiar with these objects may skip the section without loss.

These currents generally have the form [96, 5, 26]

$$\mathfrak{J}^s = \sum_{k=0}^{s} a^s_k \; {:}\underline{\partial}^{\otimes k}\phi^c\,\underline{\partial}^{\otimes s-k}\phi^{c(*)}{:} \; - \text{traces}, \tag{2.6-25}$$

where it is understood that the indices of the derivatives are totally symmetrised, and the numerical prefactors are given by

$$a_k^s = \frac{1}{2}(-1)^k \binom{s}{k} \frac{\binom{s+\mathrm{d}-4}{k+\frac{\mathrm{d}}{2}-2}}{\binom{s+\mathrm{d}-4}{\frac{\mathrm{d}}{2}-2}} = \frac{1}{2}(-1)^k \binom{s}{k} \frac{(\frac{\mathrm{d}}{2}-1)_s}{(\frac{\mathrm{d}}{2}-1)_k(\frac{\mathrm{d}}{2}-1)_{s-k}}, \tag{2.6-26}$$

$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. We have indicated that in case of a complex field, the currents are constructed from the field and its adjoint. By the Wick contraction, we mean

$$:\phi(\underline{x})^c\phi(\underline{y})^c := \phi(\underline{x})^c\phi(\underline{y})^c - \langle\phi(\underline{x})^c\phi(\underline{y})^c\rangle_{\mathrm{vac}},$$

subtracting the expectation take in the vacuum state. At least on the perturbative level of the $1/N$-expansion, this is a uniquely defined quantity.

In the real field case, it is obvious that due to symmetrisation, only the currents with even spin s survive the symmetrisation. The currents are (quasi-)primary, implying that for different spin, they are in general orthogonal; ie $\langle \mathcal{J}^s \mathcal{J}^t \rangle = 0$ if $s \neq t$. In the UV case, these currents are generally conserved since they are based on the free field ϕ^c, whereas in the IR fixpoint theory, the currents are conserved only to leading order in $1/N$.

2.6.1 Subtraction of Traces

As the tensors are totally symmetric, we may contract the free indices by a vector $\underline{y} \in \mathbb{C}^{\mathrm{d}}$ (this is a very common technique; see eg [86]). Currents are then obtained by applying a partial derivative with respect to $\underline{y}$ [5]. The "current polynomial" is thus given by

$$\mathcal{J}^s[\underline{y}] = \sum_{k=0}^{s} a_k^s \; :(\underline{y}\cdot\underline{\partial})^k\phi\,(\underline{y}\cdot\underline{\partial})^{s-k}\phi^{(*)}: \; - \text{traces}, \tag{2.6-27}$$

and symmetrisation is implicit in this scheme. We indicate the "generating argument" by square brackets. The current operator is reobtained by differentiation,

$$\mathcal{J}^s = \frac{1}{s!}(\partial_{\underline{y}})^s \mathcal{J}^s[\underline{y}].$$

Subtraction of traces is technically a nontrivial procedure; this section is meant largely for the illustration that in the case of the currents $\mathcal{J}^s$, the subtraction makes such difficulties in the general case that we could not find a closed formula and it does not seem to exist in the literature. There is no deep material requisite for the rest of the thesis in this subsection.

[5] In some texts, a factor $\frac{1}{s!}$ is inserted accounting for the numerical prefactors obtained when differentiating with respect to $\underline{y}$.

In the $\underline{y}$-polynomial scheme, a traceless tensor $\mathcal{T}^s[\underline{y}]$ is characterised by the condition

$$\triangle_{\underline{y}}\mathcal{T}^s[\underline{y}] = 0,$$

so it corresponds to a *harmonic* function. The "subtraction" term contains (symmetrised) tensors of the structure $g_{ij}\mathcal{T}^{s-2}_{(k)}$, where g is the metric. These would be represented by $\underline{y}$-polynomials of the form

$$\underline{y}^2 p^{s-2}[\underline{y}],$$

where $p^{s-2}[\underline{y}]$ is a homogeneous polynomial of order $s-2$. Subtractions are meaningful and well defined if there exists a unique function $p^{s-2}[\underline{y}]$ such that

$$\triangle_{\underline{y}}(\mathcal{J}^s[\underline{y}] - \underline{y}^2 p^{s-2}[\underline{y}]) = 0. \tag{2.6-28}$$

For the proof of existence and uniqueness of the subtraction, see [7]. Alternatively, we could restrict the vector $\underline{y}$ to the complex cone $\underline{y}^2 = 0$ (this is an important technique for the proof).

As a first application, we evaluate the subtraction scheme for a symmetric tensor of the type $\frac{1}{s!}(\underline{t}\cdot\underline{y})^s$, $\underline{t}\in\mathbb{C}^{\mathrm{d}}$; by solving the relevant differential equation (2.6-28) and selecting the unique solution with finite polynomial degree, we find

$$\begin{aligned}\frac{1}{s!}(\underline{t}\cdot\underline{y})^s \;-\; \text{traces} =& \frac{(\underline{t}\cdot\underline{y})^s}{s!}\,{}_2\mathrm{F}_1\Big(-\frac{s}{2}, \frac{1}{2}-\frac{s}{2}; 2-s-\frac{\mathrm{d}}{2}; \frac{\underline{t}^2\underline{y}^2}{(\underline{t}\cdot\underline{y})^2}\Big)\\ =& \frac{|\underline{t}|^s|\underline{y}|^s}{2^s(\frac{\mathrm{d}}{2}-1)_s}\; C_s^{\frac{\mathrm{d}}{2}-1}\Big(\frac{\underline{t}\cdot\underline{y}}{|\underline{t}|\,|\underline{y}|}\Big),\end{aligned} \tag{2.6-29}$$

where $C_s^\lambda(x)$ are the Gegenbauer polynomials [6] (this may be checked by computing the divergence). This is the *unique* harmonic function of homogeneity degree s which can be constructed from the expressions $\underline{t}\cdot\underline{y}$ and $\underline{y}^2$ (containing $\underline{y}$). If there were another function with that property, the subtraction of traces procedure would not be well-defined (this is of course not a proof). Note that due to the symmetry of the expression in $\underline{t}$ and $\underline{y}$, this function is harmonic with respect to the variable $\underline{t}$ as well. If O is a real orthogonal matrix in $\mathbb{R}^{\mathrm{d}}$, then

$$\frac{1}{s!}(\underline{t}\cdot O\underline{y})^s \;-\; \text{traces} = \frac{|\underline{t}|^s|\underline{y}|^s}{2^s(\frac{\mathrm{d}}{2}-1)_s}\; C_s^{\frac{\mathrm{d}}{2}-1}\Big(\frac{\underline{t}\cdot O\underline{y}}{|\underline{t}|\,|\underline{y}|}\Big)$$

is harmonic in $\underline{t}$ and $\underline{y}$ simultaneously. We list the first few cases:

$$\begin{aligned}\frac{1}{0!}(\underline{t}\cdot\underline{y})^0 \;-\; \text{traces} &= 1\\ \frac{1}{1!}(\underline{t}\cdot\underline{y})^1 \;-\; \text{traces} &= \underline{t}\cdot\underline{y}\\ \frac{1}{2!}(\underline{t}\cdot\underline{y})^2 \;-\; \text{traces} &= \tfrac{(\underline{t}\cdot\underline{y})^2}{2} - \tfrac{\underline{t}^2\underline{y}^2}{2\mathrm{d}}\\ \frac{1}{3!}(\underline{t}\cdot\underline{y})^3 \;-\; \text{traces} &= \tfrac{(\underline{t}\cdot\underline{y})^3}{6} - \tfrac{\underline{t}^2\underline{y}^2(\underline{t}\cdot\underline{y})}{2(\mathrm{d}+2)}\end{aligned}$$

[6] also called ultraspherical polynomials

If we substitute $\partial_{\underline{y}}$ instead of $\underline{y}$, then

$$\left(e^{\underline{t}\cdot\partial_{\underline{y}}} - \text{traces}_{\underline{t}}\right) f[\underline{y}] = \left(e^{\underline{t}\cdot\partial_{\underline{y}}} f[\underline{y}] - \text{traces}_{\underline{t}}\right) = f[\underline{t}] - \text{traces}_{\underline{t}} = f[\underline{t}]$$

iff f is a harmonic function; so $\left(e^{\underline{t}\cdot\partial_{\underline{y}}} - \text{traces}_{\underline{t}}\right)$ acts as the identity on the harmonic functions.

Computing the harmonic $\underline{y}$-polynomial for the currents $\mathfrak{J}^s[\underline{y}]$ is not an easy task. In a first step, we encode the derivatives acting on the two field operators with left and right pointing arrows

$$\mathfrak{J}^s[\underline{y}] \sim \sum_k a_k^s (\underline{y} \cdot \overleftarrow{\partial})^k (\underline{y} \cdot \vec{\partial})^{s-k} - \text{traces} = \sum_k a_k^s g_1^k g_2^{s-k} - \text{traces}.$$

One one hand side, we can sum easily

$$\sum_{k=0}^{s} a_k^s g_1^k g_2^{s-k} = (-1)^{\frac{\mathrm{d}}{2}-2} \frac{\Gamma(\frac{\mathrm{d}}{2}-1)}{2} (g_1+g_2)^{\frac{\mathrm{d}}{2}-2+s} (-g_1 g_2)^{1-\frac{\mathrm{d}}{4}} P_{\frac{\mathrm{d}}{2}-2+s}^{2-\frac{\mathrm{d}}{2}} \left(\frac{g_2-g_1}{g_2+g_1}\right).$$

For the dimensions usually considered, this becomes quite a simple function; in $\mathrm{d} = 4$,

$$\sum_{k=0}^{s} a_k^s g_1^k g_2^{s-k} = \frac{(g_2+g_1)^s P_s\left(\frac{g_2-g_1}{g_2+g_1}\right)}{2}.$$

Subtraction of traces can now be performed by setting $g_1 = \underline{y} \cdot \underline{\partial}_1, g_2 = \underline{y} \cdot \underline{\partial}_2$ etc. However, the subtraction for general s is a very involved procedure, and the combinatorial difficulties are protecting the solution very well [7].

We will see that for our purposes, it is more convenient to indicate whenever traces are subtracted; this will not hinder us to obtain the results we are interested in, at least within the limitations of this text.

2.6.2 n-point Functions of Currents

When we compute the connected n-point function of these currents of different spin, we find that the spin-0 current plays a special role. In the UV fixpoint theory, all connected n-point functions are generated by a single ϕ-loop with the currents inserted in an arbitrary order; the total correlation is given by the sum over all permutations.

In the IR fixpoint theory, the behaviour is considerably different. In the leading order of $1/N$, the effective topology should be tree, ie there may be several ϕ-loops which are linked by σ-propagators and span a tree network in this manner. There

[7] The uniqueness argument does not apply here because we have available the terms $\underline{y} \cdot \partial_1$, $\underline{y} \cdot \partial_2$ and $\underline{y}^2$ for the construction of the currents $\mathfrak{J}^s[\underline{y}]$

is one exception for currents of spin $s > 0$: The σ couples to to a ϕ-loop via a $\sigma\phi^2$-coupling, and this can be written as $\sigma\mathcal{J}^0$. Consider a diagram where one $\mathcal{J}^s$ couples to a ϕ-loop which connects only to a *single* σ-propagator. This means that $\mathcal{J}^s$ and $\mathcal{J}^0$ are Wick-contracted (like $\langle\mathcal{J}^0\,\mathcal{J}^s\rangle$ is the free UV theory). But this contraction vanishes, since the currents $\mathcal{J}^s$ and $\mathcal{J}^0$ are orthogonal under Wick contraction.

The important conclusion is that (to leading order in $1/N!$) the two-point and three-point functions of higher-spin tensor currents do not contain any intermediary σ-propagators, and therefore the correlations are identical in the UV and IR, and so they are also conserved in the IR fixpoint theory (to this order!).

2.6.3 Two-point Functions in the Free UV Theory

The facts summarised here briefly will be derived explicitly below in in section 3.4.4. See also [5], whose normalisation of the underlying scalar field operators coincides with ours [8]. The normalisation of the two-point function of the free massless vector field ϕ^a is

$$\langle\phi^a(\underline{x})\phi^b(0)\rangle = \frac{\Gamma(\frac{\mathrm{d}}{2}-1)}{4\pi^{\frac{\mathrm{d}}{2}}}\frac{\mathcal{N}}{(\underline{x}^2)^{\frac{\mathrm{d}}{2}-1}}\delta^{ab}, \qquad \mathcal{N}=1. \tag{2.6-30}$$

This is in agreement with the earlier discussion of the $O(N)$ vector model. We will retain the constant $\mathcal{N}$, so that all further results can be easily adapted to a different normalisation by changing $\mathcal{N}$.

Define the matrix-valued function

$$I_{jl}(\underline{x}) = \delta_{jl} - 2\frac{x_j x_l}{\underline{x}^2}. \tag{2.6-31}$$

Note that I is always orthogonal (it acts as a mirror symmetry with respect to the plane orthogonal to $\underline{x}$). Then, symmetry considerations and conservation arguments dictate the form of the two-point function in coordinate space to be [39, III.2]

$$\langle\mathcal{J}^s(\underline{x})[\underline{y}]\mathcal{J}^s(0)[\underline{y}']\rangle = N\cdot n(s)\left((\underline{y}\cdot I(\underline{x})\underline{y}')^s \;-\; \text{traces}\right)\frac{1}{(\underline{x}^2)^{\mathrm{d}-2+s}}, \tag{2.6-32}$$

where $n(s)$ is a normalisation. By [5, 26], we find that

$$n(s) = (-1)^s\frac{2^{s-5}\,s!\Gamma(2s+\mathrm{d}-3)\Gamma(\frac{\mathrm{d}}{2}-1)^2}{\pi^{\mathrm{d}}\Gamma(s+\mathrm{d}-3)}\,\mathcal{N}^2.$$

By covariance, the two-point functions vanish for currents of different spin.

[8] The normalisation in [4] is different!

2.7 Twist-2 Conformal Partial Wave Expansion for the Free UV Theory

We will construct a type of conformal partial wave expansion (CPWE) for the free $O(N)$ vector model, which we found as UV fixpoint of the $O(N)$-symmetric ϕ^4 vector model. Once we have found an appropriate structure, we will be able to give an EAdS-presentation for the amplitudes of this model, ie a mathematical prescription how to compute them based on the geometric notions of EAdS. The expansion is based solely on bilinear twist-2 currents and we will show it for the free UV fixpoint theory.

Consider the correlation function of n operators $\frac{1}{2} :\phi(\underline{x}_j)^2:$, where $\underline{x}_j \in \mathbb{R}^{\mathrm{d}}$. In a first step, we select by cluster expansion the connected part of this amplitude. An elementary computation yields that the amplitude is given by the sum

$$\begin{aligned}&\frac{1}{2^n}\langle :\phi(\underline{x}_1)^2: \dots :\phi(\underline{x}_n)^2:\rangle_{\text{conn}} \\ &\quad = \sum_{p\in\pi(1,2,\dots,n)} \phi^{c_1}(\underline{x}_{p_1})\phi^{c_1}(\underline{x}_{p_1})\phi^{c_2}(\underline{x}_{p_2})\phi^{c_2}(\underline{x}_{p_2})\dots\dots\phi^{c_n}(\underline{x}_{p_n})\phi^{c_n}(\underline{x}_{p_n}),\end{aligned} \tag{2.7-33}$$

where π denotes the set of permutations of the cyclic (!) and inversion symmetric n-tupels (so the n-tupel $(1,2,\dots,n)$ is indistinguishable from $(2,3,\dots,n,1)$ and $(n, n-1,\dots,1)$). The contractions indicated yield massless scalar propagators. All in all, there are $\frac{(n-1)!}{2}$ summands in this expression. The right-hand-side will be proportional to N because of the colour summation.

The operator $\phi(\underline{y})$ has spin 0 and scaling dimension $\frac{\mathrm{d}}{2}-1$ by elementary arguments. This amplitude can be simplified by recognising that any bilocal operator constructed out of operators $\phi(\underline{y}), \phi(\underline{z})$ has an expansion in terms of local (Hermitian) totally symmetric traceless tensor fields (quasi-primary [9] conserved tensor currents) $\mathfrak{J}^s_{(i)}$ of even [10] integer spin s and twist 2 (ie of conformal dimension $\Delta_s = \mathrm{d}-2+s$); here $(i) = i_1,\dots,i_s$ is a multiindex. These currents have already been introduced in section 2.6. We have the equality [39, 73]

$$\phi(\underline{y})\cdot\phi(\underline{y})\phi(\underline{z})\cdot\phi(\underline{z}) = \int \mathrm{d}^{\mathrm{d}}x \sum_{\text{even } s\,=\,0}^{\infty} \mathfrak{J}^s_{(i)}(\underline{x})\; c^{s|0,0}_{(i)}(\underline{x}|\underline{y},\underline{z}), \tag{2.7-34}$$

where $c^{s|0,0}(\underline{x}|\underline{y},\underline{z})$ are tensor functions [11] of order N^0 whose indices are contracted with the indices of $\mathfrak{J}^s$. The currents $\mathfrak{J}^s$ are orthogonal for different spin, $\langle \mathfrak{J}^s_{(i)}\mathfrak{J}^t_{(j)}\rangle = 0$ for $s \neq t$ and we can find the coefficients by

$$c^{s|0,0}_{(i)}(\underline{x}|\underline{y},\underline{z}) = D^s_{(i),(j)}(\partial_{\underline{x}})\;\langle \mathfrak{J}^s_{(j)}(\underline{x})\phi(\underline{y})\cdot\phi(\underline{y})\phi(\underline{z})\cdot\phi(\underline{z})\rangle, \tag{2.7-35}$$

[9] There will in general be a large number of secondary fields in the same representation, ie derivatives of the quasi-primary fields.

[10] For bilocal combinations of complex scalars $\phi^*(\underline{y})\phi(\underline{z})$ we would get also odd spins.

[11] $c^{s|0,0}(\underline{x}|\underline{y},\underline{z})$ might be defined properly only as a distribution.

where $D^s(\partial_{\underline{x}})$ is the (tensor) differential operator fulfilling

$$D^s_{(i),(j)}(\partial_{\underline{x}})\,\langle \mathcal{J}^s_{(j)}(\underline{x})\mathcal{J}^s_{(k)}(\underline{y})\rangle = \delta^{(\mathrm{d})}_{(i),(k)}(\underline{x}-\underline{y}) \;-\; \text{traces}. \tag{2.7-36}$$

Since the two-point function is of order N^1, the operator D^s is of order N^{-1}. The functions $c^{s|0,0}$ are symmetric

$$c^{s|0,0}(\underline{x}|\underline{y},\underline{z}) = c^{s|0,0}(\underline{x}|\underline{z},\underline{y}), \qquad s \text{ even}. \tag{2.7-37}$$

Using the currents, we can write the contributions due to the permutations $(1,2,3,\dots,n)$ and $(2,1,3,\dots,n)$ in the sum (2.7-33) as

$$\begin{aligned}
&\phi(\underline{x}_1)\cdot\phi(\underline{x}_1)\phi(\underline{x}_2)\cdot\phi(\underline{x}_2)\cdots\cdots\phi(\underline{x}_n)\cdot\phi(\underline{x}_n)\\
&\qquad\qquad + \phi(\underline{x}_1)\cdot\phi(\underline{x}_1)\phi(\underline{x}_2)\cdot\phi(\underline{x}_2)\cdots\cdots\phi(\underline{x}_n)\cdot\phi(\underline{x}_n)\\
&= \sum_{\text{even } s\,=\,0}^{\infty} \int \mathrm{d}^{\mathrm{d}}y\; c^{s|0,0}_{(i)}(\underline{y}|\underline{x}_1,\underline{x}_2)\; \mathcal{J}^s_{(i)}(\underline{y})\phi(\underline{x}_3)\cdot\phi(\underline{x}_3)\cdots\cdots\phi(\underline{x}_n)\cdot\phi(\underline{x}_n),
\end{aligned}$$

where we have indicated that one of the fields ϕ making up the current $\mathcal{J}^s$ is contracted with $\phi(\underline{x}_3)$ and the other is contracted with $\phi(\underline{x}_n)$; since $\mathcal{J}^s$ is bilinear, there are two possible ways of realising this contraction scheme yielding the same result. To generalise the construction, we need to handle the CPWE of three currents.

The construction for tensor currents is hardly more complicated; since the currents $\mathcal{J}^s$ are contractions of bilinears in the fields and their derivatives, we can immediately state that there exists an expansion of the form

$$\mathcal{J}^t_{(i)}(\underline{y})\mathcal{J}^u_{(j)}(\underline{z}) = \int \mathrm{d}^{\mathrm{d}}x \sum_{\text{even } s\,=\,0}^{\infty} \mathcal{J}^s_{(k)}(\underline{x})\; c^{s|t,u}_{(k)|(i),(j)}(\underline{x}|\underline{y},\underline{z}), \tag{2.7-38}$$

with c a function or distribution symmetric in the last index/argument pair. On the left-hand-side, there are still two elementary field operators left for contraction. Notice that formula (2.7-34) is a special case of this equation for $t = u = 0$. In particular, the correlation of three currents is obtained as

$$\langle \mathcal{J}^s_{(i)}(\underline{x})\mathcal{J}^t_{(j)}(\underline{y})\mathcal{J}^u_{(k)}(\underline{z})\rangle = \int \mathrm{d}^{\mathrm{d}}\tilde{x}\,\langle \mathcal{J}^s_{(i)}(\underline{x})\mathcal{J}^s_{(l)}(\underline{\tilde{x}})\rangle\; c^{s|t,u}_{(l)|(j),(k)}(\underline{\tilde{x}}|\underline{y},\underline{z}). \tag{2.7-39}$$

The last ingredient which we need is the associativity of the contractions: We have eg

$$\mathcal{J}^s_{(k)}\left[\,\mathcal{J}^t_{(i)}\mathcal{J}^u_{(j)}\,\right]\mathcal{J}^v_{(l)} = \mathcal{J}^s_{(k)}\,\mathcal{J}^t_{(i)}\mathcal{J}^u_{(j)}\,\mathcal{J}^v_{(l)} + \mathcal{J}^s_{(k)}\,\mathcal{J}^t_{(i)}\,\mathcal{J}^u_{(j)}\,\mathcal{J}^v_{(l)}, \tag{2.7-40}$$

where on the left-hand-side the first and third contraction is with the total bracket.

The strategy for computing the n-point correlation function of $\frac{1}{2}:\phi^2:\equiv \mathcal{J}^0$ is thus as follows:

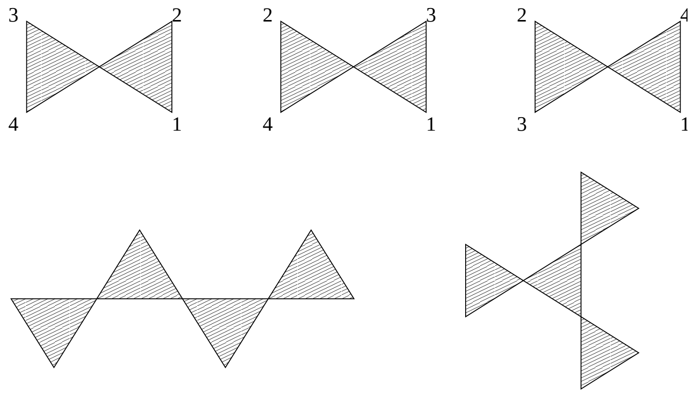

Figure 2.7: (top) All possible CCNA's for $n = 4$. (bottom) Two possible CCNA's (without numbering of external operators) for $n = 6$.

Select an arbitrary *cyclic commutative non-associative structure (CCNA)* on the set $\{1, 2, \ldots, n\}$, ie a set of $n-2$ triangles, where each corner of each triangle either carries one element of the set $\{1, 2, \ldots, n\}$ ("external operator") or touches exactly one corner of another triangle ("internal operator"). For each triangle, the order of the corners is irrelevant, ie the CCNA is not a plane structure. Each number is supposed to appear exactly once (cf. figure 2.7). The external operators represent the field operators which appear in the correlation function we are going to compute.

We will show the following facts: i) Each CCNA corresponds to a sum over certain contraction schemes of the operators $\frac{1}{2} :\phi^2:$. ii) If we sum over all CCNA's, then we obtain a multiple of the total correlation function (2.7-33). The CCNA's are then interpreted as tree Feynman graphs of an effective classical theory, with certain qualifications concerning the combinatorial prefactors.

Ad i) Assign to each internal operator j an integration coordinate $\underline{x}_j \in \mathbb{R}^{\mathrm{d}}$ and a spin variable s_j. Assign to the external operators carrying a number k the coordinate $\underline{x}_k$ of the respective field operator, and the spin 0.

Select an arbitrary triangle of the graph. Using the coordinates $\underline{x}$ and spins s of the three adjacent operators a, b, c, we write the expectation on the left-hand-side of equation (2.7-39) as contraction

$$\int \mathrm{d}^{\mathrm{d}}y\, \langle \mathcal{J}^{s_a}(\underline{x}_a)\mathcal{J}^{s_a}(\underline{y})\rangle c^{s_a|s_b,s_c}(\underline{y}|\underline{x}_b, \underline{x}_c) = \mathcal{J}^{s_a}(\underline{x}_a)\mathcal{J}^{s_b}(\underline{x}_b)\mathcal{J}^{s_c}(\underline{x}_c) \tag{2.7-41}$$

(we have suppressed the tensor indices). Now repeat the following steps: Check whether all operators appearing in this equation are external operators. We will assume that this is not the case and there is still one internal operator left, for sake of concreteness c. Thus there is another triangle bounding on this operator. This triangle has two more operators, we will call them d and e. Multiply formula (2.7-41) by $c^{s_c|s_d,s_e}(\underline{x}_c|\underline{x}_d, \underline{x}_e)$, integrate over $\underline{x}_c$ and sum over s_c. On the right-hand-side, we can use equation (2.7-38) to perform the integral, obtaining a factor $4\mathcal{J}^{s_d}(\underline{y})\mathcal{J}^{s_e}(\underline{x}_e)$. This expression has to be contracted with the remaining operators; we can resolve the contractions using formula (2.7-40). Note that this procedure generates *two* possible contraction schemes of the operators. We repeat these step until all operators in the formula are external. Since we have assigned spin 0 to the external operators, we may now substitute $\mathcal{J}^0(\underline{x}_j) = \frac{1}{2} :\phi^2(\underline{x}_j):$ throughout and resolve the contractions.

For the top left CCNA indicated in figure 2.7, we obtain the relation

$$\sum_{s \text{ even}} \int \mathrm{d}^d y \int \mathrm{d}^d z \, c^{s|0,0}(\underline{y}|\underline{x}_1, \underline{x}_2) \langle \mathcal{J}^s(\underline{y}) \mathcal{J}^s(\underline{z}) \rangle c^{s|0,0}(\underline{z}|\underline{x}_3, \underline{x}_4)$$
$$= \phi(\underline{x}_1)\cdot\phi(\underline{x}_1)\phi(\underline{x}_2)\cdot\phi(\underline{x}_2)\phi(\underline{x}_3)\cdot\phi(\underline{x}_3)\phi(\underline{x}_4)\cdot\phi(\underline{x}_4)$$
$$+ \phi(\underline{x}_1)\cdot\phi(\underline{x}_1)\phi(\underline{x}_2)\cdot\phi(\underline{x}_2)\phi(\underline{x}_3)\cdot\phi(\underline{x}_3)\phi(\underline{x}_4)\cdot\phi(\underline{x}_4).$$

Note that the position of the $\langle \mathcal{J}\mathcal{J} \rangle$-correlation on the left-hand-side is not unique; we have a choice when we begin the iteration, and it could be anywhere in the graph.

Ad ii) In the general case, there are $n-2$ c-kernels on the left-hand-side and 2^{n-3} different contraction schemes on the right-hand-side contributing. Counting powers of 2, we find that the prefactor of the all contraction schemes on the right-hand-side is always unity. However, obviously it is not sufficient to use one CCNA since we do not get all different contraction schemes necessary for the connected correlation function (2.7-33). We have to sum over *all* different CCNA's; then, by symmetry, each contraction scheme will be accounted for the same number of times. It remains to determine how many different CCNA's there are.

The answer is simple enough: draw on a plane the CCNA as binary tree with the external vertex number 1 as top node. The number of different trees is given by the Catalan number [92] $C_{n-2} = \frac{(2n-4)!}{(n-1)!(n-2)!}$. There are $(n-1)!$ different ways to number the remaining end nodes of the tree by the remaining external operators; finally, we overcount by a factor of 2^{n-2} since at each triangle, we may exchange the two subtrees or external operators hanging on, and there are $n-2$ triangles. So the total number of CCNA's is

$$\#(\text{CCNA}) = \frac{(2n-4)!}{2^{n-2}(n-2)!}.$$

Since each CCNA produces a total of 2^{n-3} different contraction schemes, summing the multiplicity of all contraction schemes results in a total of $\frac{(2n-4)!}{2(n-2)!}$ summands. The correlation function (2.7-33) requires exactly $\frac{(n-1)!}{2}$ contraction schemes, indicating that the sum over all CCNA's overcounts each contraction scheme by a factor

$$\frac{(2n-4)!}{2(n-2)!} \frac{2}{(n-1)!} = \frac{(2n-4)!}{(n-1)!(n-2)!} = C_{n-2}$$

in comparison with the full connected correlation function.

In order to formulate our results, we will write the c-kernel in a suggestive manner by help of a "generalised vertex" V. We expect that V only exists in the sense of distributions. The generalised vertex is defined by

$$V^{s,t,u}_{(i),(j),(k)}(\underline{x},\underline{y},\underline{z}) = D^t_{(j),(j')}(\partial_{\underline{y}})D^u_{(k),(k')}(\partial_{\underline{z}})c^{s|t,u}_{(i)|(j'),(k')}(\underline{x},\underline{y},\underline{z}) \tag{2.7-42}$$

and has the order N^{-2}. This definition implies

$$c^{s|t,u}_{(i)|(j),(k)}(\underline{x}|\underline{y},\underline{z}) = \int \mathrm{d}^\mathrm{d}y' \int \mathrm{d}^\mathrm{d}z' \, \langle \mathcal{J}^t_{(j)}(\underline{y})\mathcal{J}^t_{(j')}(\underline{y}')\rangle \, \langle \mathcal{J}^u_{(k)}(\underline{z})\mathcal{J}^u_{(k')}(\underline{z}')\rangle \, V^{s,t,u}_{(i),(j'),(k')}(\underline{x},\underline{y}',\underline{z}')$$

and therefore, by (2.7-39),

$$\begin{aligned} G^{s,t,u}(\underline{x},\underline{y},\underline{z}) =& \langle \mathcal{J}^s(\underline{x})\mathcal{J}^t(\underline{y})\mathcal{J}^u(\underline{z})\rangle \\ =& \int \mathrm{d}^\mathrm{d}x' \int \mathrm{d}^\mathrm{d}y' \int \mathrm{d}^\mathrm{d}z' \, \langle \mathcal{J}^s(\underline{x})\mathcal{J}^s(\underline{x}')\rangle \, \langle \mathcal{J}^t(\underline{y})\mathcal{J}^t(\underline{y}')\rangle \, \langle \mathcal{J}^u(\underline{z})\mathcal{J}^u(\underline{z}')\rangle \, V^{s,t,u}(\underline{x}',\underline{y}',\underline{z}') \end{aligned}$$

of order N^1 (we have suppressed all tensor indices).

Finally, note that all the arguments go through in the general case when we expand a correlation function of n different twist-2 currents $\langle \mathcal{J}^{s_1}(\underline{x}_1)\ldots\mathcal{J}^{s_n}(\underline{x}_n)\rangle$. We have found the following result:

Proposition 2.4. *The correlation function $\langle \mathcal{J}^{s_1}(\underline{x}_1)\ldots\mathcal{J}^{s_n}(\underline{x}_n)\rangle_{conn}$ is equivalent to C^{-1}_{n-2} times a sum of all amplitudes generated by all possible tree graphs containing symmetric, traceless quasi-primary tensor currents of all even spins s with propagators $\langle \mathcal{J}^s_{(i)}\mathcal{J}^s_{(j)}\rangle$ and effective three-current interaction vertices given by the symbols $V^{s,t,u}(\underline{x},\underline{y},\underline{z})$, with the currents $\mathcal{J}^{s_j}(\underline{x}_j)$ at the tips of the tree.*

Remark 2.5. Equivalently, the amplitudes corresponding to the trees might be obtained by using correlations of three currents $G^{s,t,u}(\underline{x},\underline{y},\underline{z})$, and integrating out the coordinates $\underline{x}$ with the inverse propagator $D^s(\partial_{\underline{x}})$ as "kernel" whenever two such correlations have a common midpoint

$$\sum_s \int \mathrm{d}^\mathrm{d}x \, G^{s,s_1,s_2}(\underline{x},\underline{y}_1,\underline{y}_2)D^s(\partial_{\underline{x}})G^{s,t_1,t_2}(\underline{x},\underline{z}_1,\underline{z}_2)$$

(suppressing tensor indices).

The operator D^s can formally be assigned an integral kernel $(G^s)^*(\underline{x},\underline{y})$; see appendix A.2 for the scalar case.

We have thus generalised a construction given by Diaz and Dorn [26] for $n = 4$ and proven a conjecture formulated by these two authors. The peculiarity of this approach is that by registering all different topologies in the CPWE, we have been able to get rid of all the higher twist currents. The disadvantage is the appearance of the factor C^{-1}_{n-2}, forbidding the interpretation of the tree diagrams we have just constructed as an effective classical field theory. However, this is not completely surprising: A quantum field theory can never be equivalent to a classical field theory.

Proposition 2.4 is the technical tool which allows us to construct an EAdS-presentation of the boundary correlations of tensor currents in sections 3.4 and 3.5.

Example: Three-point functions. Let us study the simplest case $V^{0,0,0}$ of the effective vertices, implied by the scalar three-point function

$$G^{0,0,0}(\underline{x}_1,\underline{x}_2,\underline{x}_3) = \frac{1}{2^3}\langle{:}\phi(\underline{x}_1)^2{:}{:}\phi(\underline{x}_2)^2{:}{:}\phi(\underline{x}_3)^2{:}\rangle.$$

We have from the application of Wick's theorem

$$G^{0,0,0}(\underline{x}_1,\underline{x}_2,\underline{x}_3) = \frac{N}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^3 \frac{1}{|\underline{x}_1-\underline{x}_2|^{\mathrm{d}-2}|\underline{x}_1-\underline{x}_3|^{\mathrm{d}-2}|\underline{x}_2-\underline{x}_3|^{\mathrm{d}-2}}. \tag{2.7-43}$$

Since the two-point function of the spin-0 current is

$$\frac{1}{2^2}\langle{:}\phi(\underline{x})^2{:}{:}\phi(0)^2{:}\rangle = \frac{N}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^2 |\underline{x}|^{4-2\mathrm{d}} \tag{2.7-44}$$

(implying the conformal dimension $\Delta = \mathrm{d}-2$), the inverse propagator D^0 formally has scaling dimension $\Delta^* = 2$ and with the correct normalisation (cf. appendix A.1) has the formal kernel

$$(G^0)^*(\underline{x}) = \frac{2^5\,\Gamma(\mathrm{d}-2)}{\mathcal{N}^2\Gamma(\frac{\mathrm{d}}{2}-1)^2\Gamma(2-\frac{\mathrm{d}}{2})\Gamma(\frac{\mathrm{d}}{2}-2)}\frac{1}{|\underline{x}|^4}. \tag{2.7-45}$$

Using the D'EPP relation (appendix A.3) to contract the inverse propagator with the $\underline{x}_1-\underline{x}_2$ and $\underline{x}_1-\underline{x}_3$-terms, we conclude that

$$c^{0|0,0}(\underline{x}_1|\underline{x}_2,\underline{x}_3) = \frac{\mathcal{N}\,\Gamma(\frac{\mathrm{d}}{2}-1)\Gamma(\mathrm{d}-2)}{4\pi^{\mathrm{d}}\Gamma(2-\frac{\mathrm{d}}{2})\Gamma(\frac{\mathrm{d}}{2}-1)^2}\frac{1}{|\underline{x}_1-\underline{x}_2|^2\,|\underline{x}_1-\underline{x}_3|^2\,|\underline{x}_2-\underline{x}_3|^{2(\mathrm{d}-3)}}.$$

In $\mathrm{d}=3$, this shows that the symbol c is the three-point correlation of two operators ${:}\phi(\underline{x}_2)^2{:}, {:}\phi(\underline{x}_3)^2{:}$ with scaling dimension 1 with the composite operator ${:}\phi(\underline{x}_1)^4{:}$.
In principle, we can now apply the D'EPP relation two more times on the inverse propagator kernels $(G^0)^*$ attached to the legs $\underline{x}_1,\underline{x}_2$, thus obtaining the effective vertex

$$V^{0,0,0}(\underline{x}_1,\underline{x}_2,\underline{x}_3) = \frac{2^7\,\Gamma(\mathrm{d}-2)^2(\mathrm{d}-3)(4-\mathrm{d})}{\mathcal{N}^3\Gamma(2-\frac{\mathrm{d}}{2})^2\Gamma(\frac{\mathrm{d}}{2}-1)^6\Gamma(\frac{\mathrm{d}}{2}-2)^2}\frac{1}{|\underline{x}_1-\underline{x}_2|^2\,|\underline{x}_1-\underline{x}_3|^2\,|\underline{x}_2-\underline{x}_3|^2}.$$

This vertex seems to vanish, however, in $\mathrm{d}=3$, as has been remarked by Petkou [78]. This is a misleading conclusion, however: the Schwinger parametrisation for conformal propagators puts serious restrictions on the space dimensionality in which these computations are valid, and in this case, the conditions mean that we should not expect validity unless $4<\mathrm{d}<6$. All other dimensions can be reached by analytic continuation in d. If we take the *weak* limit $\mathrm{d}\to 3$, then the factor $\mathrm{d}-3$ will vanish, but on the other hand, we get divergent contributions from the $\underline{x}$-space integrals. The precise meaning of this expression can be obtained by the following consideration: In d $= 3$, the $\underline{x}_2 - \underline{x}_3$-term in $c^{0|0,0}$ vanishes altogether. Therefore, the inverse propagators act directly on the $\underline{x}_1 - \underline{x}_2$ resp. $\underline{x}_1 - \underline{x}_3$-terms which happen to possess the correct scaling dimensions to yield precisely a delta distribution. In d $= 3$, therefore,

$$V^{0,0,0}_{\mathrm{d}=3}(\underline{x}_1, \underline{x}_2, \underline{x}_3) = \frac{256}{\mathcal{N}^3}\, \delta^{(3)}(\underline{x}_1 - \underline{x}_2)\delta^{(3)}(\underline{x}_1 - \underline{x}_3).$$

This result could have been obtained directly by recognising that in d $= 3$, the D'EPP relation can be applied directly to the scalar ϕ-loop integral, since the necessary condition $\Delta_1 + \Delta_2 + \Delta_3 = 3$ for the applicability of D'EPP holds.

Chapter 3

EAdS-Holography of the $O(N)$-symmetric ϕ^4 Vector Model

We have already reviewed the general outline of the AdS/CFT correspondence in the introduction; it is the purpose of this section to establish a holographic description applicable to the $O(N)$ ϕ^4 vector model developed in the preceding section. This is the central chapter of this thesis. We begin by introducing basic geometrical notions.

Remark. While in the discussion of the $O(N)$-symmetric ϕ^4 vector model in chapter 2 we have retained $\hbar$, we will set $\hbar \equiv 1$ throughout chapter 3, with the exception of the fundamental discussion in section 3.2.

3.1 Geometry of Euclidean Anti-de-Sitter Space

The $\mathrm{d}+1$-dimensional Euclidean Anti-de-Sitter space [1] can be defined by its embedding into a $\mathrm{d}+2$-dimensional ambient space. Let $\tilde{\eta} = \mathrm{diag}(+, \ldots, +, -)$ be the metric of the Minkowski space $\mathbb{R}^{\mathrm{d}+1,1}$, with scalar product $(,)$. With this metric, the hyperboloid

$$\mathrm{EAdS}_{\mathrm{d}+1} = \{\tilde{x} \in \mathbb{R}^{\mathrm{d}+1,1} \mid \tilde{\eta}_{\tilde{\mu}\tilde{\nu}} \tilde{x}^{\tilde{\mu}} \tilde{x}^{\tilde{\nu}} = -1, \tilde{x}^{\mathrm{d}+1} > 0\} \tag{3.1-1}$$

with the induced metric defines Euclidean Anti-de-Sitter space with unit curvature. We adorn all vectors and indices in the embedding space with a tilde. We define the modulus

$$|\tilde{x}| = \sqrt{-(\tilde{x}, \tilde{x})}, \tag{3.1-2}$$

which is real in the neighbourhood of the EAdS-hyperboloid $|\tilde{x}| = 1$, $\tilde{x}^{\mathrm{d}+1} > 0$. A very convenient coordinate system is given by the *Poincaré coordinates* $(x^0, \ldots, x^{\mathrm{d}})$ defined by

$$x^0 = \frac{1}{\tilde{x}^0 + \tilde{x}^{\mathrm{d}+1}} > 0, \qquad x^i = \frac{\tilde{x}^i}{\tilde{x}^0 + \tilde{x}^{\mathrm{d}+1}}. \qquad (1 \leq i \leq \mathrm{d}) \tag{3.1-3}$$

[1]A very helpful collection of formulae concerning Euclidean AdS can be found in [50, 58].

We will generally denote by $\underline{x}$ the coordinates $(x^1,\dots,x^{\mathrm{d}})$ and refer to them as the "horizontal" coordinates. The coordinate transform can be inverted by

$$\tilde{x}^0 = \frac{1-(x^0)^2-\underline{x}^2}{2x^0}, \qquad \tilde{x}^i = \frac{x^i}{x^0}, \qquad \tilde{x}^{\mathrm{d}+1} = \frac{1+(x^0)^2+\underline{x}^2}{2x^0}, \tag{3.1-4}$$

where $1 \le i \le \mathrm{d}$. The metric in Poincaré coordinates is given by

$$g_{\mu\nu} = \frac{1}{(x^0)^2}\delta_{\mu\nu}, \tag{3.1-5}$$

with determinant $\det g = (x^0)^{-2(\mathrm{d}+1)}$. This is a positive definite metric. The Christoffel symbols in Poincaré coordinates are (with $g^\nu_\mu \equiv \delta^\nu_\mu$)

$$\Gamma^\sigma_{\mu\nu} = x^0 g_{\mu\nu} g^\sigma_0 - \frac{1}{x^0}\left(g^\sigma_\mu g^0_\nu + g^0_\mu g^\sigma_\nu\right). \tag{3.1-6}$$

Anti-de-Sitter space is invariant under proper Lorentz transformations $SO(\mathrm{d}+1,1)$ in the ambient space $\mathbb{R}^{\mathrm{d}+1,1}$. These transformations leave invariant the scalar product $(\tilde{y},\tilde{z}) = \tilde{\eta}_{\tilde{\mu}\tilde{\nu}}\tilde{y}^{\tilde{\mu}}\tilde{z}^{\tilde{\nu}}$, $\tilde{y},\tilde{z} \in \mathbb{R}^{\mathrm{d}+1,1}$. Let u,v be coordinates in the Poincaré patch, and $\tilde{u},\tilde{v}$ the corresponding coordinates in $\mathbb{R}^{\mathrm{d}+1,1}$. Then

$$(\tilde{u},\tilde{v}) = -\frac{(u^0)^2+(v^0)^2+(\underline{u}-\underline{v})^2}{2u^0v^0} \le -1.$$

is invariant under $SO(\mathrm{d}+1,1)$ (the product $(\,,\,)$ is *not* a bilinear form of the Poincaré coordinates u,v). Note that $(\tilde{u},\tilde{u}) = -1$. Naturally, the squared chordal distance

$$\sigma(u,v) \equiv (\tilde{u}-\tilde{v},\tilde{u}-\tilde{v}) = -2-2(\tilde{u},\tilde{v}) = \frac{(u^0-v^0)^2+(\underline{u}-\underline{v})^2}{u^0v^0} \ge 0$$

is invariant. Finally, choosing 4 points u,v,y,z in the Poincaré patch, we have the useful relation

$$\begin{aligned}(\tilde{y}-\tilde{z},\tilde{u}-\tilde{v}) =& (\tilde{y},\tilde{u})-(\tilde{y},\tilde{v})-(\tilde{z},\tilde{u})+(\tilde{z},\tilde{v}) \\ =& -\frac{1}{2}\Big(\sigma(y,u)-\sigma(y,v)-\sigma(z,u)+\sigma(z,v)\Big).\end{aligned}$$

The d'Alembertian on $\mathrm{EAdS}_{\mathrm{d}+1}$ in Poincaré coordinates is

$$\Box^g = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\partial^\mu = (x^0)^2\partial_0^2 + (1-\mathrm{d})x^0\partial_0 + (x^0)^2\triangle, \tag{3.1-7}$$

where we have abbreviated the lateral derivatives as $\triangle = \sum_{i=1}^{\mathrm{d}}\partial_i^2$. The action of this operator on functions of the squared chordal distance $\sigma(x,y)$ only is found to be

$$\begin{aligned}\Box^g_x f(\sigma) =& (1+\mathrm{d})(\sigma+2)f'(\sigma) + \sigma(\sigma+4)f''(\sigma) \\ =& \left[\sigma(\sigma+4)\right]^{\frac{1-\mathrm{d}}{2}}\partial_\sigma\left[\sigma(\sigma+4)\right]^{\frac{1+\mathrm{d}}{2}}\partial_\sigma f(\sigma) \\ =& r^{-\mathrm{d}}\left(1+\frac{r^2}{4}\right)^{\frac{1-\mathrm{d}}{2}}\partial_r r^{\mathrm{d}}\left(1+\frac{r^2}{4}\right)^{\frac{1+\mathrm{d}}{2}}\partial_r f(r^2),\end{aligned}$$

where we have substituted $r = \sqrt{\sigma}$. From this form, one can see that for x in the vicinity of $y \in \mathrm{EAdS}_{\mathrm{d}+1}$, $r(x,y)$ behaves as a Euclidean distance in a flat space. For let $f(\sigma)$ be a function with support close enough to 0. Then

$$\begin{aligned}\int \mathrm{d}^{\mathrm{d}+1}x \sqrt{g} f(\sigma) &= \int \frac{\mathrm{d}^{\mathrm{d}+1}x}{(x^0)^{\mathrm{d}+1}} f\left(\frac{(x^0-y^0)^2 + (\underline{x}-\underline{y})^2}{x^0 y^0}\right) \\ &\approx \int \frac{\mathrm{d}^{\mathrm{d}+1}x}{(y^0)^{\mathrm{d}+1}} f\left(\frac{(x^0-y^0)^2 + (\underline{x}-\underline{y})^2}{(y^0)^2}\right) \\ &= \int \mathrm{d}^{\mathrm{d}+1}x\, f\left((x^0-y^0)^2 + (\underline{x}-\underline{y})^2\right).\end{aligned}$$

This estimate is good as long as $x^0 \approx y^0$, so $\sigma \ll 1$. A particular case is the native δ-distribution on Anti-de-Sitter space: Obviously,

$$\delta^{\mathrm{EAdS}}(x,y) = \frac{1}{\sqrt{g}} \delta^{(\mathrm{d}+1)}(x,y) = \delta^{(\mathrm{d}+1)}_{\|\cdot\|^2}(\sigma(x,y)) \tag{3.1-8}$$

where in the first equality we inserted the definition of the δ-distribution on EAdS space through the Euclidean flat space δ-distribution; in the second equality, $\delta^{(\mathrm{d}+1)}_{\|\cdot\|^2}$ denotes the flat space distribution depending on the Euclidean distance squared.

Symmetries. In $\mathbb{R}^{\mathrm{d}+1,1}$, the symmetry group $SO_0(\mathrm{d}+1,1)$ is generated by the algebra of generators G_{ab}, acting via rotations $G_{ab}, 0 \le a < b \le \mathrm{d}$ and Lorentz boosts $G_{a,\mathrm{d}+1}, 0 \le a \le \mathrm{d}$. How does the group action look like in Poincaré coordinates?

- Rotations in the (a,b)-plane with $G_{ab}, 1 \le a < b \le \mathrm{d}$ are mapped on identical rotations M_{ab}.
- The generators G_{0a} and $G_{a,\mathrm{d}+1}, 1 \le a \le \mathrm{d}$ combine: $N^{\mathrm{tr}}_a = G_{0a} + G_{a,\mathrm{d}+1}$ are acting as transverse translations in Poincaré coordinates. The action of $N^{\mathrm{sc}}_a = G_{0a} - G_{a,\mathrm{d}+1}$ with parameter $\underline{c}$ is a special conformal transformation ("conformal translation")
$$(x^0, \underline{x}) \mapsto \frac{\left(x^0,\ \underline{x} - ((x^0)^2 + \underline{x}^2)\underline{c}\right)}{1 - 2\underline{x}\cdot\underline{c} + ((x^0)^2 + \underline{x}^2)\underline{c}^2}.$$
- A boost generated by $G_{0,\mathrm{d}+1}$ with rapidity θ is mapped on a dilation with the rapidity θ, ie a scaling factor e^θ.

The symmetry group may be implemented by the matrices

$$G \in \{\tilde{g} \in GL(\mathrm{d}+2,\mathbb{R}) \mid {}^t\tilde{g}\tilde{\eta}\tilde{g} = \tilde{\eta},\ \det\tilde{g} = 1,\ \tilde{g}_{\mathrm{d}+1,\mathrm{d}+1} \ge 1\}, \tag{3.1-9}$$

acting on the EAdS hyperboloid embedded in $\mathbb{R}^{\mathrm{d}+1,1}$. In even dimensions, it may advantageous to use the extended EAdS group

$$G' \in \{\tilde{g} \in GL(\mathrm{d}+2,\mathbb{R}) \mid {}^t\tilde{g}\tilde{\eta}\tilde{g} = \tilde{\eta},\ \tilde{g}_{\mathrm{d}+1,\mathrm{d}+1} \ge 1\} \tag{3.1-10}$$

including a reflection on the first $\mathrm{d}+1$ axes [31].

Remark 3.1. There is another important symmetry operation keeping the chordal distance invariant: The inversion I. It acts as

$$I : (x^0, \underline{x}) \mapsto \frac{(x^0, \underline{x})}{(x^0)^2 + \underline{x}^2}. \tag{3.1-11}$$

One easily checks that in the notation of the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$, the involution acts as a reflection on the $\tilde{x}^0 = 0$-plane,

$$I : (\tilde{x}^0, \tilde{x}^1, \cdots, \tilde{x}^{\mathrm{d}+1}) \mapsto (-\tilde{x}^0, \tilde{x}^1, \cdots, \tilde{x}^{\mathrm{d}+1}). \tag{3.1-12}$$

Obviously, it is an involution, obeying $I^2 = 1$. The special conformal transformations are expressible by inversion through ordinary translations, via

$$e^{ia\cdot N^{\mathrm{sc}}} = I e^{-ia\cdot N^{\mathrm{tr}}} I. \tag{3.1-13}$$

However, it is clear that the inversion does not preserve orientation; so it is only an element of the group $O(\mathrm{d}+1, 1)$.

3.2 Correspondence via Partition Functions

Explicit constructions implementing the AdS/CFT correspondence are often based on the equivalence of correlation functions calculated in the bulk (AdS) theory on one hand and the boundary theory on the other. The initial suggestions how to implement this equivalence originate in early papers by Witten [103] and Gubser, Klebanov and Polyakov [51]. They proposed that there exists a relation between the generating functionals of both theories. We will discuss the different options in this section, and apply this to the $O(N)$ vector model. We will formulate a proposition (proposition 3.2 on page 65) about the mechanism which we believe is at the heart of the holographic theory of this model. This will lay the foundation for the detailed analysis in the subsequent parts.

The construction suggested by Witten is based on the definition

$$\langle e^{\frac{1}{\hbar}O(f)} \rangle_{\mathrm{CFT}} = \frac{\mathscr{Z}^{\mathrm{cl}}_{\varphi\{f\}}}{\mathscr{Z}^{\mathrm{cl}}_{\varphi\{0\}}}. \tag{3.2-14}$$

We explain the ingredients of this formula. f is a test function living on the boundary of AdS, and O is an operator in the boundary conformal field theory smeared with the test function f. The left hand side instructs us to compute the generating functional for correlations of the operator O in the vacuum state. On the defining side, we have a classical Lagrangian field theory living on AdS space. This field theory contains a (scalar) field φ "corresponding" to the operator O. φ is distinguished because it has nontrivial boundary conditions: The boundary value of the field φ is fixed to be f (it will be necessary to make detailed instructions on what is meant by "boundary value"); this is indicated by the notation $\varphi\{f\}$. The functional $\mathscr{Z}^{\mathrm{cl}}_{\varphi\{f\}}$ denotes the generating function of a classical field theory

$$\mathscr{Z}^{\mathrm{cl}}_{\varphi\{f\}} = e^{-\frac{1}{\hbar}S_{\varphi\{f\}}},$$

where $S_{\varphi\{f\}}$ is the action of the classical solution $\varphi\{f\}$ of the field equations with boundary value f. The action generally has to be regularised, ie one needs to subtract a total divergence to render the Lagrangian density absolutely integrable. The boundary of AdS space can be reached in a finite time from any point inside AdS; this implies that changing the boundary conditions does make a difference. In general, there is a simple relationship between the mass of the field φ and the scaling dimension of the boundary operator O. If tensor fields are used, then there exists naturally a relation between the spins of the bulk field and the boundary operator[70].

Following [32], we will refer to Witten's prescription as the "dual prescription" for φ, due to the marked difference to the usual field theoretic source term which would be implemented by a term $f \cdot \partial\varphi$ in the action ($\partial\varphi$ is the boundary value).

In the most simple examples, one can show by direct computation that the suggested correspondence is able to reproduce simple boundary correlations from the associated bulk graphs. Our first task will be to consider some simple variations of this elementary procedure.

Quantum correspondence. It is suggestive to try and use not a classical, but a quantum theory in the bulk. We define the quantum partition function

$$\mathscr{Z}_{\varphi\{f\}} = \int_{\{f\}} \mathscr{D}\varphi\; e^{-\frac{1}{\hbar}S_\varphi} \equiv \int \mathscr{D}\varphi\; \delta(\partial\varphi - f)\, e^{-\frac{1}{\hbar}S_\varphi}, \tag{3.2-15}$$

where the integration is restricted to field configurations with prescribed boundary values. In the second equality, the Dirac delta distribution expresses this fact symbolically. The quantum correspondence naturally is defined by

$$\langle e^{\frac{1}{\hbar}O(f)}\rangle_{\mathrm{CFT}} = \frac{\mathscr{Z}_{\varphi\{f\}}}{\mathscr{Z}_{\varphi\{0\}}}. \tag{3.2-16}$$

This path integral incorporates fluctuations around the classical solution $\varphi\{f\}$. The boundary value f appears as a source term coupled to the field by the (dual) "bulk-to-boundary" propagator. If the boundary CFT has a (formal) expansion in terms of Feynman diagrams, then it is expected that the correspondence acts crudely in a graph-by-graph manner; this is due to the fact that the Feynman diagrams are weighted with multiples of the coupling constants etc. and we expect to find a similar weighting on the bulk side.

Note that there is a connection between the quantum and the classical generating functional. It is a basic fact of field theory that the classical generating functional has a diagrammatic expansion which looks like the Feynman expansion of the corresponding quantised field theory; however, it contains only "tree-graphs", ie graphs without loops (this can be shown eg by recursively substituting the field equation into the action). If we define

$$e^{\frac{1}{\hbar}W_{\varphi\{f\}}} = \mathscr{Z}_{\varphi\{f\}},$$

then $W_{\varphi\{f\}}$ is a power series in $\hbar$ (with some logarithmic part from vacuum fluctuations) and generates the *connected* correlation functions; $\hbar$ serves in $W_{\varphi\{f\}}$ as loop-counting parameter for the connected diagrams. Defining

$$\mathscr{Z}^{(\alpha)}_{\varphi\{f\}} = \int_{\{f\}} \mathscr{D}\varphi \; e^{-\frac{1}{\alpha\hbar}S_\varphi}, \tag{3.2-17}$$

the classical generating functional can be reobtained by

$$\mathscr{Z}^{\mathrm{cl}}_{\varphi\{f\}} = \lim_{\alpha\to 0+} \left(\mathscr{Z}^{(\alpha)}_{\varphi\{f\}}\right)^{\alpha}. \tag{3.2-18}$$

In the limit, only the exponent of order $1/\hbar$ corresponding to tree diagrams survives (the vacuum diagrams logarithmic in $\hbar$ vanish). The commonplace description of "letting $\hbar$ go to zero" is actually a "red herring" [42], because $\hbar$ just serves to define the unit of mass.

The success of the classical correspondence in the literature demands that we justify such a modification. A strong argument is the algebraic version of AdS/CFT found by Rehren [83, 82] which implies that a quantum correspondence should exist (even if the direct connection to the Lagrangian approach is not absolutely clear). Indeed, if we believe in the fundamental character of AdS/CFT , then (3.2-16) is simply a "quantisation" of the "classical" correspondence (3.2-14); or, rather, (3.2-14) is the "classical limit" of the fully quantum correspondence (3.2-16). In the weak-coupling limit a fully quantised field theory converges against the corresponding weakly-coupled classical field theory; the reason is that if we expand the correlation functions as a series in the coupling constant, then the leading order term is precisely the correlation given by the corresponding classical theory. Equality is reached in general only when the coupling constant vanishes and the theory becomes free. It is therefore quite a reasonable assumption that the classical Witten-type correspondence is a good approximation as long as the bulk theory is weakly coupled (but never completely valid except in the limiting case of a free bulk theory).

Different boundary prescriptions. Another modification is obtained by abandoning Witten's boundary value prescription and insisting on using the field theoretic prescription for taking boundary values,

$$\mathscr{Z}^{\mathrm{cl}}_{\varphi[f]} = e^{-\frac{1}{\hbar}\left(S_{\varphi[f]} + if\cdot\partial\varphi[f]\right)}.$$

Here, $\varphi[f]$ denotes the classical solution in the presence of the "boundary source term" $if \cdot \partial\varphi$ (this is also in need of a proper definition). Along the boundary, the field is now allowed to vary (this is the equivalent to Neumann boundary conditions). It turns out that this rebellious act is very sensible.

Resorting to diagrammatics again, one finds (eg [32]) that the only modification necessary is the substitution of a different bulk-to-bulk and bulk-to-boundary propagator ("field theoretic prescription" for φ), and the insertion of i. The AdS/CFT correspondence reads

$$\langle e^{-\frac{i}{\hbar}O(f)}\rangle_{\mathrm{CFT}} = \frac{\mathscr{Z}^{\mathrm{cl}}_{\varphi[f]}}{\mathscr{Z}^{\mathrm{cl}}_{\varphi[0]}}; \tag{3.2-19}$$

however, the boundary theory (and the operator O) are generally different from (3.2-14). Note that it is conceivable to use different prescriptions for different fields at the same time.

The field theoretic prescription is easily transferred to the quantum correspondence. The partition function is

$$\mathscr{Z}_{\varphi[f]} = \int \mathscr{D}\varphi\; e^{-\frac{1}{\hbar}(S_\varphi + if\cdot\partial\varphi)}, \tag{3.2-20}$$

and the correspondence defined by

$$\langle e^{-\frac{i}{\hbar}O(f)}\rangle_{\mathrm{CFT}} = \frac{\mathscr{Z}_{\varphi[f]}}{\mathscr{Z}_{\varphi[0]}}. \tag{3.2-21}$$

Comparing to the dual prescription, we can see that (3.2-20) and (3.2-15) are related by functional Fourier transform [32]: Formally [2],

$$\mathscr{Z}_{\varphi[g]} = \int \mathscr{D}f\; e^{-\frac{i}{\hbar}f\cdot g}\,\mathscr{Z}_{\varphi\{f\}} \tag{3.2-22}$$

We want to check whether such relation also holds in the classical case. Substitute on both sides $\hbar \mapsto \alpha\hbar$, exponentiate by α and take the limit $\alpha \to 0$:

$$\begin{aligned}\mathscr{Z}^{\mathrm{cl}}_{\varphi[g]} = \lim_{\alpha\to 0+}\left(\int \mathscr{D}f\; e^{-\frac{i}{\alpha\hbar}f\cdot g}\,\mathscr{Z}^{(\alpha)}_{\varphi\{f\}}\right)^{\alpha}& \\ \neq \int \mathscr{D}f\; e^{-\frac{i}{\hbar}f\cdot g}\lim_{\alpha\to 0+}\left(\mathscr{Z}^{(\alpha)}_{\varphi\{f\}}\right)^{\alpha} &= \int \mathscr{D}f\; e^{-\frac{i}{\hbar}f\cdot g}\,\mathscr{Z}^{\mathrm{cl}}_{\varphi\{f\}}.\end{aligned}$$

In the classical case, the two prescriptions are *not* related by Fourier transform. What is the meaning of the right-hand side? Interpreting f as a dynamical field, it is a quantum theory of a single scalar field whose vertices are given by the tree diagrams of the classical theory. This is, to say the least, very strange.

To summarise, we have the choice of boundary conditions (dual or field theoretic), and we have the choice of classical vs. quantum. Since we will proceed from a given boundary theory (the $O(N)$ vector model) to its holographic realisation by some (a priori unknown) bulk theory, this choice should not be ours. We just have to decide which version of the correspondence "fits" the boundary model at hand. This will be the task of the next three subsections.

[2] There might arise the need for regularising the transformation.

3.2.1 Criticism of the Classical Correspondence

There are strong arguments limiting the classical version of the correspondence based on positivity arguments, which we will review in this subsection (there appears a similar discussion in [85]).

Consider an interacting quantum field theory in the bulk AdS. It has been shown by Bertola et al [10] (see also [33]) that by computing correlation functions for the bulk theory and letting the localisation regions of the operators approach the boundary while scaling the correlations at the same time in a specific manner, one may obtain correlation functions for a boundary theory obeying all the requirements of a sensible quantum field theory in the axiomatic sense. Intuitively, this should be clear also from the fact that we are doing nothing else but restrict the support of the correlations to the boundary of AdS (or alternatively, to a "brane", ie a submanifold hovering over the conformal boundary of AdS). This establishes a correspondence in the "field-theoretic prescription", along the lines of formula (3.2-21). The "dual prescription" is then defined procedurally by the functional Fourier transform of the generating function, as indicated, even though its interpretation is not clear from the axiomatic viewpoint.

The assumption of a classical theory in the bulk, however, introduces severe problems: Let us rewrite formula (3.2-18) as

$$\mathscr{Z}^{\mathrm{cl}}_{\varphi\{f\}} = \lim_{n\to\infty} \left(\int_{\{f\}} \mathscr{D}\varphi \; e^{-\frac{n}{\hbar} S_\varphi} \right)^{\frac{1}{n}} . \tag{3.2-23}$$

It is obvious that without taking the final power $\frac{1}{n}$, the right-hand-side is a perfectly well-defined path integral for the partition function (since the positivity of the correlation functions does not rely on a specific value for $\hbar$, so we may scale $\hbar$ as we like); each term in the Feynman expansion of the partition function is adorned with a weight $n^{c-\ell}$, where c is the number of connected components of the respective graph and ℓ is the loop number. Taking the power $\frac{1}{n}$, we introduce an additional prefactor n^{-c}, suppressing the disconnected Feynman graphs contributing to the partition function. One easily checks that this prefactor cannot be ascribed to a rescaling of the underlying coupling constants and propagators.

The correlations of the holographic boundary theory (3.2-21) receive the same factors $n^{c-l}n^{-c}$. The factor n^{c-l} can be absorbed in some parameter rescaling of the boundary theory (this must be possible because the holographic correspondence should not rely on some particular value of the Planck constant of the bulk theory). Since the connectedness structure of the correlations is the same in the boundary theory (this can be shown eg by cluster expansion), the number of disconnected components is the same in the boundary ansd the bulk correlation functions, and the factor n^{-c} has the same interpretation in the boundary theory, suppressing correlation functions with several disconnected pieces (clusters). As in the bulk, it cannot be ascribed to any rescaling of the underlying coupling constants and propagators. This leads to a violation of positivity in the boundary correlations which has been studied by

Kniemeyer [58]. The logical conclusion must be that if the bulk theory is classical, then the boundary theory must be classical as well. This is an important result.

We want to raise the question whether it really is necessary to take the final power of $\frac{1}{n}$, enforcing the treatment as "classical" theory. A rather more natural statement would be to acknowledge that for very large n, the correlation function

$$\mathscr{Z}^{\mathrm{wc}}_{\varphi\{f\}} = \int_{\{f\}} \mathscr{D}\varphi\, e^{-\frac{n}{\hbar}S_{\varphi}}. \tag{3.2-24}$$

describes a *weakly coupled* quantum system; with the usual implication that tree graphs and disconnected graphs are overweighted, but without violation of positivity. In effect, we are dealing with an effective Planck constant $\frac{\hbar}{n}$ in this system, and we have to ask whence this parameter n arises. A natural candidate in the framework of the AdS/CFT correspondence is the number of colours N of the boundary gauge theory. So the statement envisioned is then that in the large-N limit, the boundary theory is corresponding to a (weakly coupled) bulk theory with effective Planck constant proportional to $\frac{1}{N}$ (this rescaling of the Planck constant effectively rescales all the bulk couplings). If we want the correspondence to hold beyond the leading order in $1/N$, the tree approximation cannot be expected to hold any longer, and the bulk theory will have to include loop corrections. The assumption of a weakly coupled quantum system would also imply that the different boundary prescriptions for the bulk fields are related by a functional Fourier transform of the boundary source terms, as indicated in (3.2-22).

3.2.2 Propagators in Different Prescriptions

Since we will rely on the perturbative approach for the holographic reconstruction of the bulk theory from the boundary $O(N)$ vector model, it is imperative to review some basic properties of propagators in bulk (E) AdS theories. We will not dwell on technical arguments here, as these are presented in depth in chapter 6, but focus rather on the conceptual side.

For *Lagrangian* theories in bulk AdS space, we have defined at the beginning of section 3.2 the meaning of different prescriptions - field theoretic and dual. These prescriptions make a statement about the implementation of the *source terms* on the conformal boundary, either in the "field-theoretic" fashion $e^{i\mathcal{T}^s J^s}$, where $\mathcal{T}^s$ is the bulk field and J^s is the source term, or in the "dual" fashion, by a formal δ-distribution $\delta(\mathcal{T}^s - J^s)$ [3]. It is important to realise that the functional integral itself is quite independent of these prescriptions: we have (and in fact must make) the choice of a domain of integration. Selecting a path integral domain of integration consisting of functions with a particular boundary behaviour, we have control over the boundary conditions of the bulk-to-bulk propagators, without introducing any source terms on the boundary or prescribed boundary values at all. Once we have

[3] It is rather adequate to speak of "boundary values" instead of "boundary source terms" in that case.

specified the domain of integration, we can choose how to introduce source terms on the boundary.

If we choose the dual prescription and fix the boundary values, then the δ-distributions will *further* restrict the domain of integration of the path integral. Out of this restriction arise certain consistency conditions between the domain of integration of the (unconstrained) path integral and the subsequent restriction by a δ-distribution: the domain of integration has to include functions taking the prescribed boundary values. Otherwise, the path integral will just vanish. The effective domain of integration will be situated "around" the classical solution of the field equations in the bulk, with the prescribed boundary values. Since the fluctuations of the bulk fields are in this case subdued at the boundary, intuitively we would expect that the propagators in the bulk which are generated by the fluctuations of the fields are falling off towards the boundary faster than in the unconstrained case.

For the bulk-to-boundary propagators, the reasoning is different: Since in the dual prescription, the fields are held fixed stiffly at the boundary, the bulk-to-boundary propagators tend to fall off much faster in the *horizontal* direction in a given z^0-slice over the conformal boundary than when the boundary source terms are field-theoretic.

We have to study in detail the influence of all these factors on the propagators. For the propagators between sources in the bulk, we have to find out what is there boundary behaviour; for the propagators coupling to source terms on the boundary, there is also a question of normalisation [103, 32].

The free equations of motion of the bulk fields depend only on the kinetic term of the Lagrangian and not on any type of boundary condition or source term prescriptions, and therefore are always the same. In particular, we expect an identical set of Feynman rules for all prescriptions and boundary conditions when we perturb the free theory by an interaction term in the Lagrangian. In the short-distance regime (ie, much closer than the curvature radius of AdS), the curvature of AdS becomes negligible, and all bulk-to-bulk propagators are converging to the corresponding flat-space propagator.

Field-theoretic prescription for boundary source terms. We begin by studying the case of field-theoretic boundary source terms. A free spin s bulk tensor field $\mathcal{T}^s(z)$ of square mass $m^2 \geq -\frac{\mathrm{d}^2}{4} - s$ has the equation of motion (cf. (3.4-119) below)

$$(\Box_z^{\mathrm{EAdS}} + m^2)\mathcal{T}^s(z) = 0. \tag{3.2-25}$$

The bulk-to-bulk propagators are given by the Green's functions of the inhomogeneous equation of motion. Note that the normalisation of the Green's function depends on the normalisation of the kinetic term in the action. We find that there are basically two types of Green's functions (all others may composed of linear combinations of these two), with the asymptotic behaviour

$$G_{\text{bu}}^{\Delta_\pm\ s}(z,u) \sim \left(\frac{z^0 u^0}{(\underline{z}-\underline{u})^2+(z^0)^2+(u^0)^2}\right)^{\Delta_\pm}$$
$$\text{when } u^0 \gg z^0 \text{ or } u^0 \ll z^0 \text{ or } (\underline{z}-\underline{u})^2 \gg (z^0)^2+(u^0)^2, \quad (3.2\text{-}26)$$

where

$$\Delta_\pm = \frac{\mathrm{d}}{2} \pm \sqrt{\frac{\mathrm{d}^2}{4}+s+m^2} = \frac{\mathrm{d}}{2} \pm \beta_0. \quad (3.2\text{-}27)$$

We call β_0 the "effective mass" of the field on the curved Anti-de-Sitter space. We have suppressed the tensor indices. These solutions are behaving near the conformal boundary $u^0 \to 0$ as $G_{\text{bu}}^{\Delta_\pm\ s} \sim (u^0)^{\Delta_\pm}$. We say that the dimensions Δ_+ and Δ_- fulfilling $\Delta_+ + \Delta_- = \mathrm{d}$ are *conjugate*. For $\Delta_+ = \Delta_- = \frac{\mathrm{d}}{2}$, the propagators are identical. A field-theoretic boundary source term $J^s(\underline{u})$ has to contain a scale factor

$$\int \mathrm{d}^{\mathrm{d}}u\, J^s(\underline{u}) \lim_{u^0\to 0}(u^0)^{-\Delta_\pm}\mathfrak{T}^s(u) \quad (3.2\text{-}28)$$

to make sense in the limit. Removing u towards the boundary, we find solutions which can be identified as field-theoretic bulk-to-boundary propagators,

$$G_{\text{bubo}}^{\text{ft}\ \Delta_\pm\ s}(z,\underline{u}) = \lim_{u^0\to 0}(u^0)^{-\Delta_\pm}G_{\text{bu}}^{\Delta_\pm\ s}(z,u) \sim \left(\frac{z^0}{(\underline{z}-\underline{u})^2+(z^0)^2}\right)^{\Delta_\pm}. \quad (3.2\text{-}29)$$

For tensor fields, the limit operation includes a projection of the respective tangent spaces. Near the boundary $z^0 \to 0$, they behave as

$$\begin{aligned} G_{\text{bubo}}^{\text{ft}\ \Delta_\pm\ s}(z,\underline{u}) &\sim \frac{(z^0)^{\Delta_\pm}}{(\underline{z}-\underline{u})^{2\Delta_\pm}} && z^0 \ll |\underline{z}-\underline{u}|, \\ G_{\text{bubo}}^{\text{ft}\ \Delta_\pm\ s}(z,\underline{u}) &\sim (z^0)^{-\Delta_+} && \underline{u}=\underline{z}. \end{aligned} \quad (3.2\text{-}30)$$

We will discuss these propagators in section 3.4. One finds that a bulk tensor field with propagator $G_{\text{bu}}^{\Delta_\pm\ s}$ corresponds to a boundary operator with conformal dimension $\Delta_\pm$; the two-point function for the operator on the boundary is

$$G_{\text{bo}}^{\text{ft}\ \Delta_\pm\ s}(\underline{z},\underline{u}) = \lim_{z^0\to 0}(z^0)^{-\Delta_\pm}G_{\text{bubo}}^{\text{ft}\ \Delta_\pm\ s}(z,\underline{u}) \sim \frac{1}{(\underline{z}-\underline{u})^{2\Delta_\pm}}. \quad (3.2\text{-}31)$$

This result could have been also obtained by the equality of the eigenvalues of the quadratic Casimir operators of the conformal symmetry group in the bulk and boundary (see section 3.3.1 below).

The masses of bulk fields are therefore related to the scaling dimensions of the boundary operators associated by (3.2-21). Up to the normalisation, the choice of scaling dimension $\Delta_\pm$ for the boundary operator fixes uniquely the propagator which we have to use in the bulk.

What are the respective domains of integration for the path integral inducing these propagators? In section 6, we implement a “naïve ” approach to the (Euclidean) path integral where all field configurations are taken from a Hilbert space with a scalar product constructed from the regularised symmetric “Klein-Gordon” form (the kinetic term of the Lagrangian); it turns out that in order to obtain propagators $G^{\mathrm{ft}\,\Delta_+}$, one has to impose additional constraints on the boundary behaviour of the configurations (roughly speaking, they should fall off as $(u^0)^{\Delta_+}$). We will address the path integral which implements a bulk-to-bulk propagator with boundary behaviour $(u^0)^{\Delta_-}$ as Neumann type path integral whereas the path integral which implements a bulk-to-bulk propagator with boundary behaviour $(u^0)^{\Delta_+}$ as Dirichlet type path integral. Below, we will show that this terminology makes very much sense.

The path integral may even implement a “mixed” boundary behaviour

$$G^{\alpha\,s}_{\mathrm{bu}} \sim \alpha G^{\Delta_+\,s}_{\mathrm{bu}} + (1-\alpha) G^{\Delta_-\,s}_{\mathrm{bu}}, \qquad \alpha \in \mathbb{R}. \tag{3.2-32}$$

For $\alpha \neq 0, 1$, the corresponding conformal operators on the boundary no longer have a definite scale dimension; they are mixtures of operators with scaling dimension Δ_+ and Δ_-. On the level of the path integral, these propagators are implementable by giving different weights to the configurations with boundary behaviour $(u^0)^{\Delta_\pm}$. Admittedly, this latter construction is somewhat artificial.

Dual prescription for boundary source terms. If the boundary source terms for the field $\mathcal{T}^s$ are implemented with the dual prescription, then one can formally realise the fixing of boundary values by a δ-distribution on the boundary, which in the case of the Neumann path integral with configurations behaving as $(u^0)^{\Delta_-}$ near the boundary reads

$$\prod_{\underline{u} \in \mathbb{R}^{\mathrm{d}}} \delta\left(J^s(\underline{u}) - \lim_{u^0 \to 0} (u^0)^{-\Delta_-} \mathcal{T}^s(u) \right) \tag{3.2-33}$$

Because the fluctuations of the field near the boundary are suppressed by the fixing of boundary values and the propagators are generated by these fluctuations, the bulk-to-bulk propagator in this case behaves as $(u^0)^{\Delta_+}$ near the boundary; we consequently obtain the bulk-to-bulk propagator $G^{\Delta_+\,s}_{\mathrm{bu}}$ [4].

In order to determine the dual bulk-to-boundary propagator, we have to find the kernel $G^{\mathrm{dl}\,\Delta_+\,s}_{\mathrm{bubo}}(u, \underline{x})$ which fulfills

$$J^s(\underline{u}) = \lim_{u^0 \to 0} (u^0)^{-\Delta_-} \left(\int \mathrm{d}^{\mathrm{d}}x \, G^{\mathrm{dl}\,\Delta_+\,s}_{\mathrm{bubo}}(u, \underline{x}) J^s(\underline{x}) \right) \tag{3.2-34}$$

[4] By the index Δ_+ in the dual prescripiton, we do not refer to the behaviour of the underlying path integral, as in [32], but to the boundary behaviour of the propagator. So their relation $\Gamma_- = G_+$ reads $G^{\mathrm{dl}\,\Delta_+\,s}_{\mathrm{bu}} = G^{\mathrm{ft}\,\Delta_+\,s}_{\mathrm{bu}}$ here, and we may drop the prescription index on the bulk-to-bulk propagators.

for arbitrary, sufficiently regular J^s. The term in brackets is the classical solution of the free field equations whose boundary values are fixed by (3.2-33), as this classical solution distinguishes the minimum of the action. Note that the normalisation of $G_{\text{bubo}}^{\text{dl}\,\Delta_+\,s}$ does thus not depend on the normalisation of the action. Solving this equation, one shows that the dual and the field-theoretic bulk-to-boundary propagator are multiples [5]

$$G_{\text{bubo}}^{\text{dl}\,\Delta_+\,s} = c \cdot G_{\text{bubo}}^{\text{ft}\,\Delta_+\,s}. \tag{3.2-35}$$

In the scalar case, for a conventionally normalised action $\frac{1}{2}\phi(m^2 - \Box^{\text{EAdS}})\phi$, the factor $c = \Delta_+ - \Delta_-$. For the boundary-to-boundary propagators, one gets using the methods of [32]

$$G_{\text{bo}}^{\text{dl}\,\Delta_+\,s} = c^2 \cdot G_{\text{bo}}^{\text{ft}\,\Delta_+\,s} \tag{3.2-36}$$

with the same constant c.

If we try to implement a naïve δ-distribution also in the case of the Dirichlet path integral over a function space with boundary behaviour $(u^0)^{\Delta_+}$, then the formalism turns out to be inconsistent: The resulting propagator (Green's function) must be vanishing faster than $(u^0)^{\Delta_+}$ at the boundary, and such a Green's function does not exist. So the domain of path integration must always contain functions falling off like $(u^0)^{\Delta_-}$; *we are forced to adopt the Neumann path integral if the dual prescription is implemented by a simple δ-distribution.* We will at the end of this section propose a method how this difficulty can be overcome.

By functional Fourier transform (Box 3.1) with respect to J^s, the field-theoretic source term (3.2-27) on the branch Δ_- (Neumann path integral) is formally transformed into the δ-function (3.2-33), and vice versa; at the same time, the scaling dimension Δ_- of the corresponding boundary operator is changed to the conjugate dimension $\Delta_+ = \mathrm{d} - \Delta_-$. Extrapolating this consideration to the Dirichlet path integral, we might therefore *define* the dual prescription with scaling dimension $\Delta_\pm$ by functional Fourier transform of the field-theoretic prescription with scaling dimension $\Delta_\mp$.

From the fact that dual and field-theoretic propagators are related by functional Fourier transform, it follows strictly that they are related by convolution [32]

$$\begin{aligned} G_{\text{bubo}}^{\text{dl}\,\Delta_\mp\,s} G_{\text{bo}}^{\text{ft}\,\Delta_\pm\,s} &= G_{\text{bubo}}^{\text{ft}\,\Delta_\pm\,s}, \\ -G_{\text{bubo}}^{\text{ft}\,\Delta_\mp\,s} G_{\text{bo}}^{\text{dl}\,\Delta_\pm\,s} &= G_{\text{bubo}}^{\text{dl}\,\Delta_\pm\,s}. \end{aligned} \tag{3.2-37}$$

In particular, the normalisation of the dual boundary-to-boundary propagator does depend on the normalisation of the action. By the same method of proof,

$$\begin{aligned} G_{\text{bu}}^{\Delta_\mp\,s} &= G_{\text{bu}}^{\Delta_\pm\,s} - G_{\text{bubo}}^{\text{ft}\,\Delta_\pm\,s}(G_{\text{bo}}^{\text{ft}\,\Delta_\pm\,s})^{-1} G_{\text{bobu}}^{\text{ft}\,\Delta_\pm\,s}, \\ G_{\text{bu}}^{\Delta_\mp\,s} &= G_{\text{bu}}^{\Delta_\pm\,s} + G_{\text{bubo}}^{\text{dl}\,\Delta_\mp\,s}(G_{\text{bo}}^{\text{dl}\,\Delta_\mp\,s})^{-1} G_{\text{bobu}}^{\text{dl}\,\Delta_\mp\,s}. \end{aligned} \tag{3.2-38}$$

Here, G_{bobu} and G_{bubo} are adjoints. In the rest of the work, we take the conservative view that functional Fourier transform on the boundary acts between the "natural"

[5] Equation (3.2-34) can *not* be solved by a multiple of $G_{\text{bubo}}^{\text{ft}\,\Delta_-\,s}$, as one might think naïvely .

Box 3.1: Functional Fourier Transform of Boundary Source Terms

We explain the action of the functional Fourier transform on the path integral for the free scalar field. With field-theoretic boundary values (3.2-28) and source terms T^0 in the bulk, the Neumann or Dirichlet path integral for the field $\mathcal{T}^0$ is

$$\begin{aligned}\mathscr{Z}_{[J^0]}[T^0] = \mathscr{Z}_0^{-1} \int \mathscr{D}(\mathcal{T}^0) \exp\Big\{ -\frac{1}{2}\int \mathrm{d}^{\mathrm{d}+1}z\, \mathcal{T}^0 (G^0)^{-1}\mathcal{T}^0 \\ + \int \mathrm{d}^{\mathrm{d}+1}z\, \mathcal{T}^0 T^0 + \int \mathrm{d}^{\mathrm{d}}u\, J^0(\underline{u}) \lim_{u^0\to 0}(u^0)^{-\Delta_\pm}\mathcal{T}^0(u)\Big\} \\ = \exp\Big\{\frac{1}{2}\int \mathrm{d}^{\mathrm{d}+1}z\, T^0 G_{\mathrm{bu}}^{\Delta_\pm\,0} T^0 + \int \mathrm{d}^{\mathrm{d}+1}z\, T^0 G_{\mathrm{bubo}}^{\mathrm{ft}\,\Delta_\pm\,0} J^0 + \frac{1}{2}\int \mathrm{d}^{\mathrm{d}}u\, J^0 G_{\mathrm{bo}}^{\mathrm{ft}\,\Delta_\pm\,0} J^0\Big\}.\end{aligned}$$

Here, $(G^0)^{-1}$ denotes the quadratic kernel of the path integral, and the propagators act by convolution. By substituting $J^0 \to -iJ^0$ and performing the functional Fourier transform with a normalisation constant n_f, we obtain on the left hand side

$$\begin{aligned} n_f \int \mathscr{D}(J^0) \exp(iJ^0K^0)\mathscr{Z}_{[-iJ^0]}[T^0] \\ = \mathscr{Z}_1^{-1} \int \mathscr{D}(\mathcal{T}^0) \prod_{\underline{u}\in\partial\,\mathrm{EAdS}} \delta\left(K^0(\underline{u}) - \lim_{u^0\to 0}(u^0)^{-\Delta_\pm}\mathcal{T}^0(u)\right) \\ \exp\Big\{ -\frac{1}{2}\int \mathrm{d}^{\mathrm{d}+1}z\, \mathcal{T}^0 (G^0)^{-1}\mathcal{T}^0 + \int \mathrm{d}^{\mathrm{d}+1}z\, \mathcal{T}^0 T^0\Big\} \\ = \exp\Big\{\frac{1}{2}\int \mathrm{d}^{\mathrm{d}+1}z\, T^0 G_{\mathrm{bu}}^{\Delta_\mp\,0} T^0 + \int \mathrm{d}^{\mathrm{d}+1}z\, T^0 G_{\mathrm{bubo}}^{\mathrm{dl}\,\Delta_\mp\,0} K^0 + \frac{1}{2}\int \mathrm{d}^{\mathrm{d}}u\, K^0 G_{\mathrm{bo}}^{\mathrm{dl}\,\Delta_\mp\,0} K^0\Big\}\end{aligned}$$

by the definition of dual boundary source terms, and on the right hand side

$$\begin{aligned} n_f \int \mathscr{D}(J^0) \exp(iJ^0K^0)\mathscr{Z}_{[-iJ^0]}[T^0] \\ = \exp\Big\{\frac{1}{2}\int \mathrm{d}^{\mathrm{d}+1}z\, T^0 G_{\mathrm{bu}}^{\Delta_\pm\,0} T^0 - \frac{1}{2}\int \mathrm{d}^{\mathrm{d}}u\, (K^0 - G_{\mathrm{bobu}}^{\mathrm{ft}\,\Delta_\pm\,0}T^0)(G_{\mathrm{bo}}^{\mathrm{ft}\,\Delta_\pm\,0})^{-1}(K^0 - G_{\mathrm{bobu}}^{\mathrm{ft}\,\Delta_\pm\,0}T^0)\Big\}\end{aligned}$$

by the rules of Gaussian integration respectively. Equating these two expressions, we get the first half of the formulas (3.2-37) and (3.2-38). The second half is obtained by applying again the inverse functional Fourier transform, substituting $J^0 \to iJ^0$ and equating both sides. In addition, we extract the relation

$$G_{\mathrm{bo}}^{\mathrm{dl}\,\Delta_\mp\,0} = -(G_{\mathrm{bo}}^{\mathrm{ft}\,\Delta_\pm\,0})^{-1}.$$

cases $G^{\mathrm{ft}\,\Delta_-\,s}$ and $G^{\mathrm{dl}\,\Delta_+\,s}$. In the field-theoretic setting, we normalise the propagators consequently by (3.2-29) and (3.2-31), given that the boundary source terms are formally implemented by (3.2-28). As discussed before, the dual prescription in terms of an independent path integral with a naïve δ-distribution does not make sense unless $\Delta = \Delta_+$. See however the paragraph at the end of this section.

Unitarity Bound. There exists a physical lower bound to the scaling dimensions of the symmetric tensor operators of spin s on the boundary, the unitarity bound [64]

$$\Delta^s_{ub} = \begin{cases} \frac{\mathrm{d}}{2} - 1 & \text{for } s = 0 \\ \mathrm{d} - 2 + s & \text{for } s \geq 1. \end{cases} \tag{3.2-39}$$

By the equivalence of representations in the bulk and on the boundary (cf. section 3.3.1), the same bound is required in the bulk for the bulk theory to be unitary. Since for spin $s \geq 1$ the unitarity bound $\Delta^s_{ub} > \frac{\mathrm{d}}{2}$ in $2 < \mathrm{d} < 4$, the $\Delta_- \leq \frac{\mathrm{d}}{2}$ branch of scaling dimensions can never occur. For tensor fields, the unitarity bound limits the mass from below: If m^2 in (3.2-27) gets too small, then also Δ_+ will lie below the unitarity bound, so there is no possible set of boundary conditions which makes sense for these fields. We take this as the definition of a "massless" tensor field in the bulk: Equating $\Delta_+ = \Delta^s_{ub}$, we obtain the minimal mass

$$m_s^2 = (\mathrm{d} - 2 + s)(s - 2) - s \qquad\qquad (s \geq 1).$$

That this mass is not zero (as we expect from a "massless" field) is due to the constant background curvature of Anti-de-Sitter space. We will find that in the theories we are discussing, there is set of higher spin gauge fields in the bulk lying precisely on this lower limit (they are corresponding holographically to the quasi-primary bilinear tensor currents introduced in section 2.6). For spin 0, there is a certain choice; as long as

$$m^2 \leq 1 - \frac{\mathrm{d}^2}{4},$$

both Δ_- and Δ_+ are an option. It is quite impressive to see that there is such a direct equivalence between these physical limits in the holographic bulk AdS theory and the theory on the boundary.

How to implement boundary source terms in the Dirichlet path integral: a suggestion. This is a side result; since there are some arguments which may be controversial, it will not be used in the rest of the text.

Before discussing to the EAdS case, we consider first a simple analogy in flat space. In a flat space path integral over a region U with a boundary ∂U, we have the choice to implement the path integral over a space of functions with either Dirichlet or Neumann boundary conditions, yielding propagators which obey these respective boundary conditions (corresponding to propagators with behaviour $(u^0)^{\Delta_+}$ resp. $(u^0)^{\Delta_-}$ in the AdS setting).

We want to introduce field-theoretic boundary source terms. We define an auxiliary hypersurface H_ε which is a constant distance ε away from the boundary ∂U, and place the source terms J on H_ε. Then, we let H_ε approach the boundary ($\varepsilon \to 0$). In the Neumann type path integral, the bulk-to-bulk propagator $G(\underline{u}, \underline{x})$ for $\underline{u}, \underline{x} \in U$ asymptotically is constant near the boundary, so we couple the field ϕ to the source terms by a summand

$$\lim_{\varepsilon \to 0} \int_{H_\varepsilon} \mathrm{d}x \, \phi(\underline{x}) J(\underline{x}) \tag{3.2-40}$$

in the action; whereas in the Dirichlet case, they have to be coupled by

$$\lim_{\varepsilon \to 0} \int_{H_\varepsilon} \mathrm{d}x \, \varepsilon^{-1} \phi(\underline{x}) J(\underline{x}) \tag{3.2-41}$$

since the the two-point function with Dirichlet boundary conditions behaves as $G(\underline{x}, \underline{u}) \sim \mathrm{dist}(\underline{x}, \partial U)$ for $\underline{x}, \underline{u} \in U$ and $\underline{x}$ near the boundary. So effectively, by the rule of l'Hospital the sources are coupled to the normal derivative of the field at the boundary, by a source term in the action

$$\lim_{\varepsilon \to 0} \int_{H_\varepsilon} \mathrm{d}x \, [\partial_{\underline{n}(\underline{x})} \phi(\underline{x})] J(\underline{x}). \tag{3.2-42}$$

We suggest to take this as *definition* for field-theoretic boundary source terms in the Dirichlet path integral. If we demand dual boundary source terms instead, then in the Neumann path integral, we have to fix the boundary values of the field by the usual factor

$$\lim_{\varepsilon \to 0} \prod_{\underline{x} \in H_\varepsilon} \delta(\phi(\underline{x}) - J(\underline{x})) \tag{3.2-43}$$

in the path integral. This seems straightforward, but we want to suggest that the procedure is subtle: The hypersurface H_ε splits the region U into an outer region U_{out} (with boundaries ∂U and H_ε) and an inner region U_{in} (with boundary ∂U), and the path integral falls apart into two separate pieces, under the condition that discontinuities in the derivative of the field at H_ε are allowed. This point has to be stressed because we will have to make a similar demand in the Dirichlet case. We argue that the piece in U_{out} contributes only to the two-point function of the source terms J. In the limit $\varepsilon \to 0$, the outer region U_{out} gets so slim that even the lowest modes have a very high excitation energy; so there is no propagation in the outer region U_{out}, and the outer path integral does not contribute anything (except a constant factor due to the Casimir effect; but this contributes only to the overall normalisation). In effect, the region U_{in} takes over the role of U, and the boundary condition at ∂U due to the path integral is completely extinguished in favour of the boundary condition at H_ε which now takes the role of ∂U.

If we had instead implemented the δ-factors not on H_ε, but directly on the boundary ∂U, then we would have a double boundary condition: by the Neumann path integral, on ∂U, the normal derivative of the field vanishes. At the same time, the value of the field would be fixed by the δ-distributions. In general, there exists no classical field configuration solving the field equations and fulfilling both boundary conditions at the same time; so the method of letting the field fluctuate around the minimum of the action (the classical configuration) on which the path integral is based would not be applicable.

In order to introduce dual boundary source terms in the Dirichlet path integral, we should consequently *fix the normal derivative* by a factor

$$\lim_{\varepsilon\to 0} \prod_{\underline{x}\in H_\varepsilon} \delta([\partial_{\underline{n}(\underline{x})}\phi(\underline{x})] - J(\underline{x})). \tag{3.2-44}$$

We want to argue that the same mechanism as in the Neumann path integral applies here: The path integral falls apart into two separate pieces over the regions U_{in} and U_{out}, and the integral over U_{out} in the limit $\varepsilon \to 0$ does not contribute any more to the correlations. If we demand that the derivative is fixed and the path integral falls apart into two separate pieces, we have to admit that at H_ε, the field amplitude may be discontinuous; the normal derivative, however, is fixed on both sides. We are aware that this assumption is debatable. We feel that it is justified, because in the Neumann case, we had to make a similar assumption concerning the discontinuity of the derivative. In the limit $\varepsilon \to 0$, only the derivative of the field at the boundary is fixed – not its value.

It is evident that the field fluctuations near the boundary ∂U in the Neumann path integral with field-theoretic boundary source terms are of the same magnitude as the fluctuations near the boundary ∂U in the Dirichlet path integral with dual boundary source terms; the field is free to fluctuate in both cases (for $J \equiv 0$, they coincide). This implies that the bulk-to-bulk propagators in both cases are identical, they have Neumann boundary conditions. Likewise, the field fluctuations near the boundary ∂U in the Dirichlet path integral with field-theoretic boundary source terms are of the same magnitude as the fluctuations near the boundary ∂U in the Neumann path integral with dual boundary source terms; in both cases, the fluctuations are suppressed (for $J \equiv 0$, they coincide). This implies that the bulk-to-bulk propagators in both cases are identical, they have Dirichlet boundary conditions.

We expect a similar argument to hold in the EAdS setting. We want to suggest that the field-theoretic source term (3.2-28) should be re-interpreted in the Dirichlet path integral case Δ_+; the way it is written, it corresponds to (3.2-41), and we should find a way to formulate it paralleling (3.2-42). For the scalar field, we were in fact able to derive a completely sufficient prescription for the implementation of field-theoretic and dual boundary source terms which harmonises with a path integral of both Neumann (configurations behave as $(u^0)^{\Delta_-}$) and Dirichlet (configurations behave as $(u^0)^{\Delta_+}$) type. We explain this construction.

For the path integral of Dirichlet type, we propose to define the field-theoretic source terms in the action for the scalar field by

$$\lim_{\varepsilon\to 0} \int_{H_\varepsilon} \mathrm{d}u\, J^0(u)\, (u^0)^{\Delta_- - \Delta_+ + 1} \partial_{u^0} \left[(u^0)^{-\Delta_-} \mathfrak{T}^0(u) \right], \tag{3.2-45}$$

where H_ε is the hypersurface $u^0 = \varepsilon$ in EAdS. Since in the path integral with Dirichlet type boundary conditions the field configurations have an asymptotic behaviour near the boundary as $\mathcal{T}^0(u) \sim (u^0)^{\Delta_+} t(\underline{u})$, the source term (3.2-45) effectively implements

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{H_\varepsilon} \mathrm{d}u\, J^0(u)\ (u^0)^{\Delta_- - \Delta_+ + 1}\partial_{u^0}\left[(u^0)^{-\Delta_-}\mathcal{T}^0(u)\right]\\
=&\lim_{\varepsilon\to 0}\int_{H_\varepsilon} \mathrm{d}u\, J^0(u)\ (\Delta_+ - \Delta_-)t(\underline{u})\\
=&(\Delta_+ - \Delta_-)\lim_{\varepsilon\to 0}\int_{H_\varepsilon} \mathrm{d}u\, J^0(u)\ (u^0)^{-\Delta_+}\mathcal{T}^0(u).
\end{aligned}$$

This proves that (3.2-45) is equivalent to (3.2-28) on the branch Δ_+ in the field-theoretic case, up to the prefactor $\Delta_+ - \Delta_-$. The bulk-to-boundary propagator therefore is in this normalisation

$$G^{\mathrm{ft}\ \Delta_+\ 0}_{\mathrm{bubo}}(z,\underline{u}) = (\Delta_+ - \Delta_-)\lim_{u^0\to 0}(u^0)^{-\Delta_+} G^{\Delta_+\ 0}_{\mathrm{bu}}(z,u). \tag{3.2-46}$$

If the field-theoretic boundary source terms in the Neumann path integral are defined in this manner and the action is normalised conventionally as $\frac{1}{2}\phi(m^2 - \Box^{\mathrm{EAdS}})\phi$, then equation (3.2-35) holds with $c = 1$, and similarly equation (3.2-36) which can be derived from (3.2-35) and (3.2-38) [6]. Going over to dual boundary source terms, we have to introduce a factor

$$\lim_{\varepsilon\to 0}\prod_{u\in H_\varepsilon}\delta\left(J^0(u) - (u^0)^{\Delta_- - \Delta_+ + 1}\partial_{u^0}\left[(u^0)^{-\Delta_-}\mathcal{T}^0(u)\right]\right) \tag{3.2-47}$$

into the path integral. By analog arguments as in the flat space example, EAdS space is separated into two regions $\mathrm{EAdS}_{\mathrm{in}}$ and $\mathrm{EAdS}_{\mathrm{out}}$, separated by the hypersurface H_ε. The path integral in the region $\mathrm{EAdS}_{\mathrm{out}}$ has Dirichlet boundary conditions at $\partial\,$EAdS and "fixed normal derivative" boundary conditions at H_ε, and in the limit $\varepsilon \to 0$ ceases to contribute. Since the field fluctuations near H_ε in $\mathrm{EAdS}_{\mathrm{in}}$ are not suppressed by (3.2-47), we have to take into account field configurations with the boundary behaviour $(u^0)^{\Delta_-}$, implying that the bulk-to-bulk propagator is $G^{\Delta_-\ 0}_{\mathrm{bu}}$. However, we have to ascertain how the dual boundary term (3.2-47) in the path integral evaluates on these configurations, since originally it was supposed to act on Dirichlet configurations with a boundary behaviour $(u^0)^{\Delta_+}$. This is a subtle question. We assume that the relevant field configurations $\mathcal{T}^0$ near the boundary asymptotically solve the free equations of motion (in the flat case, this was the condition for the applicability of the l'Hospital rule). This implies that they have an asymptotic expansion

$$\mathcal{T}^0(u) \sim (u^0)^{\Delta_-} g_0(\underline{u}) + (u^0)^{\Delta_- + 2} g_2(\underline{u}) + \ldots \tag{3.2-48}$$
$$+ (u^0)^{\Delta_+} h_0(\underline{u}) + (u^0)^{\Delta_+ + 2} h_2(\underline{u}) + \ldots \tag{3.2-49}$$

[6] If we set $c = 1$ and normalise the propagators correspondingly, this seems to contradict with (3.13) and (3.14) of [32] which rely on an analyticity argument involving a sign change of the factor c. However, this analyticity argument is simply not valid in our approach since the field-theoretic boundary terms in the Neumann and Dirichlet path integral are defined *differently*.

It is essential that this in an expansion in $(u^0)^2$. When inserting $\mathcal{T}^0(u)$ into the δ-terms, we get

$$\lim_{\varepsilon\to 0}\prod_{u\in H_\varepsilon}\delta\left(J^0(u)-(u^0)^{\Delta_--\Delta_++1}\partial_{u^0}\left[(u^0)^{-\Delta_-}\mathcal{T}^0(u)\right]\right)$$
$$=\lim_{\varepsilon\to 0}\prod_{u\in H_\varepsilon}\delta\left(J^0(u)-\left[2(u^0)^{\Delta_--\Delta_++2}g_2(\underline{u})+\dots+(\Delta_+-\Delta_-)h_0(\underline{u})+\dots\right]\right).$$

The g_0-term is extinguished by the derivative. We have to check that the whole g-series vanishes in the limit $u^0\to 0$. By the unitarity bound for scalar fields (3.2-39), $\Delta_-\geq\frac{\mathrm{d}}{2}-1$, so

$$\Delta_--\Delta_++2=\Delta_--(\mathrm{d}-\Delta_-)+2\geq 2\left(\frac{\mathrm{d}}{2}-1\right)-\mathrm{d}+2=0; \tag{3.2-50}$$

so (excluding the case where Δ_- lies exactly on the unitarity bound), the contribution g_2 and all higher terms from the g-series vanishes and h_0 is the only summand which survives the limit $u^0\to 0$; it is finite. We have shown that the δ-distribution (3.2-47) is well defined in the Dirichlet path integral.

In order to determine the dual bulk-to-boundary propagator, we have to find the kernel $G^{\mathrm{dl}\,\Delta_-\,0}_{\mathrm{bubo}}(u,\underline{x})$ which fulfills

$$J^0(\underline{u})=\lim_{u^0\to 0}(u^0)^{\Delta_--\Delta_++1}\partial_{u^0}(u^0)^{-\Delta_-}\left(\int \mathrm{d}^{\mathrm{d}}x\,G^{\mathrm{dl}\,\Delta_-\,0}_{\mathrm{bubo}}(u,\underline{x})J^0(\underline{x})\right) \tag{3.2-51}$$

for arbitrary, sufficiently regular J^0. The term in brackets is the classical solution of the free field equations whose boundary values are fixed by (3.2-47), as this classical solution minimises the action; since it solves the field equation in the bulk, it has an asymptotic expansion near the boundary of the form (3.2-48) with a non-vanishing h-series [58]. The normalisation of $G^{\mathrm{dl}\,\Delta_-\,0}_{\mathrm{bubo}}$ is independent of the normalisation of the action. Furthermore, equations (3.2-37) are valid independently, since they rest exclusively on the fact that propagators with conjugate scaling dimension and prescription are related by functional Fourier transform. This allows us to compute for a conventionally normalised action

$$G^{\mathrm{dl}\,\Delta_-\,0}_{\mathrm{bubo}}=G^{\mathrm{dl}\,\Delta_-\,0}_{\mathrm{bubo}}G^{\mathrm{ft}\,\Delta_+\,0}_{\mathrm{bo}}(G^{\mathrm{ft}\,\Delta_+\,0}_{\mathrm{bo}})^{-1}\overset{(3.2\text{-}37)}{=}G^{\mathrm{ft}\,\Delta_+\,0}_{\mathrm{bubo}}(G^{\mathrm{ft}\,\Delta_+\,0}_{\mathrm{bo}})^{-1}$$
$$\underset{\text{with }c=1}{\overset{(3.2\text{-}35)\text{ and }(3.2\text{-}36)}{=}}G^{\mathrm{dl}\,\Delta_+\,0}_{\mathrm{bubo}}(G^{\mathrm{dl}\,\Delta_+\,0}_{\mathrm{bo}})^{-1}\overset{(3.2\text{-}37)}{=}-G^{\mathrm{ft}\,\Delta_-\,0}_{\mathrm{bubo}}. \tag{3.2-52}$$

We summarise our ideas in a tentative

Definition. In the scalar field path integral over EAdS with a domain of integration appropriate to Neumann boundary conditions for the propagators, boundary source terms in the field-theoretic prescription are implemented by (3.2-28) with Δ_-. Boundary source terms in the dual prescription are implemented by a factor (3.2-33) in the Neumann path integral. They are related by functional Fourier transform with respect to the source term.

In the scalar field path integral over EAdS with a domain of integration appropriate to Dirichlet boundary conditions for the propagators, boundary source terms in the field-theoretic prescription are implemented by (3.2-45). Boundary source terms in the dual prescription are implemented by a factor (3.2-47) in the Dirichlet path integral. They are related by functional Fourier transform with respect to the source term.

This is based on the assumption that the path integral separates into two distinct regions at the hypersurface H_ε where the boundary source terms are implemented. We are confident that this assumption can be more firmly grounded than we have done it, and that the definition can be extended to fields with spin $s > 0$.

In the rest of the thesis, we will however retain the normalisation of the propagators according to equations (3.2-29), (3.2-31) and (3.2-37).

3.2.3 UV/IR Duality and Holographic Duality

We have so far discussed the path integral approach to the AdS/CFT correspondence and possible variants in this framework; in the preceding subsection, we have stated the properties of the propagators. In this subsection, we will combine this analysis with the insights on the renormalisation group fixpoints of the $O(N)$ vector model obtained in chapter 2. We will thus come to the central proposition of this thesis.

As we have explained in the last subsection, the different boundary prescriptions (dual and field-theoretic) for a field φ in a Lagrangian bulk theory lead to the same set of Feynman diagrams for this theory, but to a different set of propagators for the field φ. The bulk theories obtained from different prescriptions are related by functional Fourier transform with respect to the boundary value of φ. In section 2.5, we have observed a similar phenomenon for the two conformal scaling limits of the $O(N)$ vector model: The free UV and interacting IR fixpoint theories were also related by a functional Fourier transform, cf. proposition 2.3 on page 31. This leads to the striking conclusion that if there exists a holographic description for one of the fixpoints, then we can construct the holographic pendant of the other fixpoint by functional Fourier transform. If the bulk theories are Lagrangian, then the functional Fourier transform is implemented solely by exchanging the relevant propagators from one prescription to the other.

Let us briefly characterise what we should expect from the holographic duals, based on very simple arguments in the line of [55]: The free UV fixpoint theory possesses the family of conserved quasi-primary tensor currents of even spin which has been given in section 2.6. It has been conjectured that the UV theory as boundary theory is holographically dual to the higher spin theory in bulk AdS, containing symmetric tensor fields $\mathcal{T}^s$ of all even spins $s \geq 0$ (cf. section 1.1.1).

The correlations between $:\phi^2:$-operators or current operators $\mathcal{J}^s$ on the boundary consist of a single ϕ-loop. The simplest correlation is the two-point function; it consists of two parallel massless propagators. The propagators are proportional to $|\underline{k}|^{-2}$, so their coordinate space behaviour is $|\underline{x}|^{2-\mathrm{d}}$; we find

$$\langle :\phi(0)^2 :: \phi(\underline{x})^2 : \rangle_{\mathrm{UV}} \sim \frac{1}{(\underline{x}^2)^{\mathrm{d}-2}},$$

so the scaling dimension of $:\phi^2:$ is $\Delta_{:\phi^2:} = \mathrm{d} - 2$. In the dimensional range $2 < \mathrm{d} < 4$, this is below $\frac{\mathrm{d}}{2}$, which by the reasoning of section 3.2.2 can be identified with the branch Δ_- of the scaling dimension. On this branch, the relation between the boundary operator scaling dimension and the bulk mass is

$$\Delta_{:\phi^2:} = \frac{\mathrm{d}}{2} - \beta_0 = \frac{\mathrm{d}}{2} - \sqrt{\frac{\mathrm{d}^2}{4} + m^2}$$

leading to an effective mass $\beta_0 = 2 - \frac{\mathrm{d}}{2}$ and a mass squared $m^2 = -2(\mathrm{d}-2)$ of the scalar bulk field. For the currents $\mathcal{J}^s$ with spin $s > 0$ and scaling dimensions $\Delta_{\mathcal{J}^s} = \mathrm{d} - 2 + s$, in the range of dimensions considered $\Delta_{\mathcal{J}^s} > \frac{\mathrm{d}}{2}$ lies on the branch Δ_+. The dimensions of the boundary operators and of the bulk propagators are therefore summarised by

$$\Delta_s^{\mathrm{UV}} = \mathrm{d} - 2 + s \tag{3.2-53}$$

for all s. When computing boundary correlations by the AdS/CFT correspondence, we have to use propagators $G^{\Delta_s^{\mathrm{UV}}\, s}$ for the fields in the bulk to achieve the proper amplitude; the choice of dual or field-theoretic prescription is a matter of the realisation.

In contrast, in the interacting IR limit theory (where we examine the correlations of the field σ) the σ-propagator is proportional to $k^{4-\mathrm{d}}$; this gives a coordinate space behaviour

$$\langle \sigma(0)\sigma(\underline{x}) \rangle_{\mathrm{IR}} \sim \frac{1}{(\underline{x}^2)^2}.$$

The scaling dimension of σ is therefore $\Delta_\sigma = 2$; and because this is larger than $\frac{\mathrm{d}}{2}$ in $2 < \mathrm{d} < 4$, we have to use the branch Δ_+. For the corresponding scalar bulk field, the effective mass $\beta_0 = 2 - \frac{\mathrm{d}}{2}$ and the Lagrangian mass $m^2 = -2(\mathrm{d}-2)$ are identical to those of $:\phi^2:$ in the UV case, however. The dimensions $\Delta_{:\phi^2:}$ and Δ_σ are conjugate,

$$\Delta_{:\phi^2:} + \Delta_\sigma = \mathrm{d}. \tag{3.2-54}$$

The boundary currents of spin $s \geq 2$ retain all their scaling dimensions from the UV fixpoint theory, to leading order in $1/N$ (although there are higher order corrections); but since the spin does not change and $s > 1$ throughout, the fields are still on the upper branch $\Delta_+ > \frac{\mathrm{d}}{2}$, by unitarity. So in the IR fixpoint model, the scaling dimensions are summarised by

$$\Delta_s^{\mathrm{IR}} = \begin{cases} 2 & \text{for } s = 0 \\ \mathrm{d} - 2 + s & \text{for } s \geq 2. \end{cases} \tag{3.2-55}$$

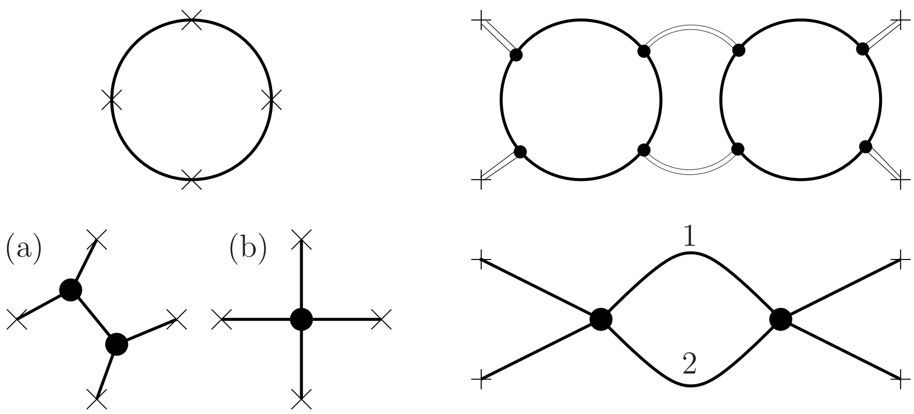

Figure 3.1: **(left top)** Typical graph in free UV fixpoint theory. Crosses are $:\phi^2:$ or $\mathcal{J}^s$ currents. Lines are propagators of the field ϕ. **(left bottom)** Two possible corresponding bulk graphs (a) and (b) with effective topology derived from $1/N$ expansion. Crosses are sources on the boundary, circles are vertices in AdS. All propagators in (b) are bulk-to-boundary propagators, in (a) there is one bulk-to-bulk propagator.
(right top) Typical graph in interacting IR fixpoint theory. Plusses are source terms for σ, double lines are σ-propagators. **(right bottom)** Corresponding bulk graph with effective topology derived from $1/N$ expansion. Plusses are sources on the boundary. There are two bulk-to-bulk propagators between the vertices, and four bulk-to-boundary propagators. The numbers are referred to in the text.

Since UV/IR duality of the $O(N)$ vector model fixpoints involves a functional Fourier transform from $:\phi^2:$ to σ only, we conclude that in the bulk, the two holographic theories are related by changing the boundary prescription for the scalar field *only*. Since the dual prescription is confined to the branch Δ_+, we conclude that in the holographic IR fixpoint theory, the scalar bulk field with scaling dimension $\Delta_0^{\mathrm{IR}} = 2$ should be realised in the dual prescription, and in the holographic UV fixpoint theory the scalar field with dimension $\Delta_0^{\mathrm{UV}} = \mathrm{d} - 2$ should be realised in the field-theoretic prescription. However, the issue of prescriptions is somewhat elusive: Formally, we may use (3.2-35) and (3.2-36) to switch between different prescriptions without changing the boundary operator scaling dimension; in particular, it is perfectly sensible to exchange the dual prescription on the IR side for the field-theoretic prescription (the propagators get a different normalisation then). We will examine this question of *realisation* in a section 3.6. *In the meanwhile, we assert that, up to normalisation, the (bulk-to-bulk) propagator for the scalar field in the UV fixpoint holographic theory has boundary behaviour* $(z^0)^{\mathrm{d}-2}$, *while for the IR fixpoint holographic theory, it behaves as* $(z^0)^2$. This is a very important result, and our construction of the holographic correspondence for the UV and IR fixpoint theories has here its foundation.

We also have seen in section 2.3 that the effective graph topology in the boundary theories is that of the $1/N$ expansion; the ϕ-loops take the role of effective vertices. By the usual rules of AdS/CFT , we should expect that in the bulk theories, $1/N$

appears as a loop counting parameter, ie the effective Planck constant is proportional $1/N$; we are not too specific about the detailed structure of the corresponding bulk graph, so that for example in figure 3.1 (left) on page 64 we leave open several possiblities for the realisation in the bulk. The loop number is fixed, however, by the order in N.

While certainly this seems quite intuitive from the perspective of graph-by-graph holographic AdS-presentation, it does not seem to agree very well with UV/IR duality: The IR boundary theory contains an increasing amount of (connected) diagrams contributing in the subleading orders of $1/N$ as in figure 3.1 (right) and therefore, the bulk graphs contributing to the holographic amplitudes should contain an increasing amount of loops. On the other hand, the connected diagrams in the UV fixpoint theory on the boundary are *all* of order $1/N$ and contain exactly one ϕ-loop as in figure 3.1 (left); and should therefore in the corresponding holographic bulk theory have a tree topology (one may guess that the ϕ-loop is substituted by one effective vertex with as many legs as there are insertions on the loop).

However, UV/IR duality demands that we should obtain the holographic UV theory in the bulk by using a scalar propagator with boundary behaviour $(z^0)^{\mathrm{d}-2}$ in place of the scalar propagator with boundary behaviour $(z^0)^2$ endemic to the IR fixpoint theory – the graph topology does *not* change when going over from the IR fixpoint theory. *Every* holographic bulk graph contributing to the IR fixpoint theory appears in the holographic UV fixpoint theory. We thus have the

Proposition 3.2. *Assume that the holographic bulk theories corresponding to the UV and IR fixpoint theories of the $O(N)$-symmetric ϕ^4 vector model on the boundary are Lagrangian, and that the scalar operators of the boundary theories are coupled linearly to fundamental fields of the holographic bulk theories, either via the field-theoretic or the dual prescription. Assume that structure of the boundary theories under the $1/N$-expansion is reflected in the bulk theories by a Planck quantum of action proportional to $1/N$, so that the ϕ-loops of the boundary theory correspond holographically to subdiagrams in the bulk with a tree topology.*
Then, the UV fixpoint holographic theory in the bulk must contain loop graphs, because the IR fixpoint holographic theory does. Both holographic theories are quantum (and not classical). There exists a dynamical mechanism suppressing loop graphs in the UV fixpoint holographic theory in order to be consistent with the free UV fixpoint theory on the boundary.

Proof. We summarise the arguments: By UV/IR duality (proposition 2.3 on page 31), the boundary UV/IR theories are linked by functional Fourier transform. Since the bulk theories are Lagrangian, the functional Fourier transform on the boundary sources (3.2-22) acts in a simple manner: It exchanges the boundary prescription for the scalar bulk field. Since the IR fixpoint theory on the boundary is interacting and has a nontrivial $1/N$-expansion, the corresponding holographic bulk theory has a perturbation expansion containing loops. The assertion for the UV fixpoint holographic bulk theory follows. ☐

If the bulk theories are not Lagrangian (ie they have an expansion in graphs but these graphs do not derive from a Lagrangian), then the action of the functional Fourier transform when we go over from the IR to the UV hologram or vice versa via UV/IR duality generates a set of contributing graphs with different topologies, invalidating the premises of the proposition.

The statement that all loops are suppressed dynamically in the UV hologram is very strong: A loop may involve as many propagators as one could fancy, and the vertices in the loop are coupled to other portions of the graph or even external sources and thereby modulate the amplitude of the loop "alone". A "graph-global" mechanism effecting that the amplitude of a Feynman graph vanishes as soon as a propagator creating a loop is inserted somewhere is hardly imaginable (such a mechanism would let a graph like in figure 3.1 (right bottom) vanish in the UV holographic theory since propagators 1 and 2 form a closed loop).

One may however envision a "graph-local" mechanism extinguishing loops. Any loop graph in the bulk involves bulk-to-bulk propagators; these propagators usually induce a summation over fields of all possible spins, and different kinds of vertices for them. *If there were a mechanism suppressing bulk-to-bulk propagation in the graphs, then all loops would be suppressed, since a loop contains at least one bulk-to-bulk propagator.* The UV fixpoint holographic theory would effectively contain only bulk-to-boundary propagators.

This does not mean that all bulk-to-bulk propagators vanish separately: rather, bulk-to-bulk propagation which results from the sum over the tensor fields of all possible spins (HS bulk fields) should cancel:

$$\sum_s \tilde{V}_1^{s,\cdots}(\mathrm{D}^{z_1},\dots)\, G_{\mathrm{bu}}^{\mathrm{UV}\,s}(z_1,z_2)\, \tilde{V}_2^{s,\cdots}(\overleftarrow{\mathrm{D}}^{z_2},\dots) = 0, \tag{3.2-56}$$

where $\tilde{V}_i$ are vertices in the bulk acting on the propagators via the covariant EAdS-derivative D^{z_j} (D^{z_2} acts towards the left here), and $G_{\mathrm{bu}}^{\mathrm{UV}\,s}(z_1,z_2)$ is the bulk-to-bulk propagator for the intermediate tensor field with spin s appropriate to the UV fixpoint holographic theory, ie with boundary behaviour $(z^0)^{\mathrm{d}-2+s}$. If one measures correlations

$$\langle \mathcal{T}^s(z)\dots\rangle_{\mathrm{bulk}} \sim \int \mathrm{d}^{\mathrm{d}+1}z'\, G_{\mathrm{bu}}^{\mathrm{UV}\,s}(z,z')\cdots$$

of a HS tensor field $\mathcal{T}^s$ with other operators in the bulk, a cancellation does not take place (since there is no summation over spins implied).

In addition, there might exist auxiliary bulk fields whose propagator just cancels the effective bulk-to-bulk propagator due to the HS fields; in that case, the summation in equation (3.2-56) would have to extend over these additional fields as well, and we would have to declare their couplings via additional vertices. However, these fields must not be detected at the boundary: Otherwise, they would form a part of the boundary UV fixpoint theory. There are candidates for such fields: For example the bulk critical scalar of effective mass 0. The critical scalar can never reach the boundary, so no trace of it can be detected from the boundary. Another possibility is that they are tensor fields of odd spin which are discussed in [87]. However, such speculations make sense only after we have determined the net propagation due to the summation over the even spin tensor fields. We summarise this in the

Working Hypothesis 3.3. *One possible mechanism for the dynamical suppression of loop graphs in the Lagrangian UV fixpoint hologram is the total cancellation of bulk-to-bulk propagation if the relevant bulk-to-bulk propagators are summed over tensor fields of all even spins and possibly one or several auxiliary bulk fields.*

If graph-internal bulk-to-bulk propagation is effectively suppressed in the UV holographic fixpoint theory, how does the AdS-presentation of the (connected) correlations look like? All sources on the boundary are connected to the bulk graphs by bulk-to-boundary propagators; no internal bulk-to-bulk propagators in the graph are allowed, and therefore, there has to be exactly *one* vertex in the bulk where all these propagators end. So in figure 3.1 (left bottom), graph (a) would be suppressed and in fact graph (b) is the only allowed bulk graph. Naturally, there should exist vertices of all orders (since there are boundary correlations of arbitrary order) and these vertices must be expected to contain high derivatives, possibly of infinite order - they are nonlocal. We will see that the twist-2 CPWE (section 2.7) is a tool for the construction of these nonlocal vertices.

Going over to the IR fixpoint holographic theory, we have to use the propagator with boundary behaviour $(z^0)^{\Delta_0^{\mathrm{IR}}}$ for the bulk scalar which is holographically dual to the boundary scalar σ. As a consequence, the bulk-to-bulk propagators for the HS tensor fields in the bulk non longer cancel: rather, we have

$$\begin{aligned}\sum_s \tilde{V}_1^{s,\cdots}(\mathrm{D}^{z_1},\dots)\, G_{\mathrm{bu}}^{\mathrm{IR}\; s}(z_1,z_2)\, \tilde{V}_2^{s,\cdots}(\overleftarrow{\mathrm{D}}^{z_2},\dots)\\ = \tilde{V}_1^{0,\cdots}(\mathrm{D}^{z_1},\dots)\, \Big(G_{\mathrm{bu}}^{\Delta_0^{\mathrm{IR}}\;0}(z_1,z_2) - G_{\mathrm{bu}}^{\Delta_0^{\mathrm{UV}}\;0}(z_1,z_2)\Big)\, \tilde{V}_2^{0,\cdots}(\overleftarrow{\mathrm{D}}^{z_2},\dots).\end{aligned} \tag{3.2-57}$$

This is the effective bulk-to-bulk propagator which will contribute to the loops in the IR fixpoint holographic bulk theory. It solves the equation of motion for the bulk-to-bulk propagator, but without the usual singularities on the diagonal: These cancel exactly when subtracting $G_{\mathrm{bu}}^{\Delta_0^{\mathrm{IR}}\;0}$ and $G_{\mathrm{bu}}^{\Delta_0^{\mathrm{UV}}\;0}$. We expect the the vertices $\tilde{V}_j^{0,\cdots}(\mathrm{D}^{z_j},\dots)$ coupling to the scalar field do not contain any derivatives D^{z_j} in the end; since we were not able to complete their construction and could not prove this statement, we retain the argument D^{z_j}. Apart from this unusual effective propagator, the IR bulk theory is a quantum field theory with a nonlocal interaction, including vertices of arbitrary order.

Witten's original suggestion for a classical bulk theory to correspond to a boundary CFT appears now in a different light: It is the "missing", or rather suppressed, bulk-to-bulk propagation, which gives the theory its pseudo-classical appearance.

Summary. We have by comparison of UV/IR duality in the boundary $O(N)$ vector model and the duality of the possible boundary prescriptions in holographic

Lagrangian AdS theories made an important conclusion: Under the assumption of the validity of the $1/N$ expansion in the bulk, a Lagrangian UV holographic theory in the bulk is a quantum (not classical) theory. We have for the time being explained this by a hypothesis demanding that there should be effectively no bulk-to-bulk propagation in the UV holographic theory. This hypothesis fixes uniquely the structure of the contributing graphs in the free UV fixpoint holographic theory: The connected n-point functions ($n > 2$) contain exactly one vertex in AdS. There must be vertices of arbitrary order.

In the remainder of this chapter, we will examine whether the conditions in the proposition (notably the $1/N$ expansion in the bulk) can be met, and assemble material for the final test of the working hypothesis. There is much (and technical) work to be done, as the necessary objects (propagators and vertices) are only known in specific cases; we need them in a form which can be handled efficiently. Is there really a mechanism which effectively makes bulk-to-bulk propagation vanish? If so, then on the level of perturbation theory, we are a large step further in the understanding of the holographic correspondence of the $O(N)$ vector model.

In section 3.6, we will show that by a semi-classical path integral for the bulk partition function, bulk-to-bulk propagation can be excluded efficiently. However, it is not clear whether this construction violates unitarity resp. Osterwalder-Schrader reflection positivity (in the Euclidean setting); and therefore, the status of the working hypothesis is unclear.

3.3 Group Representation Theory and the AdS/CFT Correspondence

The analysis of the AdS/CFT correspondence in the framework of the representation theory of the conformal group has been championed by Dobrev [30, 29]; a summary can be found in [31]. The work is in the Euclidean framework. In this section, we will shortly summarise the ideas present in the literature. Note that we slightly adapt the notation of the literature, in order to prevent clashes with the notation used throughout this text. We have added this section because many concepts which are in use have been derived in the group theoretical framework. There is no material in this section which is the author's own work.

3.3.1 Induced Representations of the Conformal Group

Since the symmetry group $SO_0(\mathrm{d}+1,1)$ of Euclidean AdS and the conformal symmetry group of its conformal boundary, the compactified $\mathbb{R}^{\mathrm{d}}$, are identical, it is suggestive to compare the representations on both spaces and find intertwiners between these representations. The construction of the representations proceeds by the technique of *induced representations*; we summarise the results here with the utmost brevity and urge the reader to consult the original publications quoted at the beginning of the section.

In the sequel, we will need the following subgroups of the Euclidean conformal group: The Euclidean Lorentz (rotation) group $M = SO(\mathrm{d})$, the one-dimensional dilatation group A, the Abelian group of Euclidean translations $N^{\mathrm{tr}} \cong \mathbb{R}^{\mathrm{d}}$, and the Abelian group of special conformal transformations $N^{\mathrm{sc}} \cong \mathbb{R}^{\mathrm{d}}$.

Representations on the boundary. The representations used in the conformal boundary theory are called elementary representations (ER) after [29]; we give them in the noncompact picture, which is relevant for physics. They are obtained by induction from the so-called parabolic subgroup $P = MAN^{\mathrm{sc}}$. This is natural since N^{tr} is locally isomorphic to G/MAN^{sc}, so we identify the conformal boundary with N^{tr}. The representation space of the representation T_χ is, following Dobrev,

$$C_\chi = \{f \in C^\infty(\mathbb{R}^{\mathrm{d}}, V_\mu)\},$$

where $\chi = [\mu, \Delta]$, Δ is the conformal weight, μ is a unitary irreducible representation of the Euclidean Lorentz group $SO(\mathrm{d})$, and V_μ is the finite-dimensional representation space of μ. We also have to demand a special asymptotic behaviour of these functions as $\to \infty$, with the leading term $f(\underline{x}) \sim (\underline{x}^2)^{-\Delta}$.

The irreducible representation T_χ acts like

$$(T_\chi(g)f)(\underline{x}) = |a|^{-\Delta}\, D^\mu(m) f(\underline{x}'), \tag{3.3-58}$$

where a is the associated scale factor, and m is the rotation obtained from nonglobal Bruhat decomposition $g = n^{\mathrm{tr}} m a n^{\mathrm{sc}}$, more precisely $g^{-1} n^{\mathrm{tr}}_{\underline{x}} = n^{\mathrm{tr}}_{\underline{x}'} m^{-1} a^{-1} (n^{\mathrm{sc}})^{-1}$ (see [31]). The matrix $D^\mu(m)$ is the representation matrix of m in the representation μ. The "coordinates" $\underline{x}$ and $\underline{x}'$ are related by a geometric point transform $\underline{x}' = g^{-1}\underline{x}$.

Representations in the Bulk. The bulk EAdS can be identified with $N^{\mathrm{tr}}A \simeq G/K$, where $K = SO(\mathrm{d}+1)$ is the maximal compact subgroup; we have to discuss the representations induced by the maximal compact subgroup K. This is shown by Iwasawa decomposition $G = N^{\mathrm{tr}}AK$. The representation spaces are

$$\hat{C}^\tau = \{\phi \in C^\infty(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+, \hat{V}_\tau)\}, \tag{3.3-59}$$

where τ is an irreducible representation of K, and $\hat{V}_\tau$ is its finite-dimensional representation space. The action of the representation is

$$(\hat{T}^\tau(g)\phi)(\underline{x}, |a|) = \hat{D}^\tau(k)\phi(\underline{x}', |a'|), \tag{3.3-60}$$

where the group elements are related by the Iwasawa decomposition $g^{-1} n^{\mathrm{tr}}_{\underline{x}} a = n^{\mathrm{tr}}_{\underline{x}'} a' k^{-1}$, and $\hat{D}^\tau(k)$ is the representation matrix of k in $\hat{V}_\tau$; again, we obtain the geometric point transformation $(|a|, \underline{x}) = g^{-1}(|a'|, \underline{x}')$. While we may choose any irrep τ of the maximal compact subgroup K, the bulk representations which are usually discussed in the perturbative approach to AdS/CFT mostly are selected from so-called "minimal representations" $\tau = \tau(\mu)$, uniquely specified by a choice of the boundary representation μ.

However, $\hat{T}^\tau$ is not irreducible; so an ER on the boundary can in general only be equivalent to a subrepresentation of $\hat{T}^\tau$. A good method to single out such subrepresentations is to select the eigenspaces of the Casimir operators of the conformal group. The Casimir operators are in general differential operators acting on the functions in the representation space. The selection of an irreducible subrepresentation amounts therefore to solving the corresponding differential equations; this is not a simple task.

Pairs of dual representations. Since we have established the content of these representations, we will now ask in which sense they are equivalent. To begin, we find that the boundary representations always arrive in pairs: For the representation $\chi = [\mu, \Delta]$, we find that $\chi^* = [\mu^*, \mathrm{d} - \Delta]$ is the representation conjugated by Weyl reflection. Here, μ^* is the "mirror image" of μ.

The conformal two-point function G_χ on the boundary is given by

$$G_\chi(\underline{x}) \sim \frac{1}{(\underline{x}^2)^\Delta} D^\mu(m(\underline{x}))$$

with $m(\underline{x})_{ij} = \frac{2}{\underline{x}^2} x_i x_j - \delta_{ij} \in M$ a rotation. It serves as intertwiner between these representations:

$$G_\chi : C_{\chi^*} \to C_\chi,\ T_\chi(g) \circ G_\chi = G_\chi \circ T_{\chi^*}(g), \qquad \forall g \in G$$

where G_χ is the convolution operator with kernel $G_\chi(x)$ defined by

$$(G_\chi f)(\underline{x}_1) = \int \mathrm{d}^\mathrm{d} x_2\, G_\chi(\underline{x}_1 - \underline{x}_2) f(\underline{x}_2) \qquad f \in C_{\chi^*}.$$

Therefore, we have partial equivalence $C_\chi \simeq C_{\chi^*}$. In particular, the values of all the Casimirs coincide. At generic points, the representations are even equivalent (so $G_\chi G_{\chi^*} = \mathbb{1}_\chi$ and $G_{\chi^*} G_\chi = \mathbb{1}_{\chi^*}$).

The dual representations $\chi = [\mu, \Delta]$ and $\chi^* = [\mu^*, \mathrm{d} - \Delta]$ can be contracted naturally. For consider a tensor operator O in the representation χ. If J is a source function transforming under the representation χ^*, then the integral

$$\int \mathrm{d}^\mathrm{d} x\, J(\underline{x}) \cdot O(\underline{x}),$$

where $\cdot$ denotes the natural contraction in the representation μ, is invariant under the action of the conformal group. This is the situation we come across in Witten's proposal for the implementation of the AdS/CFT correspondence.

Equivalent and partially equivalent representations. Since we are interested in possible equivalences between the bulk and boundary representartions, we further restrict the bulk representations: Namely, given $\chi = [\mu, \Delta]$ with Δ real, we define that $\hat{C}^\tau_\chi$ is the maximal subrepresentation of $\hat{C}^\tau$ which has the same Casimir values as C_χ and has matching asymptotic behaviour $\phi(\underline{x}, |a|) \sim |a|^{-\Delta}\varphi(\underline{x})$ for $|a| \to 0$. It is clear that there must be at least partial equivalence between C_χ and $\tilde{C}^\tau_\chi$. It can be established that the two representations are indeed equivalent if Δ is generic (see [30] for a list of the exceptional values). For these values, it is also established that $\hat{C}^\tau_\chi = \hat{C}^\tau_{\chi^*} \equiv \hat{C}^\tau_{\chi,\chi^*}$.

There are two intertwiners: The bulk-to-boundary intertwiner $L^\tau_\chi : \hat{C}^\tau_{\chi,\chi^*} \to C_\chi$ acts like a projection

$$(L^\tau_\chi \phi)(\underline{x}) = \lim_{|a|\to 0} |a|^{-\Delta} \Pi^\tau_\mu \phi(|a|, \underline{x}),$$

where Π^τ_μ is the standard projection operator from the K-representation space $\hat{V}_\tau$ to the M-representation space V_μ. The intertwining property of L^τ_χ is

$$L^\tau_\chi \circ \hat{T}^\tau(g) = T_\chi(g) \circ L^\tau_\chi \qquad \forall g \in G.$$

The boundary-to-bulk intertwiner $\hat{L}^\tau_\chi : C_\chi \to \hat{C}^\tau_{\chi,\chi^*}$, on the other hand, is constructed as integral convolution operator

$$(\hat{L}^\tau_\chi f)(|a|, \underline{x}) = \int \mathrm{d}^{\mathrm{d}} x' \, |a|^{\Delta - \mathrm{d}} K^\tau_\chi \left(\frac{\underline{x} - \underline{x}'}{|a|} \right) f(\underline{x}'),$$

where $K^\tau_\chi : V_\mu \to \hat{V}_\tau$ is some linear operator. The intertwining property of $\hat{L}^\tau_\chi$ is

$$\hat{T}^\tau(g) \circ \hat{L}^\tau_\chi = \hat{L}^\tau_\chi \circ T_\chi(g) \qquad \forall g \in G.$$

In particular, $\hat{L}^\tau_\chi L^\tau_\chi = \hat{L}^\tau_{\chi^*} L^\tau_{\chi^*} = \mathbb{1}_{\hat{C}^\tau_{\chi,\chi^*}}$ and $L^\tau_\chi \hat{L}^\tau_\chi = \mathbb{1}_{C_\chi}$. It can be shown that at generic points, we can reconstruct the bulk field completely from its boundary values.

Dual intertwiners can be related to each other by the boundary propagator: We have up to a prefactor

$$L^\tau_{\chi^*} \sim G_{\chi^*} \circ L^\tau_\chi. \tag{3.3-61}$$

Bulk-to-bulk Propagators. Different Prescriptions. We are relating the group theoretical results to the discussion of the propagators in section 3.2.2. We found that the propagators have a distinguished boundary behaviour, which for a given mass m^2 in the bulk is characterised by $(z^0)^{\Delta_\pm}$, with $\Delta_\pm$ given in (3.2-27). Can they be characterised by the representation method? The answer is that for generic points, there is indeed a simple characterisation. Both types of propagators act according to

$$G^{\Delta_\pm}_{\chi,\chi^*} : \hat{C}^\tau_{\chi,\chi^*} \to \hat{C}^\tau_{\chi,\chi^*}. \tag{3.3-62}$$

For a crossed contraction of the bulk-to-boundary intertwiners, we have up to a multiple

$$\hat{L}^{\tau}_{\chi^*} \circ L^{\tau}_{\chi} \sim G^{\Delta_-}_{\chi,\chi^*} - G^{\Delta_+}_{\chi,\chi^*} \sim \hat{L}^{\tau}_{\chi} \circ L^{\tau}_{\chi^*} \tag{3.3-63}$$

(this equation will be shown in a functional integral setup in section 6.1.1 below). The integral kernels of the propagators $G^{\Delta_\pm}_{\chi,\chi^*}(|a|,\underline{x};\,|b|,\underline{y})$ (as function of $|a|,\underline{x}$, with $|b|,\underline{y}$ fixed, or vice versa) are not elements of C^{τ}_{χ,χ^*}, since they solve the free equation of motion only up to contact terms $\delta^{(\mathrm{d})}(\underline{x}-\underline{y})\delta(|a|-|b|)$, and the free equations of motion are precisely the Casimir differential equations required to hold for elements of C^{τ}_{χ,χ^*}. However, they still can be identified by their asymptotic behaviour, $G^{\Delta_-}_{\chi,\chi^*}(|a|,\underline{x};\,|b|,\underline{y}) \sim |a|^{\Delta_-} \sim |a|^{\mathrm{d}-\Delta}$ for $|a| \to 0$ and $G^{\Delta_+}_{\chi,\chi^*}(|a|,\underline{x};\,|b|,\underline{y}) \sim |a|^{\Delta}$ for $|a| \to 0$, if we identify the Δ from this section with Δ_+ from (3.2-27). It is tempting to denote the field-theoretic propagator by $G^{\tau}_{\chi^*} \equiv G^{\Delta_-}_{\chi,\chi^*}$ and say that it propagates the representation χ^*, whereas the dual propagator might be called $G^{\tau}_{\chi} \equiv G^{\Delta_+}_{\chi,\chi^*}$ and said to propagate the "shadow field" in the representation χ [60].

From what has been said before, however, there is really no stringent reason to make a distinction between those two representations *in the bulk*. The limiting behaviour does *not* imply relations like "$L^{\tau}_{\chi} \circ G^{\Delta_-}_{\chi,\chi^*} = 0$" or similar: $G^{\Delta_-}_{\chi,\chi^*}$, acting as convolution operator according to (3.3-62), will certainly not equal 0, and therefore will not map into the kernel of neither L^{τ}_{χ} not $L^{\tau}_{\chi^*}$.

3.3.2 The Lifting Programme

Now we have formulated the equivalence between the bulk and boundary representations, we proceed to describe how this equivalence is exploited for the good of the AdS/CFT correspondence. The term "lifting" of correlations means more than the mere EAdS-presentation of amplitudes which is an essential, but technical, prerequisite for the establishment of a holographic correspondence on the perturbative level (cf. subsection 1.1.2): The amplitudes of the boundary theory are now seen as boundary limits of the correlations of an intrinsic (physical) bulk theory in the sense of Dobrev's representation theory (summarised in the preceding section), where the coordinates of the correlation functions are taken towards the boundary while the functions are scaled according to their scaling dimensions. If a theory is "lifted" from the boundary into the bulk, then the boundary theory can be reobtained from the "lifted" bulk theory by restriction to the boundary.

The lifting procedure we are reporting about here (as extracted from the literature) is based on representation theory. The obvious object to lift are the correlation functions of operators in various representations of the conformal group. On a deeper level, it is a natural question what is a connection between this lifting and the (technical) EAdS-presentation which can be developed independently, based on Feynman diagrammatic methods.

There are various implementations; we will stick to the formulation by Rühl [86], with further elaboration by Leonhardt, Manvelyan et al [60, 61]; however cf. Petkou [78, 79].

Lifting of correlation functions. The starting point is a CFT on the conformal boundary of EAdS, the one-point compactification of $\mathbb{R}^{\mathrm{d}}$. Consider a correlation function

$$G(f_1^*, f_2^*, \ldots, f_n^*) = \langle O_1(f_1^*) O_2(f_2^*) \ldots O_n(f_n^*) \rangle_{\mathrm{CFT}}, \tag{3.3-64}$$

where O_j are (tensor) operators transforming under the elementary representation χ_j of the conformal group, smeared by the test functions f_j^* transforming under the dual representation χ_j^*. We assume that the bulk theory is given by some diagrammatic expansion, without making further assumptions on the nature of this expansion; it might be a classical perturbative field theory (as in Witten's original suggestion), some perturbative quantum theory, or even a skeleton expansion [7]. The statement of the lifting hypothesis is an equality of the sort

$$G(f_1^*, f_2^*, \ldots, f_n^*) \stackrel{!}{=} \prod_{j=1}^{n} \int \mathrm{d}^{\mathrm{d}} x_j \int_{\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}+1} y_j\, f^*(\underline{x}_j) G_{\chi_j}^{\mathrm{bobu}}(\underline{x}_j;\, y_j)$$
$$\langle \varphi_1^*(y_1) \varphi_2^*(y_2) \ldots \varphi_n^*(y_n) \rangle_{\text{bulk, amputated}},$$

where $G_{\chi_j}^{\mathrm{bobu}}(\underline{x}_j;\, y_j)$ is some bulk-to-boundary propagator for the representation χ_j, φ_j^* are bulk fields, and we demand the contraction of the indices of the finite dimensional representation spaces. That $G_{\chi_j}^{\mathrm{bobu}}$ should propagate χ_j is clear since it is to be contracted with f_j^* transforming under χ_j^*. Therefore, as a tensor operator, φ_j^* transforms under χ_j^*, implying that its *amputated correlations* (missing the propagators) are acting as integral kernels between functions transforming according to χ_j^*. However it cannot be said that these kernels "transform" in some way according to the representation χ_j, since they contain singular "delta"-contributions. We will assume for simplicity that the representation of φ_j^* is equivalent to one of the representation spaces $C_{\chi_j^*}^{\tau_j}$ introduced in the preceding section, so that $G_{\chi_j}^{\mathrm{bubo}} = L_{\chi_j}^{\tau_j}$). Acting with $\hat{L}_{\chi_j^*}^{\tau_j}$, and using (3.3-63), we obtain

$$\prod_{j=1}^{n} \int \mathrm{d}^{\mathrm{d}} x_j\, \hat{L}_{\chi_j^*}^{\tau_j}(y_j;\, \underline{x}_j) G(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$$
$$\stackrel{!}{\sim} \prod_{j=1}^{n} \int_{\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}+1} z_j \, (G_{\chi_j^*}^{\tau_j}(y_j;\, z_j) - G_{\chi_j}^{\tau_j}(y_j;\, z_j))\, \langle \varphi_1^*(z_1) \varphi_2^*(z_2) \ldots \varphi_n^*(z_n) \rangle_{\text{bulk, ampt.}}$$
$$= \langle \varphi_1^*(y_1) \varphi_2^*(y_2) \ldots \varphi_n^*(y_n) \rangle_{\text{bulk}} + \text{shadow field contributions.}$$

The "shadow field contributions" are the contributions by the unwanted propagator $G_{\chi_j}^{\tau_j}$. If one manages to sort them out from the right-hand side, then we have the bulk correlations of the field φ^*. One may achieve this by analysing the scaling behaviour when y_j is being taken towards infinity; this is rather a procedural than a precise definition and involves analysis of hypergeometric functions and Kummer relations [86].

[7] It is an interesting idea to lift not the correlation functions themselves, but rather the bootstrap equations which define the boundary CFT.

A particular case is the two-point function; it can be used to determine explicitly the bulk-to-bulk propagators. Since $G(O(\underline{x}_1), O(\underline{x}_2)) = G_\chi(\underline{x}_1, \underline{x}_2)$ for O an operator transforming under χ, we get

$$\int \mathrm{d}^{\mathrm{d}}x\, \hat{L}^\tau_\chi(y_1;\, \underline{x}) \hat{L}^\tau_{\chi^*}(y_2;\, \underline{x}) \sim G^\tau_{\chi^*}(y_1;\, y_2) - G^\tau_\chi(y_1;\, y_2), \tag{3.3-65}$$

so that the bulk-to-bulk propagators can be disentangled by selecting the appropriate asymptotic behaviour.

3.4 EAdS-Presentation of Three-Point Functions of Bilinear Quasi-Primary Tensor Currents in the Free UV Theory

After having discussed the representation-theoretical view of the AdS/CFT correspondence, we will now develop in detail a perturbative approach.

As a first step, we will construct an EAdS-presentation of the three-point function of bilinear tensor currents in the free UV fixpoint theory, since three-point functions are the simplest non-trivial objects which have a holographic EAdS-presentation. This is the first step of the programme developed in section 3.2.3, as it reveals the structure of the holographic three-vertex. We consider quasi-primary bilinear twist-2 singlets of the form (see 2.6-25)

$$\mathcal{J}^s = \sum_{k=0}^{s} a^s_k \; :\underline{\partial}^{\otimes k} \phi^c\, \underline{\partial}^{\otimes s-k} \phi^{c(*)}: \; - \text{traces}, \tag{3.4-66}$$

where the tensor indices are silent and we have assumed total symmetrisation. The constants a^s_k are given in (2.6-26). If ϕ is a real field, then currents of odd spin are all vanishing.

Definition 3.4. An *EAdS-presentation* of the correlation function of $n \geq 3$ bilinear quasi-primary twist-2 tensor currents is an integral representation of the form

$$\begin{aligned} G^{s_1,\dots,s_n}_{(l_1),\dots,(l_n)}(\underline{x}_1, \dots, \underline{x}_n) =& \langle \mathcal{J}^{s_1}_{(l_1)}(\underline{x}_1) \dots \mathcal{J}^{s_n}_{(l_n)}(\underline{x}_n) \rangle \\ \overset{!}{=}& \int \frac{\mathrm{d}^{\mathrm{d}}z\; \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} \tilde{V}^{s_1,\dots,s_n\;(\mu_1),\dots,(\mu_n)}(\mathrm{D}_{z_1}, \dots, \mathrm{D}_{z_n}) \\ & G^{s_1}_{\text{bubo }(\mu_1),(l_1)}(z_1, \underline{x}_1) \dots G^{s_n}_{\text{bubo }(\mu_n),(l_n)}(z_n, \underline{x}_n) \Big|_{z_i = z}, \end{aligned} \tag{3.4-67}$$

where z is integrated all over EAdS, $G^s_{\text{bubo }(\mu),(l)}(z, \underline{x})$ is the spin s bulk-to-boundary propagator from the boundary point $\underline{x}$ to the bulk point z with the boundary conditions appropriate to the scaling dimension of the current $\mathcal{J}^s$, and $\tilde{V}$ acts as a differential operator via the covariant derivatives D^{z_j} on the propagators and is of order N^1.

The EAdS-presentation has *formal conformal covariance* if it is conformally covariant already prior to integration of z.

In the UV fixpoint theory, the scaling dimensions of the currents are $\Delta(\mathcal{J}^s) = \mathrm{d}-2+s$. In order to define the propagators, we will have to decide for some normalisation. We choose the normalisation (3.2-29) and (3.2-31) appropriate to the *field-theoretic* prescription for all spins; if we later realise the system by a concrete path integral, we will have to change the normalisation appropriate to the prescriptions necessary. We denote the respective propagators by $G^{\text{ft UV } s}_{\text{bubo } (\mu),(l)}(z, \underline{x})$.

While the bulk-to-boundary and bulk-to-bulk propagators of the tensor fields have been reported in the literature (for the vector bulk-to-bulk propagator one may consult eg [32], the tensor bulk-to-boundary propagators are examined in [60]), we feel that it is necessary to see in a direct computation how these propagators arise. To our knowledge, there is to this date no conclusive account of the generic bulk vertex $\tilde{V}$ even in the case of $n = 3$ (there are some newer results on vertices involving the spin 2 tensor from the holographic renormalisation group [72]). For $n = 3$, there are no alternatives to this structure which do not contain loops in EAdS, so it is a good object to begin with.

We will perform the analysis in Euclidean Anti-de-Sitter space, whose geometry is discussed in section 3.1. The concept of Wick rotation makes sense in this space [13], and the amplitudes could be continued analytically to AdS proper. After discussing the notation of vectors, tangent vectors and related concepts in EAdS and its embedding space $\mathbb{R}^{\mathrm{d}+1,1}$, we will show that the simple concept of Schwinger parametrisation for the correlations, which has been elaborated by the author in a previous publication [53], does not lead to proper conformally covariant amplitudes. We will then discuss alternatives for their derivation.

3.4.1 Conformal Invariants in the Embedding Space

Embedding Geometry. If we want to obtain a truly covariant expression on EAdS resembling a Feynman graph, we should be introducing a "vertex"-like object situated at $(z^0, \underline{z})$. This will include covariant derivatives acting on the boundary-to-bulk propagators. Since these propagators will certainly propagate tensor fields, the Christoffel symbols which appear in the covariant derivative whenever it acts on these tensors will have to be taken care of. These are rather inconvenient to handle.

A possible way out is to perform the calculation in the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$ which is a flat space (see section 3.1), and only in the end restrict the expressions to the embedded hyperboloid defining Euclidean Anti-de-Sitter space. The resulting tensors and vectors will have indices running from 0 to $\mathrm{d}+1$, however. A similar use of tensor representations on de-Sitter space obtained by restriction of tensors from the embedding space has been suggested already by Dirac [27, 28]. Einstein and Mayer [34, 35] have ventured to obtain a sort of "holographic" description of Kaluza-Klein theory on a 4-dimensional submanifold embedded in a 5-dimensional spacetime necessitated by Kaluza-Klein theory, yielding similar tensor fields with one "odd" orthogonal direction [8]. Ultimately, these orthogonal components should not to contribute at all to the correlations.

Since $\mathbb{R}^{\mathrm{d}+1,1}$ is a flat space, the covariant derivative is replaced by the partial derivative (as long as we stick to the Euclidean coordinates for $\mathbb{R}^{\mathrm{d}+1,1}$). The vertex coordinate $z = (z^0, \underline{z})$ must then be replaced by its embedded coordinate

$$\tilde{z}^0 = \frac{1-(z)^2}{2z^0}, \qquad \tilde{z}^i = \frac{z^i}{z^0}, \qquad \tilde{z}^{\mathrm{d}+1} = \frac{1+(z)^2}{2z^0}, \tag{3.4-68}$$

with $(z)^2 = (z^0)^2 + \underline{z}^2$. The scalar product of two points on the embedded hyperboloid which are specified in Poincaré coordinates is according to section 3.1

$$(\tilde{y}, \tilde{z}) = -\frac{(y^0)^2 + (z^0)^2 + (\underline{y} - \underline{z})^2}{2y^0 z^0} \leq -1. \tag{3.4-69}$$

In what follows, we will frequently use different geometric objects; while the same letter denotes always the same object, a tilde $\tilde{z}$ means that we consider the object in $\mathbb{R}^{\mathrm{d}+1,1}$, an underscore $\underline{x}$ means that this object lives on the conformal boundary (say, a boundary point or a tangent vector on the boundary) and letters without any decoration point to an object in EAdS. What are the coordinates of these objects in $\mathbb{R}^{\mathrm{d}+1,1}$ depends on their type, and we will for every single object give a coordinatisation in the Euclidean embedding space.

A boundary point $\underline{x} \in \mathbb{R}^{\mathrm{d}}$ can be represented by a lightlike ray asymptotically tangential to the hyperboloid; these rays are characterised by vectors

$$\tilde{x}^{\tilde{\mu}} = s(\underline{x}) \left(\frac{1-\underline{x}^2}{2}, \ \underline{x}, \ \frac{1+\underline{x}^2}{2} \right)^{\tilde{\mu}} \tag{3.4-70}$$

(we decorate indices in the Euclidean coordinate system for $\mathbb{R}^{\mathrm{d}+1,1}$ with a tilde). Here, $s(\underline{x})$ is an arbitrary scale factor on the ray which will change under conformal transformations; since every bulk-to-boundary propagator is thought to be connected to a boundary operator with a certain scaling dimension, the appearance of the factor $s(\underline{x})$ has to be expected. The scale factor will play a role, however, when pushing forward the tangent vectors of the boundary into EAdS.

For scalar products of mixed boundary/bulk vectors $z \in$ EAdS, $\underline{x}, \underline{y} \in \mathbb{R}^{\mathrm{d}}$, we get

$$(\tilde{x}, \tilde{z}) = -\, s(\underline{x}) \frac{(z^0)^2 + (\underline{z} - \underline{x})^2}{2z^0}, \qquad (\tilde{x}, \tilde{y}) = -\, s(\underline{x}) s(\underline{y}) \frac{(\underline{x} - \underline{y})^2}{2}, \tag{3.4-71}$$

and obviously $(\tilde{x}, \tilde{x}) = 0$. On the boundary space, let $\underline{v} \in T_{\underline{x}}$ be a tangent vector at $\underline{x}$; it can be taken into EAdS be the relation

$$\underline{v} \cdot \frac{\partial}{\partial \underline{x}} = \underline{v} \cdot \frac{\partial \tilde{x}^{\tilde{\mu}}}{\partial \underline{x}} \frac{\partial}{\partial \tilde{x}^{\tilde{\mu}}} \stackrel{!}{=} \tilde{v}^{\tilde{\mu}} \frac{\partial}{\partial \tilde{x}^{\tilde{\mu}}}, \tag{3.4-72}$$

[8]The author is indebted to Prof. H. Gönner for pointing out this connection.

whence its components in the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$ are

$$\tilde{v}^{\tilde{\mu}} = s(\underline{x})\,(-\underline{v}\cdot\underline{x},\ \underline{v},\ \underline{v}\cdot\underline{x})^{\tilde{\mu}} + (\underline{v}\cdot\underline{\nabla}s(\underline{x}))\,\tilde{x}^{\tilde{\mu}}. \tag{3.4-73}$$

For $\underline{v}$, $\underline{w} \in T_{\underline{x}}$ tangent vectors at $\underline{x}$, we have

$$(\tilde{v},\tilde{x}) = 0, \qquad (\tilde{v},\tilde{w}) = s(\underline{x})^2\underline{v}\cdot\underline{w}. \tag{3.4-74}$$

This implies that vectors $\underline{v} \in T_{\underline{x}}$ are tangent to the light cone at $\tilde{x}$ in EAdS; from (3.4-73), one can see that boundary tangent vectors properly correspond to the equivalence class $\tilde{v} + \mathbb{R}\tilde{x}$ in the bulk.
The tangent vectors to EAdS space can be represented in a similar fashion: If $t^{\mu}, \mu = 0\ldots\mathrm{d}$, is a tangent vector to the point z living on the Poincaré patch, then we have

$$t^{\mu}\frac{\partial}{\partial z^{\mu}} = t^{\mu}\frac{\partial\tilde{z}^{\nu}}{\partial z^{\mu}}\frac{\partial}{\partial\tilde{z}^{\nu}} \overset{!}{=} \tilde{t}^{\nu}\frac{\partial}{\partial\tilde{z}^{\nu}}, \tag{3.4-75}$$

whence its components in the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$ are

$$\tilde{t}^{\nu} \equiv t^{\mu}\frac{\partial\tilde{z}^{\nu}}{\partial z^{\mu}} = \Big(-t^0\frac{1-\underline{z}^2+(z^0)^2}{2(z^0)^2} - \frac{\underline{t}\cdot\underline{z}}{z^0},\ \frac{\underline{t}z^0-\underline{z}t^0}{(z^0)^2},\ -t^0\frac{1+\underline{z}^2-(z^0)^2}{2(z^0)^2} + \frac{\underline{t}\cdot\underline{z}}{z^0}\Big)^{\nu},$$

with the usual EAdS scalar product

$$(\tilde{t},\tilde{t}) = t^{\mu}t_{\mu} = \frac{(t^0)^2+\underline{t}^2}{(z^0)^2}. \tag{3.4-76}$$

We list possible expressions which can be constructed from the scalar product: if $\underline{v} \in T_{\underline{x}}$, we can form

$$v^{l}{}_{l}(\tilde{x},\tilde{z}) \equiv \underline{v}\cdot\partial_{\underline{x}}(\tilde{x},\tilde{z}) = (\tilde{v},\tilde{z}) = s(\underline{x})\frac{\underline{v}\cdot(\underline{z}-\underline{x})}{z^0} - (\underline{v}\cdot\underline{\nabla}s(\underline{x}))\frac{(z^0)^2+(\underline{z}-\underline{x})^2}{2z^0}.$$

We indicate the derivative by simply appending the relevant index. The other derivative ${}_{\tilde{\mu}}(\tilde{z},\tilde{x}) = \tilde{x}_{\tilde{\mu}}$ is trivial. Two scalar products may be contracted by

$$(\tilde{x},\tilde{z})_{\tilde{\mu}}{}^{\tilde{\mu}}(\tilde{z},\tilde{y}) = (\tilde{x},\tilde{y}) = -s(\underline{x})s(\underline{y})\,\frac{(\underline{x}-\underline{y})^2}{2},$$

and thereby

$$(\tilde{y},\tilde{z})^{\tilde{\mu}}{}_{\tilde{\mu}}(\tilde{z},\tilde{v}) = (\tilde{y},\tilde{v}) = s(\underline{x})s(\underline{y})\ \underline{v}\cdot(\underline{y}-\underline{x}) - (\underline{v}\cdot\underline{\nabla}s(\underline{x}))s(\underline{y})\frac{(\underline{x}-\underline{y})^2}{2}$$

and, for $\underline{w} \in T_{\underline{y}}$,

$$\begin{aligned}(\tilde{w},\tilde{z})_{\tilde{\mu}}{}^{\tilde{\mu}}(\tilde{z},\tilde{v}) = (\tilde{w},\tilde{v}) &= s(\underline{y})s(\underline{x})\ \underline{w}\cdot\underline{v} + (\underline{v}\cdot\underline{\nabla}s(\underline{x}))s(\underline{y})\ \underline{w}\cdot(\underline{x}-\underline{y})\\ &\quad + s(\underline{x})(\underline{w}\cdot\underline{\nabla}s(\underline{y}))\ \underline{v}\cdot(\underline{y}-\underline{x}) - (\underline{v}\cdot\underline{\nabla}s(\underline{x}))(\underline{w}\cdot\underline{\nabla}s(\underline{y}))\frac{(\underline{x}-\underline{y})^2}{2}.\end{aligned}$$

We finally mention the boundary point $\underline{\infty}$ (the point needed for conformal compactification of the boundary of EAdS); in the embedding space, it can be represented by the ray in direction

$$\tilde{\infty}^{\tilde{\mu}} = \lim_{\underline{x}\to\infty} \frac{\tilde{x}^{\tilde{\mu}}}{\underline{x}^2} = \left(-\frac{1}{2},\ \underline{0},\ \frac{1}{2}\right)^{\tilde{\mu}}. \tag{3.4-77}$$

The expressions involving boundary points and their tangent vectors are in general *not* invariants on EAdS, since they do not have the required form; a conformal symmetry transformation will change the scale factor $s(\underline{x})$ locally, and therefore, the push-forward (3.4-73) of the boundary tangent vectors $\underline{v}$ will contain factors of the form $(\underline{v} \cdot \underline{\nabla} s(\underline{x}))\,\tilde{x}$. While the scale factor $s(\underline{x})$ whenever it appears can be attributed to a physical scale dependence of the underlying physical quantities (eg the boundary operators whose correlations we are computing), the derivative $\underline{\nabla} s(\underline{x})$ does not have such an interpretation. The logical consequence is that any expression linear in the boundary tangent vector $\tilde{v}$ should be invariant under the transformation $\tilde{v} \mapsto \tilde{v} + r \cdot \tilde{x}$, $r \in \mathbb{R}$. In other words, if $T_{\tilde{x}}$ is the tangent space at $\tilde{x}$, then there should be an equivalence relation $T_{\tilde{x}} \ni \tilde{x} \sim 0$ efficient. For generic functions $G(\tilde{v})$ containing several instances of $\tilde{v}$, conformal covariance means that they vanish under the application of the differential operator

$$\tilde{x}^{\tilde{\mu}} \partial_{\tilde{v}^{\tilde{\mu}}} G(\tilde{v}) = 0. \tag{3.4-78}$$

Note that this notion may be even transferred to points situated *within* EAdS: If $z \in \mathrm{EAdS}$, then the tangent vector $\tilde{z} \in T_{\tilde{z}}$ points in a direction orthogonal to EAdS should not have a physical meaning, and therefore, covariant expressions should not depend on this component of the tangent vectors.

For $\tilde{v} \in T_{\tilde{x}}$ and $\tilde{a} \in T_{\tilde{z}}$, a typical invariant expression is

$$(\tilde{v}, \tilde{a}) - \frac{(\tilde{v}, \tilde{z})(\tilde{x}, \tilde{a})}{(\tilde{x}, \tilde{z})}. \tag{3.4-79}$$

For $\tilde{v} \sim \tilde{x}$ or $\tilde{a} \sim \tilde{z}$, this will vanish. Or, for $\underline{x}$ a boundary point and $\tilde{z}, \tilde{u} \in \mathbb{R}^{\mathrm{d}+1}$,

$$\frac{(\tilde{v}, \tilde{u})}{(\tilde{x}, \tilde{u})} - \frac{(\tilde{v}, \tilde{z})}{(\tilde{x}, \tilde{z})} \tag{3.4-80}$$

fulfills the same purpose. For *one* additional point (without a tangent vector) besides $\underline{v} \in T_{\underline{x}}$, the construction of such invariants is not possible.

We will use the scalar product as the basic building block for the EAdS-presentation of the correlations. While in the following, we will set $s(\underline{x}) = 1$ and ignore the scale factor, we have to take care that all expression which we obtain ultimately are formed of invariants like (3.4-79), independent of derivatives $\underline{\nabla} s(\underline{x})$ of the boundary scale factor $s(\underline{x})$.

3.4.2 Formally Non-covariant Generating Function Approach for 3-Point Correlations

We proceed now to show that a simple generating-function argument based on Schwinger parametrisation – although it looks intriguingly natural – does not lead to covariant results. In this sense, this section is a dead end and not a prerequisite for the more successful truely covariant approach which we will pursue in the sequel; however, it illustrates nicely the problems which we have to expect.

In coordinate space, the generating function for the correlation function of three currents is (including the combinatorial factor $\frac{1}{2} \cdot 2^3$; for charged fields, this factor does not arise, instead we have to sum over the two different orderings 1-2-3 and 1-3-2) given by a slight modification of the scalar three-point function (2.7-43),

$$\mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{w}_1, \underline{w}_2, \underline{w}_3] \qquad (3.4\text{-}81)$$
$$= \frac{N}{2} \left(\frac{\Gamma(\frac{\mathrm{d}}{2} - 1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}} \right)^3 \frac{1}{|\underline{x}_1 - \underline{x}_2 - \underline{w}_3|^{\mathrm{d}-2} |\underline{x}_2 - \underline{x}_3 - \underline{w}_1|^{\mathrm{d}-2} |\underline{x}_3 - \underline{x}_1 - \underline{w}_2|^{\mathrm{d}-2}},$$

where we had to insert $\mathcal{N}^3$ to take into account the general normalisation (2.6-30) of ϕ. The currents are generated by letting the derivatives in (3.4-66) act on the vector indices $\underline{w}_j$ and setting $\underline{w}_j \equiv 0$ ultimately; we indicate the "generating arguments" by square brackets. For example, a tensor current of spin s at $\underline{x}_1$ is generated by acting with

$$\mathcal{J}^s[\partial_{\underline{w}_3}, \partial_{\underline{w}_2}] = \sum_{k=0}^{s} a_k^s \left(-\frac{\partial}{\partial \underline{w}_3} \right)^{\otimes k} \left(\frac{\partial}{\partial \underline{w}_2} \right)^{\otimes s-k} - \text{ traces.}$$

The sign factor on the $\underline{w}_3$-derivative had to be included since the $\underline{w}_j$ are directed variables, "pointing" clockwise around the loop.

In [53, below 6-38] (for a brief summary see appendix D), I show that in the wave number domain, the generating function can be displayed as

$$\mathcal{G}(\underline{k}_1, \underline{k}_2, \underline{k}_3)[\underline{w}_1, \underline{w}_2, \underline{w}_3] = N4(2\pi)^{\mathrm{d}} \delta^{(\mathrm{d})} \Big(\sum_j \underline{k}_j \Big) \left(\frac{\mathcal{N}}{(2\pi)^{\frac{\mathrm{d}}{2}}} \right)^3 \int_0^\infty \mathrm{d}^3\tau \left(\frac{1}{4\pi T} \right)^{\frac{\mathrm{d}}{2}}$$
$$\exp \left(\underline{k}_1 \cdot \underline{k}_2 \frac{(\tau_1 + \tau_2)\tau_3}{T} + \text{cycl. perm.} \right)$$
$$\exp -\frac{i}{T} \left(\underline{w}_3 \cdot (\underline{k}_1 \tau_2 - \underline{k}_2 \tau_1) + \text{cycl. perm.} \right) - \frac{1}{4T} \Big(\sum_{j=1}^{3} \underline{w}_j \Big)^2, \qquad (3.4\text{-}82)$$

where the positive scalars τ_j are Schwinger parameters ("moduli"), and total loop modulus is $T = \tau_1 + \tau_2 + \tau_3$. Now, $\underline{w}_j$ generates a momentum running clockwise around the loop, so we have to include factors of -1 (see figure 3.2 for the conventions). Note that there are regeularisation issues for currents with high spin; our derivation will be

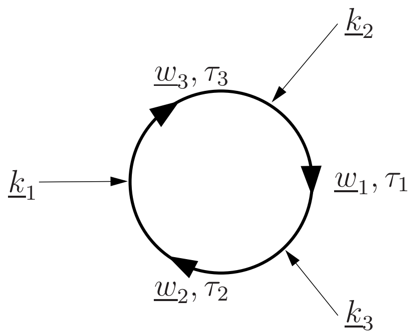

Figure 3.2: Feynman diagram for a massless field ϕ coupling three tensor currents.

formal (the regularisation scheme which is best suited to our purpose is dimensional regularisation, as we are dealing anyhow with arbitrary $2 < \mathrm{d} < 4$). The generating function can be derived by parametrising the propagators running around the loop as

$$\frac{1}{k^2} = \int_0^\infty \mathrm{d}\tau \, e^{-\tau k^2} \tag{3.4-83}$$

and integrating out the loop momentum (see appendix D for few details).

The formal variables $\underline{w}_j$ are not very much suited for a generating function, as eg $\underline{w}_3$ is acted on by the derivative operators generating the currents $\underline{k}_1$ and $\underline{k}_2$ at the same time. This is easily remedied by going over to a different parametrisation of the generating function. We simply split up $\underline{w}_j$ according to

$$\underline{w}_3 = -\underline{v}_1 + \underline{v}_2', \qquad \underline{w}_1 = -\underline{v}_2 + \underline{v}_3', \qquad \underline{w}_2 = -\underline{v}_3 + \underline{v}_1'.$$

and let $\underline{v}_j^{(\prime)}$ denote the generating variables of the current j. These have to be substituted in to the generating function. The variables $\underline{v}_j^{(\prime)}$ are formally tangent vectors at $\underline{x}_j$. The current $\mathcal{J}^s$ coupling to $\underline{k}_1$ is generated by

$$\mathcal{J}^s[\partial_{\underline{v}_1}, \partial_{\underline{v}_1'}] = \sum_{k=0}^{s} a_k^s \, \left(\partial_{\underline{v}_1}\right)^{\otimes k} \, \left(\partial_{\underline{v}_1'}\right)^{\otimes s-k} \; - \; \text{traces}, \tag{3.4-84}$$

and similarly for the other currents.

We again go over to coordinate space; this is done by Fourier transform with $(2\pi)^{-\frac{\mathrm{d}}{2}} \int \mathrm{d}^{\mathrm{d}} k_j \, e^{-i\underline{k}_j \cdot (\underline{z} - \underline{x}_j)}$. Since we have in the meantime integrated out the loop momentum, we obtain a result differing from (3.4-81). The momentum conserving delta distribution is taken care of by the newly introduced coordinate $\underline{z}$; we are performing a triangle-star diagram transform. The overall result is then

$$\mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{v}_{(j)}, \underline{v}_{(j)}'] = N 4 (4\pi)^{-2\mathrm{d}} \mathcal{N}^3 \int \mathrm{d}^{\mathrm{d}} z \int_0^\infty \mathrm{d}^3 \tau \left(\frac{T}{\tau_1 \tau_2 \tau_3} \right)^{\mathrm{d}} \tag{3.4-85}$$
$$\exp\left[-T \frac{(\underline{z} - \underline{x}_1)^2}{4 \tau_2 \tau_3} + \frac{\underline{v}_1 \cdot (\underline{x}_2 - \underline{x}_1 + \underline{v}_2')}{2\tau_3} + \frac{\underline{v}_1' \cdot (\underline{x}_3 - \underline{x}_1)}{2\tau_2} - \frac{\underline{v}_2^2 + \underline{v}_3'^2}{4\tau_1} + \text{cycl. perm.} \right].$$

We substitute the Schwinger parameters τ_j by new parameters α_j according to

$$\tau_j = \frac{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3}{\alpha_j},$$

with inverse

$$\alpha_j = \frac{\tau_1\tau_2\tau_3}{\tau_j(\tau_1+\tau_2+\tau_3)}.$$

For the Jacobian of the transformation we get

$$\mathrm{d}^3\tau = \frac{(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)^3}{(\alpha_1\alpha_2\alpha_3)^2}\,\mathrm{d}^3\alpha = \alpha_1\alpha_2\alpha_3 A^3\,\mathrm{d}^3\alpha\,,$$

with the abbreviation

$$A = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} = \frac{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3}. \tag{3.4-86}$$

We obtain

$$\begin{aligned}\mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{v}_{(j)}, \underline{v}'_{(j)}] &= N4(4\pi)^{-2\mathrm{d}}\mathcal{N}^3 \int \mathrm{d}^{\mathrm{d}}z \int_0^\infty \mathrm{d}^3\alpha\,(\alpha_1\alpha_2\alpha_3)^{1-\mathrm{d}} A^{3-\mathrm{d}} \\ \exp\left[-\frac{(\underline{z}-\underline{x}_1)^2}{4\alpha_1} + \frac{\underline{v}_1\cdot(\underline{x}_2-\underline{x}_1+\underline{v}'_2)}{2A\alpha_1\alpha_2} + \frac{\underline{v}'_1\cdot(\underline{x}_3-\underline{x}_1)}{2A\alpha_1\alpha_3} - \frac{\underline{v}_2^2+\underline{v}_3'^2}{4A\alpha_2\alpha_3} + \text{cycl. perm.}\right]&.\end{aligned} \tag{3.4-87}$$

Since we will have to subtract the traces anyhow in order to generate the currents, the $\underline{v}_j^2$ and $\underline{v}_j'^2$-terms are of no consequence.

A form of this expression which comes very close to a proper EAdS-vertex structure can be obtained by using the Laplace type transform

$$A^{3-\mathrm{d}} e^{C/2A} = \frac{2^{7-2\mathrm{d}}}{\Gamma(\mathrm{d}-3)} \int_0^\infty \frac{\mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} (z^0)^{3\mathrm{d}-6} e^{-\frac{A(z^0)^2}{4}}\,{}_0\mathrm{F}_1\left(\mathrm{d}-3; (z^0)^2\frac{C}{8}\right),$$

valid in $\mathrm{d} > 3$. Once we have used this representation, we also rescale all the Schwinger parameters $\alpha_j \mapsto z^0\alpha_j/2$. This yields

$$\begin{aligned}\mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{v}_{(j)}, \underline{v}'_{(j)}] &= N\frac{2^{3-3\mathrm{d}}\mathcal{N}^3}{\pi^{2\mathrm{d}}\Gamma(\mathrm{d}-3)} \int \frac{\mathrm{d}^{\mathrm{d}}z\;\mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} \\ &\int_0^\infty \mathrm{d}^3\alpha\,(\alpha_1\alpha_2\alpha_3)^{1-\mathrm{d}}\,{}_0\mathrm{F}_1\left(\mathrm{d}-3; \frac{C}{2}\right) \exp\left(-\sum_j \frac{(\underline{z}-\underline{x}_j)^2 + (z^0)^2}{2z^0\alpha_j}\right),\end{aligned}$$

with

$$C = \frac{\underline{v}_1\cdot(\underline{x}_2-\underline{x}_1+\underline{v}'_2)}{\alpha_1\alpha_2} + \frac{\underline{v}'_1\cdot(\underline{x}_3-\underline{x}_1)}{\alpha_1\alpha_3} - \frac{\underline{v}_2^2+\underline{v}_3'^2}{2\alpha_2\alpha_3} + \text{cycl. perm.}$$

independent of $\underline{z}$ and z^0. The integrand has an expansion as Taylor series in C,

$$ {}_0\mathrm{F}_1\left(\mathrm{d}-3; \frac{C}{2}\right) = \sum_{s=0}^{\infty} \frac{1}{2^s s! (\mathrm{d}-3)_s} C^s, \tag{3.4-88} $$

starting at C^0. The measure $\mathrm{d}^{\mathrm{d}}z\ \mathrm{d}z^0\ (z^0)^{-\mathrm{d}-1}$ is the EAdS volume element, and we see that the exponent displays the structure of an invariant EAdS-distance. This looks already like a satisfactory result which has been reached solely by the application of Schwinger parametrisation. We now examine whether this formula is conformally covariant (on the formal level, ie *before* integration of z).

Failure of Formal Conformal Covariance. Utilising the scalar product available in the embedding space which has been introduced in section 3.4.1, the generating function is rewritten in the form

$$ \begin{aligned} \mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{v}_{(j)}, \underline{v}'_{(j)}] = N \frac{2^{3-3\mathrm{d}} \mathcal{N}^3}{\pi^{2\mathrm{d}} \Gamma(\mathrm{d}-3)} \int \frac{\mathrm{d}^{\mathrm{d}}z\ \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} \\ \int_0^{\infty} \mathrm{d}^3\alpha\ (\alpha_1\alpha_2\alpha_3)^{1-\mathrm{d}}\ {}_0\mathrm{F}_1\left(\mathrm{d}-3; \frac{C}{2}\right) \exp\left(\sum_j \frac{(\tilde{z}, \tilde{x}_j)}{\alpha_j}\right), \end{aligned} \tag{3.4-89} $$

with the piece C in the pseudo-invariant form (valid for $s(\underline{x}) \equiv 1$)

$$ C \;=\; \frac{(\tilde{v}_1, \tilde{x}_2) + (\tilde{v}_1, \tilde{v}'_2)}{\alpha_1\alpha_2} + \frac{(\tilde{v}'_1, \tilde{x}_3)}{\alpha_1\alpha_3} + \text{cycl. perm.} + \text{quadratic terms in } \underline{v}_j, \underline{v}'_j. \tag{3.4-90} $$

We have left out the quadratic terms $\underline{v}_j^2$, $\underline{v}_j'^2$, because the will get subtracted anyhow when the operators $\mathcal{J}^s[\partial_{\underline{v}}, \partial_{\underline{v}'}]$ in the form (3.4-84) generating the currents are applied, since these include subtraction of traces.

The examination of C reveals that a boundary point $\tilde{x}_j$ or vector $\tilde{v}_j$ of leg j is always accompanied by a factor α_j^{-1} (except in the left-out quadratic terms). The immediate integration of α_j is cumbersome, because C appears as argument of the hypergeometric series and we would have to expand the powers of C. By a generating function argument, we can ban all factors α_j^{-1} into an exponential function; C is then generated by acting with the differential operator

$$ C = (\partial_{\tilde{a}_1}, \partial_{\tilde{z}_2}) + (\partial_{\tilde{a}_1}, \partial_{\tilde{a}'_2}) + (\partial_{\tilde{a}'_1}, \partial_{\tilde{z}_3}) + \text{cycl. perm.} \tag{3.4-91} $$

on the generating exponentials

$$ \prod_{j=1}^{3} \exp \frac{(\tilde{z}_j, \tilde{x}_j) + (\tilde{a}_j, \tilde{v}_j) + (\tilde{a}'_j, \tilde{v}'_j)}{\alpha_j} $$

and setting afterwards $\tilde{z}_j = \tilde{z}$, $\tilde{a}_j = \tilde{a}'_j = 0$. The exponential terms left over are precisely the exponentials necessary in the generating function (3.4-89),

$$\mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{v}_{(j)}, \underline{v}'_{(j)}] = N \frac{2^{3-3\mathrm{d}}\mathcal{N}^3}{\pi^{2\mathrm{d}}\Gamma(\mathrm{d}-3)} \int \frac{\mathrm{d}^{\mathrm{d}}z\, \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}}$$
$$\int_0^\infty \mathrm{d}^3\alpha\, (\alpha_1\alpha_2\alpha_3)^{1-\mathrm{d}}\ {}_0\mathrm{F}_1\left(\mathrm{d}-3; \frac{C}{2}\right) \prod_{j=1}^{3} \exp \frac{(\tilde{z}_j, \tilde{x}_j) + (\tilde{a}_j, \tilde{v}_j) + (\tilde{a}'_j, \tilde{v}'_j)}{\alpha_j}\Bigg|_{\substack{\tilde{z}_j = \tilde{z} \\ \tilde{a}_j = \tilde{a}'_j = 0}}.$$

Now we may integrate out the Schwinger parameters α_j. The resulting powers would have to be identified as the propagators. Using the expansion (3.4-88), the desired correlations can be obtained by differentiating with respect to $\underline{v}_j$ and $\underline{v}'_j$, and selecting and applying the relevant differential operators out of the expansion in C.

If we take $s(\underline{x}) \neq 1$, the conformal invariance of this term is (formally) lost; by this we mean that the amplitudes generated by application of the differential operators $\mathcal{J}^s[\partial_{\underline{v}}, \partial_{\underline{v}'}]$ are invariant, but only *after* integration of z. We will examine two simple cases. For $s = (0,0,0)$, we have to take into account only the C^0-term, since there are no external indices on the relevant generating operator

$$\mathcal{J}^0[\partial_{\underline{v}}, \partial_{\underline{v}'}] = \frac{1}{2}.$$

Integration of the Schwinger parameters results in

$$G^{0,0,0}(\underline{x}_1, \underline{x}_2, \underline{x}_3) = N 2^{-3\mathrm{d}}\pi^{-2\mathrm{d}}(\mathrm{d}-3)\Gamma(\mathrm{d}-2)^2\mathcal{N}^3 \int \frac{\mathrm{d}^{\mathrm{d}}z\, \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} \prod_j (-\tilde{z}, \tilde{x}_j)^{2-\mathrm{d}}.$$

It is consequent to identify the factors $(-\tilde{z}, \tilde{x}_j)^{2-\mathrm{d}}$ with the boundary-to-bulk propagators for the scalar fields; including the scale factors, we have simply

$$G^{\mathrm{UV}\,0}_{\mathrm{bubo}}(\tilde{z}, \underline{x}) \sim s(\underline{x})^{2-\mathrm{d}}(-\tilde{z}, \tilde{x}_j)^{2-\mathrm{d}}, \tag{3.4-92}$$

where the boundary operator $\mathcal{J}^0(\underline{x}) = \frac{1}{2} :\phi(\underline{x})^2:$ has scaling dimension $\Delta_0^{\mathrm{UV}} = \mathrm{d}-2$, implying the prefactor $s(\underline{x})^{-\Delta}$ (we do not worry about prescriptions or normalisations). Since there are no tangent vectors involved, conformal covariance is guaranteed as it stands.

The (unsymmetrised!) vector-scalar-scalar case $s = 1, 0, 0$ with a charged ϕ uses the generating operator

$$\mathcal{J}^1[\partial_{\underline{v}}, \partial_{\underline{v}'}] = \frac{1}{2}\partial_{\underline{v}'} - \frac{1}{2}\partial_{\underline{v}},$$

resulting in

$${}_{\text{non-sym}}G^{1,0,0}_l(\underline{x}_1, \underline{x}_2, \underline{x}_3) = -N 2^{-3-3\mathrm{d}}\pi^{-2\mathrm{d}}\Gamma(\mathrm{d}-1)\Gamma(\mathrm{d}-2)\mathcal{N}^3 \int \frac{\mathrm{d}^{\mathrm{d}}z\, \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}}$$
$$(-\tilde{z}, \tilde{x}_1)^{1-\mathrm{d}\,\tilde{\mu}}(\tilde{z}, \tilde{x}_1)_l \left(\partial_{\tilde{z}_2^{\tilde{\mu}}} - \partial_{\tilde{z}_3^{\tilde{\mu}}}\right)(-\tilde{z}_2, \tilde{x}_2)^{2-\mathrm{d}}(-\tilde{z}_3, \tilde{x}_3)^{2-\mathrm{d}}\Big|_{\tilde{z}_2 = \tilde{z}_3 = \tilde{z}}.$$

(mind that this vanishes upon symmetrisation). If we interpret, in parallel to the scalar case, the term $(-\tilde{z}, \tilde{x}_1)^{1-\mathrm{d}}\,{}^{\tilde{\mu}}(\tilde{z}, \tilde{x}_1)_l$ as propagator for the vector field and $\left(\partial_{\tilde{z}_2^{\tilde{\mu}}} - \partial_{\tilde{z}_3^{\tilde{\mu}}}\right)$ as the (unsymmetrised) EAdS-vertex, then we find an unpleasant surprise: Including all scale factors, the propagator has a behaviour

$$G^{\mathrm{UV}\;1\;\tilde{\mu}}_{\mathrm{bubo}\;l}(\tilde{z}, \underline{x}) \sim s(\underline{x})^{1-\mathrm{d}}(-\tilde{z}, \tilde{x})^{1-\mathrm{d}} \left({}^{\tilde{\mu}}(\tilde{z}, \tilde{x})_l + \tilde{x}^{\tilde{\mu}} \nabla_l s(\underline{x})\right). \tag{3.4-93}$$

The gradient of the scale factor does not vanish from the propagator; and since the contraction of the gradient term with the vertex and the other propagators yields a term proportional to

$$\left(\frac{(\tilde{x}_1, \tilde{x}_2)}{(\tilde{z}, \tilde{x}_2)} - \frac{(\tilde{x}_1, \tilde{x}_3)}{(\tilde{z}, \tilde{x}_3)}\right) \nabla_l s(\underline{x})$$

which does not vanish, formally conformal covariance is lost for good.

We also want to mention that for higher spins, we would meet other strange effects: A boundary field of spin s couples to all bulk representations with spins $\tilde{s} \leq s$, and $s - \tilde{s}$ even. In addition, the bulk tensor representations involve the "odd" tangential direction $\tilde{z} \in T_{\tilde{z}}$ in an essential manner (this can be checked already for the spin 1 case displayed). This is in grotesque disagreement with the group theoretical foundations which have been summarised in section 3.3.1.

We come to the following conclusion: The conformal covariance is broken on the formal level because already the z-integration is not formally conformally covariant. Going back, this can already be seen directly from formula (3.4-85) which is still purely on the boundary, by writing it in the embedding space notation. We take $\tilde{z}$ to be the embedding space point corresponding to the boundary point $\underline{z}$. Then the generating formula (3.4-85) can be written

$$\begin{aligned}
\mathcal{G}(\underline{x}_1, \underline{x}_2, \underline{x}_3)[\underline{v}_{(j)}, \underline{v}'_{(j)}] &= N 4 (4\pi)^{-2\mathrm{d}} \mathcal{N}^3 \int \mathrm{d}^{\mathrm{d}} z \int_0^\infty \mathrm{d}^3\tau \left(\frac{T}{\tau_1\tau_2\tau_3}\right)^{\mathrm{d}} \\
\exp\left[T \frac{(\tilde{z}, \tilde{x}_1)}{2\tau_2\tau_3} + \frac{(\tilde{v}_1, \tilde{x}_2) + (\tilde{v}_1, \tilde{v}'_2)}{2\tau_3} + \frac{(\tilde{v}'_1, \tilde{x}_3)}{2\tau_2} + \text{quadratic terms} + \text{cycl. perm.}\right].
\end{aligned}$$

One easily checks on simple examples that this does not have the required invariance if gradient terms $(\underline{v} \cdot \underline{\nabla} s(\underline{x}))\,\tilde{x}$ are added to $\tilde{v}$ resp. $\tilde{v}'$ (to be precise, equation (3.4-78) does not hold).

The lesson we have learned is that we have to insist on proper conformal covariance right from the moment when we introduce the horizontal vertex integration $\int \mathrm{d}^{\mathrm{d}} z$. It is based on a subtle interplay between the different summands contributing to the correlation functions of the tensor currents. The simple tool of Schwinger parametrisation is effectively a shorthand for the numerical prefactors coming along with these summands, and while it may be that ultimately, some method is found to generate a formally conformally covariant EAdS-presentation of the correlations by using integral representations of these prefactors, ordinary Schwinger parametrisation does not seem to do the job.

3.4.3 Formally Covariant Correlations on the Boundary

Since we have found in the preceding section that formal conformal covariance of the EAdS-presentation of the three-point function (ie conformal covariance without integrating out the EAdS vertex coordinate z in (3.4-67)) is not easy to achieve and it does not do to follow blindly those manipulations of the correlations which seem to offer themselves for the purpose, we need to compute explicitly – by force – directly and step-by-step the EAdS-presentation.

To repeat, the notion of formal conformal covariance means that if the three-point function is written in an "embedding space" notation, then for any point $\tilde{x} \in \mathbb{R}^{\mathrm{d}+1,1}$ on the lightlike rays $(\tilde{x}, \tilde{x}) = 0$ characterising the conformal boundary, the tangent space vectors $\tilde{v} \in T_{\tilde{x}}$ are in equivalence classes $\tilde{v} + \mathbb{R}\tilde{x}$, and the EAdS-presentation (3.4-67) respects these classes on a formal level, ie without the vertex integration $\mathrm{d}z$ performed. The necessary criterion for this is

$$\tilde{x}^{\tilde{\mu}} \partial_{\tilde{v}^{\tilde{\mu}}} G(\tilde{v}) = 0, \tag{3.4-94}$$

with $G(\tilde{v})$ the correlation. For the EAdS-presented amplitude involving points $\tilde{z} \in \mathbb{R}^{\mathrm{d}+1,1}$ lying on EAdS, the tangent vectors $\tilde{v} \in T_{\tilde{z}}$ should be taken from equivalence classes $\tilde{v} + \mathbb{R}\tilde{z}$.

As a preliminary step one might ask for a form of the three-point correlations purely on the boundary which displays explicitly the conformal covariance. We know that they must be covariant since this was one of their defining conditions; however, this is difficult to see when looking at their definition (2.6-25). In this section, we will obtain a form of the correlations which displays their covariance immediately.

The three-point function of the currents can be generated by application of the derivatives $\tilde{v}_j^{\tilde{\mu}} \partial_{\tilde{x}_j^{\tilde{\mu}}}$ to the propagators of the scalar three-point function, written covariantly as

$$\begin{aligned}
&\langle :\varphi(\underline{x}_1)^2::\varphi(\underline{x}_2)^2::\varphi(\underline{x}_3)^2: \rangle \\
&\qquad = 4N \left(\frac{\Gamma(\frac{\mathrm{d}}{2} - 1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}} \right)^3 (-2)^{3-\frac{3\mathrm{d}}{2}} \, (\tilde{x}_1, \tilde{x}_2')^{1-\frac{\mathrm{d}}{2}} (\tilde{x}_2, \tilde{x}_3')^{1-\frac{\mathrm{d}}{2}} (\tilde{x}_3, \tilde{x}_1')^{1-\frac{\mathrm{d}}{2}} \Big|_{\tilde{x}_j' = \tilde{x}_j} .
\end{aligned}$$

The current at $\underline{x}_j$ is generated by inserting

$$\mathcal{J}^s[\underline{v}_j] = \sum_{k=0}^{s} a_k^s \left(\tilde{v}_j^{\tilde{\mu}} \partial_{\tilde{x}_j^{\tilde{\mu}}} \right)^k \left(\tilde{v}_j^{\tilde{\nu}} \partial_{\tilde{x}_j'^{\tilde{\nu}}} \right)^{s-k} - \text{ traces}, \tag{3.4-95}$$

cf. (2.6-25) for the original definition of the currents. This may be checked by the rules of section 3.4.1, in particular (3.4-71). Note that the quadratic terms $(\tilde{v}_j, \tilde{v}_j)$ are automatically absent, since we use the partial derivative with respect to $\tilde{x}_j$ (and not $\underline{x}_j$ or similar).

One easily checks that the derivatives $\tilde{v}_j^{\tilde{\mu}} \partial_{\tilde{x}_j^{\tilde{\mu}}}$ indeed have all the necessary properties of derivations (chain rule, ...), and commute. In order to compute their action on the propagators explicitly, define the following expressions for $\underline{v} \in T_{\underline{x}}$ a tangent vector at $\underline{x}$:

$$\begin{aligned} I_{12} =& -\frac{1}{(\tilde{x}_1, \tilde{x}_2)} & K_{12} =& (\tilde{v}_1, \tilde{v}_2) - \frac{(\tilde{v}_1, \tilde{x}_2)(\tilde{x}_1, \tilde{v}_2)}{(\tilde{x}_1, \tilde{x}_2)} \\ J_{12} =& \frac{(\tilde{v}_1, \tilde{x}_2)}{(\tilde{x}_1, \tilde{x}_2)} & J_{21} =& \frac{(\tilde{v}_2, \tilde{x}_1)}{(\tilde{x}_1, \tilde{x}_2)}. \end{aligned} \tag{3.4-96}$$

I_{12} and K_{12} are symmetric in 1,2, whereas J_{12} and J_{21} are not. While I_{12} and K_{12} are already respecting the equivalence classes $\tilde{v} + \mathbb{R}\tilde{x}$, J_{12} and J_{21} are not. They fulfill the following simple set of differentiation rules:

$$\begin{aligned} \tilde{v}_1^{\tilde{\mu}} \partial_{\tilde{x}_1^{\tilde{\mu}}} I_{12} =& - I_{12} J_{12} & \tilde{v}_2^{\tilde{\mu}} \partial_{\tilde{x}_2^{\tilde{\mu}}} I_{12} =& - I_{12} J_{21} \\ \tilde{v}_1^{\tilde{\mu}} \partial_{\tilde{x}_1^{\tilde{\mu}}} J_{12} =& - J_{12}^2 & \tilde{v}_2^{\tilde{\mu}} \partial_{\tilde{x}_2^{\tilde{\mu}}} J_{12} =& - I_{12} K_{12} \\ \tilde{v}_1^{\tilde{\mu}} \partial_{\tilde{x}_1^{\tilde{\mu}}} K_{12} =& - J_{12} K_{12} = \tilde{v}_2^{\tilde{\mu}} \partial_{\tilde{x}_2^{\tilde{\mu}}} K_{12}. \end{aligned} \tag{3.4-97}$$

To have better control over the algebraic behaviour governing these derivatives, we write them in terms of two auxiliary variables y_{12} and y_{21} by setting

$$\tilde{v}_1^{\tilde{\mu}} \partial_{\tilde{x}_1^{\tilde{\mu}}} \longleftrightarrow \partial_{y_{21}} \qquad \tilde{v}_2^{\tilde{\mu}} \partial_{\tilde{x}_2^{\tilde{\mu}}} \longleftrightarrow \partial_{y_{12}}. \tag{3.4-98}$$

Assuming that $I_{12} \equiv I_{12}(y_{12}, y_{21})$ and similarly $J_{12/21}$ and K_{12}, the relations (3.4-97) are written

$$\begin{aligned} \partial_{y_{21}} I_{12} =& - I_{12} J_{12} & \partial_{y_{12}} I_{12} =& - I_{12} J_{21} \\ \partial_{y_{21}} J_{12} =& - J_{12}^2 & \partial_{y_{12}} J_{12} =& - I_{12} K_{12} \\ \partial_{y_{21}} K_{12} =& - J_{12} K_{12} = \partial_{y_{12}} K_{12}. \end{aligned} \tag{3.4-99}$$

This nonlinear system of differential equations has the special solution (obeying symmetry under exchange of indices 1,2)

$$I_{12} = -\frac{1}{Y_{12}} \qquad J_{12} = \frac{y_{12}}{Y_{12}} \qquad J_{21} = \frac{y_{21}}{Y_{12}} \qquad K_{12} = \frac{C_{12}^K}{Y_{12}}, \tag{3.4-100}$$

where

$$Y_{12} = y_{21} y_{12} + C_{12}^K, \tag{3.4-101}$$

and C_{12}^K a constant. Since every differentiation increases the number of symbols $I/J/K$ by 1, multiple derivatives act as

$$(\partial_{y_{21}})^{n_1} (\partial_{y_{12}})^{n_2} I_{12}^n = \frac{\mathrm{Poly}_{n_1+n_2}(y_{21}, y_{12})}{Y_{12}^{n_1+n_2+n}},$$

the polynomial $\mathrm{Poly}_{n_1+n_2}(y_{21}, y_{12})$ (of maximal order $n_1 + n_2$) having C_{12}^K as counting parameter for K_{12}.

As mentioned before, I_{12} and K_{12} are invariants in the classes $\tilde{v}_j \mapsto \tilde{v}_j + \tilde{x}_j$. However, the J's are not. When multiple points $\tilde{x}_j$ are invoked and we define the corresponding symbols J_{ij}, one sees that certain linear combinations of the J_{ij}'s referring to different points are invariant: For $\tilde{x}_3$ anywhere in EAdS or on the conformal boundary, the difference

$$J_{12} - J_{13} = \frac{(\tilde{v}_1, \tilde{x}_2)}{(\tilde{x}_1, \tilde{x}_2)} - \frac{(\tilde{v}_1, \tilde{x}_3)}{(\tilde{x}_1, \tilde{x}_3)} \tag{3.4-102}$$

is invariant under $\tilde{v}_1 \mapsto \tilde{v}_1 + \tilde{x}_1$. So the correlations are covariant if we can give them as polynomials in the I's, K's, and differences of the J's! As an aside, for the two-point function which can be analysed similarly, no third point can be invoked, and therefore we conclude that it must be a function of K_{12} and I_{12} alone. Details will be given in section 3.4.4.

The scalar correlation including all normalisations is in terms of the symbols I_{ij}

$$G^{0,0,0}(\underline{x}_1, \underline{x}_2, \underline{x}_3) = \frac{N}{2} \left(\frac{\Gamma(\frac{\mathrm{d}}{2} - 1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}} \right)^3 2^{3 - \frac{3\mathrm{d}}{2}} \left(I_{12} I_{23} I_{31}\right)^{\frac{\mathrm{d}}{2} - 1} \tag{3.4-103}$$

(we see here that due to the inclusion of "$-$" in the definition of I_{ij}, it is positive and the powers are convenient to handle even for non-integer dimensions). Combining this with the current generators (3.4-95) higher-spin tensor correlations are given by

$$\begin{aligned} G^{s_1,s_2,s_3}(\underline{x}_1, \underline{x}_2, \underline{x}_3) =& 4N \left(\frac{\Gamma(\frac{\mathrm{d}}{2} - 1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}} \right)^3 2^{3 - \frac{3\mathrm{d}}{2}} \sum_{k_1=0}^{s_1} a^{s_1}_{k_1} \sum_{k_2=0}^{s_2} a^{s_2}_{k_2} \sum_{k_3=0}^{s_3} a^{s_3}_{k_3} . \\ & \left[(\tilde{v}_2^{\tilde{\mu}} \partial_{\tilde{x}_2^{\tilde{\mu}}})^{s_2 - k_2} (\tilde{v}_1^{\tilde{\mu}} \partial_{\tilde{x}_1^{\tilde{\mu}}})^{k_1} I_{12}^{\frac{\mathrm{d}}{2} - 1} \right] \left[(\tilde{v}_3^{\tilde{\mu}} \partial_{\tilde{x}_3^{\tilde{\mu}}})^{s_3 - k_3} (\tilde{v}_2^{\tilde{\mu}} \partial_{\tilde{x}_2^{\tilde{\mu}}})^{k_2} I_{23}^{\frac{\mathrm{d}}{2} - 1} \right] \\ & \left[(\tilde{v}_1^{\tilde{\mu}} \partial_{\tilde{x}_1^{\tilde{\mu}}})^{s_1 - k_1} (\tilde{v}_3^{\tilde{\mu}} \partial_{\tilde{x}_3^{\tilde{\mu}}})^{k_3} I_{31}^{\frac{\mathrm{d}}{2} - 1} \right] - \text{traces}. \end{aligned} \tag{3.4-104}$$

We will employ the y_{12}/y_{21} differential calculus to compute these efficiently; as a preview of the general results of this section, we will find for the vector-scalar-scalar correlation (non-symmetrised)

$$G^{1,0,0}_{\text{non-symm}}(\underline{x}_1, \underline{x}_2, \underline{x}_3) \sim (J_{12} - J_{13}) \left(I_{12} I_{23} I_{31}\right)^{\frac{\mathrm{d}}{2} - 1}, \tag{3.4-105}$$

which is clearly confomally covariant. For the tensor-scalar-scalar correlation of a spin s-tensor, we will find

$$G^{s,0,0}_{\text{non-symm}}(\underline{x}_1, \underline{x}_2, \underline{x}_3) \sim (J_{12} - J_{13})^s \left(I_{12} I_{23} I_{31}\right)^{\frac{\mathrm{d}}{2} - 1} - \text{traces}$$

which is similarly conformally covariant (and will vanish for odd spin s upon symmetrisation).

In order to prove these results, we have to compute the action of the derivatives in the current operator $\mathcal{J}^s$ as given in (3.4-104). Before we may consider the sum over differentiation schemes, we have to compute the action of these differentiations on the propagators.

For the propagator between current 1 and 2, the relevant derivatives are (written in the y_{ij} differentiation scheme)

$$
\begin{aligned}
&(\partial_{y_{12}})^{s_2-k_2}(\partial_{y_{21}})^{k_1} I_{12}^{\frac{\mathrm{d}}{2}-1}\\
=&(\partial_{y_{12}})^{s_2-k_2}\left(\frac{\mathrm{d}}{2}-1\right)_{k_1}(-1)^{\frac{\mathrm{d}}{2}-1+k_1}\frac{y_{12}^{k_1}}{Y_{12}^{\frac{\mathrm{d}}{2}-1+k_1}}\\
=&\sum_{n_{12}=0}^{\min(k_1,s_2-k_2)}(-1)^{\frac{\mathrm{d}}{2}-1+k_1+s_2-k_2-n_{12}}\left(\frac{\mathrm{d}}{2}-1\right)_{k_1+s_2-k_2-n_{12}}\\
&\binom{s_2-k_2}{n_{12}}\frac{k_1!}{(k_1-n_{12})!}\frac{y_{12}^{k_1-n_{12}}y_{21}^{s_2-k_2-n_{12}}Y_{12}^{n_{12}}}{Y_{12}^{\frac{\mathrm{d}}{2}-1+k_1+s_2-k_2}}\\
=&\sum_{n_{12}=0}^{\min(k_1,s_2-k_2)}\sum_{m_{12}=0}^{n_{12}}(-1)^{\frac{\mathrm{d}}{2}-1+k_1+s_2-k_2-n_{12}}\left(\frac{\mathrm{d}}{2}-1\right)_{k_1+s_2-k_2-n_{12}}\\
&\binom{n_{12}}{m_{12}}\binom{s_2-k_2}{n_{12}}\frac{k_1!}{(k_1-n_{12})!}\frac{y_{12}^{k_1-m_{12}}y_{21}^{s_2-k_2-m_{12}}(C_{12}^K)^{m_{12}}}{Y_{12}^{\frac{\mathrm{d}}{2}-1+k_1+s_2-k_2}}\\
=&\sum_{m_{12}=0}^{\min(k_1,s_2-k_2)}\sum_{n_{12}=m_{12}}^{\min(k_1,s_2-k_2)}(-1)^{k_1+s_2-k_2-n_{12}+m_{12}}\left(\frac{\mathrm{d}}{2}-1\right)_{k_1+s_2-k_2-n_{12}}\\
&\binom{n_{12}}{m_{12}}\binom{s_2-k_2}{n_{12}}\frac{k_1!}{(k_1-n_{12})!}I_{12}^{\frac{\mathrm{d}}{2}-1}(I_{12}K_{12})^{m_{12}}J_{12}^{k_1-m_{12}}J_{21}^{s_2-k_2-m_{12}}.
\end{aligned}
$$

The summation over n_{12} can be done (this is the only nontrivial summation in the process), yielding

$$
\begin{aligned}
(\partial_{y_{12}})^{s_2-k_2}(\partial_{y_{21}})^{k_1} I_{12}^{\frac{\mathrm{d}}{2}-1} =& \sum_{m_{12}=0}^{\min(k_1,s_2-k_2)}(-1)^{s_2-k_2+k_1}\left(\frac{\mathrm{d}}{2}-1+m_{12}\right)_{k_1-m_{12}}\\
&\left(\frac{\mathrm{d}}{2}-1\right)_{s_2-k_2}\binom{s_2-k_2}{m_{12}}\frac{k_1!}{(k_1-m_{12})!}I_{12}^{\frac{\mathrm{d}}{2}-1}(I_{12}K_{12})^{m_{12}}J_{12}^{k_1-m_{12}}J_{21}^{s_2-k_2-m_{12}}.
\end{aligned}
$$

This has to be done for each of the three propagators linking the three currents; and finally, we may multiply the results, including also the prefactors a_k^s from the summation inherent in the definition (3.4-66) of the current operators. The result is

$$
\begin{aligned}
G^{s_1,s_2,s_3}(\underline{x}_1,\underline{x}_2,\underline{x}_3) &= 4N\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^3 2^{3-\frac{3\mathrm{d}}{2}} \\
&\sum_{k_1=0}^{s_1} a^{s_1}_{k_1}\sum_{k_2=0}^{s_2} a^{s_2}_{k_2}\sum_{k_3=0}^{s_3} a^{s_3}_{k_3}\sum_{m_{12}=0}^{\min(k_1,s_2-k_2)}\sum_{m_{23}=0}^{\min(k_2,s_3-k_3)}\sum_{m_{31}=0}^{\min(k_3,s_1-k_1)} \\
&(-1)^{s_2-k_2+k_1}\left(\frac{\mathrm{d}}{2}-1+m_{12}\right)_{k_1-m_{12}}\left(\frac{\mathrm{d}}{2}-1\right)_{s_2-k_2} \\
&\binom{s_2-k_2}{m_{12}}\frac{k_1!}{(k_1-m_{12})!}I_{12}^{\frac{\mathrm{d}}{2}-1}(I_{12}K_{12})^{m_{12}}J_{12}^{k_1-m_{12}}J_{21}^{s_2-k_2-m_{12}} \\
&(-1)^{s_3-k_3+k_2}\left(\frac{\mathrm{d}}{2}-1+m_{23}\right)_{k_2-m_{23}}\left(\frac{\mathrm{d}}{2}-1\right)_{s_3-k_3} \\
&\binom{s_3-k_3}{m_{23}}\frac{k_2!}{(k_2-m_{23})!}I_{23}^{\frac{\mathrm{d}}{2}-1}(I_{23}K_{23})^{m_{23}}J_{23}^{k_2-m_{23}}J_{32}^{s_3-k_3-m_{23}} \\
&(-1)^{s_1-k_1+k_3}\left(\frac{\mathrm{d}}{2}-1+m_{31}\right)_{k_3-m_{31}}\left(\frac{\mathrm{d}}{2}-1\right)_{s_1-k_1} \\
&\binom{s_1-k_1}{m_{31}}\frac{k_3!}{(k_3-m_{31})!}I_{31}^{\frac{\mathrm{d}}{2}-1}(I_{31}K_{31})^{m_{31}}J_{31}^{k_3-m_{31}}J_{13}^{s_1-k_1-m_{31}} - \text{traces}.
\end{aligned}
$$

At this point, it is advantageous to insert the full expression (2.6-26) for the coefficients a^s_k; after resolving various factorials, we are left with

$$
\begin{aligned}
G^{s_1,s_2,s_3}(\underline{x}_1,\underline{x}_2,\underline{x}_3) &= \frac{N}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^3 2^{3-\frac{3\mathrm{d}}{2}}(I_{12}I_{23}I_{31})^{\frac{\mathrm{d}}{2}-1} \\
&\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\sum_{k_3=0}^{s_3}\sum_{m_{12}=0}^{\min(k_1,s_2-k_2)}\sum_{m_{23}=0}^{\min(k_2,s_3-k_3)}\sum_{m_{31}=0}^{\min(k_3,s_1-k_1)} \\
&(-1)^{s_2-k_2}\frac{\left(\frac{\mathrm{d}}{2}-1\right)_{s_2}}{\left(\frac{\mathrm{d}}{2}-1\right)_{m_{31}}}\frac{s_2!}{m_{12}!(s_2-k_2-m_{12})!(k_1-m_{12})!} \\
&(I_{12}K_{12})^{m_{12}}J_{12}^{k_1-m_{12}}J_{21}^{s_2-k_2-m_{12}} \\
&(-1)^{s_3-k_3}\frac{\left(\frac{\mathrm{d}}{2}-1\right)_{s_3}}{\left(\frac{\mathrm{d}}{2}-1\right)_{m_{12}}}\frac{s_3!}{m_{23}!(s_3-k_3-m_{23})!(k_2-m_{23})!} \\
&(I_{23}K_{23})^{m_{23}}J_{23}^{k_2-m_{23}}J_{32}^{s_3-k_3-m_{23}} \\
&(-1)^{s_1-k_1}\frac{\left(\frac{\mathrm{d}}{2}-1\right)_{s_1}}{\left(\frac{\mathrm{d}}{2}-1\right)_{m_{23}}}\frac{s_1!}{m_{31}!(s_1-k_1-m_{31})!(k_3-m_{31})!} \\
&(I_{31}K_{31})^{m_{31}}J_{31}^{k_3-m_{31}}J_{13}^{s_1-k_1-m_{31}} - \text{traces}.
\end{aligned}
$$

We exchange the order of summations

$$\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\sum_{k_3=0}^{s_3}\sum_{m_{12}=0}^{\min(k_1,s_2-k_2)}\sum_{m_{23}=0}^{\min(k_2,s_3-k_3)}\sum_{m_{31}=0}^{\min(k_3,s_1-k_1)}$$
$$=\sum_{m_{12}=0}^{\min(s_1,s_2)}\sum_{m_{23}=0}^{\min(s_2-m_{12},s_3)}\sum_{m_{31}=0}^{\min(s_3-m_{23},s_1-m_{12})}\sum_{k_1=m_{12}}^{s_1-m_{31}}\sum_{k_2=m_{23}}^{s_2-m_{12}}\sum_{k_3=m_{31}}^{s_3-m_{23}}$$

and shift $k_1 \mapsto k_1 + m_{12}, k_2 \mapsto k_2 + m_{23}, k_3 \mapsto k_3 + m_{31}$; this reduces the k_{ij}-summations to simple binomial sums involving the J_{ij}-symbols. We have

Lemma 3.5. *The correlation of three quasi-primary bilinear twist-2 currents in the free massless UV theory is given by*

$$G^{s_1,s_2,s_3}(\underline{x}_1,\underline{x}_2,\underline{x}_3) = \langle \mathcal{J}^{s_1}(\underline{x}_1)\mathcal{J}^{s_2}(\underline{x}_2)\mathcal{J}^{s_3}(\underline{x}_3)\rangle \tag{3.4-106}$$

$$\begin{aligned}=&\frac{N}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^3 2^{3-\frac{3\mathrm{d}}{2}}\sum_{m_{12}=0}^{\min(s_1,s_2)}\sum_{m_{23}=0}^{\min(s_2-m_{12},s_3)}\sum_{m_{31}=0}^{\min(s_3-m_{23},s_1-m_{12})}\\
&(-1)^{m_{12}+m_{23}+m_{31}}\frac{\left(\frac{\mathrm{d}}{2}-1\right)_{s_1}}{\left(\frac{\mathrm{d}}{2}-1\right)_{m_{23}}}\frac{\left(\frac{\mathrm{d}}{2}-1\right)_{s_2}}{\left(\frac{\mathrm{d}}{2}-1\right)_{m_{31}}}\frac{\left(\frac{\mathrm{d}}{2}-1\right)_{s_3}}{\left(\frac{\mathrm{d}}{2}-1\right)_{m_{12}}}\\
&\frac{s_1!}{m_{31}!(s_1-m_{12}-m_{31})!}\frac{s_2!}{m_{12}!(s_2-m_{23}-m_{12})!}\frac{s_3!}{m_{23}!(s_3-m_{31}-m_{23})!}\\
&(I_{12}I_{23}I_{31})^{\frac{\mathrm{d}}{2}-1}(I_{12}K_{12})^{m_{12}}(I_{23}K_{23})^{m_{23}}(I_{31}K_{31})^{m_{31}}\\
&(J_{12}-J_{13})^{s_1-m_{12}-m_{31}}(J_{23}-J_{21})^{s_2-m_{23}-m_{12}}(J_{31}-J_{32})^{s_3-m_{31}-m_{23}} - \textit{traces.}\end{aligned} \tag{3.4-107}$$

For the m_{ij}, the summation bounds effectively mean that $0 \leq m_{ij}$ and

$$m_{12}+m_{31} \leq s_1 \qquad m_{23}+m_{12} \leq s_2 \qquad m_{31}+m_{23} \leq s_3.$$

Since m_{ij} counts the powers of $I_{ij}K_{ij}$ and K_{ij} contains a summand $(\tilde{v}_i,\tilde{v}_j) = \underline{v}_i \cdot \underline{v}_j$, m_{ij} is the number of "delta links" between the tensor indices of the currents and these bounds are an expression of the fact that there is a limited number of tensor indices at each current available.

As I_{ij} and K_{ij} are properly conformally covariant, and the J_{ij} terms appear only as paired differences $J_{ij} - J_{ik}$ with the same first endpoint, it follows that the total correlation is properly covariant (invariant under the operator $\tilde{x}_i^{\tilde{\mu}}\partial_{\tilde{v}_i^{\tilde{\mu}}}$). This is just what we expected for the correlations of the bilinear tensor currents $\mathcal{J}^s$. We will take equation (3.4-106) as the starting point for the computation of the formally covariant EAdS-presentation of the three-point function.

To conclude this section, we evaluate formula (3.4-106) for the $s, 0, 0$ correlation:

$$\begin{aligned}G^{s,0,0}(\underline{x}_1,\underline{x}_2,\underline{x}_3) =&\frac{N}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^3 2^{3-\frac{3\mathrm{d}}{2}}\left(\frac{\mathrm{d}}{2}-1\right)_s\\
&\times (I_{12}I_{23}I_{31})^{\frac{\mathrm{d}}{2}-1}(J_{12}-J_{13})^s - \text{traces.}\end{aligned} \tag{3.4-108}$$

3.4.4 Two-point Function of Tensor Currents, Bulk-to-Boundary Propagators and Action of Covariant EAdS-Derivative

A second ingredient which we need are propagators of the currents, notably bulk-to-boundary propagators. Again, they should be displayed in a form where conformal covariance is immediately visible.

In this section, we will discuss how the bulk-to-boundary propagator can be obtained by a simple procedure from the boundary-to-boundary propagator and derive a set of characteristic properties of this function. The propagator itself can be found in the literature, but the characterisation by the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$ is novel. We also discuss the action of the covariant EAdS-derivative on these propagators, as it will be needed in the sequel for the construction of the vertices. In particular, it will turn out that the covariant EAdS derivative is a comparatively simple object if written in the embedding space notation.

The two-point function of currents can be computed by a way completely similar to the three-point functions of the preceding section. The result displays immediately the conformal covariance of the propagators:

$$\begin{aligned} G^{s_1,s_2}(\underline{x}_1,\underline{x}_2) =& \delta^{s_1 s_2}\cdot\frac{N}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^2\frac{(-2)^{s_1}s_1!\Gamma(2s_1+\mathrm{d}-3)}{\Gamma(s_1+\mathrm{d}-3)} \\ & 2^{2-\mathrm{d}-s_1}\, I_{12}^{\mathrm{d}-2}(I_{12}K_{12})^{s_1} - \text{traces}. \end{aligned} \tag{3.4-109}$$

This coincides with the formulas reported in the literature (as summarised in section 2.6.3); the scaling dimension of these currents is $\Delta(\mathcal{J}^s) = \mathrm{d}-2+s$.

Definition 3.6. The propagator (3.4-109) defines the normalisation of the boundary-to-boundary propagator in the *field-theoretic* prescription

$$G^{\mathrm{ft}\ s}_{\mathrm{bo}}(\underline{x}_1,\underline{x}_2) = G^{s,s}(\underline{x}_1,\underline{x}_2). \tag{3.4-110}$$

This fixes the normalisation of the bulk fields, and hence the normalisation of the bulk-to-bulk propagator.

The bulk-to-boundary propagators can be obtained by the group-theoretical analysis of Dobrev et al (cf. section 3.3.1 and references therein); if we write them in the language of the embedding space, then they have the form (3.4-109), with the qualification that we simply lift one of the points from the boundary into the bulk $z \in \mathrm{EAdS}$,

$$\begin{aligned} G^{\mathrm{ft\ UV}\ s}_{\mathrm{bubo}}(\tilde{z},\underline{x})[\tilde{a},\underline{v}] =& \frac{1}{2}\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^2\frac{(-2)^{s}s!\Gamma(2s+\mathrm{d}-3)}{\Gamma(s+\mathrm{d}-3)} \\ & 2^{2-\mathrm{d}-s}\, I_{xz}^{\mathrm{d}-2}(I_{xz}K_{va})^{s} - \text{traces} \end{aligned} \tag{3.4-111}$$

(the invariants in this expression are defined in complete parallel to the boundary invariants (3.4-96); we list them below in (3.4-116)). The normalisation which we choose is that appropriate to the field-theoretic prescription (3.2-29), which the exemption of the factor N which we left out.

Near the boundary, these propagators have the behaviour [9]

$$\begin{aligned} \left\| G_{\text{bubo}}^{\text{ft UV}\ s}(z, \underline{x})[\tilde{a}, \underline{v}] \right\|_a &\sim (z^0)^{\mathrm{d}-2+s} & z^0 &\ll |\underline{x} - \underline{z}| \\ \left\| G_{\text{bubo}}^{\text{ft UV}\ s}(z, \underline{x})[\tilde{a}, \underline{v}] \right\|_a &\sim (z^0)^{-\mathrm{d}+2-s} & \underline{x} &= \underline{z}. \end{aligned}$$

This is in line with the generic behaviour (3.2-30). Quite generally, a bulk-to-boundary propagator for a bulk field corresponding to a boundary operator of scaling dimension Δ has the form

$$G_{\text{bubo}}^{\Delta\ s}(\tilde{z}, \underline{x})[\tilde{a}, \underline{v}] \sim I_{xz}^{\Delta} K_{va}^{s} - \text{traces}. \tag{3.4-112}$$

Generically, bulk-to-boundary propagators can be characterised by the following algebraic properties in the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$: Tensor propagators do not have contributions orthogonal to the EAdS hyperboloid:

$$\tilde{z}^{\tilde{\mu}} \partial_{\tilde{a}^{\tilde{\mu}}} G_{\text{bubo}}^{\Delta\ s}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] = 0 \tag{3.4-113a}$$

by construction; this is a property of K_{va}. The propagator is traceless on the bulk side: We have

$$\begin{aligned} \Box_{\tilde{a}} G_{\text{bubo}}^{\Delta\ s}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] &\sim (\partial_{\tilde{a}^{\tilde{\mu}}} K_{va})(\partial_{\tilde{a}_{\tilde{\mu}}} K_{va}) \times \text{ other terms} - \text{traces} \\ &\sim \underline{v} \cdot \underline{v} \times \text{ other terms} - \text{traces} = 0. \end{aligned} \tag{3.4-113b}$$

Terms proportional $(\tilde{x}, \tilde{x}) = 0$ and $(\tilde{x}, \tilde{v}) = 0$ appearing in the contraction vanish at once. Mind that we could as well take the EAdS trace $\partial_{a^\mu} \partial_{a_\mu}$, since by (3.4-113a) only tangent contributions occur. The propagators are homogeneous in the embedding space,

$$G_{\text{bubo}}^{\Delta\ s}(\alpha \tilde{z}; \underline{x})[\tilde{a}, \underline{v}] = \alpha^{-\Delta} G_{\text{bubo}}^{\Delta\ s}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}]. \tag{3.4-113c}$$

They also obey the free, massless equation of motion in the embedding space

$$\begin{aligned} \Box_{\tilde{z}} G_{\text{bubo}}^{\Delta\ s}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] &\sim (\partial_{\tilde{z}^{\tilde{\mu}}} K_{va})(\partial_{\tilde{z}_{\tilde{\mu}}} K_{va}) \times \text{ other terms} - \text{traces} \\ &\sim \underline{v} \cdot \underline{v} \times \text{ other terms} - \text{traces} = 0 \end{aligned} \tag{3.4-113d}$$

(contributions from the I-terms vanish immediately). Finally, there is a mixed equation

$$\partial_{\tilde{a}^{\tilde{\mu}}} \partial_{\tilde{z}_{\tilde{\mu}}} G_{\text{bubo}}^{\Delta\ s}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] = 0. \tag{3.4-113e}$$

Due to equation (3.4-113a), we may contract the bulk end of the propagator with a vector $a \in T_z$ (tangent to EAdS at $z \in$ EAdS) without loss; so it is admissible to

[9] By the norm of a bulk tensor $T^s[\tilde{a}]$ of spin s we mean

$$\|T^s[\tilde{a}]\|_a^2 = g^{\mu_1 \nu_1} \dots g^{\mu_s \nu_s} T^s_{\mu_1 \dots \mu_s} T^s_{\nu_1 \dots \nu_s}.$$

write $G^{\Delta\ s}_{\text{bubo}}(z,\underline{x})[a,\underline{v}]$. For the propagators to be truly EAdS, There should also be an equation of motion holding within the EAdS hyperboloid $(\tilde{z},\tilde{z}) = -1$. We have to restrict all expressions to the EAdS hyperboloid and use the covariant derivative D_μ applicable to that hypersurface.

Equations (3.4-113a) to (3.4-113e) actually have a second propagator solution,

$$|\tilde{z}|^{\mathrm{d}-2\Delta} G^{\mathrm{d}-\Delta\ s}_{\text{bubo}}(\tilde{z},\underline{x})[\tilde{a},\underline{v}], \tag{3.4-114}$$

with the modulus $|\tilde{z}| = \sqrt{-(\tilde{z},\tilde{z})}$; on the EAdS-hyperboloid, $|\tilde{z}| = 1$. This is the bulk-to-boundary propagator corresponding to a boundary operator with the conjugate scaling dimension $\mathrm{d}-\Delta$. Notice that the dimension Δ enters the system of equations

So in particular, since both $G^{\Delta\ s}_{\text{bubo}}$ and $G^{\mathrm{d}-\Delta\ s}_{\text{bubo}}$ are solutions with homogeneity degree $-\Delta$ in (3.4-113c), they will be solutions when we impose a homogeneity degree $\Delta-\mathrm{d}$ in (3.4-113c). Thus there are two alternative formulations for the propagators, up to factors of $|\tilde{z}|$.

The covariant calculus for EAdS gets particularly simple if we use the notation of the embedding space $\mathbb{R}^{\mathrm{d}+1,1}$ to take down all expressions, even if the covariant derivatives themselves are supposed to be on EAdS. In the following, we assume that $z \in$ EAdS, $y \in \mathbb{R}^{\mathrm{d}+1,1}$ are arbitrary points, and the tangent vectors $a, b \in T_z$ are placeholders for free indices (so the indices they are attached to are acted upon by the Christoffel symbols (3.1-6) of the covariant derivative). Since we have taken the scalar product as the underlying object, it turns out that the only covariant derivative we really have to compute is

$$a^\mu\,\mathrm{D}_\mu(\tilde{y},\tilde{b}) = (\tilde{y},\tilde{z})(\tilde{a},\tilde{b}). \tag{3.4-115a}$$

D_μ acts on z and not on y in this equation; the derivative is computed by treating the scalar product as function $(\tilde{y},\tilde{b}) = f^\mu_{\tilde{y}}(z) b_\mu$. All other expressions may be derived thereof by the chain and the product rule:

$$a^\mu\,\mathrm{D}_\mu(\tilde{b},\tilde{b}) = 2(\tilde{b},\tilde{z})(\tilde{a},\tilde{b}) = 0, \tag{3.4-115b}$$

since $(\tilde{b},\tilde{z}) = 0$ as $b \in T_z$ is tangent to EAdS. Trivially,

$$a^\mu\,\mathrm{D}_\mu(\tilde{y},\tilde{z}) = (\tilde{y},\tilde{a}). \tag{3.4-115c}$$

For more complex expressions, we need more invariants of the type (3.4-96). Convenient definitions with $a, b \in T_z$, $\underline{v} \in T_{\underline{x}}$ are

$$\begin{aligned} I_{xz} &= -\frac{1}{(\tilde{x},\tilde{z})} & K_{va} &= (\tilde{v},\tilde{a}) - \frac{(\tilde{v},\tilde{z})(\tilde{x},\tilde{a})}{(\tilde{x},\tilde{z})} \\ J_{bx} &= \frac{(\tilde{b},\tilde{x})}{(\tilde{z},\tilde{x})} & J_{ax} &= \frac{(\tilde{a},\tilde{x})}{(\tilde{z},\tilde{x})}. \end{aligned} \tag{3.4-116}$$

We compute for these

$$a^\mu\,\mathrm{D}_\mu\, I_{xz} = -a^\mu\,\mathrm{D}_\mu \frac{1}{(\tilde{z},\tilde{x})} = \frac{(\tilde{a},\tilde{x})}{(\tilde{z},\tilde{x})^2} = -J_{ax} I_{xz} \tag{3.4-117a}$$

and

$$a^\mu \, \mathrm{D}_\mu \, K_{vb} = a^\mu \, \mathrm{D}_\mu \left((\tilde{v}, \tilde{b}) - \frac{(\tilde{v}, \tilde{z})(\tilde{x}, \tilde{b})}{(\tilde{z}, \tilde{x})} \right) = -J_{bx} K_{va}. \tag{3.4-117b}$$

Finally,

$$a^\mu \, \mathrm{D}_\mu \, J_{bx} = a^\mu \, \mathrm{D}_\mu \frac{(\tilde{b}, \tilde{x})}{(\tilde{z}, \tilde{x})} = (\tilde{a}, \tilde{b}) - J_{ax} J_{bx}. \tag{3.4-117c}$$

So we have a very simple set of differentiation rules. For the contraction of two tensor indices (which we denote by $\diamond_a$), the basic rule is (observing the the contraction is in the tangent space T_z)

$$(\tilde{a}, \tilde{y}) \diamond_a (\tilde{a}, \tilde{u}) \equiv \partial_{a^\mu} (\tilde{a}, \tilde{y}) \partial_{a_\mu} (\tilde{a}, \tilde{u}) = (\tilde{y}, \tilde{u}) + (\tilde{y}, \tilde{z})(\tilde{z}, \tilde{u}). \tag{3.4-118}$$

for $(\tilde{z}, \tilde{z}) = -1$. From this, one gets quickly at

$$\begin{aligned} K_{av} \diamond_a J_{ax} &= = 0 & \underline{v} &\in T_{\underline{x}} \\ K_{av} \diamond_a K_{aw} &= \underline{v} \cdot \underline{w} = (\tilde{v}, \tilde{w}) & \underline{w} &\in T_{\underline{x}} \\ J_{ax} \diamond_a J_{ay} &= \frac{(\tilde{x}, \tilde{y})}{(\tilde{x}, \tilde{z})(\tilde{y}, \tilde{z})} + 1, \end{aligned}$$

for $(\tilde{z}, \tilde{z}) = -1$. The EAdS Laplacian may then be computed from piecing together $\Box^{\mathrm{EAdS}} = (a^\nu \, \mathrm{D}_\nu) \diamond_a (a^\rho \, \mathrm{D}_\rho)$. For a propagator of the general form $G^{\Delta \; s}_{\mathrm{bubo}}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] \sim I^{\Delta}_{xz} K^s_{va}$ − traces, the equation of motion reads

$$\left[(\Delta^2 - \mathrm{d}\Delta - s) - \Box^{\mathrm{EAdS}} \right] G^{\Delta \; s}_{\mathrm{bubo}}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] = 0. \tag{3.4-119}$$

The constant

$$m^2_{\Delta, s} = \Delta^2 - \mathrm{d}\Delta - s \tag{3.4-120}$$

is the mass of the tensor field. In the general case, for a given mass m^2 there are two possible values of Δ,

$$\Delta_\pm = \frac{\mathrm{d}}{2} \pm \sqrt{\frac{\mathrm{d}^2}{4} + s + m^2}. \tag{3.4-121}$$

In the UV case, $\Delta(\mathcal{J}^s) = \mathrm{d} - 2 + s$ and

$$m^2_s = (\mathrm{d} - 2 + s)(s - 2) - s \tag{3.4-122}$$

is the holographic mass of the bulk tensor of spin s. In particular for the bulk scalar $s = 0$, the mass $m^2_0 = 4 - 2\mathrm{d}$ is the holographic mass value found in the literature (see also section 6).

By the application of the rules for the covariant derivative, one shows also the EAdS conservation law

$$\partial_{a^\mu} \, \mathrm{D}^\mu \, G^{\mathrm{ft \; UV} \; s}_{\mathrm{bubo}}(\tilde{z}; \underline{x})[\tilde{a}, \underline{v}] = 0. \tag{3.4-123}$$

By content, this is *different* from (3.4-113e).

To summarise, we have found and characterised the boundary-to-boundary and bulk-to-boundary propagators of the HS tensor fields, and we have seen that the action of the covariant EAdS derivative is if not trivial, then quite managable if written in the embedding space geometry, employing the usual invariants.

3.4.5 EAdS-Presentation of Three-Point Functions

After so much preparation, it is time to address the fundamental problem of constructing a fully satisfactory EAdS-presentation for the three-point function of the bilinear tensor currents, with emphasis on formal conformal covariance. We will not totally reach this ambitious goal within the limitations of this thesis (although this is planned for a later publication). We will nevertheless lay down the general line of argument to that purpose, which we have developed.

As material, we have developed in section 3.4.3 the form (3.4-106) of the three-point function purely on the boundary, which displays directly conformal covariance; and we have in the preceding section discussed the bulk-to-boundary propagators, and their properties, and covariant presentation. This resulted in the simple expression (3.4-111) for the bulk-to-boundary propagators.

To give an EAdS-presentation of the correlation of three currents, we must find a vertex differential operator $\tilde{V}$, such that

$$\begin{aligned} G^{s_1,s_2,s_3}_{(l_1),(l_2),(l_3)}(\underline{x}_1,\underline{x}_2,\underline{x}_3) =& \langle \mathcal{J}^{s_1}_{(l_1)}(\underline{x}_1)\mathcal{J}^{s_2}_{(l_2)}(\underline{x}_2)\mathcal{J}^{s_3}_{(l_3)}(\underline{x}_3)\rangle \\ =& \int \frac{\mathrm{d}^{\mathrm{d}}z\ \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} \tilde{V}^{s_1,s_2,s_3\ (\mu_1),(\mu_2),(\mu_3)}(\mathrm{D}^{z_1},\mathrm{D}^{z_2},\mathrm{D}^{z_3}) \\ & G^{\text{ft UV } s_1}_{\text{bubo } (\mu_1),(l_1)}(z_1,\underline{x}_1) G^{\text{ft UV } s_2}_{\text{bubo } (\mu_2),(l_2)}(z_2,\underline{x}_2) G^{\text{ft UV } s_3}_{\text{bubo } (\mu_3),(l_3)}(z_3,\underline{x}_3)\Big|_{z_i=z}. \end{aligned} \tag{3.4-124}$$

We assign as the bulk endpoint of propagator j a point $z_j \in \mathrm{EAdS}$. The propagators can be given in the usual form, where the bulk and boundary ends are contracted with placeholder vectors $a_j \in T_{z_j}$ and $\underline{v}_j \in T_{\underline{x}_j}$ (but we will in time use the general vector $\tilde{a}_j \in T_{\tilde{z}_j}$ for contraction). We take the propagators in the field-theoretic prescription in order to have definite normalisation.

The General Strategy. We will begin with a generic, very general vertex (see (3.4-126) below) containing a family of indetermined parameters $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$, and work backwards by connecting this vertex to the propagators and integrating out the vertex coordinate z. Then, the result is compared to the boundary correlation function $G^{s_1,s_2,s_3}_{(l_1),(l_2),(l_3)}(\underline{x}_1,\underline{x}_2,\underline{x}_3)$ in the form (3.4-106). In that way, we determine the parameters $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$ and specify the vertex. It is clear that there might be alternatives to the general vertex which we select as starting point; we have no means to eliminate that possibility currently.

Vertex Structure. The first step is to obtain clarity about the general structure of the vertices which we expect. As indicated, the vertex should "saturate" all the loose indices at the endpoints of the propagators. There are two possibilities: Two indices might be contracted with the EAdS metric g as in

$$G^{\text{ft UV } s_1}_{\text{bubo } \mu_1\mu_2\ldots\mu_{s_1}}(z_1,\underline{x}_1)[\underline{v}_1]\ g^{\mu_1\nu_1}\ G^{\text{ft UV } s_2}_{\text{bubo } \nu_1\nu_2\ldots\nu_{s_2}}(z_2,\underline{x}_2)[\underline{v}_2],$$

or an index may be contracted with a covariant derivative, as in

$$G^{\text{ft UV } s_1}_{\text{bubo } \mu_1\mu_2\ldots\mu_{s_1}}(z_1, \underline{x}_1)[\underline{v}_1]\, \mathrm{D}^{\mu_1}\, G^{\text{ft UV } s_2}_{\text{bubo } \nu_1\ldots\nu_{s_2}}(z_2, \underline{x}_2)[\underline{v}_2],$$

acting on z_2 in this case, but taking into account that the unsaturated indices have to be transformed by the application of Christoffel symbols (in our case, ν_1 to ν_{s_2}). It is obvious that the order of the contractions and differentiations matters, since contracted indices do not have to be transformed by Christoffel symbols any more. Also, there are conflicts because we have three propagators acting on each other with covariant derivatives, so the order must be set up for the whole vertex at the same time. Typically, a vertex will contain many different summands, each consisting of a contraction/differentiation scheme, and a weight factor.

That the order of the differentiations and index contractions has to be specified when giving the vertex data is rather inconvenient. We now set up a protocol which allows to circumvent this trouble. We make the assumption that each summand contributing to the vertex is constructed as follows:

1. Each propagator is acted on with a couple of covariant derivatives; their indices are *not* contracted with the indices of any other propagator, but left dangling. In the end, all indices are symmetrised; this may be effected by contracting again with the placeholder vector a. Thus, for every propagator, we obtain a structure of the type
$$a^{\mu_1} \ldots a^{\mu_n} a^{\nu_1} \ldots a^{\nu_s}\, \mathrm{D}_{\mu_1} \ldots \mathrm{D}_{\mu_n}\, G^{\text{ft UV } s}_{\text{bubo } \nu_1\ldots\nu_s}(z, \underline{x})[\underline{v}]. \tag{3.4-125}$$

2. In the end, the free indices of all propagators are contracted pairwise according to some predetermined scheme, so that no free index is left. We agree that we do not contract indices from the same propagator (ie, take the trace) [10].

3. Each vertex consists of a *finite* number of summands; so in particular, the number of derivatives is limited.

The symmetrisation of derivatives allows us to use the simple rules (3.4-115a) to (3.4-115c) for the computation of the covariant derivatives. Note that we use still the notation of the embedding space, since it allows a very economic treatment.

The contraction scheme which we have to perform is actually fixed by the number of free indices f_j which each propagator (3.4-125) has after the covariant derivatives have been applied. For let c_{ij} be the number of contractions between propagator i and propagator j. Then, since all free indices have to be contracted,

$$f_1 = c_{12} + c_{31} \qquad f_2 = c_{23} + c_{12} \qquad f_3 = c_{31} + c_{23}.$$

This system can be solved for the c_{ij}, giving

$$c_{12} = \frac{f_1 + f_2 - f_3}{2} \qquad c_{23} = \frac{f_2 + f_3 - f_1}{2} \qquad c_{31} = \frac{f_3 + f_1 - f_2}{2}.$$

[10] It might be necessary to modify that rule in the end.

There are certain conditions for the f_j in order for this system to have a solution in nonnegative integers. If there are d_j derivatives acting on propagator j with spin s_j, then $f_j = d_j + s_j$. The weight which the vertex will give to the differentiations (d_1, d_2, d_3) will be denoted $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$. Gathering everything together, the action of the total vertex $\tilde{V}^{s_1,s_2,s_3}$ in the bulk is

$$\begin{aligned}
&\tilde{V}^{s_1,s_2,s_3\,(\mu_1),(\mu_2),(\mu_3)}(\mathrm{D}^{z_1},\mathrm{D}^{z_2},\mathrm{D}^{z_3})\,G^{\text{ft UV}\,s_1}_{\text{bubo}\,(\mu_1),(l_1)}(z_1,\underline{x}_1)G^{\text{ft UV}\,s_2}_{\text{bubo}\,(\mu_2),(l_2)}(z_2,\underline{x}_2)G^{\text{ft UV}\,s_3}_{\text{bubo}\,(\mu_3),(l_3)}(z_3,\underline{x}_3)\Big|_{z_i=z}\\
&=\sum_{d_1,d_2,d_3} C^{s_1,s_2,s_3}_{d_1,d_2,d_3}\left(g^{\mu\nu}\partial_{a_1^\mu}\partial_{a_2^\nu}\right)^{\frac{s_1+d_1+s_2+d_2-s_3-d_3}{2}}\left(g^{\mu\nu}\partial_{a_2^\mu}\partial_{a_3^\nu}\right)^{\frac{s_2+d_2+s_3+d_3-s_1-d_1}{2}}\left(g^{\mu\nu}\partial_{a_3^\mu}\partial_{a_1^\nu}\right)^{\frac{s_3+d_3+s_1+d_1-s_2-d_2}{2}}\\
&\qquad\left[(a_1^\mu\,\mathrm{D}_\mu)^{d_1}G^{\text{ft UV}\,s_1}_{\text{bubo}}(\tilde{z},\underline{x}_1)[\tilde{a}_1,\underline{v}_1]\right]\;\left[(a_2^\mu\,\mathrm{D}_\mu)^{d_2}G^{\text{ft UV}\,s_2}_{\text{bubo}}(\tilde{z},\underline{x}_2)[\tilde{a}_2,\underline{v}_2]\right]\\
&\qquad\qquad\left[(a_3^\mu\,\mathrm{D}_\mu)^{d_3}G^{\text{ft UV}\,s_3}_{\text{bubo}}(\tilde{z},\underline{x}_3)[\tilde{a}_3,\underline{v}_3]\right]. \qquad (3.4\text{-}126)
\end{aligned}$$

It is necessary to understand that we might be wrong in our assumptions and the actual vertices have a structure which does not fall under this protocoll.

Covariant Derivatives of Propagators. Our next step is to obtain clarity about the possible terms which can arise in evaluating the derivatives (3.4-125) of the propagators. The bulk-to-boundary propagators (3.4-111) are generically of the structure

$$G^{\text{ft UV}\,s}_{\text{bubo}}(\tilde{z},\underline{x})[\tilde{a},\underline{v}] \sim I_{xz}^{\mathrm{d}-2}(I_{xz}K_{va})^s - \text{traces}$$

with the symbols I_{xz} and K_{va} defined in (3.4-116), so we need the covariant derivatives $a^\mu\,\mathrm{D}_\mu\, I_{xz}$ and $a^\mu\,\mathrm{D}_\mu\, K_{va}$, and $a^\mu\,\mathrm{D}_\mu\, J_{ax}$ since J_{ax} is generated by differentiation. These are listed in (3.4-117a) to (3.4-117c).

In order to have better control over the action of the derivatives, we encode once more the derivative $a^\mu\,\mathrm{D}_\mu$ by ∂_y and assume that I_{xz}, J_{ax} and K_{va} are functions of y. We get the system of equations

$$\begin{aligned}
\partial_y I_{xz}(y) &= -J_{ax}(y)I_{xz}(y)\\
\partial_y J_{ax}(y) &= (\tilde{a},\tilde{a}) - J_{ax}(y)^2\\
\partial_y K_{va}(y) &= -J_{ax}(y)K_{va}(y),
\end{aligned}$$

with the particular solution [11]

$$\begin{aligned}
I_{xz}(y) &= \operatorname{sech}(y\sqrt{(\tilde{a},\tilde{a})})\\
J_{ax}(y) &= \sqrt{(\tilde{a},\tilde{a})}\tanh(y\sqrt{(\tilde{a},\tilde{a})})\\
K_{va}(y) &= C_K\operatorname{sech}(y\sqrt{(\tilde{a},\tilde{a})}),
\end{aligned}$$

with C_K a counting variable for K_{va}. By the rules of symbolic differentiation,

$$\begin{aligned}
(a^\mu\,\mathrm{D}_\mu)^n G^{\text{ft UV}\,s}_{\text{bubo}}(\tilde{z},\underline{x})[\tilde{a},\underline{v}] &\sim (a^\mu\,\mathrm{D}_\mu)^n I_{xz}^{\mathrm{d}-2}(I_{xz}K_{va})^s\\
&\sim \mathrm{Poly}_n(J_{ax},(\tilde{a},\tilde{a}))\cdot I_{xz}^{\mathrm{d}-2}(I_{xz}K_{va})^s \qquad (3.4\text{-}127)\\
&\sim C_K^s\partial_y^n\left(\operatorname{sech}(y\sqrt{(\tilde{a},\tilde{a})})\right)^{\mathrm{d}-2+2s},
\end{aligned}$$

[11] The hyperbolic secant $\operatorname{sech} x = (\cosh x)^{-1}$.

where Poly_n is a polynomial of degree n. By an adaptation of a rule for the symbolic differentiation for the cosine [1, 01.07.20.0004.01],

$$\partial_y^n \left(\mathrm{sech}(y\sqrt{(\tilde{a},\tilde{a})})\right)^q = \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{r=0}^{n-k}\sum_{l=0}^{j}(-1)^{k+j+r}$$
$$\frac{2^{-j-n+k}(2l+2r-j-n+k)^n(q)_{n+1}}{j+n-k+q}\frac{1}{r!(n-k-r)!\; l!(j-l)!\; (k-j)!}$$
$$\sqrt{(\tilde{a},\tilde{a})}^k\left(\sqrt{(\tilde{a},\tilde{a})}\tanh(y\sqrt{(\tilde{a},\tilde{a})})\right)^{n-k}\left(\mathrm{sech}(y\sqrt{(\tilde{a},\tilde{a})})\right)^q$$

We could not resolve this multiple sum any further, but it is clear that only even k contribute (since there should only appear powers of $(\tilde{a},\tilde{a})$ and not its square root). The notable insight is that the algebra of symbols I_{xz}, J_{ax}, K_{va} and $(\tilde{a},\tilde{a})$ closes under covariant differentiation in a relatively simple way and that we may expect an expression of the form (3.4-127) when evaluating the covariant deriavtives.

Contractions at the Vertex. In the next step, the dangling indices a are contracted at the vertex. We have to contract in the tangent space T_z (not in $T_{\tilde{z}}$); however, for K_{va}, these contractions are identical, because K_{va} has no components pointing in $\tilde{z}$-direction, $\tilde{z}^{\tilde{\mu}}\partial_{\tilde{a}^{\tilde{\mu}}}K_{va} = 0$. We denote contractions by $\diamond_a$, so that

$$f(\tilde{a}) \diamond_a g(\tilde{a}) \equiv [\partial_{a^\mu} f(\tilde{a})][\partial_{a_\mu} g(\tilde{a})]. \tag{3.4-128}$$

Note that a complete contraction of a tensor with n indices will generate a factor $n!$ by this way of defining contractions. Carefully working out the action of the contractions, we get the following set of rules:

$$\begin{aligned}
(\tilde{a},\tilde{a}) \diamond_a (\tilde{a},\tilde{a}) =& 4(\tilde{a},\tilde{a})\\
K_{va} \diamond_a (\tilde{a},\tilde{a}) =& 2K_{va}\\
J_{ax} \diamond_a (\tilde{a},\tilde{a}) =& 2J_{ax}\\
K_{v_1a} \diamond_a K_{v_2a} =& K_{v_1v_2} + \frac{(J_{v_1z}-J_{v_1x_2})\,(J_{v_2z}-J_{v_2x_1})}{I_{x_1x_2}}\\
K_{v_1a} \diamond_a J_{ax_2} =& -\frac{(J_{v_1z}-J_{v_1x_2})I_{x_2z}}{I_{x_1x_2}}\\
J_{ax_1} \diamond_a J_{ax_2} =& 1-\frac{I_{x_1z}I_{x_2z}}{I_{x_1x_2}}.
\end{aligned} \tag{3.4-129}$$

The term 1 in the last contraction comes from the fact that neither J_{ax_1} nor J_{ax_2} are covariant themselves; so they have contributions pointing in $\tilde{z}$-direction, and we have to take the trace in the tangent space T_z explicitly. The scalar product $(\tilde{a},\tilde{a})$ can be contracted with itself, and since only the tangent indices to EAdS are contracted, the corresponding rule is

$$\partial_{a^\mu}\partial_{a_\mu}(\tilde{a},\tilde{a}) = 2(\mathrm{d}+1). \tag{3.4-130}$$

Note that by these rules, the contraction of a propagator with itself always results in zero ($K_{v_1a} \diamond_a J_{ax_1} = 0$ etc.).
Gathering everything together, the action of the total vertex $\tilde{V}^{s_1,s_2,s_3}$ in the bulk is

$$
\begin{aligned}
&\tilde{V}^{s_1,s_2,s_3\,(\mu_1),(\mu_2),(\mu_3)}(\mathrm{D}^{z_1},\mathrm{D}^{z_2},\mathrm{D}^{z_3})\;G^{\text{ft UV }s_1}_{\text{bubo }(\mu_1),(l_1)}(z_1,\underline{x}_1)G^{\text{ft UV }s_2}_{\text{bubo }(\mu_2),(l_2)}(z_2,\underline{x}_2)G^{\text{ft UV }s_3}_{\text{bubo }(\mu_3),(l_3)}(z_3,\underline{x}_3)\Big|_{z_i=z}\\
=&\sum_{d_1,d_2,d_3} C^{s_1,s_2,s_3}_{d_1,d_2,d_3}\left(g^{\mu\nu}\partial_{a_1^\mu}\partial_{a_2^\nu}\right)^{\frac{s_1+d_1+s_2+d_2-s_3-d_3}{2}}\left(g^{\mu\nu}\partial_{a_2^\mu}\partial_{a_3^\nu}\right)^{\frac{s_2+d_2+s_3+d_3-s_1-d_1}{2}}\left(g^{\mu\nu}\partial_{a_3^\mu}\partial_{a_1^\nu}\right)^{\frac{s_3+d_3+s_1+d_1-s_2-d_2}{2}}\\
&\left[(a_1^\mu\,\mathrm{D}_\mu)^{d_1}G^{\text{ft UV }s_1}_{\text{bubo}}(\tilde{z},\underline{x}_1)[\tilde{a}_1,\underline{v}_1]\right]\;\left[(a_2^\mu\,\mathrm{D}_\mu)^{d_2}G^{\text{ft UV }s_2}_{\text{bubo}}(\tilde{z},\underline{x}_2)[\tilde{a}_2,\underline{v}_2]\right]\\
&\left[(a_3^\mu\,\mathrm{D}_\mu)^{d_3}G^{\text{ft UV }s_3}_{\text{bubo}}(\tilde{z},\underline{x}_3)[\tilde{a}_3,\underline{v}_3]\right]\\
=&\sum_{d_1,d_2,d_3} C^{s_1,s_2,s_3}_{d_1,d_2,d_3}\left(g^{\mu\nu}\partial_{a_1^\mu}\partial_{a_2^\nu}\right)^{\frac{s_1+d_1+s_2+d_2-s_3-d_3}{2}}\left(g^{\mu\nu}\partial_{a_2^\mu}\partial_{a_3^\nu}\right)^{\frac{s_2+d_2+s_3+d_3-s_1-d_1}{2}}\left(g^{\mu\nu}\partial_{a_3^\mu}\partial_{a_1^\nu}\right)^{\frac{s_3+d_3+s_1+d_1-s_2-d_2}{2}}\\
&\mathrm{Poly}^{(1)}_{d_1}(J_{a_1x_1},(\tilde{a}_1,\tilde{a}_1))\cdot I^{\mathrm{d}-2}_{x_1z}(I_{x_1z}K_{v_1a_1})^{s_1}\\
&\mathrm{Poly}^{(2)}_{d_2}(J_{a_2x_2},(\tilde{a}_2,\tilde{a}_2))\cdot I^{\mathrm{d}-2}_{x_2z}(I_{x_2z}K_{v_2a_2})^{s_2}\\
&\mathrm{Poly}^{(3)}_{d_2}(J_{a_3x_3},(\tilde{a}_3,\tilde{a}_3))\cdot I^{\mathrm{d}-2}_{x_3z}(I_{x_3z}K_{v_3a_3})^{s_3}-\text{traces}\\
=&(I_{x_1z}I_{x_2z}I_{x_3z})^{\mathrm{d}-2}\sum_{\{n^{KK}_{ij}\},\{n^{JJ}_{ij}\},\{n^{KJ}_{ij}\}:\;i<j} C'_{\{n^{KK}_{ij}\},\{n^{JJ}_{ij}\},\{n^{KJ}_{ij}\}}\\
&\prod_{i<j}\left(I_{x_iz}K_{v_iv_j}I_{x_jz}+\frac{I_{x_iz}\left(J_{v_iz}-J_{v_ix_j}\right)\left(J_{v_jz}-J_{v_jx_i}\right)I_{x_jz}}{I_{x_ix_j}}\right)^{n^{KK}_{ij}}\\
&\left(1-\frac{I_{x_iz}I_{x_jz}}{I_{x_ix_j}}\right)^{n^{JJ}_{ij}}\left(-\frac{I_{x_iz}(J_{v_iz}-J_{v_ix_j})I_{x_jz}}{I_{x_ix_j}}\right)^{n^{KJ}_{ij}}\left(-\frac{I_{x_jz}(J_{v_jz}-J_{v_jx_i})I_{x_iz}}{I_{x_jx_i}}\right)^{n^{KJ}_{ji}}-\text{traces.}
\end{aligned}
$$

In the last equality, $\{n^{KK}_{ij}\},\{n^{JJ}_{ij}\},\{n^{KJ}_{ij}\}$ are parameter families counting how often the contractions $K_{v_ia_i}\diamond_a K_{v_ja_j}$, $J_{a_ix_i}\diamond J_{a_jx_j}$ etc. occur. Since the number of K's is fixed, these parameters obey the additional restriction

$$n^{KK}_{12}+n^{KK}_{13}+n^{KJ}_{12}+n^{KJ}_{13}=s_1, \tag{3.4-131}$$

and similarly for the other legs. C' is just some other family of weights applicable to the contracted vertex and depends in some complicated manner on the $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$.
The last expression which represents the contracted vertex can be even further rationalised: It consists only of three types of basic building blocks

$$I_{x_iz}K_{v_iv_j}I_{x_jz},\qquad\qquad \frac{I_{x_iz}I_{x_jz}}{I_{x_ix_j}},\qquad\qquad J_{v_iz}-J_{v_ix_j}.$$

So ultimately, the vertex in the bulk reduces after all differentiations and contractions have been completed to a form

$$\begin{aligned}
&\tilde{V}^{s_1,s_2,s_3\,(\mu_1),(\mu_2),(\mu_3)}(\mathrm{D}^{z_1},\mathrm{D}^{z_2},\mathrm{D}^{z_3})G^{\text{ft UV }s_1}_{\text{bubo }(\mu_1),(l_1)}(z_1,\underline{x}_1)G^{\text{ft UV }s_2}_{\text{bubo }(\mu_2),(l_2)}(z_2,\underline{x}_2)\\
&G^{\text{ft UV }s_3}_{\text{bubo }(\mu_3),(l_3)}(z_3,\underline{x}_3)\Big|_{z_i=z}\\
=&\sum_{k_{12},k_{23},k_{31},i_{12},i_{23},i_{31},\{j_{pq}\}} c_{k_{12},k_{23},k_{31},i_{12},i_{23},i_{31},\{j_{pq}\}}\; I^{-i_{12}}_{x_1x_2}I^{-i_{23}}_{x_2x_3}I^{-i_{31}}_{x_3x_1}\\
&I^{\mathrm{d}-2+k_{12}+k_{31}+i_{12}+i_{31}}_{x_1z}\;I^{\mathrm{d}-2+k_{23}+k_{12}+i_{23}+i_{12}}_{x_2z}\;I^{\mathrm{d}-2+k_{31}+k_{23}+i_{31}+i_{23}}_{x_3z}\\
&K^{k_{12}}_{v_1v_2}K^{k_{23}}_{v_2v_3}K^{k_{31}}_{v_3v_1}\prod_{p\neq q}(J_{v_pz}-J_{v_px_q})^{j_{pq}}-\text{traces}.
\end{aligned}\tag{3.4-132}$$

Again, c are some counting weights which depend in an obscure manner on $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$, and $\{j_{pq}\}$ is a family of counting variables. Note that the covariant character of the vertex is present on the formal level in this expression. This is an indication that the general form of the vertex which we assumed is correct.

Integration of Vertex Coordinate. For a proper EAdS-presentation (3.4-124), we must now integrate out the z-coordinate in (3.4-132) and match this with the boundary correlation $G^{s_1,s_2,s_3}_{(l_1),(l_2),(l_3)}(\underline{x}_1,\underline{x}_2,\underline{x}_3)$ in the form of lemma 3.5 on page 90. This will give us the symbols c, from which we compute back the symbols C' and finally $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$. This fixes the vertex $\tilde{V}$ (but probably not uniquely).

The generic z-integration of terms like (3.4-132) is developed in appendix B. It is straightforward, if one organises the factors well; there is only one truely tricky sum (B.2-6). Due to lack of time, the final determination of $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$ could not be completed in this work; we are hopeful to solve that problem in the near future. In the meanwhile, we will be content to quote a simple example illustrating the general procedure.

The $G^{1,0,0}_{\text{non-symm}}$-correlation. This correlation function will naturally vanish after symmetrisation; however, the unsymmetrised terms do not vanish and are a convenient vehicle for the explanation of the mechanism we have in mind.

There are three bulk-to-boundary involved; the spin-1 bulk-to-boundary propagator from (3.4-111) is

$$G^{\text{ft UV }1}_{\text{bubo}}(\tilde{z}_1,\underline{x}_1)[\tilde{a},\underline{v}_1]=-\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^2 2^{1-\mathrm{d}}(\mathrm{d}-2)\;\cdot\;I^{\mathrm{d}-2}_{x_1z_1}\left(I_{x_1z_1}K_{v_1a}\right),$$

and the scalar propagators ($j=2,3$)

$$G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_j,\underline{x}_j)=\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^2 2^{1-\mathrm{d}}\;\cdot\;I^{\mathrm{d}-2}_{x_jz_j}.$$

Since the spin-1 propagator has one free index, there must be one covariant derivative acting on either of the other propagators; introducing indeterminate parameters $C^{1,0,0}_{0,1,0}$ and $C^{1,0,0}_{0,0,1}$, the vertex is

$$\tilde{V}^{1,0,0\,\mu_1}(\mathrm{D}^{z_2},\mathrm{D}^{z_3}) = g^{\mu_1\nu}\left(C^{1,0,0}_{0,1,0}\,\mathrm{D}_{z_2^\nu} + C^{1,0,0}_{0,0,1}\,\mathrm{D}_{z_3^\nu}\right). \tag{3.4-133}$$

The EAdS-presentation should have the structure

$$\begin{aligned} &G^{\text{ft UV }1}_{\text{bubo}}(\tilde{z}_1,\underline{x}_1)[\tilde{a},\underline{v}_1] \diamond_a \left(C^{1,0,0}_{0,1,0}a^\nu\,\mathrm{D}_{z_2^\nu} + C^{1,0,0}_{0,0,1}a^\nu\,\mathrm{D}_{z_3^\nu}\right) G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_2,\underline{x}_2) G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_3,\underline{x}_3) \\ &\quad = G^{\text{ft UV }1}_{\text{bubo}\,\mu}(\tilde{z}_1,\underline{x}_1)[\underline{v}_1] g^{\mu\nu}\left(C^{1,0,0}_{0,1,0}\,\mathrm{D}_{z_2^\nu} + C^{1,0,0}_{0,0,1}\,\mathrm{D}_{z_3^\nu}\right) G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_2,\underline{x}_2) G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_3,\underline{x}_3) \end{aligned}$$

(we can make the educated guess $C^{1,0,0}_{0,0,1} = -C^{1,0,0}_{0,1,0}$). The derivatives acting on the propagators yield by (3.4-117a)

$$a^\mu\,\mathrm{D}_{z_2^\mu}\,G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_2,\underline{x}_2) = -\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^2 2^{1-\mathrm{d}}(\mathrm{d}-2)\;\cdot\;J_{ax_2}I^{\mathrm{d}-2}_{x_2z_2},$$

and similarly for propagator 3. For the contractions therefore, by (3.4-129),

$$\begin{aligned} &G^{\text{ft UV }1}_{\text{bubo}}(\tilde{z}_1,\underline{x}_1)[\tilde{a},\underline{v}_1] \diamond_a a^\mu\,\mathrm{D}_{z_2^\mu}\,G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_2,\underline{x}_2) G^{\text{ft UV }0}_{\text{bubo}}(\tilde{z}_3,\underline{x}_3) \\ &\quad = \left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^6 2^{3-3\mathrm{d}}(\mathrm{d}-2)^2\;\cdot\;I^{\mathrm{d}-2}_{x_1z_1}\left(I_{x_1z_1}K_{v_1a}\right)\diamond_a J_{ax_2}I^{\mathrm{d}-2}_{x_2z_2} \\ &\qquad = -\left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}}\right)^6 2^{3-3\mathrm{d}}(\mathrm{d}-2)^2\;\cdot\;I^{\mathrm{d}-1}_{x_1z_1}\frac{J_{v_1z}-J_{v_1x_2}}{I_{x_1x_2}}I^{\mathrm{d}-1}_{x_2z_2}, \end{aligned}$$

and similarly for the $C^{1,0,0}_{0,0,1}$-term. We must therefore integrate

$$I_2 = \int \frac{\mathrm{d}z^0\,\mathrm{d}^\mathrm{d}z}{(z^0)^{\mathrm{d}+1}} I^{\mathrm{d}-1}_{x_1z_1}\frac{J_{v_1z}-J_{v_1x_2}}{I_{x_1x_2}}I^{\mathrm{d}-1}_{x_2z_2}I^{\mathrm{d}-2}_{x_3z_3}. \tag{3.4-134}$$

By the general integral formula (B.2-9) of appendix B, the result is

$$I_2 = 2^{\frac{3\mathrm{d}}{2}-3}\pi^{\frac{\mathrm{d}}{2}}\frac{\Gamma(\frac{\mathrm{d}}{2})^2\Gamma(\frac{\mathrm{d}}{2}-1)}{\Gamma(\mathrm{d})\Gamma(\mathrm{d}-1)}\left(I_{x_1x_2}I_{x_2x_3}I_{x_3x_1}\right)^{\frac{\mathrm{d}}{2}-1}\left(J_{v_1x_2}-J_{v_1x_3}\right). \tag{3.4-135}$$

For the I_3-term multiplying $C^{1,0,0}_{0,0,1}$ a similar computation gives $I_3 = -I_2$. This looks astonishing: It seems that the $C^{1,0,0}_{0,1,0}$- or $C^{1,0,0}_{0,0,1}$-term *alone* would already suffice to generate a working bulk vertex, in the sense of the EAdS-presentation, since after integrating out the vertex, one obtains already a multiple of the full, unsymmetrised boundary correlation $G^{1,0,0}_{\text{non-symm}}$, as given in (3.4-105). However, the $C^{1,0,0}_{0,1,0}$ and $C^{1,0,0}_{0,0,1}$-terms of the vertex alone do not have the same symmetry as the boundary Wick contractions for $G^{1,0,0}_{\text{non-symm}}$. On second thought, we had to expect that: The $C^{1,0,0}_{0,1,0}$-term (3.4-134) is conformally invariant as it stands, and so is the boundary function (3.4-135) we have obtained after integrating out the vertex. It is the *only* conformally invariant possibility on the boundary to couple a vector to two scalars (unsymmetrised), and therefore the form of the result cannot be different.

The total correlation is

$$G^{1,0,0}_{\text{non-symm}} = (C^{1,0,0}_{0,1,0} - C^{1,0,0}_{0,0,1}) \left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}} \right)^6 2^{3-3\mathrm{d}}(\mathrm{d}-2)$$
$$2^{\frac{3\mathrm{d}}{2}-3}\pi^{\frac{\mathrm{d}}{2}} \frac{\Gamma(\frac{\mathrm{d}}{2})^2\Gamma(\frac{\mathrm{d}}{2}-1)}{\Gamma(\mathrm{d})\Gamma(\mathrm{d}-2)} (I_{x_1x_2}I_{x_2x_3}I_{x_3x_1})^{\frac{\mathrm{d}}{2}-1} (J_{v_1x_2} - J_{v_1x_3}) \, .$$

By symmetry considerations, one has to choose $C^{1,0,0}_{0,0,1} = -C^{1,0,0}_{0,1,0}$. Comparing with the boundary result (3.4-108), we find

$$C^{1,0,0}_{0,1,0} = N \left(\frac{\Gamma(\frac{\mathrm{d}}{2}-1)\mathcal{N}}{4\pi^{\frac{\mathrm{d}}{2}}} \right)^{-3} \frac{\Gamma(\mathrm{d})\Gamma(\mathrm{d}-2)}{\pi^{\frac{\mathrm{d}}{2}}\Gamma(\frac{\mathrm{d}}{2})^2\Gamma(\frac{\mathrm{d}}{2}-1)}. \tag{3.4-136}$$

3.4.6 Summary

We have in this section set up a programme for the determination of the vertex couplings in the bulk which are needed for a successful EAdS-presentation (3.4-67) of the boundary three-point functions of bilinear tensor currents (3.4-66). Several points were of importance:

1. It is crucial to insist on conformal invariance of the intermediate expressions at all times.
2. Notation is greatly simplified when relying on the embedding space notation throughout, along the lines of section 3.4.1.
3. The problem of bulk vertices has been reduced to a combinatorical problem of matching prefactors, described in paragraph "Integration of Vertex Coordinate" on page 100. In particular, the integration of a generic, parametrised vertex as described in paragraph "Vertex Structure" on page 95 is under control; what is missing is the matching of the generic prefactors $C^{s_1,s_2,s_3}_{d_1,d_2,d_3}$ to the boundary correlations which are given as polynomials in conformal invariants (3.4-96).
4. As a necessary prerequisite, we have also discussed the bulk-to-boundary propagators and their characterising equations in section 3.4.4.

Although we were not able to finish this programme, the outlook is very good.

Since we have discussed the EAdS-presentation of boundary three-point functions of currents in the free UV theory, we will in the next section enhance EAdS-presentation to n-point functions of currents. Subsequently, we will have to clarify the question of the field system in the bulk, as carriers of the representations we have found. This will be a step towards the question whether there is a physical grounding to the EAdS-presentation.

3.5 EAdS-Presentation of n-Point Functions in the Free UV Theory and Construction of n-valent Vertices

We have sketched in the last section how 3-point functions of bilinear twist-2 currents in the free UV fixpoint theory may be given an EAdS-presentation; in short, we almost completed an equality which can be written formally

$$G^{s_1,s_2,s_3}_{(l_1),(l_2),(l_3)}(\underline{x}_1,\underline{x}_2,\underline{x}_3) = \int \frac{\mathrm{d}^{\mathrm{d}} z\ \mathrm{d}z^0}{(z^0)^{\mathrm{d}+1}} \tilde{V}^{s_1,s_2,s_3\ (\mu_1),(\mu_2),(\mu_3)}(\mathrm{D}^{z_1},\mathrm{D}^{z_2},\mathrm{D}^{z_3}) \\ G^{\text{ft UV } s_1}_{\text{bubo } (\mu_1),(l_1)}(z_1,\underline{x}_1) G^{\text{ft UV } s_2}_{\text{bubo } (\mu_2),(l_2)}(z_2,\underline{x}_2) G^{\text{ft UV } s_3}_{\text{bubo } (\mu_3),(l_3)}(z_3,\underline{x}_3)\Big|_{z_i=z}, \tag{3.5-137}$$

where $\tilde{V}$ acts as a differential operator on the three propagators and is of order N^1. The propagators are the bulk-to-boundary propagators (3.4-111); and we have agreed to choose them from the field-theoretic prescription, in order to have a definite normalisation with respect to the boundary-to-boundary propagators. They are of order N^0, in the normalisation of section 3.4.4.

We have thus served the case $n = 3$ of definition 3.4 on page 74. In this section, we are going to continue the examination to $n \geq 4$; based on the results for the three-valent vertex, it will turn out that higher-order vertices are easier to access: They are given in terms of the three-vertices.

According to proposition 2.4 on page 41 and the following remark 2.5, a correlation function $\langle \mathcal{J}^{s_1}(\underline{x}_1)\ldots\mathcal{J}^{s_n}(\underline{x}_n)\rangle_{\text{conn}}$ is equivalent to the inverse Catalan number C^{-1}_{n-2} times a sum of all amplitudes generated by all possible tree graphs (in the context, they appeared as "cyclic commutative non-associative structures", in short CCNA's, cf. page 38) containing symmetric, traceless quasi-primary tensor currents of all even spins s bilinear in the fields ϕ, using the (EAdS-presented) three-point functions $G^{s,t,u}(\underline{x},\underline{y},\underline{z})$ as vertices and integrating out the coordinates $\underline{x}$ with the inverse propagator $D^s(\partial_{\underline{x}})$ defined by equation (2.7-36) as kernel whenever two such correlations have a common midpoint

$$\sum_s \int \mathrm{d}^{\mathrm{d}}x\, G^{s,s_1,s_2}(\underline{x},\underline{y}_1,\underline{y}_2) D^s(\partial_{\underline{x}}) G^{s,t_1,t_2}(\underline{x},\underline{z}_1,\underline{z}_2)$$

(suppressing tensor indices). The external currents $\mathcal{J}^{s_j}(\underline{x}_j)$ are inserted at the tips of the tree.

Whenever there appears such a linking term, in the EAdS-presentation this will lead to an effective bulk-to-bulk propagator of order N^{-1} (since $D^s \sim N^{-1}$)

$$G^{\text{eff } s}_{\text{bu } (\mu_1),(\mu_2)}(z_1, z_2) = \int \mathrm{d}^{\mathrm{d}} x \, G^{\text{ft UV } s}_{\text{bubo } (\mu_1),(l_1)}(z_1, \underline{x}) \; D^{s \, (l_1),(l_2)}(\partial_{\underline{x}}) \; G^{\text{ft UV } s}_{\text{bubo } (\mu_2),(l_2)}(z_2, \underline{x}). \tag{3.5-138}$$

It is tempting to interpret this as the bulk-to-bulk propagator; however, by construction $G^{\text{eff } s}_{\text{bu}}(z_1, z_2)$ should obey the equations of motion w.r.t. z_1 and z_2 on all EAdS; there is no singularity on the diagonal $z_1 = z_2$ as one expects for any decent propagator. By the general arguments of section 3.2.2 (cf. equation (3.2-38)), the effective propagator $G^{\text{eff } s}_{\text{bu}}(z_1, z_2)$ is the difference of the bulk-to-bulk-propagators of different boundary scaling dimensions; it is a completely regular solution of the equation of motion, and not a Green's function. With the dimension $\Delta^{\text{UV}}_s = \mathrm{d} - 2 + s$,

$$G^{\text{eff } s}_{\text{bu}}(z_1, z_2) = G^{\Delta^{\text{UV}}_s \; s}_{\text{bu}}(z_1, z_2) - G^{(\mathrm{d}-\Delta^{\text{UV}}_s) \; s}_{\text{bu}}(z_1, z_2) \tag{3.5-139}$$

($G^{(\mathrm{d}-\Delta^{\text{UV}}_s) \; s}_{\text{bu}}$ never appears as independent propagator because it violates the unitarity bound). The naïve EAdS-presentation of the twist-2 CPWE does not yield the correct propagators for an interpretation as effective (classical) Lagrangian field theory in the bulk, involving the higher spin tensor fields as basic fields. In addition, we could not get the combinatorics right because of the inverse Catalan number C^{-1}_{n-2} appearing in the prefactor demanded by the twist-2 CPWE.

This is not fatal to us, because, by the philosophy of section 3.2.3, we have to find a single (probably nonlocal) bulk vertex $\tilde{V}^{s_1 \dots s_n}(\mathrm{D}^{z_1}, \dots, \mathrm{D}^{z_n})$ which EAdS-presents the n-point correlations. Such a bulk vertex can be obtained from the twist-2 CPWE. We state the procedure in

Proposition 3.7. *The n-valent bulk vertex $\tilde{V}^{s_1 \dots s_n}(\mathrm{D}^{z_1}, \dots, \mathrm{D}^{z_n})$, necessary for the* EAdS*-presentation of the n-point function of quasi-primary bilinear tensor currents in the free UV fixpoint theory according to definition 3.4 on page 74, can be constructed as follows:*

(i) EAdS*-presenting each three-point function arising in the twist-2 CPWE (section 2.7) of the n-point function of the currents by (3.5-137) in the bulk,*

(ii) amputating those propagators which constitute external legs,

(iii) summing over the spins of the internal propagators and integrating out the bulk coordinates of those vertices which are not connected to an external leg.

(iv) summing over all the different CCNA's which constitute the total twist-2 CPWE, with the combinatorial prefactor C^{-1}_{n-2}.

Since the n-point vertices are generated by EAdS-presentation of ϕ-loops with n operator insertions, and each such insertion from the boundary UV theory carries an additional coupling $-i$, there will be a factor $(-i)^n$ which we have to supplement explicitly.

The truth of this proposition follows by construction. Since the boundary ϕ-loops and their twist-2 CPWE are all $\sim N^1$, the resulting n-valent bulk vertices will also be $\sim N^1$. For three-valent vertices, the result is simply given by the EAdS-presented three-valent vertex which we have constructed in section 3.4.

As a more complex example, the quartic scalar bulk vertex is

$$\tilde{V}^{s_1,s_2,s_3,s_4}(z_1,z_2,z_3,z_4) =$$
$$\frac{1}{2}\sum_t \Big\{ \delta^{\text{AdS}}(z_1-z_2)\delta^{\text{AdS}}(z_3-z_4)\ \tilde{V}^{t,s_1,s_2\,(\tilde{\mu})}(y,z_1,z_2)G^{\text{eff}\ t}_{(\tilde{\mu}),(\tilde{\nu})}(\text{bu}\ y,u)\tilde{V}^{t,s_3,s_4\,(\tilde{\nu})}(u,z_3,z_4)\Big|_{\substack{y=z_1\\u=z_3}}$$
$$+\,\delta^{\text{AdS}}(z_1-z_3)\delta^{\text{AdS}}(z_2-z_4)\ \tilde{V}^{t,s_1,s_3\,(\tilde{\mu})}(y,z_1,z_3)G^{\text{eff}\ t}_{\text{bu}\ (\tilde{\mu}),(\tilde{\nu})}(y,u)\tilde{V}^{t,s_2,s_4\,(\tilde{\nu})}(u,z_2,z_4)\Big|_{\substack{y=z_1\\u=z_2}}$$
$$+\,\delta^{\text{AdS}}(z_1-z_4)\delta^{\text{AdS}}(z_2-z_3)\ \tilde{V}^{t,s_1,s_4\,(\tilde{\mu})}(y,z_1,z_4)G^{\text{eff}\ t}_{\text{bu}\ (\tilde{\mu}),(\tilde{\nu})}(y,u)\tilde{V}^{t,s_2,s_3\,(\tilde{\nu})}(u,z_2,z_3)\Big|_{\substack{y=z_1\\u=z_2}}\Big\}.$$

The explicit computation of the vertices shall not be undertaken here, as we are still missing the precise form of the three-valent vertex. Note that the vertices contain derivatives acting on the external propagators to be engrafted onto the vertex; the way we have written it, these are simply to be connected from the right, and the δ-distributions are to be acknowledged on the final integration of coordinates in the finished bulk graph.

One might argue that the sum over spins should make the internal propagators in the n-valent vertices vanish, by the philosophy of section 3.2.3. However, this is not so: The sum over bulk-to-bulk propagators of different spins was to vanish only for the true propagators $G^{\text{UV}\ s}_{\text{bu}}$, and possibly an additional (hypothetical) field must be included into the sum to this effect.

The total scalar 4-point function in the UV theory is then given by

$$G^{\text{UV}\ 0,0,0,0}(\underline{x}_1,\underline{x}_2,\underline{x}_3,\underline{x}_4) = i^4\prod_{j=1}^{4}\int\frac{\mathrm{d}z_j^0\ \mathrm{d}^d z_j}{(z_j^0)^{\text{d}+1}}$$
$$(-i)^4\tilde{V}^{0,0,0,0}(z_1,z_2,z_3,z_4)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_1,\underline{x}_1)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_2,\underline{x}_2)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_3,\underline{x}_3)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_4,\underline{x}_4).$$

Under the hypothesis that the bulk-to-bulk propagation cancels in total, there is a second term which might possibly be contained, but which vanishes completely,

$$G^{\text{UV}\ 0,0,0,0}_{\text{cancel}}(\underline{x}_1,\underline{x}_2,\underline{x}_3,\underline{x}_4) = i^4\prod_{j=1}^{4}\int\frac{\mathrm{d}z_j^0\ \mathrm{d}^d z_j}{(z_j^0)^{\text{d}+1}}\int\frac{\mathrm{d}y^0\ \mathrm{d}^d y}{(y^0)^{\text{d}+1}}\int\frac{\mathrm{d}u^0\ \mathrm{d}^d u}{(u^0)^{\text{d}+1}}$$
$$\sum_s\Big((-i)^3\tilde{V}^{s,0,0}(y,z_1,z_2)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_1,\underline{x}_1)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_2,\underline{x}_2)\ G^{\text{UV}\ s}_{\text{bu}}(y,u)$$
$$(-i)^3\tilde{V}^{s,0,0}(u,z_3,z_4)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_3,\underline{x}_3)G^{\text{ft UV}\ 0}_{\text{bubo}}(z_4,\underline{x}_4) + \text{perms.}\Big),$$

where we have suppressed the tensor indices and written the trivalent vertices in the same nonlocal notation as the four-valent vertex. $G^{\mathrm{UV}\,s}_{\mathrm{bu}}(y,u)$ is the bulk-to-bulk propagator for the intermediate tensor field with spin s. The sum over spins s possibly includes any hypothetical field needed to make the second summand vanish in total, by equation (3.2-56).

By their construction, the n-valent vertices are very non-local in nature. Their internal structure reminds one very much of string theory. We can simulate a "tree-level" interaction of several strings by inserting n vertex operators on a worldsheet with the topology of a sphere. The worldsheet in the vicinity of vertex operator j would then be interpreted as "string j". There is no saying in which order the strings resp. vertex operators interact with each other; even if the string worldsheet is drawn suggestively as a system of tubes linked by regions connecting three tubes each, then by deforming this worldsheet, we can modify the network in such a manner that the order of interactions is different. In a way, the n-valent vertices we have constructed are mirroring this structure since they contain a sum over all different interaction schemes (CCNA's).

The non-local nature of string vertices is further reflected in the effective propagator $G^{\mathrm{eff}\,s}_{\mathrm{bu}}$ which serves as internal propagator inside the vertices: $G^{\mathrm{eff}\,s}_{\mathrm{bu}}(z_1,z_2)$ does not contain singularities on the diagonal $z_1 = z_2$, since it is the difference (3.5-138) of two propagators which have the same singularity behaviour on the diagonal (but different boundary conditions).

So it is not totally unrealistic to imagine that the bulk theory we are constructing may be obtained as an infinite-tension limit of an underlying string theory in (E)AdS.

3.6 Analysis of the Holographic Bulk Theories

We are now approaching the topic of the physical model underlying the EAdS-presentation of boundary correlation functions.

3.6.1 Bulk-to-bulk Propagators of Tensor Fields

While we have discussed the bulk-to-boundary propagators, a full theory in the bulk must certainly contain bulk-to-bulk propagators. We will, without giving them explicitly, shortly discuss how they may be evaluated.

As starting point, consider the "propagator" (3.5-138) which features in the EAdS-presentation of the twist-2 CPWE and which is given by the difference of the field-theoretic and dual prescription for the boundary current in the intermediate channel. We denoted this effective propagator of order $\sim N^{-1}$ by

$$G_{\rm bu}^{{\rm eff}\ s}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}] = \int {\rm d}^{\rm d}x\, G^{{\rm ft\ UV}\ s}_{{\rm bubo}\ (l_1)}(\tilde{z},\underline{x})[\tilde{a}]\ D^{s\ (l_1),(l_2)}(\partial_{\underline{x}})\ G^{{\rm ft\ UV}\ s}_{{\rm bubo}\ (l_2)}(\tilde{u},\underline{x})[\tilde{b}]; \quad (3.6\text{-}140)$$

it is independent of the specific boundary prescriptions for the spin-s tensor field in the bulk (up to a sign). In this notation, the bulk ends of the propagators are contracted with the $\tilde{a}$ resp. $\tilde{b}$-vectors, whereas the boundary ends are contracted directly with the inverse propagator D^s. We have found the bulk-to-boundary propagators (3.4-111); the difficulty now lies in the application of the inverse propagator D^s. This very technical operation has been performed in [60, 62], resulting in a representation of the AdS space propagator in terms of Legendre functions of the second kind.

Note that, by the structure of equation (3.6-140), the propagator $G_{\rm bu}^{{\rm eff}\ s}$ fulfills the equations (3.4-113a) to (3.4-113e) for each end separately (ie, for the pairs of points and tangent vectors $\tilde{z} \in \mathbb{R}^{\rm d+1,1}, \tilde{a} \in T_{\tilde{z}}$ and $\tilde{u} \in \mathbb{R}^{\rm d+1,1}, \tilde{b} \in T_{\tilde{u}}$ separately), which hold for the bulk end of the bulk-to-boundary propagators. A true propagator $G_{\rm bu}^{\Delta\ s}$ with boundary behaviour $(z^0)^\Delta$ should fulfill these equations as well, up to δ-terms on the diagonal (when the endpoints coincide). We have thus the following set of equations, which we write with an undetermined propagator function $G_{\rm bu}^s$:

$$\begin{aligned}
&\tilde{z}^{\tilde{\mu}}\partial_{\tilde{a}^{\tilde{\mu}}}G^s_{\rm bu} = 0, \quad \Box_{\tilde{z}}G^s_{\rm bu} = 0 + \text{diag. terms}, \quad \Box_{\tilde{a}}G^s_{\rm bu} = 0, \quad \partial_{\tilde{z}^{\tilde{\mu}}}\partial_{\tilde{a}_{\tilde{\mu}}}G^s_{\rm bu} = 0 + \text{diag. terms}\\
&\tilde{u}^{\tilde{\mu}}\partial_{\tilde{b}^{\tilde{\mu}}}G^s_{\rm bu} = 0, \quad \Box_{\tilde{u}}G^s_{\rm bu} = 0 + \text{diag. terms}, \quad \Box_{\tilde{b}}G^s_{\rm bu} = 0, \quad \partial_{\tilde{u}^{\tilde{\mu}}}\partial_{\tilde{b}_{\tilde{\mu}}}G^s_{\rm bu} = 0 + \text{diag. terms};
\end{aligned} \quad (3.6\text{-}141)$$

and the homogeneity relation

$$G^s_{\rm bu}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}] = (|\tilde{z}||\tilde{u}|)^{-\Delta}G^s_{\rm bu}\left(\frac{\tilde{z}}{|\tilde{z}|},\frac{\tilde{u}}{|\tilde{u}|}\right)[\tilde{a},\tilde{b}]. \quad (3.6\text{-}142)$$

Note that all these equations hold in the embedding space. Intrinsic EAdS equations could be formulated using the material of section 3.4.4 to relate the embedding space- and the covariant EAdS-derivative, using the homogeneity of the propagator in the embedding space Euclidean coordinates. We know furthermore that G^s_{bu} is of order $\sim N^{-1}$, homogeneous of degree s in $\tilde{a}$ and $\tilde{b}$; moreover, $G^s_{\text{bu}}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}]$ should be symmetric in the argument pairs $(\tilde{z},\tilde{a})$ and $(\tilde{u},\tilde{b})$ by construction.

By (3.4-114), we expect that this system of equations has *two* linearly independent propagator solutions, with either the boundary behaviour $(z^0)^\Delta$ or the conjugate $(z^0)^{\text{d}-\Delta}$; the generic solution is

$$\begin{aligned} G^s_{\text{bu}}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}] =& \alpha G^{\Delta\, s}_{\text{bu}}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}] \\ &+ (1-\alpha)(|\tilde{z}||\tilde{u}|)^{\text{d}-2\Delta} G^{\text{d}-\Delta\, s}_{\text{bu}}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}]. \end{aligned} \tag{3.6-143}$$

By way of its definition, there may no other vectors appear in G^s_{bu}, so that it can be written as a function of scalar products of its arguments,

$$G^s_{\text{bu}}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}] = (|\tilde{z}||\tilde{u}|)^{-\Delta} g^s\left(\frac{(\tilde{z},\tilde{u})}{|\tilde{z}||\tilde{u}|}, \frac{(\tilde{a},\tilde{z})}{|\tilde{z}|}, \frac{(\tilde{b},\tilde{z})}{|\tilde{z}|}, \frac{(\tilde{a},\tilde{u})}{|\tilde{u}|}, \frac{(\tilde{b},\tilde{u})}{|\tilde{u}|}, \tilde{a}^2, \tilde{b}^2, (\tilde{a},\tilde{b})\right). \tag{3.6-144}$$

The orthogonality conditions $\tilde{z}^{\tilde{\mu}}\partial_{\tilde{a}^{\tilde{\mu}}} G^s_{\text{bu}} = 0$ and resp. for $(\tilde{u},\tilde{b})$ are the easiest to fulfill; we use the scheme which is by now well-known and introduce the invariants

$$\begin{aligned} K_{ab} =& (\tilde{a},\tilde{b}) - \frac{(\tilde{a},\tilde{u})(\tilde{z},\tilde{b})}{(\tilde{z},\tilde{u})} \qquad K_{aa} = (\tilde{a},\tilde{a}) - \frac{(\tilde{a},\tilde{z})^2}{(\tilde{z},\tilde{z})} \qquad K_{bb} = (\tilde{b},\tilde{b}) - \frac{(\tilde{b},\tilde{u})^2}{(\tilde{u},\tilde{u})} \\ J_a =& \frac{(\tilde{a},\tilde{u})}{(\tilde{z},\tilde{u})} - \frac{(\tilde{a},\tilde{z})}{(\tilde{z},\tilde{z})} \qquad J_b = \frac{(\tilde{b},\tilde{z})}{(\tilde{z},\tilde{u})} - \frac{(\tilde{b},\tilde{u})}{(\tilde{u},\tilde{u})}. \end{aligned} \tag{3.6-145}$$

Furthermore, define the scale invariant bifunction

$$X = \frac{(\tilde{z},\tilde{u})}{|\tilde{z}||\tilde{u}|} \leq -1. \tag{3.6-146}$$

We find that the most general form possible for the solution is

$$G^s_{\text{bu}}(\tilde{z},\tilde{u})[\tilde{a},\tilde{b}] = (|\tilde{z}||\tilde{u}|)^{-\Delta} g^s\left(X, K_{ab}, K_{aa}, K_{bb}, |\tilde{z}|J_a, |\tilde{u}|J_b\right), \tag{3.6-147}$$

with the usual homogeneity and symmetry requirements for $(\tilde{z},\tilde{a})$ and $(\tilde{u},\tilde{b})$. One can see that this must be the most general ansatz since it is a function of six arguments fulfilling two differential equations, whereas the functions (3.6-144) has eight arguments and does not fulfill any differential equation [12]. We have six equations left, and this should be just sufficient to determine G^s_{bu}, up to a multiple.

For spin 0 one finds, solving these equations up to the normalisation [13],

$$G^0_{\text{bu}} = (|\tilde{z}||\tilde{u}|)^{-\Delta}\left(X^2-1\right)^{\frac{1-\text{d}}{4}}\left(C_1 P^{\frac{1-\text{d}}{2}}_{\Delta-\frac{\text{d}+1}{2}}(-X) + \frac{C_2}{\Gamma(\Delta-\text{d}+1)} Q^{\frac{1-\text{d}}{2}}_{\Delta-\frac{\text{d}+1}{2}}(-X)\right). \tag{3.6-148}$$

[12] A general argument for the invariants which are fundamental for propagators on maximally symmetric spaces can be found in [3].

[13] For $s=0$ and $\Delta = \text{d}-2$, this result can be computed directly from formulas (3.6-140) and (3.5-138), by employing a Schwinger representation not only for the bulk-to-boundary propagator, but also for the kernel of the wave operator D^0. This would also give the proper normalisation.

The C_1-term is the difference of propagators with different boundary conditions, without the singular terms on the diagonal $X = -1$, similar to $G^{\text{eff}\,0}_{\text{bu}}$ obtained from the EAdS-presentation of the twist-2 CPWE. The C_2-term is the scalar bulk-to-bulk propagator with a boundary behaviour $(z^0)^\Delta$; It obeys the differential equation only for $X < -1$, with a singular contribution at $X = -1$. This is in accordance with the results (6.1-18) and (6.1-17) which will be found in a later chapter from a direct functional integral approach.

There is a useful integral representation for the C_1-term valid for $|\Delta - \frac{\text{d}}{2}| < \frac{\text{d}}{2}$, by (C.3-10),

$$(X^2-1)^{\frac{1-\text{d}}{4}} P^{\frac{1-\text{d}}{2}}_{\Delta-\frac{\text{d}+1}{2}}(-X) = \sqrt{\frac{2}{\pi}} \frac{1}{\Gamma(\Delta)\Gamma(\text{d}-\Delta)} \int_0^\infty \frac{\text{d}\tau}{\tau} \tau^{\frac{\text{d}}{2}} e^{X\tau} K_{\Delta-\frac{\text{d}}{2}}(\tau). \quad (3.6\text{-}149)$$

The modified Bessel function of the second kind $K_{\Delta-\frac{\text{d}}{2}}(\tau)$ can be expressed by modified Bessel functions of the first kind,

$$\frac{2}{\Gamma(\frac{\text{d}}{2}-\Delta)\Gamma(1-\frac{\text{d}}{2}+\Delta)} K_{\Delta-\frac{\text{d}}{2}}(\tau) = I_{\Delta-\frac{\text{d}}{2}}(\tau) - I_{\frac{\text{d}}{2}-\Delta}(\tau). \quad (3.6\text{-}150)$$

When the Bessel functions $I_{\Delta-\frac{\text{d}}{2}}(\tau)$ or $I_{\frac{\text{d}}{2}-\Delta}(\tau)$ are substituted into (3.6-149), we obtain certain linear combinations of the C_1- and the C_2-terms; the necessary integral is given in (C.3-11) in the appendix. Each of these generates one particular boundary behaviour of the propagator. Since we want to reserve the term "propagator" for the objects which feature in the actual bulk theory and have a normalisation which is adapted to the normalisation of the vertices $\tilde{V}^{s_1,\ldots,s_n}$, we will for now be content to give the normalised Green's functions $H^{\Delta,\,s}_{\text{bu}}$ in the cases

$$H^{\text{eff}\,0}_{\text{bu}} = -\frac{(|\tilde{z}||\tilde{u}|)^{2-\text{d}}}{(2\pi)^{\frac{\text{d}}{2}}\Gamma(2-\frac{\text{d}}{2})\Gamma(\frac{\text{d}}{2}-1)} \int_0^\infty \frac{\text{d}\tau}{\tau} \tau^{\frac{\text{d}}{2}} e^{X\tau} K_{2-\frac{\text{d}}{2}}(\tau) \quad (3.6\text{-}151\text{a})$$

$$\begin{aligned} H^{\text{d}-2,\,0}_{\text{bu}} &= \frac{(|\tilde{z}||\tilde{u}|)^{2-\text{d}}}{2(2\pi)^{\frac{\text{d}}{2}}} \int_0^\infty \frac{\text{d}\tau}{\tau} \tau^{\frac{\text{d}}{2}} e^{X\tau} I_{\frac{\text{d}}{2}-2}(\tau) \\ &= -\frac{\Gamma(\frac{\text{d}-1}{2})}{4\pi^{\frac{\text{d}+1}{2}}} \frac{(\tilde{z},\tilde{u})}{((\tilde{z},\tilde{u})^2 - |\tilde{z}|^2|\tilde{u}|^2)^{\frac{\text{d}-1}{2}}} \end{aligned} \quad (3.6\text{-}151\text{b})$$

$$H^{2,\,0}_{\text{bu}} = \frac{(|\tilde{z}||\tilde{u}|)^{2-\text{d}}}{2(2\pi)^{\frac{\text{d}}{2}}} \int_0^\infty \frac{\text{d}\tau}{\tau} \tau^{\frac{\text{d}}{2}} e^{X\tau} I_{2-\frac{\text{d}}{2}}(\tau). \quad (3.6\text{-}151\text{c})$$

The Green's function on the branch $\Delta_- = \text{d} - 2 < \frac{\text{d}}{2}$ is an algebraic function, as given. The actual propagators will be multiples of these Green's functions.

For the normalisation of the Green's functions, we have choosen the following conventions: By computing the action of the d'Alembertian, one shows that

$$\left(-\Box^{\text{EAdS}}_{\tilde{z}} + \frac{2(2-\text{d})}{|\tilde{z}|^2}\right) H^{\text{d}-2,\,0}_{\text{bu}} = -\Box_{\tilde{z}} H^{\text{d}-2,\,0}_{\text{bu}} = |\tilde{z}|^{-\text{d}} |\tilde{u}|^{2-\text{d}} \delta^{(\text{d}+1)}_{\text{EAdS}}\left(\frac{\tilde{z}}{|\tilde{z}|}, \frac{\tilde{u}}{|\tilde{u}|}\right), \quad (3.6\text{-}152)$$

and equivalently for the dual prescription. There will be a divergent contribution whenever $\tilde{z}$ and $\tilde{u}$ are lying on the same ray in the embedding space. On the EAdS-hypersurface $|\tilde{z}| = |\tilde{u}| = 1$, $\tilde{z}^{d+1}, \tilde{u}^{d+1} > 0$, we obtain the usual EAdS scalar Green's function.

The first nontrivial case is spin 1 (we choose $\Delta = \mathrm{d} - 1$). For spin 1, the general form (3.6-147) can be reduced to

$$G^1_{\mathrm{bu}} = |\tilde{z}|^{1-\mathrm{d}}|\tilde{u}|^{1-\mathrm{d}}\Big\{g_A(X)\cdot(K_{ab} + (\tilde{z},\tilde{u})J_aJ_b) - Xg_B(X)\cdot K_{ab}\Big\} \qquad (3.6\text{-}153)$$

in terms of the invariants (3.6-145). We have chosen the specific parametrisation with a view to the solution.

While the second-order equations in $\tilde{a}$ and $\tilde{b}$ are automatically fulfilled for a tensor of order 1 (ie, a vector), we have to take care of the $\Box_{\tilde{z}}$- and $\Box_{\tilde{u}}$- and the mixed equations (3.4-113d) to (3.4-113e). The solution of these equations is complicated, but standard, taking care that they are fulfilled for any $\tilde{z}, \tilde{u}, \tilde{a}, \tilde{b}$, and we obtain the general solution

$$g_A(X) = C_1(X^2-1)^{-\frac{\mathrm{d}+3}{4}}P^{-\frac{\mathrm{d}+3}{2}}_{\frac{\mathrm{d}-3}{2}}(-X) + C_2(X^2-1)^{-\frac{\mathrm{d}+3}{4}}Q^{\frac{\mathrm{d}+3}{2}}_{\frac{\mathrm{d}-3}{2}}(-X), \qquad (3.6\text{-}154a)$$

$$g_B(X) = C_1\frac{\mathrm{d}+1}{2}(X^2-1)^{-\frac{\mathrm{d}+3}{4}}P^{-\frac{\mathrm{d}+3}{2}}_{\frac{\mathrm{d}-1}{2}}(-X) - C_2(X^2-1)^{-\frac{\mathrm{d}+3}{4}}Q^{\frac{\mathrm{d}+3}{2}}_{\frac{\mathrm{d}-1}{2}}(-X).$$

We do not give the correct normalisation in this place, because we are just interested in the mechanism for the solution of these equations. It is in principle possible to obtain an integral representation of the kind (3.6-151a) to (3.6-151c) for these expressions. Note that when these solutions are substituted into (3.6-153) to obtain the EAdS propagators, all radii become $|\tilde{z}| = |\tilde{u}| = 1$. The asymptotic behaviour of these two linearly independent solutions reveals that the C_1-term describes the effective propagator $G^{\mathrm{eff}\,1}_{\mathrm{bu}}$ won by lifting of the twist-2 CPWE (see (3.5-138)); by linear combinations of the C_1- and C_2-terms, one obtains the true propagators with behaviour $(z^0)^{\mathrm{d}-1}$ resp. $(z^0)^1$; they diverge at $X = -1$, ie $\underline{z} = \underline{u}$ and $z^0 = u^0$ simultaneously. The propagator with dimension $\Delta_- = 1$ already violates in $2 < \mathrm{d} < 4$ the unitarity bound $\Delta^1_{ub} = \mathrm{d} - 1$ for spin 1.

Summary. We have found characteristic equations for the propagators in the embedding space. The advantage of working in the embedding space is that we can work without having to use the covariant derivative, using the partial derivative of the embedding space exclusively. The mass of the fields is contained in a homogeneity condition; and the system of equations has two propagator solutions, corresponding to the boundary behaviour $(z^0)^\Delta$ and $(z^0)^{\mathrm{d}-\Delta}$. For higher spins, the solution of the propagator equations is involved.

3.6.2 Higher Spin Gauge Symmetries

The bulk theory we are discussing is a gauge theory. For the spin-1 massless tensor, the gauge symmetry has exactly the form of the Maxwell gauge symmetry (vector potential), for the spin-2 tensor (graviton field), it is the Einstein diffeomorphism symmetry; for the higher spin tensors T in the bulk, it is a generalisation. The corresponding gauge transformations are of the form

$$T_{\mu_1\ldots\mu_s} \to T_{\mu_1\ldots\mu_s} + \partial_{(\mu_1}\Lambda_{\mu_2\ldots\mu_s)},$$

with Λ a (traceless) field determining the gauge transformation. Since the subject of gauge transformations is not followed in this work (with the exception of general bulk covariance of the propagators), we will not dwell on this point and refer the reader to the literature. A systematic discussion can be found in [99]; the Goldstone fields in case of a broken symmetry are analysed in [87], and some hands-on computations (concerning gauge fixing) can be examined in [70].

3.6.3 Expansion Rules for the UV Holographic Theory

We summarise the rules of EAdS-presentation which we have found. Since we are going to discuss Lagrangian theories, we will redistribute the factors of i which appeared in this graphical expansion, in such manner that the Lagrangian theories we are going to obtain have a real action. The factors of i which had to be inserted when applying the derivative $i\partial_J$ etc. on the generating function are abolished. The physical content is left completely unchanged by this step. We obtain the following rules:

(F) The bulk theory contains a tensor field $\mathcal{T}^s$ for all even spins s, which may be coupled to boundary sources J^s. Boundary correlations are generated by ∂_J.

(P) Bulk-to-boundary propagators $G^{\text{ft UV } s}_{\text{bubo}}$ for these fields have been analysed in section 3.4.4; two sources on the boundary may also be coupled by a boundary-to-boundary propagator (ibid).

(V) There is a set of bulk vertices $\tilde{V}^{s_1,\ldots,s_n}$ for all valencies $n \geq 3$, linking tensor propagators of every possible spin $s_1, \ldots, s_n$ (and we have found no general rule saying that some subset of these couplings should vanish in general). For $n = 3$, these have been discussed in section 3.4, for $n \geq 4$ in section 3.5 (cf. proposition 3.7 on page 104).

(G) The correlation functions are represented by graphs. A two-point function is represented by a propagator connecting the sources. A (connected) n-point function is represented by a graph containing exactly one vertex, connected to $n \geq 3$ sources. Disconnected correlation functions are obtained by summing over all factorisations into connected correlations.

These rules have been derived with no reference whatsoever to the programme laid down in section 3.2.3; so we are free to interpret them. We can promote (lift) this set of rules into a field theory in the bulk if we supplement the following rule:

(Bu) There are also source terms T^s in the bulk; their correlations are generated by ∂_{T^s}. They are coupled to other bulk source terms and to the vertices by bulk-to-bulk propagators $G_{\text{bu}}^{\text{UV}\, s}$ (section 3.6.1), and to the sources on the boundary by bulk-to-boundary propagators (section 3.4.4).

By application of the rule (Bu), we have a recipe by which to compute correlations between sources in the bulk. This step contains an ad hoc element because the correlations between sources in the bulk never was relevant to the EAdS-presentation. Our justification is that by the group-theoretic considerations of section 3.3, the boundary correlations may be obtained as limits of the bulk correlations when the sources are moved towards the boundary and a suitable scaling factor is applied.

There are competing possible interpretations of this structure; we will discuss them in turn.

3.6.4 Lagrangian Interpretation: The Cancellation of Bulk-to-bulk Propagation in the UV Fixpoint Theory

Under the assumption that the UV holographic theory in the bulk is Lagrangian, we have to find a path integral which would repreduce the rules (F) to (Bu) for the computation of the correlations. If the Planck quantum of action in this supposed Lagrangian theory is proportional to $1/N$, then the conditions for proposition 3.2 on page 65 are met, implying that the UV hologram must be a full quantum theory, and we have to rely on dynamical cancellation of loop diagrams. We will use this as a guideline.

When we give a Lagrangian bulk theory, we have to say which type of boundary source terms we use, either field-theoretic or dual. For the scalar operator of the UV holographic fixpoint theory with scaling dimension $\mathrm{d}-2$ on the lower branch Δ_-, we are forced to choose the field-theoretic convention; for the tensor fields with spin $s \geq 2$ with scaling dimension $\mathrm{d}-2+s$ on the upper branch Δ_+, we have the choice between field-theoretic and dual prescription. For simplicity, we will choose the *field-theoretic* source terms. This means that the field-theoretic propagators which we have used throughout the EAdS-presentation and whose normalisation is given in definition 3.6 on page 91 are the propagators which appear in the Lagrangian and therefore, the normalisations of the vertices etc. do not need to be modified.

The Lagrangian interpretation is supplied by rule (G') which reduces to rule (G) if evaluated:

(G') The correlation functions are represented by graphs. The graphs are derived from an action which is symbolically of the form

$$S[\mathcal{T}] = \int_{\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}+1}z \left\{ \sum_s \frac{1}{2} \mathcal{T}^s (G^s_{\mathrm{bu}})^{-1} \mathcal{T}^s - V(\mathcal{T}) \right\}, \tag{3.6-155}$$

with the interaction potential

$$V(\mathcal{T}) = \sum_{n=3}^{\infty} \sum_{s_1,\dots} \frac{1}{n!} \tilde{V}^{s_1,\dots,s_n} \mathcal{T}^{s_1} \dots \mathcal{T}^{s_n}. \tag{3.6-156}$$

$(G^s_{\mathrm{bu}})^{-1}$ is the inverse propagator, and we have resisted giving "boundary conditions" for it, since it is a differential operator. It should be noted that the action is proportional to N, since $\tilde{V} \sim N$ and all propagators $G^s_{\mathrm{bu}} \sim N^{-1}$. The path integral is

$$\mathscr{Z}_{[J]}[T] = \int \mathscr{D}(\mathcal{T}) \exp\left(-S[\mathcal{T}] + \sum_s \int_{\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}+1}z\, T^s \mathcal{T}^s + \sum_s \int_{\partial\,\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}}x\, J^s \mathcal{T}^s \right). \tag{3.6-157}$$

The last term represents the field-theoretic coupling of the sources at the boundary. *Under the assumption that cancellation of bulk-to-bulk propagation takes place*, the UV theory holographic correspondence is defined by

$$\left\langle \exp\left(\sum_s \int \mathrm{d}^{\mathrm{d}}x\, \mathcal{J}^s J^s \right) \right\rangle_{\mathrm{UV}} = \frac{\mathscr{Z}_{[J]}[0]}{\mathscr{Z}_{[0]}[0]}. \tag{3.6-158}$$

This Lagrangian is symbolic only because inverse bulk-to-bulk propagator $(G^s_{\mathrm{bu}})^{-1}$ does not exist as it stands. Research has been going on for a long time on this topic, initiated by Fronsdal (eg [40, 41]) who showed that one needs a set of auxiliary tensor fields φ^{s-2} which ultimately drop out in order to get the correct equations of motion for the (free) higher spin tensor fields in the bulk. Formally, if we believe that these problems can be solved somehow, then the path integral will result in the propagators $G^{\mathrm{UV}\ s}_{\mathrm{bu}}$, with boundary behaviour $(z^0)^{\mathrm{d}-2+s}$.

As indicated, there must be a mechanism suppressing graphs containing loops, if this Lagrangian theory should reduce correctly to the EAdS-presentation summarised by the rules (F) to (G). As a possible mechanism, we have suggested in hypothesis 3.3 on page 67 that bulk-to-bulk propagation cancels when summed over all spins, and probably under inclusion of one or several additional fields; in that case, we have to supplement a set of vertices and propagators for these fields. That means we have to prove that

$$\sum_s \tilde{V}^{s,\dots}_1(\mathrm{D}^{z_1},\dots)\, G^{\mathrm{UV}\ s}_{\mathrm{bu}}(z_1,z_2)\, \tilde{V}^{s,\dots}_2(\overleftarrow{\mathrm{D}}^{z_2},\dots) = 0. \tag{3.6-159}$$

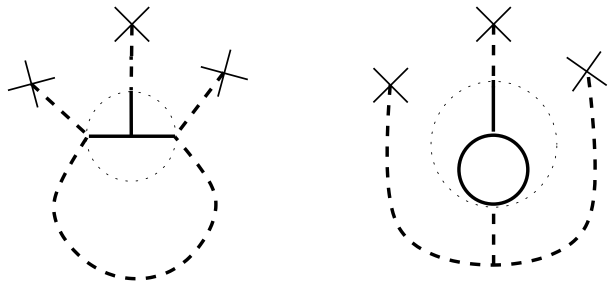

Figure 3.3: For explanation, see section 3.6.4.

On the first glance, this seems rather outlandish: As $\tilde{V}_1$ and $\tilde{V}_2$ can be vertices of arbitrary order constructed in section 3.5 and each vertex takes a different form, there seems to be an amount of constraints which is impossible to fulfill. However, as we have constructed the n-valent bulk vertices by EAdS-presentation of the twist-2 CPWE, they consist solely of the elementary three-valent vertex $\tilde{V}^{s_1,s_2,s_3}$ as building block. We conclude

Proposition 3.8. *If relation (3.6-159) holds for $\tilde{V}_1, \tilde{V}_2$ three-valent bulk vertices, then it will hold for all different n-valent vertices in the bulk obtained by* EAdS-*presentation of the twist-2 CPWE.*

This is certainly good news.

Crossed Channels. One might consider the possibility that the cancellation mechanism (3.6-159) includes crossed channels in some way or other. Since crossed channels are only defined for three-valent vertices, such an argument would be on the level of the elementary three-vertices making up all graphs. However, since by channel crossing, one changes the topology, it happens sometimes that one generates a (forbidden) n-valent bulk vertex which contains a closed loop, as in figure 3.3 on page 114. Left is a bulk graph in the UV fixpoint holographic theory. Crosses are boundary sources; they are linked by bulk-to-boundary propagators (dashed lines) to the slightly dotted circle enclosing a 5-valent vertex $\tilde{V}$. The vertex arises by the mechanism described in section 3.5 out a sum of tree diagrams with effective propagators $G^{\text{eff } s}_{\text{bu}}$ (drawn through lines) generated by the twist-2 CPWE of the boundary 5-point function of bilinear currents. One particular summand of this five-point function is displayed. The dashed propagator at the bottom is a bulk-to-bulk propagator forming a loop; eventually, this diagram must vanish dynamically in the UV fixpoint holographic theory.

On the right, one of the crossed channels of this bulk-to-bulk propagator is displayed. The topology of the effective graph defining the *vertex* has changed; this new vertex contains a loop and cannot arise out of any EAdS-presentation of a boundary correlation; so it is not a valid vertex of the UV fixpoint holographic theory. We conclude that the left diagram must vanish on its own and that the dynamical mechanism responsible for cancellation of bulk-to-bulk propagation does not involve crossed channels (since this operation does not always create admissible graphs).

The natural setup for examining bulk-to-bulk propagation is therefore the holographic four-point function, in restriction to the s-channel. Currently we do not have all necessary tools at hand in order to compute the net bulk-to-bulk propagation and deliver an explicit answer to the question whether or how bulk-to-bulk propagation cancels.

Nullifier Field. As a "brute force" method, one might consider adding for every bulk tensor field of spin s a "nullifier" tensor field $\mathcal{T}'^s$ with the same mass and spin s and in the same boundary prescription, which couples to the same vertices, only with an additional factor i in the coupling (alternatively, one could endow it with a propagator carrying an additional factor -1) [14]. This certainly puts an end to bulk-to-bulk propagation. However, this would also imply that there is a second family of boundary currents $\mathcal{J}'^s$ with integer (even) spin, behaving very similar to $\mathcal{J}^s$ in the (mixed) connected n-point functions

$$\langle \mathcal{J}^{s_1} \dots \mathcal{J}^{s_j} \mathcal{J}'^{t_1} \dots \mathcal{J}'^{t_n} \rangle_{\text{conn}} = i^n \langle \mathcal{J}^{s_1} \dots \mathcal{J}^{s_j} \mathcal{J}^{t_1} \dots \mathcal{J}^{t_n} \rangle_{\text{conn}}, \qquad (n \geq 3),$$

but having a vanishing mixed two-point function $\langle \mathcal{J}'^s \mathcal{J}^s \rangle = 0$. A second family with these properties does not exist. And since the two-point function $\langle \mathcal{J}'^s \mathcal{J}'^s \rangle = \langle \mathcal{J}^s \mathcal{J}^s \rangle$ does not fit into the above scheme, the fields $\mathcal{T}^s$ and $\mathcal{T}'^s$ can be distinguished by an observer on the boundary, and it is not possible to argue that all source terms couple to $\frac{1}{2}(\mathcal{T}^s - i\mathcal{T}'^s)$.

Semi-classical Path Integral We can write down a generating function for the correlations defined by rules (F) to (Bu); we will in turn interpret the generating function as a semi-classical path-integral.

The classical field equation for the tensor fields in the bulk with the source terms J^s on the boundary is formally

$$\left((G^s_{\text{bu}})^{-1}\mathcal{T}^s\right)(z) = T^s(z) + \delta(z^0) J^s(\underline{z}). \tag{3.6-160}$$

We supplement this with the boundary behaviour $(z^0)^{\mathrm{d}-2+s}$ required by the scaling dimensions of the boundary operators. This equation has then the unique solution

$$\mathcal{T}^s_{[J]}[T] = G^{\text{UV}\,s}_{\text{bu}}\, T^s + G^{\text{ft UV}\,s}_{\text{bubo}}\, J^s, \tag{3.6-161}$$

where the propagators act as convolution operators. The generating function is

$$\mathscr{Z}_{[J]}[T] = \mathscr{Z}_0^{-1} \exp\Big(- S[\mathcal{T}] + \sum_s \int_{\text{EAdS}} \mathrm{d}^{\mathrm{d}+1} z\, T^s \mathcal{T}^s + \sum_s \int_{\partial\,\text{EAdS}} \mathrm{d}^{\mathrm{d}} x\, J^s \mathcal{T}^s \Big)\Big|_{\mathcal{T}^s = \mathcal{T}^s_{[J]}[T]}, \tag{3.6-162}$$

[14] Krotov and Morozov [59] have taken a similar approach to construct a holographic theory. However, in their case, the vanishing mechanism is simply enforced by having two identical real scalar fields, with the couplings of the second field differing by a factor i.

with the action (3.6-155). One reads off the correlations by substituting $\mathcal{T}^s_{[J]}[T]$ and observing that for field-theoretic propagators, both

$$\begin{aligned}(G^s_{\mathrm{bu}})^{-1} G^{\mathrm{UV}\ s}_{\mathrm{bu}}\ T^s &= T^s \\ (G^s_{\mathrm{bu}})^{-1} G^{\mathrm{ft\ UV}\ s}_{\mathrm{bubo}}\ J^s &= J^s.\end{aligned}$$

Looking at the interaction potential, it is clear that source terms T^s in the bulk and J^s on the boundary couple to the vertices by propagators $G^{\mathrm{UV}\ s}_{\mathrm{bu}}$ resp. $G^{\mathrm{ft\ UV}\ s}_{\mathrm{bubo}}$. Likewise, the boundary-to-boundary propagator is $G^{\mathrm{ft\ UV}\ s}_{\mathrm{bo}}$, with the positive sign. So the proof of the statement is immediate [15].

We suggest to interpret this generating function as a semi-classical path integral with a domain of integration consisting of a single field configuration.

For the holographic theory corresponding to the IR fixpoint theory, we have to perform a functional Fourier transform with respect to the sources J^0 on the boundary. Since J^0 acts as boundary source term in the solution $\mathcal{T}^s_{[J]}[T]$ and therefore controls its boundary behaviour, the functional Fourier transform acts as a path integral which varies over all possible solutions of the free field equation (3.6-160) with different boundary values. We obtain thus

$$\begin{aligned}\mathscr{Z}_{[K^0],\, s\geq 2:\, [J^s]}[T] = \mathscr{Z}_0^{-1} \int \mathscr{D}(J^0) \exp\Big(- S[\mathcal{T}] + \sum_s \int_{\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}+1} z\, T^s \mathcal{T}^s \\ + \sum_s \int_{\partial\,\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}} x\, J^s \mathcal{T}^s + i \int_{\partial\,\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}} x\, K^0 J^0 \Big)\Big|_{\mathcal{T}^s = \mathcal{T}^s_{[J]}[T]} \,. \qquad (3.6\text{-}163)\end{aligned}$$

The holographic correspondence is defined by

$$\Big\langle \exp\Big(\sum_{s\geq 2} \int \mathrm{d}^{\mathrm{d}} x\, \mathcal{J}^s J^s + \int \mathrm{d}^{\mathrm{d}} x\, \sigma K^0 \Big) \Big\rangle_{\mathrm{IR}} = \frac{\mathscr{Z}_{[K^0],\, s\geq 2:\, [J^s]}[0]}{\mathscr{Z}_{[0],\, s\geq 2:\, [0]}[0]}. \qquad (3.6\text{-}164)$$

The factor i which appears here in the coupling between J^0 and K^0 leads to imaginary boundary correlations for the field σ. We have to look back to (2.1-5) right at the beginning, when we introduced the auxiliary field σ: By the way we introduced it, the odd correlations of σ are imaginary.

A comment on the status of this integral is in required. The quadratic Gaussian kernel for J^0 is given by the two-point function $G^{\mathrm{ft\ UV}\ 0}_{\mathrm{bo}}$. This is a positive definite function, so the integral is only formally defined. The two-point function for the boundary terms iK^0 is $-(G^{\mathrm{ft\ UV}\ 0}_{\mathrm{bo}})^{-1}$. A possible solution is to substitute $J^0 \to -iJ^0$. The J^0-path integral is the bulk analog of the σ-path integral in the boundary theory; so we should not be astonished that it is only a formal device.

[15] By formally integrating out the nullifier field of the preceding paragraph, one could find this result as well; however, the formal difficulties arising out of the manipulation of different propagators etc. are rather confusing.

There are actually two bulk-to-bulk scalar propagators in the IR fixpoint holographic theory: For the bulk-to-bulk propagator between vertices, we have to convolute the bulk-to-boundary propagators with the inverse boundary-to-boundary propagator,

$$G^{\mathrm{eff}\,0}_{\mathrm{bu}} = G^{\mathrm{ft\ UV}\,0}_{\mathrm{bubo}}(-G^{\mathrm{ft\ UV}\,0}_{\mathrm{bo}})^{-1}G^{\mathrm{ft\ UV}\,0}_{\mathrm{bobu}}. \tag{3.6-165}$$

The bulk-to-bulk propagator between source terms T^0 on the other hand includes the "direct" propagator which is a part of the Lagrangian; by (3.2-38),

$$G^{\mathrm{UV}\,0}_{\mathrm{bu}} + G^{\mathrm{ft\ UV}\,0}_{\mathrm{bubo}}(-G^{\mathrm{ft\ UV}\,0}_{\mathrm{bo}})^{-1}G^{\mathrm{ft\ UV}\,0}_{\mathrm{bobu}} = G^{\mathrm{IR}\,0}_{\mathrm{bu}}, \tag{3.6-166}$$

since the scalar in the UV theory have scaling dimension $\Delta_0^{\mathrm{UV}} = \mathrm{d}-2$ on the lower branch Δ_- and in the IR theory, $\Delta_0^{\mathrm{IR}} = 2$ on the upper branch.

We summarise our results in

Theorem 3.9. *The correlation functions of the UV fixpoint holographic theory can be computed from the semi-classical path integral in the bulk $\mathscr{Z}_{[J]}[T]$ defined in equation (3.6-162). The domain of the path "integration" is given by the unique solution (3.6-161) of the free equation of motion (3.6-160) in the bulk (including the source terms in the bulk and on the boundary); the AdS/CFT correspondence is installed by (3.6-158).*
The correlation functions of the IR fixpoint holographic theory can be computed from the semi-classical path integral in the bulk $\mathscr{Z}_{[K^0],\, s\geq 2:\, [J^s]}[T]$ defined in equation (3.6-163). The domain of the path integration ranges over the solutions (3.6-161) of the free equation of motion (3.6-160) in the bulk, where the variational degrees of freedom are the boundary source terms for the scalar field. The AdS/CFT correspondence is installed by (3.6-164).

These results are somewhere in the middle between the "classical correspondence" and the statement of proposition 3.2 on page 65 that the UV fixpoint holographic theory is a quantum field theory if it is Lagrangian and the $1/N$-expansion applies in the bulk. For a purely classical theory, the interactions should be part of the equation of motion (3.6-160).

So if one is content with this semi-classical path integral, then the problem of cancellation of bulk-to-bulk propagation does not arise. There are axiomatic issues, however, which we now discuss.

3.6.5 Axiomatic Interpretation

Precluding the question of a Lagrangian strategy, we may ask whether the UV fixpoint holographic bulk theory whose correlations we are instructed to compute by rules (F) to (Bu) on page 111 is well defined in the axiomatic sense [16]. We may

[16] The notion of "field-theoretic" and "dual" boundary source terms for the bulk fields does not make sense in this setting as there is no path integral involved, however, we will keep these terms to differentiate between the propagators which have to be used.

use the semi-classical path integral described in theorem 3.9 on page 117 to compute these correlations efficiently. A similar analysis must be performed for the IR fixpoint holographic theory.

A simple set of axioms for the correlation functions (Schwinger functions) of scalar quantum field theories on Euclidean Anti-de-Sitter space, as well as the meaning of the Wick rotation for this space, has been given in [13]; these rules correspond directly to the Osterwalder-Schrader axioms [74, 75] for Euclidean quantum field theories on flat space.

We shortly relate these axioms for the case of EAdS (we arrange them following [91]). Let $G_n(z_1, \dots, z_n)$ denote the Schwinger functions of n scalar fields at $z_j \in$ EAdS. By the Osterwalder-Schrader axioms, these functions must fulfill

(OS1) Covariance under the full EAdS group $SO(\mathrm{d}+1, 1)$.

(OS2) Symmetry in the arguments.

(OS3) Analyticity at all non-coinciding points ($z_j \neq z_k$); in particular, this implies that

$$G_n(I(z_1), \dots, I(z_n)) = \overline{G_n(z_1, \dots, z_n)},$$

where I is the inversion at the unit sphere (3.1-11).

(OS4) Cluster Property. For any two sets of non-coincident points $z'_1, \dots z'_m \in$ EAdS and $z_1, \dots z_n \in$ EAdS,

$$\lim_{\alpha\to\infty} G_{m+n}(z'_1, \dots, z'_m,\ \alpha z_1, \dots, \alpha z_n) = G_m(z'_1, \dots, z'_m) G_n(z_1, \dots z_n).$$

(OS5) Reflection Positivity. Let $\mathscr{S}(\mathrm{EAdS}^n)$ be the space of Schwartz functions on the n-fold tensor product of EAdS. If $f_0 \in \mathbb{C}$ and, for $1 \le m \le M$, $f_m \in \mathscr{S}(\mathrm{EAdS}^m)$ has its support in $\{(z_1, \dots, z_m) : 1 > (z_1^0)^2 + \underline{z}_1^2 > \dots > (z_m^0)^2 + \underline{z}_m^2\}$, then, with the convention $G_0 = 1$,

$$\sum_{m,n=0}^{M} \int \mathrm{d}z'_1 \cdots \mathrm{d}z'_m \ \mathrm{d}z_1 \cdots \mathrm{d}z_n \ \overline{f_m(z'_1, \dots, z'_m)} f_n(z_1, \dots, z_n)$$
$$G_{m+n}(I(z'_m), \dots, I(z'_1), z_1, \dots, z_n) \ge 0, \quad (3.6\text{-}167)$$

where I is the inversion at the unit sphere (3.1-11). So the meaning of reflection positivity on EAdS is rather one of "inversion positivity".

If a Euclidean theory on EAdS does not obey reflection positivity, then after the Wick rotation, the resulting quantum field theory on AdS does not obey the equivalent of Wightman positivity for AdS, making the construction of an underlying Hilbert space impossible [85].

We must demand these axioms to hold for the proposed UV fixpoint holographic theory; there are some complications because it is a tensor field theory, so the axioms must be adapted to test functions and correlations (Schwinger functions) with tensor indices. There might be positivity problems because the tensor fields are gauge fields by assumption. But we can already get results if we restrict the examination to the scalar sector of the bulk theory.

The axioms (OS1) to (OS4) can be checked without much difficulty. Axiom (OS5) is nontrivial. The most basic test for reflection positivity is of course the case $M = 1$, where it leads to the condition: If $f \in \mathscr{S}(\mathrm{EAdS})$ has its support in $\{z : 1 > (z^0)^2 + \underline{z}^2\}$, then

$$0 \leq \int \mathrm{d}z' \, \mathrm{d}z \, \overline{f(z')} f(z) G_2(I(z'), z) = \int \mathrm{d}z' \, \mathrm{d}z \, \overline{f(z')} G_{\mathrm{bu}}^{\mathrm{UV}\,0}(I(z'), z) f(z). \tag{3.6-168}$$

Since the scalar propagator $G_{\mathrm{bu}}^{\mathrm{UV}\,0}$ is *positive*, we find that for $M = 1$, reflection positivity is fulfilled. For $M \geq 2$, the test involves the integration of vertices. The analysis is by no means a formality: The effective description in theorem 3.9 on page 117 is not a path integral in the usual sense, so the formal arguments of [85] which guarantee the positivity of a Euclidean theory defined by a path integral are not applicable.

For the IR theory, the two-point function $G_{\mathrm{bu}}^{\mathrm{IR}\,0}$ in the bulk is positive, so for $M = 1$ reflection positivity in the bulk holds at least to the leading order in $1/N$. Since the semi-classical path integral for the IR fixpoint holographic theory is only formally defined, this theory carries the hidden germ of non-positivity. It is a question whether this ultimately breaks through (when $M > 1$) or whether positivity prevails.

We did not complete this analysis, but we want to point out that the bulk theories can only be regarded as a sensible theories if the Osterwalder-Schrader axioms are completely fulfilled.

3.6.6 A Free UV Bulk Theory

K.-H. Rehren has suggested [17] that the UV holographic fixpoint theory in the bulk may be set up as free theory, since the boundary theory is free. The free $O(N)$ boundary theory contains the massless $O(N)$ vector (conformal scalar) field ϕ with scaling dimension $\frac{\mathrm{d}}{2} - 1$. One could argue that the UV holographic fixpoint theory contains a single $O(N)$ vector (EAdS scalar) field φ with mass $1 - \frac{\mathrm{d}^2}{4}$; by (3.4-120), this would exactly reproduce the correlations of the boundary field ϕ if we choose the propagator on the branch Δ_-. The bilinear tensor currents $\mathcal{J}^s$ on the boundary would then be realised as bilinears in the bulk field φ; and they could possibly be lifted into the bulk easily by finding corresponding EAdS bilinears in φ which reduce to the currents $\mathcal{J}^s$ in the limit where the field operators approach the boundary.

For the free theory, this is certainly a very appealing point of view. Could it be that this theory is equivalent to the UV fixpoint holographic theory which we have constructed by EAdS-presentation? The simplest test for this hypothesis is to study

[17] private communication

the two-point function in the bulk. In the free theory suggested by Rehren, the two-point function between operators $:\varphi^2:$ in the bulk should be equal (or a a multiple of) the two-point function of the scalar field $\mathcal{T}^0$ of our previously constructed theory. Since $\mathcal{T}^0$ has boundary scaling dimension $\mathrm{d}-2$ and φ has boundary scaling dimension $\frac{\mathrm{d}}{2}-1$, we should verify an equality of the sort

$$(G_{\mathrm{bu}}^{\frac{\mathrm{d}}{2}-1,0})^2 \sim G_{\mathrm{bu}}^{\mathrm{d}-2,0}.$$

between the corresponding bulk-to-bulk propagators. We did not determine the scalar propagator $G^{\frac{\mathrm{d}}{2}-1,0}$; but we have checked that the the square root $(G_{\mathrm{bu}}^{\mathrm{d}-2,0})^{1/2}$ does not solve the wave equation for an EAdS scalar of mass $1-\frac{\mathrm{d}^2}{4}$ (or any other mass). So if there is an equivalence, it is not on such simple level; there might eg be smearing involved. In contrast, for the bulk-to-boundary propagators, the corresponding identity

$$(G_{\mathrm{bubo}}^{\frac{\mathrm{d}}{2}-1,0})^2 \sim G_{\mathrm{bubo}}^{\mathrm{d}-2,0} \tag{3.6-169}$$

does hold, since their functional form (3.4-112) is very simple. The bulk-to-boundary propagators of the tensor fields $\mathcal{T}^s$ should presumably obey a similar relation (ie one should be able to construct them from $G_{\mathrm{bubo}}^{\frac{\mathrm{d}}{2}-1,0}$ and its derivatives).

The IR fixpoint holographic theory is then still attainable by functional Fourier transform with respect to the boundary source terms J^0. However, when we transform the free bulk theory in this way, we obtain a theory whose interactions are localised exclusively on the boundary.

This is a general construction: *Any* conformal boundary theory constructed by perturbation around a conformal free field with arbitrary scaling dimension (as advocated in [32, 33]) has an immediate AdS/CFT holographic correspondent, by representing the (free) conformal boundary fields through boundary correlations of (free) fields in AdS. The interactions (vertices) are in such an approach always localised purely on the boundary, and the propagators between the vertices are boundary-to-boundary propagators.

If we integrate out the bulk vertices of the holographic IR or UV fixpoint theory which we have constructed by EAdS-presentation, then we obtain a theory which should be very similar in appearance: All the interactions are again localised on the boundary. In a sense, while these free bulk theories are situated in EAdS, they do not realise the AdS/CFT correspondence on the descriptive level, but rather are “prior” to it.

Why do we insist on obtaining interactions in the bulk? After all, a theory is characterised by its correlations (and not by some method to compute them – one method is as good as any other). This is the point of view of the algebraic version of the correspondence [83]. The answer lies in the nature of the Maldacena conjecture [68] of the AdS/CFT correspondence. It can be pointedly formulated as a correspondence between different descriptions of the same theory in the bulk and on the boundary. This is to say that we associate different (physical) pictures with the theories which we compare (eg "string theory" vs. "CFT"). At least as long as we do not have perfect control over the physical implications of any model we might formulate, the description will continue to play a role for us. In that sense, the theory with interactions in the bulk is nearer to a string theoretical description in the bulk than the same theory formulated with interactions on the boundary only.

Chapter 4

Conclusions and Perspective

Since the second part does not pursue further the main line of argument, but rather follows a sideline and presents many technical computations, we draw the conclusions right after the main body of the text.

In the present work, we construct a realisation of the AdS/CFT correspondence for the conformal UV and IR fixpoint theories of the $O(N)$-symmetric ϕ^4 vector model in $2 < \mathrm{d} < 4$ dimensions, in terms of a semi-classical path integral in Euclidean Anti-de-Sitter space.

The construction is based on the graphical $1/N$-expansion of the boundary theories; by a procedure called "EAdS-presentation", the graphs of this expansion are transported step-by-step into Euclidean Anti-de-Sitter space, where they take the form of covariant integral representations, again with a graphical structure. The correlation functions of the fixpoint theories on the boundary have thus a diagrammatic expansion in Euclidean Anti-de-Sitter space. By formulating a set of rules governing the syntax of these diagrams in the bulk which does not refer to the boundary correlation functions, these integral representations are promoted to a prescription for computing correlation functions between operators localised in bulk EAdS. In a second step, we obtain a semi-classical "path integral" in bulk EAdS which produces precisely those correlations.

This procedure is performed explicitely for the UV fixpoint theory on the boundary; by "UV/IR duality", the results are extended also to the IR fixpoint theory.

The bulk holographic theories contain tensor fields of all even spins and vertices of arbitrary order n, starting at $n = 3$. The vertices are highly non-local, and in their structure are suggestive of string theory. The UV fixpoint theory on the boundary is the free $O(N)$ vector theory, and the corresponding holographic theory in the bulk has a very simple diagrammatic expansion, where connected correlations contain at most one vertex and all loop graphs are suppressed. The IR fixpoint theory on the boundary is interacting, and correspondingly, the bulk theory has a complex diagrammatic expansion.

The semi-classical character of the bulk path integrals is contained in their path integral domain of integration: It ranges only over configurations obeying the free equations of motion in the bulk, taking into account source terms in the bulk and on the boundary. Only the scalar field in the IR fixpoint holographic theory has variational degrees of freedom left since its boundary value is not fixed a priori. This is interesting because the entropy problem and the question of how the degrees of freedom on the boundary and in the bulk relate to each other are problems which are debated hotly, leading back to the works of Bekenstein and Hawking [8, 52] on black hole entropy which initiated the holographic era. In the semi-classical path integral as we have found it, the degrees of freedom are "shared" between bulk EAdS and its conformal boundary; either viewpoint has its own right. This is a very strong indication that we are on the right track. In the UV fixpoint holographic theory, the semi-classical "path integral" has no degrees of freedom at all and ranges only over a single configuration.

In order to complete the construction of the bulk theories, the precise form of the three-vertices in the bulk must be computed; this project could not be finished within the confines of this thesis, but we have preliminary results which are getting very close to completion. To establish the form of the vertices would indeed be novel; in particular, as the precise form of the vertices does not rely on very specific model assumptions but rather follows from rather general premises and may be applicable to similar models.

A detailed, axiomatic characterisation of the Euclidean bulk theories is eminently important. This involves testing the axiom of reflection positivity. As long as we do not have clarity on that point, we cannot decide whether the bulk theories do make sense at all as quantum theories. We must point out that this characterisation will in the IR case require additional regularisation of the amplitudes, as the boundary IR theory as we have handled it still contains residual divergences, and these will naturally appear in the bulk theory as well.

Lastly, there is the question of interpretation of the bulk theories: The AdS/CFT correspondence as advocated by Maldacena [68] relates a conformal field theory on the boundary to a bulk theory containing gravity; so do the bulk theories we have constructed bear some relation to gravity? Certainly, the UV fixpoint holographic theory is a very plain theory; we have discussed briefly a possibility how the UV fixpoint holographic bulk theory might be related explicitly to a free bulk theory. In the best case, it will be a sort of "free" field theory on a gravity background. There are some rudimentary interactions between the various tensor fields, in particular all fields are coupled to the spin-2 tensor field, which might model a simple interaction with gravitons; so this is not completely impossible. On the other hand, from the point of view of Rehren duality [83], there is no immediate reason why the bulk theories have to bear any resemblance to gravity. The question has to remain open for the time being, until a detailed investigation into the phenomenology of the bulk theories is made.

The construction of the bulk theories in this thesis has relied heavily on the particular structure of the $O(N)$-symmetric ϕ^4 vector model on the boundary. A generalisation of the results might aim to include also boundary operators which are not in the $O(N)$ singlet sector. Ultimately, this leads to the question whether the methods used in this thesis are applicable to the full Yang-Mills theory with gauge group $O(N)$ (at least pure Yang-Mills, without coupled fermions). This construction faces the difficulty that the $1/N$-expansion in this case is very complex already on the boundary; in the double-line notation of t'Hooft [94], it is a genus expansion. The resummation which could be performed comfortably in the $O(N)$-symmetric ϕ^4 vector model is seemingly impossible. So we do not see how the simple method of EAdS-presentation would be applied.

The significance of this current work from my personal point of view lies in the fact that for the specific perturbative model which I have considered, the AdS/CFT correspondence is filled with life; and since the determination of the vertices is only a small step away, we can attack a host of challenging questions which are relevant for the AdS/CFT concept as a whole, in this particular model.

Part II

Technical Supplement: Schwinger Parametrisation with Constraints

The Schwinger parametrisation is a common technique in perturbative Euclidean quantum field theory to compute Feynman amplitudes. It is based on the close connection between Green's functions of the Laplace operator and the heat kernel, which is a solution of the hyperbolic partial differential heat equation. The "time" parameter in this equation is a variable which is introduced artificially.

In particular on curved spaces, this method is very popular, as the heat kernel can be expanded systematically in the local curvature, and already the low-order terms will yield very good approximations to the Green's function [100]. One advantage of this method is that the asymptotic behaviour of the heat kernel is very well known; so boundary value problems are under very good control.

In this chapter we want to discuss the Schwinger parametrisation of simple quantum field systems under the assumption of additional constraints on the fields. Such constraints arise naturally in the AdS/CFT correspondence where the dependence of the partition function of a quantum field on anti-de Sitter space is studied as a functional of its boundary values at infinity. The type of constraint which applies in this system is actually very subtle: Because the expectation of the field operator vanishes even in the unconstrained case when the localisation of the field operator approaches conformal infinity, the constraints have to be put on suitably scaled expectation values. While this is no disaster, it shows that the concept of a constraint is a very broad one. – In the path integral approach, where the fields are varied over a family of possible configurations, constraints are linear functionals, evaluating these field configurations. The effect of these constraints on the Green's functions (propagators) of the quantum fields can be modelled by appropriate boundary conditions. The same is true for the heat kernel which underlies the Schwinger parametrisation.

The constraints realised through linear maps have not always to equal zero. In some cases (including the mentioned EAdS case) we might want to assign them a particular value. This is equivalent to demanding that the fields are having prescribed boundary values; in the discussion of the holographic conjecture in the preceding chapters, this has been termed the "dual prescription". In perturbative calculations, these boundary values will couple to the Feynman graphs via the bulk-to-boundary (or boundary-to-bulk) propagators. These again can be found as solutions of certain partial differential equations with the appropriate boundary condition. Is there something like a Schwinger representation for these kernels? After all, they are maps from the space of possible boundary values into the space of functions over the manifold supporting the theory. When the Schwinger kernel is interpreted as the solution of the heat equation, we might make the educated guess that the "bulk-to-boundary Schwinger kernel" describes *absorption* at the boundary – the localisation region of the constraint. We will see that there are conditions when this point of view can be supported; in other cases (like Euclidean AdS) it will fail, as the boundary is simply "too far away" from any region of interest. We will show that by a local rescaling of fields in the EAdS setting, the boundary can be "brought into reach" and there does exist a Schwinger kernel for the bulk-to-boundary propagator.

Because handling general constraints involves much dealing with distributions, we will perform all calculations in a Hilbert space setting which is very well adapted to the problem; however, the usual spaces $L^2(\mathcal{M}, g)$ and $\mathscr{S}(\mathcal{M}, g)$ will not figure prominently. This will make some formulae look quite unusual. However, we feel that turning away from well trodden paths is justified by the claims of the problem at hand, and accordingly the results. We have tried to distill a set of abstract assumptions characterising the important spaces and their relations; the results will the be obtained by formal manipulation of these basic assumptions. In this way, a broad set of constraint situations is covered.

The plan of the second part is as follows: In chapter 5, we formulate the path integral under constraints, and develop a formalism which tells us how to handle the Schwinger parametrisation under these constraints. This involves the simple example of a massive scalar $\phi(x)$ field on the real line, restricted by the constraint $\phi(0) = \phi_0$. In chapter 6, we discuss the application to scalar fields on EAdS.

Remark. The path integrals which we consider in this second part fall exclusively under the notion of "Neumann path integrals" in the language of section 3.2.2. However, the formalism which will be developed in chapter 5 should be general enough to be applicable also the Dirichlet path integral; we have not tried this, though.

Chapter 5

Path Integrals with Constraints

5.1 Perturbation Theory on Curved Euclidean Spaces

As our main application will be a field theory on anti-de Sitter space, it makes sense to work on curved spaces right from the beginning. In this section we will introduce perturbation theory on curved spaces via the path integral. The developments will be largely formal. Note that there are only few examples of curved spaces where it has been shown that Wick rotation makes sense, notably the spaces of constant curvature.

We begin with a Lagrangian field theory of a single scalar field ϕ on a curved Euclidean space $\mathcal{M}(g)$ with d dimensions and metric $g_{\mu\nu}$. To illustrate the principle, we include a ϕ^n-interaction. The action is [1]

$$S[\phi] = \int \mathrm{d}^{\mathrm{d}}x \, \sqrt{g} \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{c_n}{n!} \phi^n \right\},$$

and we are computing the path integral

$$\mathscr{Z}[J] = \int \mathscr{D}(\phi) \exp -\frac{1}{\hbar} \left[S[\phi] + \langle J, \phi \rangle_g \right],$$

with $\langle J, \phi \rangle_g = \int \mathrm{d}^{\mathrm{d}}x \, \sqrt{g(x)} J(x) \phi(x)$. The sources J are taken from some appropriate test function space $\mathscr{F}(\mathcal{M})$, and the fields ϕ are living in the dual space (path space) $\mathscr{F}'(\mathcal{M})$ [43]. For this reason, the path integral can only be a formal device.

The first step leading to the Feynman expansion is integration by parts of the kinetic term, yielding

$$\int \mathrm{d}^{\mathrm{d}}x \, \sqrt{g} \partial_\mu \phi \partial^\mu \phi = - \int \mathrm{d}^{\mathrm{d}}x \, \sqrt{g} \phi \Box^g \phi,$$

where $\Box^g = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu$ is the d'Alembertian on curved manifolds. The Feynman

[1] m has units $[L^{-1}]$.

expansion is a Taylor expansion in the coupling constant c_n:

$$\begin{aligned}\mathscr{Z}[J] =& \int \mathscr{D}(\phi) \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{c_n}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \sqrt{g} \frac{\phi^n}{n!} \right)^j \exp -\frac{1}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \sqrt{g} \left\{ \frac{1}{2}\phi(m^2 - \Box^g)\phi + J\phi \right\} \\ =& \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{c_n}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \frac{\sqrt{g}}{n!} \left(-\frac{\hbar}{\sqrt{g}} \frac{\partial}{\partial J} \right)^n \right)^j \int \mathscr{D}(\phi) \exp -\frac{1}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \sqrt{g} \left\{ \frac{1}{2}\phi(m^2 - \Box^g)\phi + J\phi \right\} \\ =& \mathscr{Z}[0] \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{c_n}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \frac{\sqrt{g}}{n!} \left(-\frac{\hbar}{\sqrt{g}} \frac{\partial}{\partial J} \right)^n \right)^j \exp \frac{1}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \sqrt{g} \left\{ \frac{1}{2} J(m^2 - \Box^g)^{-1} J \right\}, \end{aligned} \tag{5.1-1}$$

where the normalisation is the usual determinant

$$\mathscr{Z}[0] = \left\| \frac{m^2 - \Box^g}{2\pi\hbar} \right\|_g^{-1/2}.$$

This leaves us with the following Feynman rules in coordinate space:

- *Propagators* G are given by the integral kernel of

$$\frac{\hbar}{m^2 - \Box^g},$$

  so they fulfill the differential equation

$$(m^2 - \Box^g)\Pi = \frac{\hbar}{\sqrt{g}} \delta_{(\mathcal{M},g)}.$$

  Here, $\delta_{(\mathcal{M},g)} = g^{-1/2}\,\delta^{(\mathrm{d})}$ is the Dirac delta distribution appropriate to the manifold $\mathcal{M}$.

- Each *vertex* carries a coordinate x and corresponds to

$$-\frac{c_n}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \sqrt{g}.$$

- *Source terms* are coupled to the ends of propagators and carry

$$-\frac{1}{\hbar} \int \mathrm{d}^{\mathrm{d}}x \sqrt{g} J.$$

- There are the usual *symmetry factors* Sym^{-1} associated to overcounting of diagrams.

As a rule of thumb, all prefactors are exactly as they appear in the exponential of the path integral, with the exception of the propagator which gets an additional minus sign (and therefore is positive).

5.2 Schwinger Parametrisation of Propagator

In order to determine explicitly the Green's function, we have to find the inverse of the operator $(m^2 - \Box^g)$. We can always try to expand in (generalised) eigenfunctions; as it is self-adjoint by assumption, these will be orthogonal and we can use them as a basis. An example is flat space, where the eigenfunctions are plane waves and we are finally led to momentum space loop integrals. The inversion can be performed by the Schwinger integral

$$\frac{1}{m^2 - \Box^g} = \int \mathrm{d}\alpha \, \exp -\alpha(m^2 - \Box^g). \tag{5.2-2}$$

The integral kernel $\tilde{K}_\alpha(x,y)$ of the operator $\exp -\alpha(m^2 - \Box^g)$ is the heat kernel at "Schwinger time" $\alpha > 0$. It fulfills the differential equation

$$\frac{\partial}{\partial\alpha}\tilde{K}_\alpha(x,y) = -(m^2 - \Box^g)_x \tilde{K}_\alpha(x,y). \tag{5.2-3}$$

For $\alpha = 0$, we find the initial condition

$$\tilde{K}_0(x,y) = g^{-1/2}\delta^{(\mathrm{d})}(x,y) \tag{5.2-4}$$

($g^{-1/2}$ appears due to the ever present $\int \mathrm{d}^{\mathrm{d}}x \,\sqrt{g}$ in integrations), so that

$$\int \mathrm{d}^{\mathrm{d}}y \,\sqrt{g(y)}\tilde{K}_0(x,y)\phi(y) = \phi(x).$$

We assume that $\tilde{K}_\alpha(x,y)$ is a smooth function for $\alpha > 0$. If we interpret $\tilde{K}_\alpha(x,y)$ as a density at x, then we find for the integral

$$\tilde{V}_\alpha(y) = \int \mathrm{d}^{\mathrm{d}}x \,\sqrt{g(x)}\tilde{K}_\alpha(x,y)$$

the differential equation

$$\frac{\partial}{\partial\alpha}\tilde{V}_\alpha(y) = -\int \mathrm{d}^{\mathrm{d}}x \,\sqrt{g(x)}(m^2 - \Box^g_x)\tilde{K}_\alpha(x,y) = -m^2\tilde{V}_\alpha(y),$$

using again partial integration. With the initial condition $\tilde{V}_0(y) = 1$ from (5.2-4) the solution is

$$\tilde{V}_\alpha(y) = \exp -m^2\alpha.$$

The mass does not play a very interesting role in the heat kernel: If $K_\alpha(x,y)$ is the heat kernel for $m = 0$, then we can always obtain

$$\tilde{K}_\alpha(x,y) = e^{-m^2\alpha}K_\alpha(x,y).$$

We will therefore concentrate on $K_\alpha(x,y)$. Very few heat kernels are actually known analytically. In flat space, the heat kernel is given by the appropriate Gaussian kernel for the Wiener measure. In curved space, it is approximately so for small times and neighbourhoods.

5.3 Implementing Constraints

So far, we have not made any statements about the boundary values of the fields. In flat space, this seems hardly necessary indeed: The boundary is infinitely far away, and the conditions at "infinity" are irrelevant to what is happening in the region of observation. There are however situations where a "boundary" can be reached in a finite time and it makes sense to prescribe boundary values for the fields [42].

Assume that a constraint on field configurations ϕ is given in the form

$$B\phi = f, \tag{5.3-5}$$

where B is some linear operator mapping the field configuration space into the "constraint space" and f is a function giving the prescribed value of the constraint. For example, if the manifold $\mathcal{M}$ had a boundary, then B could be the procedure of taking the limit of the field as we approach the boundary, and f would be some function on the boundary prescribing the limiting values. In the operator language the constraint takes the form

$$B\langle\phi\ldots\rangle = f\langle\ldots\rangle, \tag{5.3-6}$$

where the dots indicate any other operators, as long as their support stays away from the support of $B\phi$, and we compute the expectation in some state.

Loosely speaking, in the path integral we want to include a factor $\delta(B\phi - f)$, where δ is an appropriate Dirac distribution. Equation (5.3-5) is quite problematic: If we take some dual space $\mathscr{F}(\mathcal{M})'$ as configuration space (eg $\mathscr{F}(\mathcal{M}) = \mathscr{S}(\mathcal{M})$ the Schwartz space), then an operation like "taking the boundary value" is clearly impossible. We will therefore make a very strong assumption: Writing the free quadratic part of the Lagrangian as a sesquilinear form $\Pi^{-1}(\phi, \phi)$, we assume that the fields are living in a Hilbert space $\phi \in \mathscr{H}$ with a scalar product defined by this sesquilinear form. Field configurations falling into this category have to be differentiable almost everywhere, and so a reasonable boundary value can be expected to exist – although it cannot be expected to be very smooth or have decent behaviour. We take the view that path integration is a formal development; the formal *result* of the path integration will be taken as a *definition* of the covariance under constraints on a curved space. Introducing the abbreviation $\langle\phi\rangle_g = \int \mathrm{d}^d x \sqrt{g}\,\phi$, the sesquilinear form will also be written

$$\Pi^{-1}(\phi, \phi) = \langle\phi^*(m^2 - \Box^g)\phi\rangle_g.$$

To implement the δ-distribution, we follow the procedure indicated in [32] and introduce a *path integral* to enforce the constraint. We treat the boundary projection as a function $\mathcal{C} \ni c \mapsto (B\phi)(c)$ on the *constraint manifold* $\mathcal{C}$. $\mathcal{C}$ is the equivalent of a "boundary manifold" where the constraints are localised. The constraint $B\phi(c) = f(c)$ would then be enforced by the path integral

$$\int \mathscr{D}(\sigma) \exp \frac{i}{\hbar}\sigma[B\phi - f], \tag{5.3-7}$$

where σ is integrated over the elements of an appropriate dual space to the space of constraint functions on the boundary. Before we can formulate the constrained path integral, we must formalise these concepts.

Definitions

Definition (Configuration Space and Boundary Space)**.** We assume that Π^{-1} is a positive, symmetric sesquilinear form (anti-linear in the first argument). Define the scalar product

$$\langle f, h\rangle \equiv \Pi^{-1}(f, h) \tag{5.3-8}$$

on its domain $Q(\Pi^{-1})$; by completion, we obtain the (real or complex) Hilbert space $\mathscr{H}$. Π^{-1} acts locally like the differential quadratic form $\langle f^*(m^2 - \Box^g)h\rangle_g$. We expect that $\mathscr{H}$ contains all vectors of interest to the theory which may possibly be subject to the operator Π^{-1} (in general, it is *not* a subspace of $L^2(\mathcal{M}, g)$). We denote the adjoint of an operator A in this Hilbert space by $A^\dagger$.

The "boundary space" or "constraint space" $\mathscr{B}$ is a Hilbert space; the "constraint map" $B : \mathscr{H} \to \mathscr{B}$ is a partial isometry with domain $\operatorname{Dom} B = \mathscr{H}_{\mathrm{bo}}$, range $\operatorname{Ran} B = \mathscr{B}$ and kernel $\operatorname{Ker} B = \mathscr{H}_{\mathrm{bu}}$. It implies the orthogonal decomposition $\mathscr{H} = \mathscr{H}_{\mathrm{bo}} \oplus \mathscr{H}_{\mathrm{bu}}$. We define the self-adjoint projection $\mathcal{P}_{\mathrm{bu}} : \mathscr{H} \to \mathscr{H}_{\mathrm{bu}} \subset \mathscr{H}$.

We introduce the space $\mathscr{B}$ because many authors like to view the boundary as separate space in its own right.

Definition (Dual Structure [101])**.** Let $\mathscr{H}'$ be the dual space of $\mathscr{H}$. As $\mathscr{H}$ is a Hilbert space, $\mathscr{H}'$ is a Hilbert space isomorphic to $\mathscr{H}$; still, as the scalar product contains derivatives, it will make things easier to retain the dual space as a separate entity. The Banach space product between vectors $f \in \mathscr{H}$ and dual vectors $g \in \mathscr{H}'$ is denoted $\langle f, g\rangle_{\mathscr{H}}$ or $\langle g, f\rangle_{\mathscr{H}}$ (these are anti-linear in the left component and linear in the right component). Note that due to the sesquilinearity of the product $\langle , \rangle_{\mathscr{H}}$, the Banach space dual is anti-linear: $(zA)' = \overline{z}A'$, $z \in \mathbb{C}$, $A : \mathscr{H} \to \mathscr{H}$. By the Riesz theorem, every functional $f \in \mathscr{H}'$ corresponds to a vector "Πf" in $\mathscr{H}$. The scalar product in $\mathscr{H}'$ is

$$\langle f, g\rangle' = \langle \Pi f, \Pi g\rangle, \qquad f, g \in \mathscr{H}'$$

and the adjoint $A^{\dagger'}$. The natural mappings

$$\begin{aligned} \Pi : \quad & \mathscr{H}' \to \mathscr{H} \\ \Pi^{-1} : \quad & \mathscr{H} \to \mathscr{H}' \end{aligned}$$

are isometric isomorphisms. We overload the symbol Π^{-1} here; it denotes the original sesquilinear form (5.3-8) as well as the operator.

Under Π^{-1}, the decomposition of $\mathscr{H}$ is mapped on the decomposition $\mathscr{H}' = \mathscr{H}'_{\mathrm{bu}} \oplus \mathscr{H}'_{\mathrm{bo}} = \Pi^{-1}\mathscr{H}_{\mathrm{bu}} \oplus \Pi^{-1}\mathscr{H}_{\mathrm{bo}}$. The associated self-adjoint projection in $\mathscr{H}'$ is $\mathcal{P}'_{\mathrm{bu}}$:

$\mathscr{H}' \to \mathscr{H}'_{\mathrm{bu}} \subset \mathscr{H}'$; it is the (Banach space!) dual of $\mathcal{P}_{\mathrm{bu}}$. The ranges and kernels of the projections annihilate each other "crosswise":

$$\langle \mathscr{H}'_{\mathrm{bo}}, \mathscr{H}_{\mathrm{bu}} \rangle_{\mathscr{H}} = 0 = \langle \mathscr{H}'_{\mathrm{bu}}, \mathscr{H}_{\mathrm{bo}} \rangle_{\mathscr{H}}. \tag{5.3-9}$$

We denote the Banach space product on the boundary by $\langle b, b' \rangle_{\mathscr{B}}$, $b \in \mathscr{B}$, $b' \in \mathscr{B}'$ (anti-linear in the first entry). The constraint map B has a (Banach space) dual $B' : \mathscr{B}' \to \mathscr{H}'_{\mathrm{bo}} \subset \mathscr{H}'$ which is an inclusion fulfilling $\langle Bf, b' \rangle_{\mathscr{B}} = \langle f, B'b' \rangle_{\mathscr{H}}$, $f \in \mathscr{H}$, $b' \in \mathscr{B}'$.

In complete parallel to the operator Π on the bulk, we define

$$\Pi_{\mathrm{bo}} = B \Pi B' : \mathscr{B}' \to \mathscr{B} \tag{5.3-10}$$

for the boundary. We demand that Π_{bo} is invertible. The scalar product on $\mathscr{B}$ will be denoted

$$\langle f_B, g_B \rangle = \langle f_B, \Pi_{\mathrm{bo}}^{-1} g_B \rangle_{\mathscr{B}} = \langle (1 - \mathcal{P}_{\mathrm{bu}}) f, (1 - \mathcal{P}_{\mathrm{bu}}) g \rangle$$

where $f_B = Bf \in \mathscr{B}$, $g_B = Bg \in \mathscr{B}$, $f, g \in \mathscr{H}$. The dual boundary $\mathscr{B}'$ has a natural scalar product $\langle b', c' \rangle' \equiv \langle B'b', B'c' \rangle' = \langle \Pi_{\mathrm{bo}} b', c' \rangle_{\mathscr{B}}$, where $b', c' \in \mathscr{B}'$.

The following diagram visualises the spaces and maps relevant in the sequel:

$$\begin{tikzcd}
 & & & & \mathscr{H}' \arrow[r, "\mathcal{P}'_{\mathrm{bu}}", "="'] \arrow[d, "\Pi"] \arrow[rrrr, bend left, "\Pi_{\mathrm{bobu}}"] & \mathscr{H}'_{\mathrm{bu}} \arrow[d, "\Pi_{\mathrm{bu}}"] & \oplus & \mathscr{H}'_{\mathrm{bo}} & \mathscr{B}' \arrow[l, "\cong", "B'"'] \arrow[d, "\Pi_{\mathrm{bo}}"] & & \\
\mathcal{D}_{\mathrm{bu}} & \subset & \mathcal{D} \arrow[urr, "\imath"] & \subset & \mathscr{H} \arrow[r, "=", "\mathcal{P}_{\mathrm{bu}}"'] & \mathscr{H}_{\mathrm{bu}} & \oplus & \mathscr{H}_{\mathrm{bo}} \arrow[r, "\cong", "B"'] & \mathscr{B} \arrow[llll, bend left, "\Pi_{\mathrm{bubo}}"] & \supset & \mathcal{D}_{\mathcal{C}} \arrow[ull, "\jmath"']
\end{tikzcd}$$

By simple algebra, one finds that the Hilbert space adjoint $A^\dagger$ of an operator $A : \mathscr{H} \to \mathscr{H}$ is related to the Banach space dual $A' : \mathscr{H}' \to \mathscr{H}'$ by

$$A^\dagger = \Pi A' \Pi^{-1}.$$

Likewise, $A^{\dagger'} = \Pi^{-1} A' \Pi$ for $A : \mathscr{H}' \to \mathscr{H}'$. Finally, there is the algebraic result

$$\begin{aligned} \mathcal{P}_{\mathrm{bu}} &= 1 - \Pi\, B' \,(B\, \Pi\, B')^{-1}\, B, \\ \mathcal{P}'_{\mathrm{bu}} &= 1 - B' \,(B\, \Pi\, B')^{-1}\, B\, \Pi. \end{aligned} \tag{5.3-11}$$

The *local* interpretation is supplied by the following

Assumption 5.1 (Local Interpretation)**.** We assume that there exists an interpretation of $\mathscr{H}$ as a set of g-measurable functions. So if $f, g \in \mathscr{H} \cap L^2(\mathcal{M}, g)$, then we know that the scalar product

$$\langle f, h \rangle_g = \int_{\mathcal{M}} \mathrm{d}x \, \sqrt{g}\, f(x)^* h(x)$$

is finite.

The local structure is implemented by a self-dual operator (in the Banach space sense) $\imath : \mathcal{D} \subset \mathscr{H} \to \mathscr{H}'$ with the defining property

$$\langle \imath f, g \rangle_{\mathscr{H}} := \langle f, g \rangle_g, \qquad f, g \in \mathcal{D}. \tag{5.3-12}$$

The scalar product $\langle , \rangle_g$ is well-defined only for $f, g \in L^2(\mathcal{M}, g)$, requiring $\mathcal{D} \subset L^2(\mathcal{M}, g)$. It is assumed to be non-degenerate on $\mathcal{D}$; ie $\langle d, d \rangle_g = 0$ implies $d = 0$ in $\mathcal{D}$ [2]. We define also the subspace $\mathcal{D}_{\mathrm{bu}} = \mathcal{D} \cap \mathscr{H}_{\mathrm{bu}}$.

Due to the non-degeneracy of $\langle , \rangle_g$ on $\mathcal{D}$, the map $\imath$ is injective. The definition of $\imath$ implies that $\Pi\imath$ and $\imath\Pi$ are self-adjoint operators in the Hilbert space $\mathscr{H}$ resp. $\mathscr{H}'$. We enlarge the definition of the g-scalar product by setting

$$\langle h, d \rangle_g := \langle h, \imath d \rangle_{\mathscr{H}}, \qquad h \in \mathscr{H},\, d \in \mathcal{D}. \tag{5.3-13}$$

We need a technical

Assumption 5.2. We assume that $\imath\mathcal{D}_{\mathrm{bu}} \cap \mathscr{H}'_{\mathrm{bo}} = \{0\}$, and that $\imath\mathcal{D}_{\mathrm{bu}} + \mathscr{H}'_{\mathrm{bo}}$ is dense in $\mathscr{H}'$. Finally, $\mathcal{D}_{\mathrm{bu}}$ is supposed to be dense in $\mathscr{H}_{\mathrm{bu}}$ and $\mathcal{P}'_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}} : \mathcal{D}_{\mathrm{bu}} \subset \mathscr{H}_{\mathrm{bu}} \to \mathscr{H}'_{\mathrm{bu}}$ self-dual in the restriction to the bulk space.

To justify the first assumption, suppose that there exists $f \in \mathcal{D}_{\mathrm{bu}}$ such that $\imath f \in \mathscr{H}'_{\mathrm{bo}}$. Then, for all $h \in \mathscr{H}_{\mathrm{bu}}$, $\langle h, f \rangle_g = \langle h, \imath f \rangle_{\mathscr{H}} = 0$; so f is weakly localised purely "on the boundary", while at the same time, it is an element of $\mathcal{D}_{\mathrm{bu}}$, so it has a vanishing boundary value $Bf = 0$ - a rather unpleasant situation. The last assumption says that that inclusion with Dirichlet boundary conditions is self-dual.

Remark. In an $L^2(\mathcal{M}, g)$-setting, we would have to solve the analog task of giving a self-adjoint extension for the unbounded operator $(m^2 - \Box^g)$ and its inverse.

In order to complete the picture, we have to give a scheme for the

Definition (Local Interpretation of Boundary)**.** We assume that there exists a "boundary manifold" $\mathcal{C}$ with a metric ∂g. The vectors in the boundary space $\mathscr{B}$ have an interpretation as ∂g-measurable functions over the manifold $\mathcal{C}$. This implies that the sesquilinear scalar product

$$\langle f, h \rangle_{\partial g} = \int_{\mathcal{C}} \mathrm{d}x \sqrt{\partial g}\; f(x)^* h(x)$$

makes sense for $f, h \in \mathscr{B} \cap L^2(\mathcal{C}, \partial g)$.

The local structure is implemented by a self-dual operator (in the Banach space sense) $\jmath : \mathcal{D}_{\mathcal{C}} \subset \mathscr{B} \to \mathscr{B}'$ with the defining property

$$\langle \jmath f, g \rangle_{\mathscr{B}} := \langle f, g \rangle_{\partial g}, \qquad f, g \in \mathcal{D}_{\mathcal{C}}. \tag{5.3-14}$$

[2] The meaning of this assumption lies in the fact that we assume the Hilbert space $\mathscr{H}$ norm always vanishes for functions which are null in the $L^2(\mathcal{M}, g)$-sense

The scalar product $\langle , \rangle_{\partial g}$ is well-defined only for $f, g \in L^2(\mathcal{C}, \partial g)$, requiring $\mathcal{D}_{\mathcal{C}} \subset L^2(\mathcal{C}, \partial g)$. It is assumed to be non-degenerate on $\mathcal{D}_{\mathcal{C}}$; ie $\langle d, d\rangle_{\partial g} = 0$ implies $d = 0$ in $\mathcal{D}_{\mathcal{C}}$.

Due to the non-degeneracy of $\langle , \rangle_{\partial g}$ on $\mathcal{D}_{\mathcal{C}}$, the map $\jmath$ is injective. The definition of $\jmath$ implies that $\Pi_{\mathrm{bo}}\jmath$ and $\jmath\Pi_{\mathrm{bo}}$ are self-adjoint operators in the Hilbert space $\mathscr{B}$ resp. $\mathscr{B}'$. We enlarge the definition of the ∂g-scalar product by setting

$$\langle b, d\rangle_{\partial g} := \langle b, \jmath d\rangle_{\mathscr{B}}, \qquad b \in \mathscr{B},\ d \in \mathcal{D}_{\mathcal{C}}. \tag{5.3-15}$$

There are no assumptions about a natural relationship between the local interpretation of the bulk functions and of the boundary functions.

Formulation of constrained path integral

Written in terms of the spaces just introduced, the path integral (5.1-1) with the δ given by (5.3-7) inserted is

$$\begin{aligned}\mathscr{Z}_{\phi\{f\}}[J] = &\sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{c_n}{\hbar}\frac{1}{n!}\left\langle\left(-\frac{\hbar}{\sqrt{g}}\frac{\partial}{\partial J}\right)^n\right\rangle_g\right)^j \int_{\Re\mathscr{B}'} \mathscr{D}(\sigma) \int_{\Re\mathscr{H}} \mathscr{D}(\phi) \\ &\exp\frac{1}{\hbar}\left\{-\frac{1}{2}\Pi^{-1}(\phi,\phi) - \langle J, \phi\rangle_g + i\langle \sigma, B\phi - f\rangle_{\mathscr{B}}\right\}.\end{aligned} \tag{5.3-16}$$

Although we are only working with real functions, we will keep a complex notation and restrict ourselves to real spaces only in the end; we assume $J \in \mathcal{D}$ real and $f \in \mathscr{B}$ real.

The Hilbert space Gaussian integral formula we want to apply is

$$\int_{\Re\mathscr{H}} \mathscr{D}(\phi) \exp\frac{1}{\hbar}\left(-\frac{1}{2}\langle\phi,\phi\rangle - \langle h', \phi\rangle_{\mathscr{H}}\right) = \mathscr{Z}[0] \exp\frac{1}{2\hbar}\langle h', \overline{h'}\rangle', \qquad h' \in \mathscr{H}'. \tag{5.3-17}$$

This is because the field ϕ is real. Under the condition $J \in \mathcal{D}$ real, the $L^2(\mathcal{M}, g)$ scalar product in (5.3-16) can be rewritten using the inclusion $\imath$, and we find

$$\begin{aligned}\mathscr{Z}_{\phi\{f\}}[J] &= \sum_{j=0}^{\infty}\frac{(\dots)^j}{j!}\int_{\Re\mathscr{B}'}\mathscr{D}(\sigma)\int_{\Re\mathscr{H}}\mathscr{D}(\phi)\exp\frac{1}{\hbar}\left\{-\frac{1}{2}\langle\phi,\phi\rangle - \langle \imath J + iB'\sigma,\, \phi\rangle_{\mathscr{H}} - i\langle\sigma, f\rangle_{\mathscr{B}}\right\} \\ &= \sum_{j=0}^{\infty}\frac{(\dots)^j}{j!}\mathscr{Z}[0]\int_{\Re\mathscr{B}'}\mathscr{D}(\sigma)\exp\frac{1}{\hbar}\left\{\frac{1}{2}\langle \imath J + iB'\sigma,\, \imath J - iB'\sigma\rangle' - i\langle\sigma, f\rangle_{\mathscr{B}}\right\}.\end{aligned}$$

The Gaussian integral which will help us to integrate σ is

$$\mathscr{Z}[0]\int_{\Re\mathscr{B}'}\mathscr{D}(\sigma)\exp\frac{1}{\hbar}\left(-\frac{1}{2}\langle\sigma,\sigma\rangle' - i\langle\sigma, f\rangle_{\mathscr{B}}\right) = \tilde{\mathscr{Z}}[0]\exp -\frac{1}{2\hbar}\langle\overline{f}, f\rangle.$$

Since $f \in \mathscr{B}$ is assumed to be real, the generating function finally reads

$$\begin{aligned}
\mathscr{Z}_{\phi\{f\}}[J] =& \tilde{\mathscr{Z}}[0] \sum_{j=0}^{\infty} \frac{(\dots)^j}{j!} \exp \frac{1}{2\hbar} \left\{ \langle \imath J,\, \imath J \rangle' - \langle f + B\,\Pi\,\imath J,\, f + B\,\Pi\,\imath J \rangle \right\} \\
\equiv& \tilde{\mathscr{Z}}[0] \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{c_n}{\hbar} \frac{1}{n!} \left\langle \left(-\frac{\hbar}{\sqrt{g}} \frac{\partial}{\partial J} \right)^n \right\rangle_g \right)^j \qquad (5.3\text{-}18) \\
& \exp \frac{1}{\hbar} \left\{ \frac{1}{2} \langle J, \Pi_{\mathrm{bu}} J \rangle_g - \langle J, \Pi_{\mathrm{bubo}} f \rangle_g - \langle f, \Pi_{\mathrm{bo}}^{-1} f \rangle_{\mathscr{B}} \right\}.
\end{aligned}$$

We read off the modified Feynman rules of the constrained theory [3]:

- The *bulk-to-bulk propagator* $G_{\mathrm{bu}} : \mathscr{H} \to \mathscr{H}$ is the integral kernel (with respect to the measure $\sqrt{g}\,\mathrm{d}x$) of

$$G_{\mathrm{bu}} = \hbar\,\Pi_{\mathrm{bu}}\imath = \hbar \left(\Pi \; - \; \Pi\, B' \,(B\,\Pi\, B')^{-1}\, B\,\Pi \right) \imath. \qquad (5.3\text{-}19)$$

  The propagator does *not* depend on the value f of the constraints; it depends only on the type of constraint, ie on the operator B. Obviously

$$\Pi_{\mathrm{bu}} = \mathcal{P}_{\mathrm{bu}}\Pi = \Pi\mathcal{P}'_{\mathrm{bu}} = \mathcal{P}_{\mathrm{bu}}\Pi\mathcal{P}'_{\mathrm{bu}}.$$

- The *boundary-to-bulk operator* $\Pi_{\mathrm{bubo}} : \mathscr{B} \to \mathscr{H}$ is given by

$$\Pi_{\mathrm{bubo}} = \Pi\, B' \,(B\,\Pi\, B')^{-1} = \left(B|_{\mathscr{H}_{\mathrm{bo}}} \right)^{-1} \qquad (5.3\text{-}20)$$

  (The boundary space is supposed to act from the right). The associated propagator can only be constructed if we specify the measure used for boundary localisation. The *bulk-to-boundary operator* $\Pi_{\mathrm{bobu}} : \mathscr{H}' \to \mathscr{B}'$ is given by the Banach space dual

$$\Pi_{\mathrm{bobu}} = \Pi'_{\mathrm{bubo}} = (B\;\Pi\; B')^{-1}\, B\,\Pi = (B')^{-1}(1 - \mathcal{P}'_{\mathrm{bu}}). \qquad (5.3\text{-}21)$$

- The term quadratic in f takes role of a boundary-to-boundary propagator; at the same time it defines a "hangover" Lagrangian. By integrating f, the path integral *without* boundary conditions is recovered. Therefore, it is adequate to introduce the operator $\Pi_{\mathrm{bo}}^{-1} : \mathscr{B} \to \mathscr{B}'$ defining the *boundary Lagrangian*

$$\Pi_{\mathrm{bo}}^{-1} = (B\;\Pi\; B')^{-1}. \qquad (5.3\text{-}22)$$

[3] We *strictly* distinguish abstract operators (or quadratic forms) Π and their respective kernels; for an operator $\Pi : \mathscr{H}' \to \mathscr{H}$, the associated kernel is

$$\Pi(x, y) = \langle \delta_x,\, \Pi\imath\delta_y \rangle_g = \langle \delta_x,\, \imath\Pi\imath\delta_y \rangle_{\mathscr{H}},$$

where δ_x is the Dirac distribution of the underlying measure. Propagators G contain an additional factor $\hbar$.

At the same time, Π_{bo}^{-1} is the kernel of the scalar product in the space $\mathscr{B}$, just as Π^{-1} is in the full space $\mathscr{H}$, and acts as boundary-to-boundary propagator in the generating function (5.3-18). Note that there appears an additional sign in the boundary-to-boundary correlations.

- Each *vertex* carries a coordinate x and corresponds to

$$-\frac{c_n}{\hbar}\int \mathrm{d}^{\mathrm{d}}x\,\sqrt{g}.$$

- *Source terms* are coupled to the bulk ends of propagators and carry

$$-\frac{1}{\hbar}\int \mathrm{d}^{\mathrm{d}}x\,\sqrt{g}J.$$

- *Constraint terms* are coupled to the boundary ends of propagators and carry

$$\frac{1}{\hbar}\left|f\right\rangle_{\mathscr{B}}.$$

- There are the usual *symmetry factors* Sym^{-1} associated to overcounting of diagrams.

We comment shortly on the condition that the space of field configurations is a Hilbert space. Usually, it is assumed that the field configurations are living in the dual of a nuclear space, like the Schwartz space. However, we take the simplistic view that the (unconstrained) Gaussian path integral is defined in a natural manner on the Hilbert space $\mathscr{H}$. After the path integral is performed, the Hilbert space $\mathscr{H}'$ accommodates the *sources* $\imath J$ smearing the operators. Now the source terms are usually restricted to be Schwartz functions or similar test function spaces. As long as we consider (unconstrained) free fields, our treatment yields the maximal space of source functions; given two sources $f, g \in \mathscr{H}'$, their correlation $\langle f, g\rangle'$ will be finite by definition. For interacting theories, the space $\mathscr{H}'$ is probably too large; to be able to integrate the vertices, we have pulled back the sources by $\imath$ onto the space $\mathscr{H}$ (they could in fact be implemented without pulling back, by giving up the claim that they should be coupled to the propagators by an $L^2(\mathcal{M}, g)$-integral). This space is presumably still too large. However, this is not a problem: From the point of view of perturbation theory, the “proper” test function space is only attainable perturbatively – and we do not lose much by making it slightly too large at the outset.

”Dual” and ”field theoretic” prescription

We want to compare the path integral with fixed boundary values developed so far (”dual prescription” according to [32]) to the ”usual” unconstrained path integral with a linear boundary source term $\langle \phi, B'k'\rangle_{\mathscr{H}}$, $k' \in \mathscr{B}'$ in the Lagrangian replacing the constraints ("field theoretic prescription"). This is equivalent to using the full propagator Π; alternatively, we could imagine integrating over all possible boundary configurations, ie to treat the boundary configurations f in (5.3-18) as dynamical fields. The duality is captured by the relation

$$\Pi = \Pi_{\text{bu}} + \Pi_{\text{bubo}}\Pi_{\text{bo}}\Pi_{\text{bobu}}. \tag{5.3-23}$$

The unconstrained propagation splits up into a part freely crossing through the bulk, and another part propagating along the boundary for some time (with a propagator determined by the "boundary Lagrangian" Π_{bo}^{-1}).

Adding a source term $\langle B\phi, k'\rangle_{\mathscr{B}}$ with $k' \in \mathscr{B}'$ to the action in the path integral (5.3-16), we find after performing the integrations that sources k' on the boundary are coupled to the the propagator Π by

$$\Pi B'k' = \Pi_{\text{bubo}}\Pi_{\text{bo}}k' \equiv \Pi_{\text{bubo}}^{\text{ft}}k'.$$

Using the lift operation $\jmath$ we can represent the boundary source term as $k' = \jmath k$, $k \in \mathscr{B}$. The boundary-to-boundary propagator is given by

$$\Pi_{\text{bo}} = B\Pi B'.$$

This is the inverse of the quadratic kernel of the boundary Lagrangian (5.3-22). Note that there exists an inverse to the relation (5.3-23), describing the transition from the field theoretic bulk-to-bulk propagators to the "dual" one:

$$\Pi_{\text{bu}} = \Pi + i^2\Pi_{\text{bubo}}^{\text{ft}}\Pi_{\text{bo}}^{-1}\Pi_{\text{bobu}}^{\text{ft}}. \tag{5.3-24}$$

We need the factor i^2, it can be realised as a coupling between the propagators.

5.4 A Simple Example

Consider a simple example illustrating nicely the application of constraints: A free massive scalar field in d = 1 dimensions (on the real line) with the constraint

$$\phi(0) = \phi_0.$$

Of course, this is a flat space theory, but it serves the purpose even better as the results will not be shrouded by technicalities. The sesquilinear form defining the Lagrangian is

$$\Pi^{-1}(f,g) = \int \mathrm{d}x\, f(x)^*(m^2 - \partial_x^2)g(x).$$

We will take $\mathscr{H}$ to be real. We define $\mathscr{H}$ as the domain of selfadjointness of the Klein-Gordon operator with vanishing boundary conditions at infinity. Vectors in $\mathscr{H}'$ may not necessarily be representable as functions (rather as distributions); however, we will write them as if they were functions. The unconstrained bulk operator Π is

$$\mathscr{H}' \ni f \mapsto (\Pi f)(y) = \langle x \mapsto \frac{e^{-m|x-y|}}{2m}, f\rangle_{\mathscr{H}} = \int \mathrm{d}x\, f(x)\frac{e^{-m|x-y|}}{2m}.$$

Applying the Klein-Gordon operator Π^{-1} on this function, one reobtains f. If contracted with $g \in \mathscr{H}'$, this will be the $\mathscr{H}'$ scalar product. The scalar products are most easily implemented in wave number space; they are

$$\langle f, g\rangle = \int \mathrm{d}k\,(m^2 + k^2)\overline{\hat{f}(-k)}\hat{g}(k), \qquad \langle f, g\rangle' = \int \mathrm{d}k\,\frac{\overline{\hat{f}(-k)\hat{g}(k)}}{m^2 + k^2}.$$

The space $\mathscr{H} = \mathscr{W}_1(\mathbb{R})$ is the first Sobolev space; it contains functions whose first derivative may be discontinuous (but nothing worse than that). These functions will be bounded, moreover.

We comment shortly on the inclusion $\imath$. The interpretation of $\mathscr{H}$ as function space is automatic by construction; it is in fact a subspace of $L^2(\mathbb{R})$ (easy to see in the Fourier domain). At the same time, $\mathscr{H}' \supset L^2(\mathbb{R})$. The inclusion $\imath : \mathscr{H} \hookrightarrow \mathscr{H}'$ is verbatim when vectors are written as functions, it is a bounded operator. This implies $\mathcal{D} = \mathscr{H}$. In view of this remark, we may thus say briefly that $\Pi(x, y) = \frac{e^{-m|x-y|}}{2m}$.

Obviously, the constraint operator has to act as

$$Bf = f(0), \qquad f \in \mathscr{H}.$$

It maps into $\mathscr{B} = \mathbb{R}$, with dual $\mathscr{B}' = \mathbb{R}$. The easiest way to proceed is by constructing

$$(B'b')(x) = b'\delta(x), \qquad b' \in \mathbb{R}.$$

This implies the dual null space $\mathscr{H}'_{\mathrm{bo}} = \mathrm{Ran}(B') = \mathbb{R}\delta(x)$. From that, we get the null space $\mathscr{H}_{\mathrm{bo}} = \Pi\mathscr{H}'_{\mathrm{bo}} = \mathbb{R}e^{-m|x|}$ and $\mathscr{H}_{\mathrm{bu}} = \mathcal{D}_{\mathrm{bu}} = \{f \in \mathscr{H} \mid \langle \mathscr{H}'_{\mathrm{bo}}, f\rangle_{\mathscr{H}} = 0\} = \{f \in \mathscr{H} \mid f(0) = 0\} = \mathring{\mathscr{W}}_1(\mathbb{R})$, a Sobolev space with internal boundary condition, indicated by the circle [104].

We investigate the boundary space. From

$$B\,\Pi\,f = \int \mathrm{d}x\, f(x)\frac{e^{-m|x|}}{2m}$$

for $f \in \mathscr{H}'$, obtain

$$\Pi_{\mathrm{bo}}^{-1} = (B\,\Pi\,B')^{-1} = (B\,\Pi\,\circ_x\,\delta(x))^{-1} = \left(\int \mathrm{d}x\,\delta(x)\frac{e^{-m|x|}}{2m}\right)^{-1} = 2m$$

which is the positive kernel of the quadratic boundary Lagrangian. The action of the projections (5.3-11) turns out to be

$$\begin{aligned}
(\mathcal{P}'_{\mathrm{bu}}f)(x) =& \left(f - B'\,(B\,\Pi\,B')^{-1}\,B\,\Pi\,f\right)(x) \\
=& f(x) - \delta(x)\int \mathrm{d}y\, e^{-m|y|}f(y), \\
(\mathcal{P}_{\mathrm{bu}}f)(x) =& f(x) - e^{-m|x|}f(0).
\end{aligned}$$

The kernel of the boundary-to-bulk propagation operator (5.3-20) is

$$\Pi_{\text{bubo}}(x) = \left(B|_{\mathscr{H}_{\text{bo}}}\right)^{-1} = e^{-m|x|}.$$

The bulk-to-bulk propagator by (5.3-19) is

$$G_{\text{bu}}(x,y) = \hbar\frac{e^{-m|x-y|}}{2m} - \hbar e^{-m|x|}\frac{e^{-m|y|}}{2m} = \hbar\frac{e^{-m|x-y|} - e^{-m(|x|+|y|)}}{2m} \geq 0.$$

We can see that

$$G_{\text{bu}}(x,y) = 0 \qquad \text{if } x\cdot y \leq 0.$$

There is no propagation across the origin where the field is pinned. The vacuum expectation is

$$\langle\phi(x)\rangle = \Pi_{\text{bubo}}(x)\phi_0 = e^{-m|x|}\phi_0. \tag{5.4-25}$$

Near the origin, the field is pinned to ϕ_0 as expected; there is a spatial relaxation on the scale of the inverse mass. The two-point function is

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle =& \langle\phi(x)\rangle\langle\phi(y)\rangle + \langle\phi(x)\phi(y)\rangle_c\\ =& e^{-m(x+y)}\phi_0^2 + \hbar\frac{e^{-my}\sinh mx}{m}, \qquad 0\leq x\leq y.\end{aligned}$$

The second part is the correlation of fluctuations. As $x\to 0$, the fluctuations of $\phi(x)$ are obviously suppressed.

There is an interesting interpretation of the propagator $G_{\text{bu}}(x,y)$: Consider the case where $x,y>0$. The first summand proportional $\exp -m|x-y|$ is the propagator without the pinning condition at 0. The second summand proportional $-\exp -m(|x|+|y|)$ can be interpreted as propagating from x to 0, and from 0 to y, with a reflection at 0. The reflection brings a factor -1 into the amplitude. This indicates an absorbing boundary.

For sake of completeness, we touch on the local structure of the boundary which is simple in this case. A "natural" metric structure ∂g on the boundary would be given by

$$\langle f,\, g\rangle_{\partial g} = fg, \qquad\qquad f,g\in\mathscr{B}.$$

Compare this to the induced Hilbert space structure, which yields

$$\langle f,\, g\rangle_{\mathscr{B}} = \langle\Pi_{\text{bo}}^{-1}f,\, g\rangle_{\mathfrak{E}} = 2mfg.$$

The boundary inclusion is trivially $\jmath : \mathscr{B} = \mathbb{R} \ni f \mapsto f \in \mathbb{R} = \mathscr{B}'$.

5.5 Schwinger Parametrisation with Constraints

In the case without boundary conditions, we had for the Schwinger parametrisation the formula

$$\Pi = \int_0^\infty \mathrm{d}\alpha\, \exp -\alpha(m^2 - \Box^g).$$

As an operator equation in $L^2(\mathbb{R}^d)$, this is obviously true for analytic vectors of $m^2 - \Box^g$.

This expression must be modified accordingly if it should fit into the constraint formalism based in the dual pair of spaces $\mathscr{H}/\mathscr{H}'$. Π^{-1} and Π_{bu}^{-1} are maps from $\mathscr{H}$ into $\mathscr{H}'$, so it does not make sense to exponentiate them. This problem of domains can be overcome by inserting an inclusion $\imath$ at a suitable position; we want to give meaning to the expression

$$\Pi_{\mathrm{bu}}\imath = \int_0^\infty \mathrm{d}\alpha \, \exp -\alpha \, (\Pi_{\mathrm{bu}}\imath)^{-1}.$$

Because we are in the first place interested in the bulk-to-bulk propagator, we will consider the restricted operator $\Pi_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}}$. Subsequently, we suggest an operator serving as inverse for $(\Pi_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}})^{-1}$, show that it is self-adjoint and apply functional calculus to do the Schwinger integral.

Consider the operator $\mathcal{S}$ fulfilling

$$\begin{aligned} \mathcal{S} : \imath\mathcal{D}_{\mathrm{bu}} + \mathscr{H}'_{\mathrm{bo}} &\to \mathcal{D}_{\mathrm{bu}} \\ \imath d_{\mathrm{bu}} + h'_{\mathrm{bo}} &\mapsto d_{\mathrm{bu}}. \end{aligned} \tag{5.5-26}$$

It is easy to see that $\mathcal{S}$ is well defined by the assumption $\imath\mathcal{D}_{\mathrm{bu}} \cap \mathscr{H}'_{\mathrm{bo}} = \{0\}$. In particular, $\mathcal{S}$ fulfills

$$\mathcal{S}\mathcal{P}'_{\mathrm{bu}}\imath d_{\mathrm{bu}} = d_{\mathrm{bu}}, \qquad d_{\mathrm{bu}} \in \mathcal{D}_{\mathrm{bu}}. \tag{5.5-27}$$

$$\langle \mathcal{S}h', d_{\mathrm{bu}}\rangle_g = \langle h', d_{\mathrm{bu}}\rangle_{\mathscr{H}}. \qquad h' \in \mathrm{Dom}(\mathcal{S}),\ d_{\mathrm{bu}} \in \mathcal{D}_{\mathrm{bu}}. \tag{5.5-28}$$

The operator $\imath\mathcal{S} : \imath\mathcal{D}_{\mathrm{bu}} + \mathscr{H}'_{\mathrm{bo}} \to \imath\mathcal{D}_{\mathrm{bu}}$ is idempotent. It is *not* a projection operator in $\mathscr{H}'$ in general, as it need not be self-adjoint: The kernel $\mathrm{Ker}(\imath\mathcal{S}) = \mathscr{H}'_{\mathrm{bo}}$ and the range $\mathrm{Ran}(\imath\mathcal{S}) = \imath\mathcal{D}_{\mathrm{bu}}$ are orthogonal only if $\imath\mathcal{D}_{\mathrm{bu}} \subset \mathscr{H}'_{\mathrm{bu}}$ (which is highly exotic). In fact, it may be even unbounded. This operator will play an important role later on.

Lemma 5.3. *The inverse* $(\Pi_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}})^{-1} \subset \mathcal{S}\Pi^{-1}$.

Proof. For $d_{\mathrm{bu}} \in \mathcal{D}_{\mathrm{bu}}$,

$$(\mathcal{S}\Pi^{-1})\,\Pi_{\mathrm{bu}}\imath d_{\mathrm{bu}} = (\mathcal{S}\Pi^{-1})\,\Pi\,\mathcal{P}'_{\mathrm{bu}}\,\imath d_{\mathrm{bu}} = \mathcal{S}\,\mathcal{P}'_{\mathrm{bu}}\imath d_{\mathrm{bu}} = d_{\mathrm{bu}},$$

so $\mathcal{S}\Pi^{-1}$ is a proper left inverse of $\Pi_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}}$. It is easy to show that if $f \in \mathrm{Ran}(\Pi_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}})$, then $f \in \mathscr{H}_{\mathrm{bu}}$ and $\Pi^{-1}f \in \mathrm{Dom}(\mathcal{S})$. For such f,

$$\Pi_{\mathrm{bu}}\imath(\mathcal{S}\Pi^{-1})f = \mathcal{P}_{\mathrm{bu}}\Pi\,(\Pi^{-1}f + h'_{\mathrm{bo}}) = \mathcal{P}_{\mathrm{bu}}(f + \Pi\,h'_{\mathrm{bo}}) = f$$

where $h'_{\mathrm{bo}} \in \mathscr{H}'_{\mathrm{bo}}$ such that $\Pi^{-1}f + h'_{\mathrm{bo}} \in \imath\mathcal{D}_{\mathrm{bu}} = \mathrm{Ran}(\imath)$; so $\mathcal{S}\Pi^{-1}$ is also a proper right inverse of $\Pi_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}}$. $\square$

Self-adjointness of $\mathcal{S}\Pi^{-1}$. In the following we examine $\mathcal{S}\Pi^{-1}$ as operator with domain

$$\mathrm{Dom}(\mathcal{S}\Pi^{-1}) = \Pi\,\mathrm{Dom}(\mathcal{S}) = \Pi\imath\mathcal{D}_{\mathrm{bu}} + \mathscr{H}_{\mathrm{bo}}.$$

Proposition 5.4. $\mathcal{S}\Pi^{-1}$ *is a self-adjoint, positive operator in the Hilbert space* $\mathscr{H}$.

Proof. The density is clear by the density of $\Pi^{-1}\,\mathrm{Dom}(\mathcal{S}\Pi^{-1}) = \imath\mathcal{D}_{\mathrm{bu}} + \mathscr{H}'_{\mathrm{bo}} \subset \mathscr{H}'$, and the fact that Π is an isomorphism. Let $d, \tilde{d} \in \mathcal{D}_{\mathrm{bu}}$ and $n, \tilde{n} \in \mathscr{H}_{\mathrm{bo}}$. Then

$$\begin{aligned}\langle \Pi\imath d + n, \mathcal{S}\Pi^{-1}(\Pi\imath\tilde{d} + \tilde{n})\rangle &= \langle \Pi\imath d + n, \tilde{d}\rangle = \langle \Pi\imath d, \tilde{d}\rangle \\ &= \langle \imath d, \tilde{d}\rangle_{\mathscr{H}} = \langle d, \tilde{d}\rangle_g = \langle \mathcal{S}\Pi^{-1}(\Pi\imath d + n), \Pi\imath\tilde{d} + \tilde{n}\rangle,\end{aligned}$$

which proves the symmetry and positivity (by the positivity of $L^2(\mathcal{M}, g)$).

The self-adjointness is shown by the (Banach space) self-duality of $\mathcal{S}$. We compute directly that the dual $\mathcal{S}' = \mathcal{S}$. For assume that given $f' \in \mathscr{H}'$, there exists $f \in \mathscr{H}$ such that $\langle f', \mathcal{S}g\rangle_{\mathscr{H}} = \langle f, g\rangle_{\mathscr{H}}$ for all $g = \imath d_{\mathrm{bu}} + h'_{\mathrm{bo}} \in \mathrm{Dom}(\mathcal{S})$. Then, by definition of the dual, $f' \in \mathrm{Dom}(\mathcal{S}')$ and $f = \mathcal{S}'f'$. Since $\mathcal{S}g = d_{\mathrm{bu}}$ is independent of h'_{bo}, such f must fulfill $f \in \mathscr{H}_{\mathrm{bu}}$, with the implication $\langle f', d_{\mathrm{bu}}\rangle_{\mathscr{H}} = \langle f', \mathcal{S}g\rangle_{\mathscr{H}} = \langle f, g\rangle_{\mathscr{H}} = \langle f, \imath d_{\mathrm{bu}}\rangle_{\mathscr{H}}$. We use that $\mathcal{P}'_{\mathrm{bu}}\imath|_{\mathcal{D}_{\mathrm{bu}}} : \mathcal{D}_{\mathrm{bu}} \subset \mathscr{H}_{\mathrm{bu}} \to \mathscr{H}'_{\mathrm{bu}}$ is self-dual by assumption 5.2; so $\langle f, \imath d_{\mathrm{bu}}\rangle_{\mathscr{H}} = \langle \imath f, d_{\mathrm{bu}}\rangle_{\mathscr{H}}$. Since $\mathcal{D}_{\mathrm{bu}}$ is dense in $\mathscr{H}_{\mathrm{bu}}$, that implies $f' - \imath f \in \mathscr{H}_{\mathrm{bu}}{}^{\perp} = \mathscr{H}'_{\mathrm{bo}}$. But this is equivalent to $f = \mathcal{S}f'$. □

Consequently, the Schwinger kernel

$$K^{\mathrm{bu}}_{\alpha} = \exp -\alpha\mathcal{S}\Pi^{-1}, \qquad \alpha > 0, \tag{5.5-29}$$

implements a strongly continuous semigroup (strongly continuous in $\mathscr{H}$!). The Schwinger parametrised representation of the propagator

$$\overline{\Pi_{\mathrm{bu}}\imath} = \int_0^{\infty} \mathrm{d}\alpha\, \exp -\alpha\mathcal{S}\Pi^{-1}. \tag{5.5-30}$$

can be defined by ordinary functional calculus for self-adjoint operators. As the integrand $\exp -\alpha\mathcal{S}\Pi^{-1}$ is a bounded operator, it is well-defined on all of $\mathscr{H}$ by closure; as it is positive furthermore, the integration delivers the maximal domain for the propagator.

Interpretation as Diffusion Process with Absorption at the Boundary. Define the bounded, self-adjoint operator $K^{\mathrm{bu}}_{\alpha} = \exp -\alpha\mathcal{S}\Pi^{-1}$, $\alpha > 0$. We can obtain an operator differential equation for K^{bu}_{α}, giving the "time" derivative

$$\frac{\partial}{\partial\alpha} K^{\mathrm{bu}}_{\alpha} = -\mathcal{S}\Pi^{-1} K^{\mathrm{bu}}_{\alpha} \tag{5.5-31}$$

with the initial condition

$$\operatorname*{s-lim}_{\alpha\to 0+} K^{\mathrm{bu}}_{\alpha} = \mathrm{id}\,. \tag{5.5-32}$$

Some properties of the evolution under Schwinger "time" α are collected in the following

Proposition 5.5. *For $\alpha > 0$ and $f \in \mathscr{H}$, the vector $K_\alpha^{\mathrm{bu}} f$ is an analytic vector for $\mathcal{S}\Pi^{-1}$. On the subdomain $\mathrm{Dom}(\mathcal{S}\Pi^{-1}) \cap \mathcal{D}_{\mathrm{bu}}$, $\mathcal{S}\Pi^{-1}$ is a positive operator with respect to the $L^2(\mathcal{M}, g)$ scalar product. For $f \in \mathscr{H}_{\mathrm{bu}}(\mathscr{H}_{\mathrm{bo}})$, $K_\alpha^{\mathrm{bu}} f$ lies in the subspace $\mathscr{H}_{\mathrm{bu}}(\mathscr{H}_{\mathrm{bo}})$. For $f \in \mathcal{D}_{\mathrm{bu}}$, $K_\alpha^{\mathrm{bu}} f$ lies in the subspace $\mathcal{D}_{\mathrm{bu}}$.*

Remark. By definition [81, X.6], the set $C^\infty(A) = \bigcap_{n=1}^\infty \mathrm{Dom}(A^n)$ contains the C^∞-vectors for an operator A. A vector $f \in C^\infty(A)$ is an analytic vector [ibid] for A if there exists $t > 0$ such that

$$\sum_{n=0}^\infty \frac{\|A^n f\|}{n!} t^n < \infty.$$

Proof. Since $(\mathcal{S}\Pi^{-1})^n K_\alpha^{\mathrm{bu}} = (\mathcal{S}\Pi^{-1})^n \exp -\alpha \mathcal{S}\Pi^{-1}$ is a bounded operator by functional calculus, certainly $K_\alpha^{\mathrm{bu}} f \in \mathrm{Dom}((\mathcal{S}\Pi^{-1})^n)$ for all $n > 0$. For the analyticity, note that by functional calculus, $\|(\mathcal{S}\Pi^{-1})^n \exp -\alpha \mathcal{S}\Pi^{-1}\| \le \left(\frac{n}{\alpha}\right)^n e^{-n}$, so

$$\sum_{n=0}^\infty \frac{\|(\mathcal{S}\Pi^{-1})^n K_\alpha^{\mathrm{bu}} f\|}{n!} t^n \le \|f\| \sum_{n=0}^\infty \frac{n^n e^{-n}}{n!} \left(\frac{t}{\alpha}\right)^n .$$

Choosing $t = \alpha/2 > 0$, it is easily shown using Stirling's approximation that the r.h.s. converges.

Given a vector $f \in \mathrm{Dom}(\mathcal{S}\Pi^{-1}) \cap \mathcal{D}_{\mathrm{bu}}$, we have

$$\begin{aligned} 0 \le \langle f, f \rangle &\overset{f \in \mathcal{D}_{\mathrm{bu}}}{=} \langle \mathcal{S}\imath f, f \rangle = \langle \mathcal{S}\Pi^{-1} \Pi \imath f, f \rangle \overset{f \in \mathrm{Dom}(\mathcal{S}\Pi^{-1})}{=} \langle \Pi \imath f, \mathcal{S}\Pi^{-1} f \rangle \\ &= \langle \imath f, \mathcal{S}\Pi^{-1} f \rangle_{\mathscr{H}} \overset{f \in \mathcal{D}}{=} \langle f, \mathcal{S}\Pi^{-1} f \rangle_g . \end{aligned}$$

To show that $\mathscr{H}_{\mathrm{bu}}$ and $\mathscr{H}_{\mathrm{bo}}$ are invariant subspaces, use that by construction we have $K_\alpha^{\mathrm{bu}} h_{\mathrm{bo}} = h_{\mathrm{bo}}$ and $\langle h_{\mathrm{bo}}, K_\alpha^{\mathrm{bu}} f_{\mathrm{bu}} \rangle = \langle K_\alpha^{\mathrm{bu}} h_{\mathrm{bo}}, f_{\mathrm{bu}} \rangle = \langle h_{\mathrm{bo}}, f_{\mathrm{bu}} \rangle = 0$ for $h_{\mathrm{bo}} \in \mathscr{H}_{\mathrm{bo}}$, $f_{\mathrm{bu}} = \mathscr{H}_{\mathrm{bu}}$, by the self-adjointness of K_α^{bu}.

For the last statement, assume that $f \in \mathcal{D}_{\mathrm{bu}}$. This implies that there exists $g \in \mathscr{H}$ such that $f = \mathcal{S}\Pi^{-1} g$. But then, $K_\alpha^{\mathrm{bu}} f = K_\alpha^{\mathrm{bu}} \mathcal{S}\Pi^{-1} g = \mathcal{S}\Pi^{-1} K_\alpha^{\mathrm{bu}} g$ lies in the domain $\mathrm{Ran}(\mathcal{S}) = \mathcal{D}_{\mathrm{bu}}$. □

Let $K_\alpha^{\mathrm{bu}}(x, y)$ be the $L^2(\mathcal{M}, g)$-integral kernel of K_α^{bu}. Can we find a partial differential equation for $K_\alpha^{\mathrm{bu}}(x, y)$? By the proposition, it is clear that $\mathscr{H}_{\mathrm{bo}}$ is an invariant eigenspace with eigenvalue 0 which does not mix with other eigenspaces under the flow of the differential equation (5.5-31). At any given time α, we can project out the piece $(1 - \mathcal{P}_{\mathrm{bu}}) K_\alpha^{\mathrm{bu}}$ mapping into this eigenspace. Applying differential equation (5.5-31), we derive the "boundary condition"

$$\frac{\partial}{\partial \alpha} B (1 - \mathcal{P}_{\mathrm{bu}}) K_\alpha^{\mathrm{bu}} = 0, \tag{5.5-33}$$

valid for *all* $\alpha > 0$ (the isometry B has been inserted for convenience). In particular for $f \in \mathscr{H}_{\mathrm{bu}}$ in the bulk, we have the initial condition $(1 - \mathcal{P}_{\mathrm{bu}}) K_0^{\mathrm{bu}} f = (1 - \mathcal{P}_{\mathrm{bu}}) \,\mathrm{id}\, f = 0$, which must therefore hold for all $\alpha > 0$.

On the other hand, choose any $d_{\text{bu}} \in \mathcal{D}_{\text{bu}}$ and take the $L^2(\mathcal{M}, g)$ scalar product with equation (5.5-31). By property (5.5-28), we have for all $f \in \mathscr{H}$

$$\frac{\partial}{\partial \alpha}\langle d_{\text{bu}},\, K_\alpha^{\text{bu}} f\rangle_g = -\langle d_{\text{bu}},\, \mathcal{S}\Pi^{-1} K_\alpha^{\text{bu}} f\rangle_g = -\langle d_{\text{bu}},\, \Pi^{-1} K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}}. \tag{5.5-34}$$

This is the equation governing the bulk behaviour of the constrained Schwinger kernel. (5.5-33) and (5.5-34) together with the initial condition (5.5-32), when written explicitly using the kernel $K_\alpha^{\text{bu}}(x, y)$, constitute a set of equations determining this kernel. They take the form of a diffusion system.

Bulk-to-boundary Schwinger kernel. Similarly to what we did in the unconstrained case, we pick a heat packet $f \in \mathcal{D}_{\text{bu}}$, let it evolve with K_α^{bu} and finally test the result with an arbitrary vector $d \in \mathcal{D}$, in the $L^2(\mathcal{M}, g)$-sense. The resulting amplitude is called

$$V_{d,\alpha}(f) = \langle d, K_\alpha^{\text{bu}} f\rangle_g.$$

Its "time" derivative is

$$\frac{\partial}{\partial \alpha} V_{d,\alpha}(f) = -\langle d, \mathcal{S}\Pi^{-1} K_\alpha^{\text{bu}} f\rangle_g = -\langle d, \imath\mathcal{S}\Pi^{-1} K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}}.$$

We now perform the operator equivalent of integration by parts, writing

$$\frac{\partial}{\partial \alpha} V_{d,\alpha}(f) = -\,\langle d,\, \Pi^{-1} K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} - \langle d,\, (\imath\mathcal{S} - 1)\Pi^{-1} K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}}.$$

The first summand consists itself of two parts. These correspond to the two terms in $\Pi^{-1} = m^2 - \Box^g$: The m^2 determines the volume dissipation by the mass (which appears in the Schwinger representation as a linear damping). The contribution from $\Box^g$ is just the local concentration change resulting from the redistribution through diffusion.

The second summand must then be the absorption at the boundary. This gives a very direct interpretation of the processes which have to be set up in order to derive a path integral for the Schwinger representation. To understand the significance of this second summand, notice that $\operatorname{Ran}(\imath\mathcal{S} - 1) = \mathscr{H}'_{\text{bo}}$. This shows that the second summand contributes only through the boundary value Bd - the bulk behaviour of d is of no consequence. Consider what happens if we integrate the Schwinger kernel *inside* the scalar product

$$\begin{aligned}\langle d, (\imath\mathcal{S} - 1)\Pi^{-1} \int_0^\infty \mathrm{d}\alpha\, K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} &= \langle d,\, (\imath\mathcal{S} - 1)\Pi^{-1}\Pi_{\text{bu}}\imath f\rangle_{\mathscr{H}} = \langle d,\, (\imath\mathcal{S} - 1)\mathcal{P}'_{\text{bu}}\imath f\rangle_{\mathscr{H}} \\ &= \langle d,\, (1 - \mathcal{P}'_{\text{bu}})\imath f\rangle_{\mathscr{H}} = \langle (1 - \mathcal{P}_{\text{bu}})d,\, \imath f\rangle_{\mathscr{H}} = \langle Bd,\, \Pi_{\text{bobu}}\imath f\rangle_{\mathscr{B}}. \end{aligned} \tag{5.5-35}$$

We will use equation (5.5-35) to characterise the novel Schwinger kernel for the bulk-to-boundary propagator obeying a relation very similar to the one of the bulk-to-bulk kernel:

$$\langle Bd,\, \Pi_{\text{bobu}}\imath f\rangle_{\mathscr{B}} \equiv \int_0^\infty \mathrm{d}\alpha\, \langle Bd,\, K_\alpha^{\text{bobu}} f\rangle_{\partial g}, \tag{5.5-36}$$

where ∂g is the metric structure on the boundary. This relation is clearly not sufficient to *define* K_α^{bobu}, but it will motivate the following developments. A natural means to obtain a formula in the form (5.5-36) is provided if

$$\langle d, (\imath\mathcal{S}-1)\Pi^{-1}\int_0^\infty \mathrm{d}\alpha\, K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} \stackrel{?}{=} \int_0^\infty \mathrm{d}\alpha\, \langle d, (\imath\mathcal{S}-1)\Pi^{-1}K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} \tag{5.5-37}$$

is true. We will comment momentarily on this condition, but let us first explore its consequences. Combining (5.5-35), (5.5-36) and (5.5-37), the action of the "Schwinger kernel" $K_\alpha^{\text{bobu}} : \mathscr{H} \to \mathscr{B}$ for to bulk-to-boundary propagator is explicitly given by

$$\begin{aligned}
\langle Bd,\, K_\alpha^{\text{bobu}} f\rangle_{\partial g} &\stackrel{?}{=} \langle d,\, (\imath\mathcal{S}-1)\Pi^{-1}K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} \\
&= \langle (1-\mathcal{P}_{\text{bu}})d,\, (\imath\mathcal{S}-1)\Pi^{-1}K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} \qquad \text{since } \operatorname{Ran}(\imath\mathcal{S}-1) = \mathscr{H}'_{\text{bo}} \\
&= \langle (1-\mathcal{P}_{\text{bu}})d,\, \imath\mathcal{S}\Pi^{-1}K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} \qquad \text{since } K_\alpha^{\text{bu}} f \in \mathscr{H}_{\text{bu}} \\
&= \langle (1-\mathcal{P}_{\text{bu}})d,\, \mathcal{S}\Pi^{-1}K_\alpha^{\text{bu}} f\rangle_{g} \\
&= -\langle (1-\mathcal{P}_{\text{bu}})d,\, \partial_\alpha K_\alpha^{\text{bu}} f\rangle_{g}.
\end{aligned} \tag{5.5-38}$$

The bulk-to-boundary Schwinger kernel is *not* a simple heat kernel; it is obtained from the bulk heat kernel by measuring the extent of absorption at the boundary.

What can go wrong in the equality (5.5-37)? Since the integral of the Schwinger kernel is basically defined by matrix elements, there are no principal obstructions against pulling it out of a scalar product. The Achilles' heel is the operator $\imath\mathcal{S}$: This may be an unbounded operator in $\mathscr{H}'$, and therefore not continuous; in this case, there is a hazard that (5.5-37) is spoilt. One might try to take the dual $(\imath\mathcal{S}-1)' = \mathcal{S}\imath - 1$ and let it act on d on the left-hand side; as the integration commutes with taking a matrix element, this would render both sides equal. The point is that in the interesting case where $Bd \neq 0$ has a non-vanishing boundary value, $d \in \mathcal{D}$ but $d \notin \mathcal{D}_{\text{bu}}$; in this case, possibly $\imath d \notin \operatorname{Dom}(\mathcal{S})$. We cannot expect that taking the dual is allowed in the generic case.

To summarise, the relation (5.5-37) might not hold because integration does not commute with the evaluation of an *unbounded* functional.

Remark 5.6. The third line of (5.5-38) offers a practical way of calculating the Schwinger kernel of the bulk-to-boundary propagator. Observe that on the right-hand side $\mathcal{S}\Pi^{-1} K_\alpha^{\text{bu}} f \in \mathcal{D}_{\text{bu}}$. Topologise $\mathscr{H}$ by the weak $\sigma(\mathscr{H}, \mathcal{D}_{\text{bu}})_g$-topology. Let $n_j \in \mathscr{H}_{\text{bu}}$ be a family of functions approximating $(1-\mathcal{P}_{\text{bu}})d \in \mathscr{H}_{\text{bo}}$ in this weak topology (such a family need not always exist). By (5.5-28),

$$\begin{aligned}
\langle Bd,\, K_\alpha^{\text{bobu}} f\rangle_{\partial g} &= \lim_j \langle n_j,\, \mathcal{S}\Pi^{-1}K_\alpha^{\text{bu}} f\rangle_g \\
&= \lim_j \langle n_j,\, \Pi^{-1}K_\alpha^{\text{bu}} f\rangle_{\mathscr{H}} = \lim_j \langle n_j,\, K_\alpha^{\text{bu}} f\rangle.
\end{aligned} \tag{5.5-39}$$

If n_j is chosen skilfully, then this last limit may be evaluated with ease. The point is that $K_\alpha^{\text{bu}} f \in \mathscr{H}_{\text{bu}}$, so when the actual kernel $K_\alpha^{\text{bu}}(x, y)$ is known, the scalar product may be simplified by integration by parts. The bulk-to-boundary propagator is thus a suitable limit of the bulk-to-bulk propagator.

5.6 A Simple Example (cont'd)

We are looking for a linear operator $\mathcal{S}$ fulfilling

$$\mathcal{S}(\imath f_{\rm bu} + h'_{\rm bo}) = \mathcal{S} \circ_x (f_{\rm bu}(x) - h'_{\rm bo}\delta(x)) \stackrel{!}{=} f_{\rm bu}$$

for functions $f_{\rm bu} \in \mathcal{D}_{\rm bu} = \mathscr{H}_{\rm bu}$ and $h'_{\rm bo} \equiv h'_{\rm bo}\delta(x) \in \mathscr{H}'_{\rm bo}$. The solution is

$$\mathcal{S}f(x) = \begin{cases} 0 & \text{if } x = 0, \\ f(x) & \text{if } x \neq 0. \end{cases} \qquad f \in \mathscr{H}'_{\rm bo} + \imath\mathcal{D}_{\rm bu}$$

The value at $x = 0$ is of no importance; it is important that we "project out" the distributional part. A little analysis shows that the (generalised) eigenvectors of $\mathcal{S}\Pi^{-1}$ may be labelled conveniently as

$$\begin{aligned} e_k(x) &= \left(\frac{2}{\pi(m^2+k^2)}\right)^{1/2} \theta(kx)\sin(kx), \qquad k \in \mathbb{R}\setminus\{0\}, \\ e_{\mathscr{B}}(x) &= \frac{1}{(2m)^{1/2}} e^{-m|x|}. \end{aligned}$$

where the normalisation is adapted to the scalar product in $\mathscr{H}$ and the corresponding eigenvalues for $\mathcal{S}\Pi^{-1}$ are $k^2 + m^2$ and 0.

We can see that $\imath\mathcal{S}$ is an unbounded operator by the following example: Let $\delta_\varepsilon \in \mathcal{D}_{\rm bu}$ be a smooth approximation of width ε to the Dirac delta distribution (it may have support on all of $\mathbb{R}$). Note that $\left|\frac{x}{\varepsilon}\right| \delta_{\varepsilon^2}(x-\varepsilon) \in \mathcal{D}_{\rm bu}$ for $\varepsilon \neq 0$. Consider the family

$$n_\varepsilon(x) = \delta(x) - \left|\frac{x}{\varepsilon}\right| \delta_{\varepsilon^2}(x-\varepsilon) \in \mathscr{H}'.$$

For $\varepsilon \to 0$, we can estimate $\|n_\varepsilon\|' \approx \frac{1}{m}(1 - e^{-m|\varepsilon|}) \approx |\varepsilon|$ using the integral kernel $\frac{1}{2m}e^{-m|x-y|}$ of the scalar product in $\mathscr{H}'$. However, $(\imath\mathcal{S}n_\varepsilon)(x) = -\left|\frac{x}{\varepsilon}\right| \delta_{\varepsilon^2}(x-\varepsilon)$ with norm $\|\imath\mathcal{S}n_\varepsilon\|' = \frac{1}{2m}$.

Let us examine what the PDE approach has to say about the kernel $K^{\rm bu}_\alpha(x,y)$. The PDE (5.5-34) reads

$$\frac{\partial}{\partial\alpha}K^{\rm bu}_\alpha(x,y) = -(m^2 - \partial_x^2)K^{\rm bu}_\alpha(x,y), \qquad x \neq 0. \tag{5.6-40}$$

The boundary condition (5.5-33) becomes

$$\frac{\partial}{\partial\alpha}K^{\rm bu}_\alpha(0,y) = 0. \tag{5.6-41}$$

Together with the initial condition $K^{\rm bu}_0(x,y) = \delta(x-y)$, the solution is determined as

$$\begin{aligned} K^{\rm bu}_\alpha(x,y) = {} & \frac{\theta(xy)}{2\sqrt{\pi\alpha}}\left(e^{-\frac{(x-y)^2}{4\alpha}} - e^{-\frac{(x+y)^2}{4\alpha}}\right)e^{-m^2\alpha} \\ & + \delta(y)\left\{\frac{e^{m|x|}}{2}\left(1 - \operatorname{erf}\frac{|x|+2m\alpha}{2\sqrt{\alpha}}\right) + \frac{e^{-m|x|}}{2}\left(1 - \operatorname{erf}\frac{|x|-2m\alpha}{2\sqrt{\alpha}}\right)\right\}. \end{aligned}$$

Let us interpret the single terms. The first line is the part which is relevant for the bulk heat equation; it is a diffusion term with a sink at $x = 0$. The second line maintains the property (5.6-41); it balances any change by a variable source at $x = 0$. Note that away from the origin $x = 0$, the second line fulfills the heat equation independently. Although this is not immediately visible, K_α^{bu} is a symmetric operator in the Hilbert space $\mathscr{H}$ (*not* in $L^2(\mathcal{M}, g)$).

The integration $G_{\mathrm{bu}}(x, y) = \int_0^\infty \mathrm{d}\alpha\, K_\alpha^{\mathrm{bu}}(x, y)$ will be divergent on the second line, so we have to select $y \neq 0$. The support of K_α^{bu} is restricted to both x and y lying on the same side of the origin; there is no diffusion across 0. It is positive. Its "Fourier transform" with respect to the generalised eigenfunctions is

$$K_\alpha^{\mathrm{bu}}(k, k') = \delta(k - k') e^{-(k^2+m^2)\alpha} + \delta_{k,\mathscr{B}}\delta_{k',\mathscr{B}}.$$

This is written down the teensiest bit laissez-faire, the second summand containing Kronecker deltas picking the boundary basis vector $e_{\mathscr{B}}$. Finally, the heat kernel of the bulk-to-boundary propagator is according to (5.5-38) given by

$$K_\alpha^{\mathrm{bobu}}(y) = -\int \mathrm{d}x\, e^{-m|x|} \partial_\alpha K_\alpha^{\mathrm{bu}}(x, y) = \frac{|y|}{2\sqrt{\pi\alpha^3}} e^{-\frac{y^2}{4\alpha} - m^2\alpha}, \qquad y \neq 0. \tag{5.6-42}$$

We will demonstrate the application of remark 5.6 to the calculation of this kernel. The elements of the boundary space $\mathscr{H}_{\mathrm{bo}}$ are multiples of $n_0(x) = e^{-m|x|}$. One sees easily that this function is well approximated in the weak $\sigma(\mathscr{H}, \mathcal{D}_{\mathrm{bu}})_{\mathrm{d}x}$-topology by

$$n_\epsilon(x) = \begin{cases} e^{-m|x|} & \text{if } |x| \geq \epsilon \\ \frac{e^{-m\epsilon}}{\sinh m\epsilon} \sinh m|x| & \text{if } |x| < \epsilon. \end{cases} \tag{5.6-43}$$

as $\epsilon \longrightarrow 0+$. This is a solution of the massive Klein-Gordon equation with eigenvalue 0 for $|x| \neq \epsilon$, $x \neq 0$, so

$$(m^2 - \partial_x^2) n_\epsilon(x) = \frac{m}{\sinh m\epsilon} \left(\delta(x - \epsilon) + \delta(x + \epsilon)\right) + c_0 \delta(x). \tag{5.6-44}$$

The actual value of the constant c_0 is not important. The Schwinger kernel of the bulk-to-boundary propagator is determined by the scalar product

$$\begin{aligned} \langle B n_0,\, K_\alpha^{\mathrm{bobu}} f \rangle_{\partial g} &= \lim_{\epsilon \to 0+} \langle n_\epsilon,\, K_\alpha^{\mathrm{bu}} f \rangle \\ &= \lim_{\epsilon \to 0+} \int \mathrm{d}x\, \frac{m}{\sinh m\epsilon} \left(K_\alpha^{\mathrm{bu}}(\epsilon, x) + K_\alpha^{\mathrm{bu}}(-\epsilon, x) \right) f(x). \end{aligned}$$

The term $\int \mathrm{d}x\, c_0 K_\alpha^{\mathrm{bu}}(0, x) f(x) = c_0 f(0)$ vanishes because $f(0) = 0$ by assumption ($f \in \mathcal{D}_{\mathrm{bu}}$). Taking the limit is now straightforward. Because $B n_0 = 1$ and $\partial g = 1$, we obtain

$$K_\alpha^{\mathrm{bobu}}(x) = \lim_{\epsilon \to 0+} \frac{m}{\sinh m\epsilon} \left(K_\alpha^{\mathrm{bu}}(\epsilon, x) + K_\alpha^{\mathrm{bu}}(-\epsilon, x) \right) = \frac{|x|}{2\sqrt{\pi\alpha^3}} e^{-\frac{x^2}{4\alpha} - m^2\alpha}$$

as before. In this sense, the bulk-to-boundary propagator is the limit

$$G_{\text{bobu}}(x) = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \left(G_{\text{bu}}(\epsilon, x) + G_{\text{bu}}(-\epsilon, x) \right) .$$

of the bulk-to-bulk propagator.

We want to point out that this model can be regarded as a super-simple realisation of AdS/CFT with a boundary consisting of one point. If one introduces a second constraint $\phi(x_1) = \phi_1$, then we have a boundary consisting of two points, so we can even get an intuition about boundary-to-boundary propagation.

Chapter 6

An Application to Field Theory on EAdS

The geometry of Euclidean Anti-de-Sitter space has been covered in section 3.1.

6.1 Scalar Field Theory on EAdS with Constraints

According to the situation in AdS/CFT correspondence with dual boundary source terms (cf. section 3.2), we have to consider a field theory on (Euclidean) AdS with fixed conditions on the boundary of EAdS. We want to construct a Neumann path integral on the Euclidean Anti-de-Sitter space introduced in section 3.1, under the restriction that we want to prescribe values of the field for $z^0 \to 0$ in a sensible way [1]. For that purpose, we will adapt the formalism developed in the preceding section. We begin by constructing the Hilbert spaces $\mathscr{H}_{\mathrm{bu}} \subset \mathscr{H}$ and $\mathscr{H}'$. We take $\mathscr{H}_{\mathrm{bu}}$ to be the a concrete domain of self-adjointness of the Klein-Gordon operator $m^2 - \hbar^2\Box^g$ and adjoin some non-normalisable solutions of the wave equation in the space $\mathscr{H}_{\mathrm{bo}}$. The Hilbert space $\mathscr{H}$ is then the abstract sum $\mathscr{H}_{\mathrm{bu}} \oplus \mathscr{H}_{\mathrm{bo}}$.

Construction of the positive symmetric form

We will construct a symmetric, positive domain by simply enumerating all (smooth) eigenfunctions of the Klein-Gordon operator on EAdS having nonnegative eigenvalues and making an appropriate choice (this amounts to selecting appropriate boundary conditions). Its domain will be a linear space generated by a subset of these eigenfunctions, under the condition that it should be invariant under the EAdS symmetry group. By definition, all the eigenfunctions are orthogonal. The advantage of this method lies in the fact that completeness is automatic.

[1] There is a related analysis of EAdS path integral function spaces by Gottschalk and Thaler in the setting of nuclear spaces[48].

We begin by determining the solutions of the Klein-Gordon equation for the real eigenvalue λ,

$$\begin{aligned}\lambda f =&(m^2 - \Box^g) f \\ =&m^2 f - \left((x^0)^2\partial_0^2 + (1-\mathrm{d})x^0\partial_0 + (x^0)^2\triangle\right) f, \qquad x^0 > 0.\end{aligned} \tag{6.1-1}$$

Because the Poincaré coordinates are horizontally translation invariant, we make the general ansatz

$$f(x) = e^{i\underline{k}\cdot\underline{x}} f_0(x^0), \qquad \underline{k} \in \mathbb{R}^{\mathrm{d}}.$$

We will later build a real Hilbert space from these functions; however, for the moment it is easier to construct the complex Hilbert space. We restrict the vectors $\underline{k}$ to real values; so we exclude solutions which are growing exponentially. Later, we will see that for our purposes, this is acceptable. We have now a one-dimensional differential equation

$$(m^2 - \lambda) f_0(x^0) = \left((x^0)^2\partial_0^2 + (1-\mathrm{d})x^0\partial_0 - (x^0)^2\underline{k}^2\right) f_0(x^0).$$

By substituting $f_0(x^0) = (x^0)^{\mathrm{d}/2} f_1(x^0)$, we get a modified Bessel differential equation

$$\left(\frac{\mathrm{d}^2}{4} + m^2 - \lambda\right) f_1(x^0) = \left((x^0)^2\partial_0^2 + x^0\partial_0 - (x^0)^2\underline{k}^2\right) f_1(x^0).$$

We abbreviate on the left hand side

$$\beta^2 = \left|\frac{\mathrm{d}^2}{4} + m^2 - \lambda\right|.$$

Because we will need it often, we define also the value at $\lambda = 0$,

$$\beta_0 = \sqrt{\frac{\mathrm{d}^2}{4} + m^2}.$$

For the eigenfunctions to form a complete set of functions, we will see later that β_0 must be real; this provides a lower bound $m^2 \geq -\mathrm{d}^2/4$ on the mass square. There are two types of solutions depending on the parameter range ($\underline{k} \neq 0$):

$$f_1(x^0) = \begin{cases} C_\beta K_\beta(|\underline{k}|x^0) + D_\beta I_\beta(|\underline{k}|x^0), & \text{if } \frac{\mathrm{d}^2}{4} + m^2 - \lambda > 0, \\ C_{i\beta} K_{i\beta}(|\underline{k}|x^0) + D_{i\beta} I_{i\beta}(|\underline{k}|x^0), & \text{if } \frac{\mathrm{d}^2}{4} + m^2 - \lambda \leq 0 \end{cases} \tag{6.1-2}$$

($C_.$ and $D_.$ are normalisations). In addition, for $\underline{k} = 0$, there are solutions $f_1(x^0) \sim (x^0)^{\pm\beta}$ resp. $f_1(x^0) \sim (x^0)^{\pm i\beta}$ (however, $\underline{k} = 0$ is a null set in wave number space, so we disregard them). The function $K_\beta(z)$ behaves as $z^{-\beta}$ for small z and falls off exponentially as $z \to \infty$. The function $I_\beta(z)$ increases exponentially for large z. We will therefore exclude it from out considerations. Similarly, the real function $K_{i\beta}(z) = K_{-i\beta}(z)$ oscillates with an ever increasing frequency

$$K_{i\beta}(z) \approx \Re\left[2^{i\beta}\Gamma(i\beta)e^{-i\beta\ln z}\right] + \mathcal{O}(z) \qquad \text{as } \beta > z \to 0$$

and drops off exponentially for large z, whereas $I_{i\beta}(z)$ again grows exponentially after an initial period of (complex) oscillations [2]; so we discard it. So we are left with a pair of solutions for $f(x)$:

$$f(x) = C_\beta \, e^{i\underline{xk}} \, (x^0)^{\mathrm{d}/2} \, K_\beta(|\underline{k}|x^0) \qquad \equiv e_{\underline{k},\beta}(x) \qquad \text{if } \frac{\mathrm{d}^2}{4} + m^2 - \lambda > 0, \tag{6.1-3a}$$

$$f(x) = C_{i\beta} \, e^{i\underline{xk}} \, (x^0)^{\mathrm{d}/2} \, K_{i\beta}(|\underline{k}|x^0) \qquad \equiv e_{\underline{k},i\beta}(x) \qquad \text{if } \frac{\mathrm{d}^2}{4} + m^2 - \lambda \le 0. \tag{6.1-3b}$$

To build a Hilbert space of functions $\mathscr{H}$ out of these eigenvectors, we must declare the scalar product. By definition,

$$\langle f, h \rangle = \Pi^{-1}(f, h) = \langle f^*, \Pi^{-1} h \rangle_{\mathscr{H}},$$

where Π^{-1} is understood as quadratic form resp. as operator from $\mathscr{H}$ into $\mathscr{H}'$. The dual Hilbert space $\mathscr{H}'$ contains certain distributions in the bulk, and it contains the *generalised boundary values* $\mathscr{H}'_{\mathrm{bo}}$. In the bulk, the action of Π^{-1} is given by the differential operator (6.1-1). The generalised boundary values must chosen in such a way that the scalar product $\langle , \rangle$ is symmetric. A way to do this has been shown by Klebanov and Witten [56]: Select a real solution $(m^2 - \Box^g)s(x) = 0$ of the Klein-Gordon equation. With this solution, the action of the Klein-Gordon operator in the bulk can be written as

$$\begin{aligned}(m^2 - \Box^g) f &= m^2 f - (x^0)^{1+\mathrm{d}} \partial_\mu (x^0)^{1-\mathrm{d}} \partial_\mu f \\ &= - s^{-1} (x^0)^{1+\mathrm{d}} \partial_\mu s^2 (x^0)^{1-\mathrm{d}} \partial_\mu \, [s^{-1} f].\end{aligned}$$

By integration by parts, the following equality holds for suitable f, h:

$$\begin{aligned}&- \int \mathrm{d}^{\mathrm{d}+1} x \, [s^{-1} f]^* \, \partial_\mu s^2 (x^0)^{1-\mathrm{d}} \partial_\mu \, [s^{-1} h] + \int_{\partial\,\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}} x \, [s^{-1} f]^* \, s^2 (x^0)^{1-\mathrm{d}} e^\mu(x) \partial_\mu \, [s^{-1} h] \\ &= - \int \mathrm{d}^{\mathrm{d}+1} x \, \partial_\mu s^2 (x^0)^{1-\mathrm{d}} \partial_\mu \, [s^{-1} f]^* \cdot [s^{-1} h] + \int_{\partial\,\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}} x \, s^2 (x^0)^{1-\mathrm{d}} e^\mu(x) \partial_\mu \, [s^{-1} f]^* \cdot [s^{-1} h] \\ &= \int \mathrm{d}^{\mathrm{d}+1} x \, s^2 (x^0)^{1-\mathrm{d}} \, \partial_\mu [s^{-1} f]^* \cdot \partial_\mu [s^{-1} h] \end{aligned} \tag{6.1-4}$$

where $e^\mu(x)$ is the normal vector on the boundary (in the local chart!). The integral over the surface of EAdS is understood as an integral over a surface which is immersed in EAdS and pushed towards infinity. The last integral gives us the possibility to define the scalar product in an intrinsically symmetric form if we choose an appropriate solution s. There is a certain arbitrariness in the choice of s; this reflects the fact that there are several possible ways to take boundary values. We choose

$$s = (x^0)^{\Delta_-}, \qquad \text{where } \Delta_\pm = \frac{\mathrm{d}}{2} \pm \beta_0 = \frac{\mathrm{d}}{2} \pm \sqrt{\frac{\mathrm{d}^2}{4} + m^2}, \tag{6.1-5}$$

[2] For sake of completeness, we note that $I_{i\beta}(z) \approx \frac{2^{-i\beta}}{\Gamma(1+i\beta)} e^{i\beta \ln z}$ for $z < \beta$.

and *declare* therefore the action of Π^{-1} as

$$\begin{aligned}\langle f, h\rangle = \Pi^{-1}(f,h) \equiv &- \int \mathrm{d}^{\mathrm{d}+1}x\, [(x^0)^{-\Delta_-} f]^*\, \partial_\mu (x^0)^{2\Delta_- +1-\mathrm{d}} \partial_\mu\, [(x^0)^{-\Delta_-} h] \\ &+ \int_{\partial\,\mathrm{EAdS}} \mathrm{d}^{\mathrm{d}}x\, [(x^0)^{-\Delta_-} f]^*\, (x^0)^{2\Delta_- +1-\mathrm{d}} e^\mu(x) \partial_\mu\, [(x^0)^{-\Delta_-} h] \\ = &\int \mathrm{d}^{\mathrm{d}+1}x\, (x^0)^{2\Delta_- +1-\mathrm{d}}\, \partial_\mu [(x^0)^{-\Delta_-} f]^* \cdot \partial_\mu [(x^0)^{-\Delta_-} h]. \end{aligned} \tag{6.1-6}$$

This scalar product comes "equipped with its own suitable boundary terms" [3]. It shown in the appendix that the Hilbert space $\mathscr{H}$ is spanned by the orthogonal vectors $e_{\underline{k},i\beta}$ and $e_{\underline{k},\beta_0}$. The first set is the equivalent of standing waves near the $x^0 = 0$ boundary of EAdS. Almost all geodesics in EAdS are arcs penetrating the system from the x^0 plane until they reach their point of return, whence they fall back towards the boundary. This structure is clearly visible in the wave functions $e_{\underline{k},i\beta}$: Passing on out of the initial oscillatory region, the waves have to tunnel into the potential wall generated by curvature, so they are damped exponentially.

Choosing the normalisations

$$C_{i\beta} = \left(\frac{2\beta \sinh \pi\beta}{(2\pi)^{\mathrm{d}}\pi^2}\right)^{\frac{1}{2}}, \qquad C_{\beta_0} = \left(\frac{2\sin\beta_0\pi}{(2\pi)^{\mathrm{d}}\pi}\right)^{\frac{1}{2}}, \tag{6.1-7}$$

we summarise the structure with

Proposition 6.1. *The Hilbert space $\mathscr{H}$ with the scalar product (6.1-6) is a sum of two orthogonal subspaces $\mathscr{H}_{\mathrm{bu}} \oplus \mathscr{H}_{\mathrm{bo}}$; the subspace $\mathscr{H}_{\mathrm{bu}}$ is spanned by the oscillating basis functions $e_{\underline{k},i\beta}$ ($\beta > 0$); the subspace $\mathscr{H}_{\mathrm{bo}}$ is spanned by the $e_{\underline{k},\beta_0}$, where we have to impose the condition $0 < \beta_0 < 1$ on the mass* [4]. *In the abstract setting, $\mathscr{H}_{\mathrm{bu}} \cong L^2(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+, \lambda_{i\beta}\, \mathrm{d}^{\mathrm{d}}\underline{k}\, \mathrm{d}\beta\,)$ with the scalar product*

$$\langle f, g\rangle = \int \mathrm{d}^{\mathrm{d}}\underline{k}\ \mathrm{d}\beta\, \lambda_{i\beta}\, \hat f^*(\underline{k}, i\beta) \hat g(\underline{k}, i\beta).$$

contains the functions $\hat f(\underline{k}, i\beta)$ which are represented concretely in coordinate space as

$$f(x) = \int \mathrm{d}^{\mathrm{d}}\underline{k}\ \mathrm{d}\beta\, \hat f(\underline{k}, i\beta) e_{\underline{k},i\beta}(x).$$

The inverse transformation is $\hat f(\underline{k}, i\beta) = \lambda_{i\beta}^{-1}\langle e_{\underline{k},i\beta}, f\rangle$.

Similarly, $\mathscr{H}_{\mathrm{bo}} \cong L^2(\mathbb{R}^{\mathrm{d}}, \mathrm{d}^{\mathrm{d}}\underline{k}\,)$ has the scalar product

$$\langle f, g\rangle = \int \mathrm{d}^{\mathrm{d}}\underline{k}\, \hat f(\underline{k}, \beta_0)^* \hat g(\underline{k}, \beta_0).$$

[3] Notice that in case $m^2 = 0$, this procedure is trivial as $s \equiv 1$.

[4] This is in agreement with [56]. For $\beta_0 = 0$, there are simply *no* boundary modes: The field "recedes" into the bulk.

Dual Spaces, Measurable Structure, Boundary Space

The other spaces of interest are now defined easily. The dual space of $\mathscr{H}_{\rm bo}$ is $\mathscr{H}'_{\rm bo} = \mathscr{H}_{\rm bo}{}' \cong L^2(\mathbb{R}^{\rm d}, {\rm d}^{\rm d}\underline{k}\,)$. We call its basis functions $e'_{\underline{k},\beta_0}$, and they are dual to the basis functions of $\mathscr{H}_{\rm bo}$, $\langle e_{\underline{k},\beta_0}, e'_{\underline{k}',\beta_0}\rangle_{\mathscr{H}} = \delta^{({\rm d})}(\underline{k}-\underline{k}')$. This implies $\Pi^{-1}e_{\underline{k},\beta_0} = e'_{\underline{k},\beta_0}$.

Likewise, the dual space of $\mathscr{H}_{\rm bu}$ is $\mathscr{H}'_{\rm bu} = \mathscr{H}_{\rm bu}{}' \cong L^2(\mathbb{R}^{\rm d}\times\mathbb{R}_+, \lambda_{i\beta}^{-1}\,{\rm d}^{\rm d}\underline{k}\;{\rm d}\beta\,)$. We call its basis functions $e'_{\underline{k},\beta}$, and they are dual to the basis functions of $\mathscr{H}_{\rm bu}$, $\langle e_{\underline{k},\beta}, e'_{\underline{k}',\beta'}\rangle_{\mathscr{H}} = \delta^{({\rm d})}(\underline{k}-\underline{k}')\delta(\beta-\beta')$. This implies $\Pi^{-1}e_{\underline{k},i\beta} = \lambda_{i\beta}e'_{\underline{k},i\beta}$. The total dual space is $\mathscr{H}' = \mathscr{H}'_{\rm bo}\oplus\mathscr{H}'_{\rm bu}$. The measurable structure on $\mathscr{H}$ is fixed *a priori* in this construction: The vectors $e_{\underline{k},i\beta}(x)$ and $e_{\underline{k},\beta_0}(x)$ were designed as functions on EAdS; this fixes a canonical action of the scalar product $\langle,\rangle_g$.

Remark 6.2. It is important to have coordinate space expressions for the scalar product $\langle,\rangle$. Let $f(x)$ be a function on EAdS-space with compact support. Then by integration by parts of (6.1-6), the boundary terms are discarded and we have

$$\hat{f}(\underline{k}, i\beta) = \lambda_{i\beta}^{-1}\langle e_{\underline{k},i\beta}, f\rangle = \langle e_{\underline{k},i\beta}, f\rangle_g.$$

Similarly, for $h(x)$ an arbitrary bulk function decreasing fast enough for $|\underline{x}|\to\infty$, the product $\langle e_{\underline{y},\beta_0}, h\rangle$ can be simplified by rolling the Klein-Gordon operator onto $e_{\underline{y},\beta_0}$. As this is an eigenvector with eigenvalue 0, *only the boundary term survives the integration by parts*, and we have

$$\begin{aligned}\langle e_{\underline{y},\beta_0}, h\rangle =& \lim_{x^0\to 0+}(x^0)^{1-{\rm d}+\Delta_-}\int {\rm d}^{\rm d}x\,\partial_0((x^0)^{-\Delta_-}e_{\underline{y},\beta_0}(x))\,h(x)\\ =& C_{\beta_0}\frac{\Gamma(\frac{{\rm d}+\beta_0}{2})\Gamma(\frac{{\rm d}-\beta_0}{2}+1)}{2^{-{\rm d}/2}\Gamma(\frac{\rm d}{2})}\lim_{x^0\to 0+}(x^0)^{-\Delta_-}\\ &\int\frac{{\rm d}^{\rm d}x}{((x^0)^2+|\underline{x}-\underline{y}|^2)^{\frac{{\rm d}+\beta_0}{2}}}\,{}_2{\rm F}_1\!\left(\frac{{\rm d}+\beta_0}{2},\frac{\beta_0}{2}-1;\,\frac{\rm d}{2};\,\frac{|\underline{x}-\underline{y}|^2}{|\underline{x}-\underline{y}|^2+(x^0)^2}\right)\,h(x)\end{aligned}$$

(where $e_{\underline{y},\beta_0}(x)$ is found in the appendix). In particular, the scalar product vanishes for compactly supported functions. Note that this kernel is not positive definite. In case the bulk function is of the form $h = \int {\rm d}^{\rm d}y\,h_{\rm bo}(\underline{y})e_{\underline{y},\beta_0} + h_{\rm bulk}$, the product simplifies to

$$h_{\rm bo}(\underline{y}) = \langle e_{\underline{y},\beta_0}, h\rangle = \frac{2^{1-\beta_0}}{C_{\beta_0}(2\pi)^{{\rm d}/2}\Gamma(\beta_0)}\lim_{y^0\to 0+}(y^0)^{-\Delta_-}|\underline{\nabla}|^{\beta_0}h(y). \tag{6.1-8}$$

Here, $|\underline{\nabla}|^{\beta_0}$ means the operator which in wave number space is given by $|\underline{k}|^{\beta_0}$. *This formula shows that the* $\mathscr{H}_{\rm bo}$*-basis vectors* $e_{\underline{k},\beta_0}$ *resp. their boundary projections are not a suitable basis for the boundary space (ie, they do not imply the usual geometric procedure of taking the boundary value).* The factor $|\underline{\nabla}|^{\beta_0}$ represents a non-local operation in coordinate space.

We identify the boundary spaces $\mathscr{B}\cong\mathscr{H}_{\rm bo} = L^2(\mathbb{R}^{\rm d},{\rm d}^{\rm d}\underline{k}\,)$ and $\mathscr{B}'\cong\mathscr{H}'_{\rm bo} = L^2(\mathbb{R}^{\rm d},{\rm d}^{\rm d}\underline{k}\,)$. The operators $B:\mathscr{H}\to\mathscr{B}$ and $B':\mathscr{B}'\to\mathscr{H}'$ are the trivial projection/inclusion. The boundary kernel of the Lagrangian acts simply as

$$\Pi_{\mathrm{bo}}^{-1}\, e_{\underline{k},\beta_0} = (B\Pi B')^{-1} e_{\underline{k},\beta_0} = e'_{\underline{k},\beta_0},$$

where $e^{(\prime)}_{\underline{k},\beta_0} \in \mathscr{B}^{(\prime)}$ are generalised boundary vectors. If we choose a different basis for the boundary space $\mathscr{B}$ assembled out of the vectors

$$b_{\underline{k}} = \frac{2^{1-\beta_0}}{C_{\beta_0}(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)} |\underline{k}|^{\beta_0} e_{\underline{k},\beta_0}, \tag{6.1-9}$$

then the boundary projection of a function $h(x)$ takes the form

$$Bh = \frac{2^{1-\beta_0}}{C_{\beta_0}(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)} \int \mathrm{d}^{\mathrm{d}}k\, (B_{\Delta_-}h)(\underline{k})\, |\underline{k}|^{\beta_0} e_{\underline{k},\beta_0} = \int \mathrm{d}^{\mathrm{d}}k\, (B_{\Delta_-}h)(\underline{k})\, b_{\underline{k}}$$

with the usual *geometric* boundary projection $B_{\Delta_-} : \mathscr{H} \to L^2(\mathbb{R}^{\mathrm{d}})$ acting as

$$(B_{\Delta_-}h)(\underline{x}) = \lim_{x^0\to 0} (x^0)^{-\Delta_-} h(x).$$

We impose on the boundary the $L^2(\mathbb{R}^{\mathrm{d}})$ scalar product

$$\Big\langle \int \mathrm{d}^{\mathrm{d}}k\, f(\underline{k}) b_{\underline{k}},\, \int \mathrm{d}^{\mathrm{d}}k\, h(\underline{k}) b_{\underline{k}} \Big\rangle_{\partial g} = \int \mathrm{d}^{\mathrm{d}}k\, f(\underline{k})^* h(\underline{k});$$

it implies a map $\jmath : \mathcal{D}_{\mathcal{C}} \subset \mathscr{B} \to \mathscr{B}'$ on the boundary with the defining property

$$\langle f,\, h\rangle_{\partial g} = \langle f,\, \jmath h\rangle_{\mathscr{B}}$$

for all $f \in \mathscr{B}, h \in \mathcal{D}_{\mathcal{C}}$. In particular, on finds

$$\jmath e_{\underline{k},\beta_0} = \frac{C^2_{\beta_0}(2\pi)^{\mathrm{d}}\Gamma(\beta_0)^2}{4^{1-\beta_0}} |\underline{k}|^{-2\beta_0} e'_{\underline{k},\beta_0} \tag{6.1-10}$$

from $\langle b_{\underline{k}}, b_{\underline{k}'}\rangle_{\partial g} = \delta^{(\mathrm{d})}(\underline{k} - \underline{k}')$. The map $\jmath$ plays the same role as $\imath$ in the bulk: It defines the measurable structure of the underlying Banach space. However, there is no direct connection between $\imath$ and $\jmath$. Rather, the boundary space structure is fixed by the requirement that bulk and boundary are subject to the same symmetry group.

The quadratic term of the boundary Lagrangian can now be written with an integral kernel in the b-basis. We have

$$\begin{aligned}\langle f,\, (B'\Pi B)^{-1} h\rangle_{\mathscr{B}} =& \Big\langle \int \mathrm{d}^{\mathrm{d}}k\, f(\underline{k}) b_{\underline{k}},\, (B'\Pi B)^{-1} \int \mathrm{d}^{\mathrm{d}}k\, h(\underline{k}) b_{\underline{k}} \Big\rangle_{\mathscr{B}} \\ =& \frac{4^{1-\beta_0}}{C^2_{\beta_0}(2\pi)^{\mathrm{d}}\Gamma(\beta_0)^2} \int \mathrm{d}^{\mathrm{d}}k\, |\underline{k}|^{2\beta_0} f(\underline{k})^* h(\underline{k}) \\ =& \frac{4^{1-\beta_0}}{C^2_{\beta_0}(2\pi)^{\mathrm{d}}\Gamma(\beta_0)^2} \langle |\underline{\nabla}|^{\beta_0} f,\, |\underline{\nabla}|^{\beta_0} h\rangle_{\partial g}.\end{aligned} \tag{6.1-11}$$

In fact, there is a slight mismatch between the boundary space $\mathscr{B}$ and the space $L^2(\mathbb{R}^{\mathrm{d}}, \partial g)$ of "local boundary values", which we will simply ignore in the sequel. The "two-point function" for the constraints is from (6.1-11) proportional to $|\underline{k}|^{2\beta_0}$ in wave number space; in coordinate space, this implies a behaviour $|\underline{x}|^{-\mathrm{d}-2\beta_0}$. Its scaling dimension is therefore $\Delta = \frac{\mathrm{d}}{2} + \beta_0 = \Delta_+$. Note that for the "unconstrained" system, the two-point function is given by the inverse $B\Pi B'$; so in the $b_{\underline{k}}$-basis, it behaves like $|\underline{k}|^{-2\beta_0}$, resp. $|\underline{x}|^{-\mathrm{d}+2\beta_0}$. The scaling dimension is $\Delta = \frac{\mathrm{d}}{2} - \beta_0 = \Delta_-$.

Remark 6.3. It may seem strange that the boundary $\mathcal{C} = \mathbb{R}^{\mathrm{d}}$ of EAdS is not emerging as a compact manifold. However, in the Hilbert space setting, the concept of compactification does not make sense, as $L^2(\mathbb{R}^{\mathrm{d}})$-functions are anyhow vanishing towards infinity.

Completeness in the $\mathbf{L^2}(\mathcal{M}, \mathbf{g})$ Sense

We present a formal (!) completeness relation: For $x, y \in$ EAdS, let $\delta_y(x) = \frac{1}{\sqrt{g}}\,\delta^{(\mathrm{d})}(\underline{x}-\underline{y})\delta(x^0 - y^0)$ the Dirac distribution appropriate to EAdS space. Although δ_y is not in $\mathscr{H}$, by formal algebra

$$\hat{\delta}_y(\underline{k}, i\beta) = \langle e_{\underline{k},i\beta}, \delta_y\rangle_g = e_{-\underline{k},i\beta}(y) = C_{i\beta}\, e^{-iy\underline{k}}\, (y^0)^{\mathrm{d}/2}\, K_{i\beta}(|\underline{k}|y^0).$$

Transforming back, we have

$$\begin{aligned}
&\int \mathrm{d}^{\mathrm{d}}\underline{k}\; \mathrm{d}\beta\, e_{\underline{k},i\beta}(z)\; \hat{\delta}_y(\underline{k}, i\beta) \\
&\qquad = \frac{2(y^0z^0)^{\mathrm{d}/2}}{(2\pi)^{\mathrm{d}}\pi^2} \int \mathrm{d}^{\mathrm{d}}\underline{k}\; \mathrm{d}\beta\,\beta \sinh \pi\beta\; e^{i(\underline{z}-\underline{y})\underline{k}}\; K_{i\beta}(|\underline{k}|z^0)K_{i\beta}(|\underline{k}|y^0) \\
&\qquad\qquad = \frac{(y^0)^{\mathrm{d}+1}}{(2\pi)^{\mathrm{d}}}\delta(z^0 - y^0) \int \mathrm{d}^{\mathrm{d}}\underline{k}\, e^{i(\underline{z}-\underline{y})\underline{k}} \\
&\qquad\qquad\qquad = (y^0)^{\mathrm{d}+1}\delta(z^0 - y^0)\delta^{(\mathrm{d})}(\underline{z} - \underline{y}) = \delta_y(z).
\end{aligned} \tag{6.1-12}$$

where we used the formal integral

$$\int_0^\infty \mathrm{d}\beta\,\beta \sinh \pi\beta\; K_{i\beta}(a)K_{i\beta}(b) = \frac{\pi^2 a}{2}\delta(a - b).$$

[49, 6.794 1, processed].

6.1.1 Schwinger Parametrised Bulk Propagator

For the g-products of the basis vectors, one computes

$$\begin{aligned}
\langle e_{\underline{k},i\beta}, e_{\underline{k}',i\beta'}\rangle_g &= \delta^{(\mathrm{d})}(\underline{k} - \underline{k}')\delta(\beta - \beta') = \lambda_{i\beta}^{-1}\langle e_{\underline{k},i\beta}, e_{\underline{k}',i\beta'}\rangle \\
\langle e_{\underline{k},\beta_0}, e_{\underline{k}',\beta_0}\rangle_g &= +\infty\; \cdot\; \delta^{(\mathrm{d})}(\underline{k} - \underline{k}'), \\
\langle e_{\underline{k},i\beta}, e_{\underline{k}',\beta_0}\rangle_g &= \text{not defined.}
\end{aligned}$$

So the domain $\mathcal{D}$, which should contain vectors d with $|\langle d, h\rangle_g| < \infty$ for all $h \in \mathscr{H}$, must be a subset of $\mathscr{H}_{\mathrm{bu}}$. Because for $d, h \in \mathscr{H}_{\mathrm{bu}}$

$$|\langle d, h\rangle_g|^2 \leq \langle d, d\rangle_g \langle h, h\rangle_g \leq \lambda_{i0}^{-2} \langle d, d\rangle \langle h, h\rangle < \infty,$$

the condition is automatically fulfilled for these h. Therefore, the remaining condition is that for all $h \in \mathscr{H}_{\mathrm{bo}}$, $\langle d, h\rangle_g$ is finite. Roughly, this means that

$$\int \mathrm{d}^{\mathrm{d}+1}x \ (x^0)^{-\mathrm{d}/2-\beta_0-1} |d(x)|$$

is finite; so $d(x) \sim \mathcal{O}((x^0)^{\mathrm{d}/2+\beta_0+\epsilon})$ near the boundary.

We construct the map $\imath$. By definition,

$$\langle e_{\underline{k}',i\beta'}, \imath e_{\underline{k},i\beta}\rangle_{\mathscr{H}} = \langle e_{\underline{k}',i\beta'}, e_{\underline{k},i\beta}\rangle_g = \delta^{(\mathrm{d})}(\underline{k} - \underline{k}')\delta(\beta - \beta'), \tag{6.1-13}$$

so $\mathcal{P}'_{\mathrm{bu}} \imath e_{\underline{k},i\beta} = e'_{\underline{k},i\beta}$, whereas the $\mathscr{H}'_{\mathrm{bo}}$-content is better determined formally in the Dirac basis

$$\langle e_{\underline{k},\beta_0}, \imath \delta_y\rangle_{\mathscr{H}} = \langle e_{\underline{k},\beta_0}, \delta_y\rangle_g = e_{-\underline{k},\beta_0}(y), \tag{6.1-14}$$

so $(1 - \mathcal{P}'_{\mathrm{bu}})\imath\delta_y = \int \mathrm{d}^d k \, e'_{\underline{k},\beta_0} e_{-\underline{k},\beta_0}(y)$. The relation $\mathcal{D} \subset \mathscr{H}_{\mathrm{bu}}$ implies $\mathcal{D}_{\mathrm{bu}} = \mathcal{D}$. We determine the operator $\mathcal{S}$ by its action

$$\mathcal{S} e'_{\underline{k},i\beta} = e_{\underline{k},i\beta}, \qquad \mathcal{S} e'_{\underline{k},\beta_0} = 0. \tag{6.1-15}$$

In the eigenbasis, $\mathcal{S}$ is rather trivial; however in coordinate space, this is a rather complicated action. The Schwinger (heat) kernel is determined as

$$\left(\exp -\alpha \mathcal{S} \Pi^{-1}\right) e_{\underline{k},i\beta} = e^{-\alpha\lambda_{i\beta}} e_{\underline{k},i\beta}. \tag{6.1-16}$$

For a δ-distribution,

$$\begin{aligned} K_\alpha^{\mathrm{bu}}(x, y) &= \left(\exp -\alpha \mathcal{S} \Pi^{-1}\right) \delta_y(x) \\ &= \int \mathrm{d}^{\mathrm{d}} k \ \mathrm{d}\beta \, e_{\underline{k},i\beta}(x) \ e^{-\alpha\lambda_{i\beta}} \ \hat{\delta}_y(\underline{k}, i\beta) \\ &= (x^0 y^0)^{\mathrm{d}/2} \int \mathrm{d}^{\mathrm{d}} k \ \mathrm{d}\beta \, e^{-(\beta_0^2+\beta^2)\alpha} \ C_{i\beta}^2 \ e^{i(\underline{x}-\underline{y})\underline{k}} \ K_{i\beta}(|\underline{k}|x^0) \ K_{i\beta}(|\underline{k}|y^0). \end{aligned}$$

This expression is studied in the appendix; it can be reduced to a contour integral

$$K_\alpha^{\mathrm{bu}}(x, y) = \frac{\Gamma(\frac{\mathrm{d}}{2} + 1)i}{2^{\mathrm{d}/2+2} \, \pi^{(\mathrm{d}+3)/2} \alpha^{1/2}} e^{-\beta_0^2\alpha} \int_{C_\epsilon} \mathrm{d}t \, \frac{(\sinh t) e^{-\frac{t^2}{4\alpha}}}{((1 + \frac{\sigma(x,y)}{2}) - \cosh t)^{\mathrm{d}/2+1}},$$

where the contour C_ϵ encloses the positive axis anti-clockwise, crossing through the origin. For even d, we may obtain the closed expression (C.2-7) by Cauchy's formula. Note that the volume damping term $e^{-\beta_0^2\alpha}$ uses the effective mass β_0. According to equation (5.5-30), the bulk-to-bulk propagator can be obtained from the Schwinger kernel by integrating

$$G_{\mathrm{bu}}(x, y) = \hbar \int_0^\infty \mathrm{d}\alpha \, K_\alpha^{\mathrm{bu}}(x, y) = \hbar \frac{e^{-i\pi\frac{\mathrm{d}-1}{2}}}{2\,\pi^{\frac{\mathrm{d}+1}{2}} \, [\sigma(\sigma + 4)]^{\frac{\mathrm{d}-1}{4}}} Q_{\beta_0-\frac{1}{2}}^{\frac{\mathrm{d}-1}{2}}(1 + \frac{\sigma}{2}), \tag{6.1-17}$$

as shown in the appendix (C.2-8). Note that this kernel depends solely on the chordal distance σ.

Unconstrained propagator ("field theoretic prescription"). The unconstrained propagator is by definition the integral kernel of Π, ie

$$G(y,x) = \hbar\langle\delta_y, \Pi\imath\delta_x\rangle_g,$$

where δ_x is the Dirac distribution appropriate to EAdS-space. We now split the right hand side in two parts:

$$\begin{aligned}\Pi\imath\delta_x =&\Pi\mathcal{P}'_{\mathrm{bu}}\imath\delta_x + \Pi(1-\mathcal{P}'_{\mathrm{bu}})\imath\delta_x\\ =&\Pi_{\mathrm{bu}}\imath\delta_x + \Pi(1-\mathcal{P}'_{\mathrm{bu}})\imath\delta_x.\end{aligned}$$

The contribution of the first summand is already known; it is the constrained bulk-to-bulk propagator

$$G_{\mathrm{bu}}(y,x) = \hbar\langle\delta_y,\, \Pi_{\mathrm{bu}}\imath\delta_x\rangle_g.$$

The second summand gives a new contribution which we still have to evaluate. Explicitly, using [49, 6.578 10]

$$\begin{aligned}\hbar\langle\delta_y,\, \Pi(1-\mathcal{P}'_{\mathrm{bu}})\imath\delta_x\rangle_g =&\hbar\int \mathrm{d}^{\mathrm{d}}k\, e_{-\underline{k},\beta_0}(x)\,\langle\delta_y,\, \Pi e'_{\underline{k},\beta_0}\rangle_g \\ =&\hbar\int \mathrm{d}^{\mathrm{d}}k\, e_{-\underline{k},\beta_0}(x)\, e_{\underline{k},\beta_0}(y)\\ =&\hbar\frac{\Gamma(\frac{\mathrm{d}}{2}-\beta_0)\Gamma(\frac{\mathrm{d}}{2}+\beta_0)\sin\beta_0\pi}{2\pi^{\frac{\mathrm{d}+1}{2}}\,[\sigma(\sigma+4)]^{\frac{\mathrm{d}-1}{4}}}P^{\frac{1-\mathrm{d}}{2}}_{\beta_0-\frac{1}{2}}(1+\frac{\sigma}{2}).\end{aligned} \tag{6.1-18}$$

Again, this depends solely on the chordal distance σ. Adding this to G_{bu}, we obtain after an amount of algebra [49, 8.736 1 and 7]

$$G(y,x) = \hbar\frac{e^{-i\pi\frac{\mathrm{d}-1}{2}}}{2\pi^{\frac{\mathrm{d}+1}{2}}\,[\sigma(\sigma+4)]^{\frac{\mathrm{d}-1}{4}}}Q^{\frac{\mathrm{d}-1}{2}}_{-\beta_0-\frac{1}{2}}(1+\frac{\sigma}{2}). \tag{6.1-19}$$

This is distinguished from G_{bu} only by the sign of β_0. These results are confirmed also by the literature [14, 15, 16].

Contact with Intertwiner Representations.

It is at this point important to make the connection to Dobrev's intertwiner representation of the AdS/CFT correspondence. Since the main point in his work is concerned with the equivalence of representations in the bulk and boundary of EAdS space, we can give the dictionary between our notation and his. Note that here, we are only dealing with scalar fields; however, to establish the general picture, this is totally sufficient.

The standard bulk-to-boundary intertwiner in the representation $\chi = [0, \Delta_-]$ is up to multiplicity given by

$$L_\chi \sim B$$

(there is only one bulk representation, so we drop the index τ). Its inverse (the boundary-to-bulk intertwiner) is given by

$$\hat{L}_\chi \sim \Pi_{\mathrm{bubo}} = \left(B|_{\mathscr{H}_{\mathrm{bo}}}\right)^{-1},$$

and we find that the corresponding representation spaces are on the boundary $C_\chi = \mathscr{B}$ [5] with basis vectors $b_{\underline{k}}$ and scalar product $\langle,\rangle_{\partial g}$ and in the bulk $C_{\chi,\tilde{\chi}} = \mathscr{H}_{\mathrm{bo}}$ with basis vectors $e_{\beta_0,\underline{k}}$. Note that for a complete correspondence in terms of function spaces, the choice of a basis is necessary.

The dual boundary space $C_{\tilde{\chi}}$ is naturally identified with the dual $\mathscr{B}'$ with basis vectors $b'_{\underline{k}}$ (the Banach dual basis to $b_{\underline{k}}$) and a scalar product making these vector orthonormal, and the natural equivalent of Dobrev's boundary propagator is

$$G_{\tilde{\chi}} : \mathscr{B} \to \mathscr{B}' \sim {\Pi_{\mathrm{bo}}}^{-1} = (B\Pi B')^{-1},$$

with inverse

$$G_\chi : \mathscr{B}' \to \mathscr{B} \sim \Pi_{\mathrm{bo}} = B\Pi B'.$$

By concatenating the propagator $G_{\tilde{\chi}}$ and L_χ, we find the dual intertwiner

$$L_{\tilde{\chi}} \sim G_{\tilde{\chi}} \circ L_\chi \sim (B\Pi B')^{-1}B = (\Pi B')^{-1},$$

and similarly the inverse propagator is obtained as $\hat{L}_{\tilde{\chi}} \sim \Pi B'$.

Note that the setup by Dobrev completely ignores the bulk space $\mathscr{H}_{\mathrm{bu}}$; all intertwiners are restricted to the boundary spaces.

Rühl's construction of bulk-to-bulk propagators. We are now in the situation to comment on the ideas of Rühl, Leonhardt and others on the construction of the bulk-to-bulk propagator by convolution of the kernels of the boundary-to-bulk propagators $\hat{L}_\chi$ and $\hat{L}_{\tilde{\chi}}$. Let $b_{\underline{x}}$ denote an element of the (generalised) orthonormal coordinate basis for the boundary space $\mathcal{C} = \mathbb{R}^{\mathrm{d}} \ni \underline{x}$ with scalar product ∂g; on the bulk side, we will not use the orthonormal coordinate vectors for the metric g for testing the kernel, but simply test with an element $h \in \mathscr{H}$ in the Π-scalar product. Since we have $\hat{L}_\chi \sim B^{-1}$, the respective kernel can be written

$$\hat{G}_\chi(h, \underline{x}) = \langle h, B^{-1} b_{\underline{x}} \rangle.$$

The kernel for $\hat{L}_{\tilde{\chi}} \sim \Pi B'$ is in this basis given by

$$\hat{G}_{\tilde{\chi}}(\tilde{h}, \underline{x}) = \langle \tilde{h}, \Pi B' \jmath\, b_{\underline{x}} \rangle$$

(the point is that we have to transport the wave functions from $\mathscr{B}$ to $\mathscr{B}'$ in the standard bases $b_{\underline{x}}$ and $b'_{\underline{x}}$; this is exactly he action of $\jmath$). We will take the liberty to

[5] We are a little bit sketchy here and should rather take $\mathcal{D}_{\mathcal{C}}$ or similarly.

use the complex conjugate of the kernel $\hat{G}_{\tilde{\chi}}$; since it is a real kernel, this is not an issue. Convolution of the kernels is equivalent to computing

$$G(h,\tilde{h}) = \int \mathrm{d}^{\mathrm{d}}\underline{x}\, \hat{G}_{\chi}(h,\underline{x})\overline{\hat{G}_{\tilde{\chi}}(\tilde{h},\underline{x})}.$$

To get rid of the integration, we will use the completeness relation

$$\int \mathrm{d}^{\mathrm{d}}\underline{x}\, b_{\underline{x}}\langle b_{\underline{x}}, f\rangle_{\partial g} = f \qquad f \in \mathscr{B}.$$

Since

$$\overline{\hat{G}_{\tilde{\chi}}(\tilde{h},\underline{x})} = \langle \Pi B' \jmath\, b_{\underline{x}}, \tilde{h}\rangle = \langle B' \jmath\, b_{\underline{x}}, \tilde{h}\rangle_{\mathscr{H}} = \langle \jmath\, b_{\underline{x}}, B\tilde{h}\rangle_{\mathscr{B}} = \langle b_{\underline{x}}, B\tilde{h}\rangle_{\partial g},$$

the convolution yields

$$\begin{aligned} G(h,\tilde{h}) = &\int \mathrm{d}^{\mathrm{d}}\underline{x}\, \langle h, B^{-1} b_{\underline{x}}\rangle \langle b_{\underline{x}}, B\tilde{h}\rangle_{\partial g} \\ =&\langle h, B^{-1}B\tilde{h}\rangle \\ =&\langle h, (1-\mathcal{P}_{\mathrm{bu}})\tilde{h}\rangle. \end{aligned}$$

In other words, the convolution yields nothing but the projection operator onto the boundary subspace $\mathscr{H}_{\mathrm{bo}}$. Using the $L^2(\mathcal{M}, g)$ coordinate basis δ_x and the boundary-to-bulk kernel $\hat{G}_{\chi}(y,\underline{x}) \equiv \langle \delta_y, B^{-1} b_{\underline{x}}\rangle_g = \hat{G}_{\chi}(\Pi\imath\delta_y, \underline{x})$ and similarly for $\hat{G}_{\tilde{\chi}}$, the convoluted kernel G is written as

$$G(x,y) \equiv G(\Pi\imath\delta_x,\, \Pi\imath\delta_y) = \langle \Pi\imath\delta_x,\, (1-\mathcal{P}_{\mathrm{bu}})\Pi\imath\delta_y\rangle = \langle \delta_x,\, (1-\mathcal{P}_{\mathrm{bu}})\Pi\imath\delta_y\rangle_g.$$

In the L^2-sense, this is just the kernel of the field-theoretic (unconstrained) propagator Π, *restricted to the boundary subspace* $\mathscr{H}_{\mathrm{bo}}$.

Following the prescription of Rühl and coworkers, we now have to select two terms, the "A-term" and the "B-term" which are characterised by their asymptotic behaviour towards spatial infinity. Remembering that $\Pi_{\mathrm{bu}} = \mathcal{P}_{\mathrm{bu}}\Pi$, it is clear that the meaning of the split procedure is

$$(1-\mathcal{P}_{\mathrm{bu}})\Pi = \Pi - \Pi_{\mathrm{bu}}. \tag{6.1-20}$$

The A-term and B-term are thus *both* to be interpreted as propagators, although in different prescriptions.

6.1.2 Schwinger Kernel of Boundary-to-bulk Propagator does not exist

By (5.3-20), the boundary-to-bulk propagator is given by the integral kernel of $\hbar(B|_{\mathscr{H}_{\mathrm{bo}}})^{-1}$. We choose the $b_{\underline{x}}$-basis on the boundary and compute

$$G_{\mathrm{bubo}}(x,\underline{y}) = \frac{\hbar\Gamma(\frac{\mathrm{d}}{2}+\beta_0)}{\pi^{\frac{\mathrm{d}}{2}}\Gamma(\beta_0)} \left(\frac{x^0}{(x^0)^2 + |\underline{x}-\underline{y}|^2}\right)^{\frac{\mathrm{d}}{2}+\beta_0} = G_{\mathrm{bobu}}(\underline{y},x), \tag{6.1-21}$$

by the duality of the respective operators. In wave number space,

$$G_{\mathrm{bubo}}((y^0,\underline{k}),\underline{k}') = \delta^{(\mathrm{d})}(\underline{k}-\underline{k}')\,\frac{\hbar\,2^{1-\beta_0}}{\Gamma(\beta_0)}|\underline{k}|^{\beta_0}(y^0)^{\mathrm{d}/2}K_{\beta_0}(|\underline{k}|y^0). \tag{6.1-22}$$

The propagator in the *field theoretic* (unconstrained) prescription is obtained by exchanging $\beta_0 \mapsto -\beta_0$. We want to ascertain whether the bulk-to-boundary propagator can be generated by a Schwinger kernel as well. We use the procedure indicated in remark 5.6 for the determination of the putative Schwinger kernel of the bulk-to-boundary propagator. The first step is to choose an element $n_{\mathrm{bo}} \in \mathscr{H}_{\mathrm{bo}}$ and a family $n_j \in \mathscr{H}_{\mathrm{bu}}$ approximating n_{bo} in the weak $\sigma(\mathscr{H},\mathcal{D}_{\mathrm{bu}})_g$-topology. We choose $n_{\mathrm{bo}} = \frac{2^{1-\beta_0}}{C_{\beta_0}(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)}|\underline{k}|^{\beta_0}e_{\underline{k},\beta_0}$, although this is only a generalised element of $\mathscr{H}_{\mathrm{bo}}$ (the reason is that $Bn_{\mathrm{bo}} = b_{\underline{k}}$ is a "normalised" plane wave on the boundary). n_{bo} is approximated by the family

$$n_\epsilon(x) = \begin{cases} \frac{2^{1-\beta_0}}{(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)}|\underline{k}|^{\beta_0}\ e^{i\underline{xk}}\ (x^0)^{\mathrm{d}/2}\ K_{\beta_0}(|\underline{k}|x^0) & \text{if } x^0 \geq \epsilon \\ \frac{2^{1-\beta_0}}{(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)}|\underline{k}|^{\beta_0}\ e^{i\underline{xk}}\ (x^0)^{\mathrm{d}/2}\ I_{\beta_0}(|\underline{k}|x^0)\frac{K_{\beta_0}(|\underline{k}|\epsilon)}{I_{\beta_0}(|\underline{k}|\epsilon)} & \text{if } x^0 < \epsilon \end{cases} \tag{6.1-23}$$

as $\epsilon \to 0+$ (please compare to the example on the real line!). Again, $n_\epsilon(x)$ solves the massive Klein-Gordon equation in EAdS with eigenvalue 0 everywhere in the bulk except at $x^0 = \epsilon$; the precise analysis yields

$$(m^2 - \Box^g)n_\epsilon(x) = \frac{2^{1-\beta_0}}{(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)}\ e^{i\underline{xk}}\ \delta(x^0-\epsilon)\frac{\epsilon^{\mathrm{d}/2+1}|\underline{k}|^{\beta_0}}{I_{\beta_0}(|\underline{k}|\epsilon)}\ +\ \text{boundary terms.}$$

The bulk-to-boundary Schwinger kernel is then by convolution

$$\begin{aligned} K_\alpha^{\mathrm{bobu}}(y,\underline{k}) &= \lim_{\epsilon\to 0+}\int \frac{\mathrm{d}^{\mathrm{d}}\underline{x}\ \mathrm{d}x^0}{(x^0)^{\mathrm{d}+1}}K_\alpha^{\mathrm{bu}}(y,x)\,(m^2-\Box^g)n_\epsilon(x) \\ &= \lim_{\epsilon\to 0+}\frac{2^{1-\beta_0}}{(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)}\int\frac{\mathrm{d}^{\mathrm{d}}\underline{x}\,|\underline{k}|^{\beta_0}}{\epsilon^{\mathrm{d}/2}I_{\beta_0}(|\underline{k}|\epsilon)}K_\alpha^{\mathrm{bu}}(y,(\underline{x},\epsilon))e^{i\underline{xk}} \\ &= \lim_{\epsilon\to 0+}\frac{2\beta_0}{(2\pi)^{\mathrm{d}/2}}\int\frac{\mathrm{d}^{\mathrm{d}}\underline{x}}{\epsilon^{\mathrm{d}/2+\beta_0}}K_\alpha^{\mathrm{bu}}(y,(\underline{x},\epsilon))e^{i\underline{xk}}, \end{aligned}$$

where we have used the $z \ll 1$ approximation $I_{\beta_0}(z) = \frac{1}{\Gamma(\beta_0+1)}\left(\frac{z}{2}\right)^{\beta_0}$ (note that $\underline{k}$ is a horizontal wave number on the boundary and y is a bulk coordinate). Because the k-dependence under the integral is trivial, this is equivalent to

$$K_\alpha^{\mathrm{bobu}}(y,\underline{x}) = \lim_{\epsilon\to 0+}\frac{2\beta_0}{\epsilon^{\mathrm{d}/2+\beta_0}}K_\alpha^{\mathrm{bu}}(y,(\underline{x},\epsilon)). \tag{6.1-24}$$

As the kernel K_α^{bu} falls off towards the boundary faster than any power, cf. (C.2-7), this expression actually vanishes! The conclusion is that the exchange of the α-integration and the scalar product between equations (5.5-35) and (5.5-38), serving to define K_α^{bobu}, is not allowed.

Let us stop for a moment and try to interpret this result. The deeper reason for the vanishing of $K_\alpha^{\rm bobu}$ can be traced to the vanishing of $\langle h_{\rm bo},\ \mathcal{S}\Pi^{-1} K_\alpha^{\rm bu} f\rangle_g$ for $f \in \mathcal{D}_{\rm bu}$, $h_{\rm bo} \in \mathscr{H}_{\rm bo}$ in expression (5.5-38). The Schwinger kernel $K_\alpha^{\rm bu}$ vanishes so smoothly at the boundary that we may integrate by parts

$$\langle h_{\rm bo},\ \mathcal{S}\Pi^{-1} K_\alpha^{\rm bu} f\rangle_g \overset{\rm p.i.}{=} \langle \mathcal{S}\Pi^{-1} h_{\rm bo},\ K_\alpha^{\rm bu} f\rangle_g = 0$$

by definition of $\mathcal{S}$. This is to be contrasted with the situation in the model on the real line. There, the Schwinger kernel $K_\alpha^{\rm bu}$ has a linear behaviour at the boundary, so integration by parts creates boundary terms. The different behaviour is of course to be attributed to the different character of the boundaries in question: In the real line model, the "boundary" has a finite distance from any internal point of the manifold; a diffusing particle modelled by the Schwinger kernel $K_\alpha^{\rm bu}$ can reach the boundary in a finite (Schwinger) time α. The interpretation of $K_\alpha^{\rm bobu}$ as "absorption rate" is then the correct one. In contrast, the boundary of EAdS is infinitely far away. No diffusing particle can reach the boundary in finite Schwinger time; therefore, there cannot be a nonzero bulk-to-boundary Schwinger kernel which would allow an interpretation of diffusing particle absorption at the boundary.

Basically, we can see two ways out of this situation. The first scenario uses the assumption that the formula (6.1-24) for the bulk-to-boundary Schwinger kernel is essentially correct; however, it should only be applied *within complete Feynman graphs in* EAdS *space with all external propagators attached to the boundary*. The limit $\epsilon \to 0$ should then be applied only after all vertex integrations over EAdS-space have been performed. The thoughtline is that all "Schwinger particles" are linked to the boundary by a diffusion process with a finite Schwinger time duration; so all processes are taking place mainly in a very thin layer next to the boundary. This indeed looks like a mechanism which maps the Schwinger diffusion processes from the interior of EAdS space onto equivalent processes *on* the boundary - if it works! The boundary value which we obtain when we take the limit $\epsilon \to 0$ is probably a distribution. However, it is not amiss to mention here that the integrals encountered are way off what is found in the usual tables.

The second alternative is a novel rescaling of the fields, ie the substitution of $\phi = (x^0)^{-1}\tilde\phi$ in the Lagrangian, where x^0 is the EAdS depth coordinate (that we should use the particular factor $(x^0)^{-1}$ was found by experiment). The kinetic operator is then $(x^0)^{-1}(m^2 - \Box^g)(x^0)^{-1}$, and vertices of n fields carry a factor $(x^0)^{-n}$ which has to be included in the volume integration. This leads to a completely new propagator; the correlation functions of operators are identically the same, however, if the operators are rescaled in a similar fashion [6]. One finds that the new Schwinger kernel $\tilde K_\alpha^{\rm bu}$ does not fall off equally fast any more when approaching the boundary. This second method leads to the Schwinger parametrisation used by Gopakumar in his article series [45, 46, 47]; by appearance, they are Gaussian and therefore much easier to handle in practical calculations. However, the rescaling of the fields and operators seems somewhat ad hoc: Why should one use precisely this scaling and not any other?

[6] Of course, composite operators scale in a more sophisticated manner.

Besides, invariance under the full symmetry group is broken in this approach, not only in the way the boundary values are taken (this is a common fault of all these approaches), but also in the *interior* of EAdS space. We have to ask the question whether there can be a physical interpretation of the *rescaled* fields. One possible interpretation is suggested by the holographic renormalisation group []: The depth coordinate z^0 corresponds there to the renormalisation scale. The factor $(z^0)^{-1}$ would then indicate that the *field strength* needs a renormalisation, depending linearly on the scale (in the free field model underlying the perturbation series).

6.2 Rescaling of the Fields

As the general method should clear by now, we will sketch very briefly how the rescaling of the fields changes the formalism. As indicated in the last section, we substitute $\phi = (x^0)^{-1}\tilde{\phi}$ in the EAdS Lagrangian in Poincaré coordinates, where x^0 is the EAdS depth coordinate. The Klein-Gordon operator becomes

$$(x^0)^{-1}(m^2 - \Box^g)(x^0)^{-1} = \frac{m^2}{(x^0)^2} - \left(\partial_0^2 - \frac{1+\mathrm{d}}{x^0}\partial_0 + \frac{1+\mathrm{d}}{(x^0)^2} + \triangle\right), \tag{6.2-25}$$

and vertices of n fields carry a factor $(x^0)^{-n}$ which has to be included in the volume integration. The Klein-Gordon equation for the real eigenvalue λ is

$$\lambda f = \frac{m^2}{(x^0)^2} f - \left(\partial_0^2 - \frac{1+\mathrm{d}}{x^0}\partial_0 + \frac{1+\mathrm{d}}{(x^0)^2} + \triangle\right) f, \qquad x^0 > 0.$$

By substituting

$$f(x) = e^{i\underline{k}\cdot\underline{x}}\,(x^0)^{\mathrm{d}/2+1}\,f_1(x^0), \qquad \underline{k} \in \mathbb{R}^{\mathrm{d}},$$

we get a Bessel differential equation for $f_1(x)$, with the general solution

$$f_1(x^0) = \begin{cases} A_\gamma J_{\beta_0}(\gamma x^0) + B_\gamma Y_{\beta_0}(\gamma x^0), & \text{if } \lambda > \underline{k}^2, \\ A_{i\gamma} I_{\beta_0}(\gamma x^0) + B_{i\gamma} K_{\beta_0}(\gamma x^0), & \text{if } \lambda \le \underline{k}^2, \end{cases} \tag{6.2-26}$$

where we have defined

$$\gamma^2 = \left|\lambda - \underline{k}^2\right|. \tag{6.2-27}$$

($A_.$ and $B_.$ are normalisations). In addition, for $\lambda = \underline{k}^2$, there are solutions $f_1(x^0) \sim (x^0)^{\pm\beta_0}$ (however, $\underline{k} = 0$ is a null set in wave number space, so we disregard them). The quadratic Klein-Gordon form is declared to be

$$\langle f, h\rangle = \int \mathrm{d}^{\mathrm{d}+1}x\,(x^0)^{2\Delta_- + 1 - \mathrm{d}}\,\partial_\mu[(x^0)^{-\Delta_- -1} f]^* \cdot \partial_\mu[(x^0)^{-\Delta_- -1} h]. \tag{6.2-28}$$

Testing the solutions (6.2-26), it turns out that a proper basis of orthogonal eigenfunctions parametrised by γ is given by the vectors

$$\mathscr{H}_{\mathrm{bu}} \ni \tilde{e}_{\underline{k},\gamma}(x) = \sqrt{\frac{\gamma}{(2\pi)^{\mathrm{d}}}}\, e^{i\underline{kx}}\,(x^0)^{\mathrm{d}/2+1}\,J_{\beta_0}(\gamma x^0), \qquad (\gamma > 0) \tag{6.2-29a}$$

$$\mathscr{H}_{\mathrm{bo}} \ni \tilde{e}_{\underline{k},i|\underline{k}|}(x) = C_{\beta_0} e^{i\underline{kx}}\,(x^0)^{\mathrm{d}/2+1}\,K_{\beta_0}(|\underline{k}|x^0). \tag{6.2-29b}$$

Their respective normalisation is $\langle \tilde{e}_{\underline{k},\gamma},\, \tilde{e}_{\underline{k}',\gamma'} \rangle = \lambda_{\underline{k},\gamma}\, \delta^{(\mathrm{d})}(\underline{k}-\underline{k}')\delta(\gamma-\gamma')$ where $\lambda_{\underline{k},\gamma} = \gamma^2 + \underline{k}^2$, and $\langle \tilde{e}_{\underline{k},i|\underline{k}|},\, \tilde{e}_{\underline{k}',i|\underline{k}'|} \rangle = \delta^{(\mathrm{d})}(\underline{k}-\underline{k}')$. This underlines the role of γ as "x^0-component of the momentum". The interpretation of these basis vectors as functions in coordinate space is automatic. A completeness relation like (6.1-12) can be proven easily. The definition of the dual basis and the boundary spaces runs also completely analogous. Note that while the basis vectors of $\mathscr{H}_{\mathrm{bu}}$ (in the bulk) are structured completely different, the boundary basis vectors in $\mathscr{H}_{\mathrm{bo}}$ are simply the scaled versions of those vectors we had in the non-scaled version.

We come to the really interesting point: The determination of the various Schwinger kernels. We have again

$$\left(\exp -\alpha \mathcal{S}\Pi^{-1}\right) \tilde{e}_{\underline{k},\gamma} = e^{-\alpha\lambda_{|\underline{k}|,\gamma}} \tilde{e}_{\underline{k},\gamma}, \tag{6.2-30}$$

and for a δ-distribution,

$$\begin{aligned}
\tilde{K}^{\mathrm{bu}}_{\alpha}(x,y) &= \left(\exp -\alpha \mathcal{S}\Pi^{-1}\right) \delta_y(x) \\
&= \int \mathrm{d}^{\mathrm{d}}k \int_0^\infty \mathrm{d}\gamma\, \tilde{e}_{\underline{k},\gamma}(x)\, e^{-\alpha\lambda_{|\underline{k}|,\gamma}}\, \hat{\delta}_y(\underline{k},\gamma) \\
&= (x^0 y^0)^{\mathrm{d}/2+1} \int \mathrm{d}^{\mathrm{d}}k \int_0^\infty \mathrm{d}\gamma\, \frac{\gamma}{(2\pi)^{\mathrm{d}}}\, e^{-(\gamma^2+\underline{k}^2)\alpha}\, e^{i(\underline{x}-\underline{y})\underline{k}}\, J_{\beta_0}(\gamma x^0)\, J_{\beta_0}(\gamma y^0) \\
&= \frac{(x^0 y^0)^{\mathrm{d}/2+1}}{2\alpha} \int \frac{\mathrm{d}^{\mathrm{d}}k}{(2\pi)^{\mathrm{d}}}\, e^{-\underline{k}^2\alpha - \frac{(x^0)^2+(y^0)^2}{4\alpha}}\, e^{i(\underline{x}-\underline{y})\underline{k}}\, I_{\beta_0}\left(\frac{x^0 y^0}{2\alpha}\right) \\
&= \frac{x^0 y^0}{2\alpha} \left(\frac{x^0 y^0}{4\pi\alpha}\right)^{\mathrm{d}/2} e^{-\frac{(\underline{x}-\underline{y})^2+(x^0)^2+(y^0)^2}{4\alpha}} I_{\beta_0}\left(\frac{x^0 y^0}{2\alpha}\right)
\end{aligned} \tag{6.2-31}$$

by [49, 6.633 2]. Integrating and attaching the proper scale factors

$$G_{\mathrm{bu}}(x,y) = \hbar\, (x^0 y^0)^{-1} \int_0^\infty \mathrm{d}\alpha\, \tilde{K}^{\mathrm{bu}}_{\alpha}(x,y), \tag{6.2-32}$$

we are able to obtain the correct result (C.2-8). Note that towards the boundary $x^0 \to 0$, this Schwinger kernel scales like a power. However, obviously $\tilde{K}^{\mathrm{bu}}_{\alpha}$ does not depend any more solely on the invariant chord length in EAdS space. This can be cured by scaling the Schwinger time parameter $\alpha \to x^0 y^0\, \alpha$ (although this procedure is somewhat arbitrary). The factor $x^0 y^0$ originating from the rescaling of the differential $\mathrm{d}\alpha$ cancels exactly the necessary factors which must be attached to the ends of the propagator, so we obtain the integral representation

$$\begin{aligned}
G_{\mathrm{bu}}(x,y) &= \hbar \int_0^\infty \frac{\mathrm{d}\alpha}{2\alpha} \left(\frac{1}{4\pi\alpha}\right)^{\mathrm{d}/2} e^{-\frac{\sigma+2}{4\alpha}} I_{\beta_0}\left(\frac{1}{2\alpha}\right) \\
&= \hbar \int_0^\infty \frac{\mathrm{d}\tilde{\alpha}}{2\tilde{\alpha}} \left(\frac{\tilde{\alpha}}{2\pi}\right)^{\mathrm{d}/2} e^{-(\frac{\sigma}{2}+1)\tilde{\alpha}} I_{\beta_0}(\tilde{\alpha}).
\end{aligned}$$

We can see that the only dependence on the coordinates is by the squared chordal distance $\sigma(x, y)$. [7]

Note that the modified scheme gives us the chance to determine the Fourier transform of the propagator directly: We have

$$\begin{aligned}G^{\mathrm{bu}}(x, y) =& \hbar (x^0 y^0)^{-1} \int \mathrm{d}^{\mathrm{d}} k \int_0^\infty \frac{\mathrm{d}\gamma}{\lambda_{|\underline{k}|,\gamma}} \tilde{e}_{\underline{k},\gamma}(x)\, \hat{\delta}_y(\underline{k}, \gamma) \\ =& \hbar \frac{(x^0 y^0)^{\frac{\mathrm{d}}{2}}}{(2\pi)^{\mathrm{d}}} \int \mathrm{d}^{\mathrm{d}} k\, e^{i(\underline{x}-\underline{y})\underline{k}} \int_0^\infty \mathrm{d}\gamma\, \frac{\gamma}{\gamma^2 + \underline{k}^2}\, J_{\beta_0}(\gamma x^0)\, J_{\beta_0}(\gamma y^0) \\ =& \hbar \frac{(x^0 y^0)^{\frac{\mathrm{d}}{2}}}{(2\pi)^{\mathrm{d}}} \int \mathrm{d}^{\mathrm{d}} k\, e^{i(\underline{x}-\underline{y})\underline{k}}\, I_{\beta_0}(|\underline{k}| \min(x^0, y^0))\, K_{\beta_0}(|\underline{k}| \max(x^0, y^0))\end{aligned} \tag{6.2-33}$$

by [49, 6.541 1].

We show that the Schwinger kernel for the bulk-to-boundary propagator does not vanish for the rescaled field. Because of the similarity of the boundary wave functions, we can re-use the family (6.1-23) after scaling with x^0, ie we use $x^0 n_\epsilon(x) \to \frac{2^{1-\beta_0}}{C_{\beta_0}(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)} |\underline{k}|^{\beta_0} \tilde{e}_{\underline{k},i|\underline{k}|}$; accordingly, we get

$$\tilde{K}^{\mathrm{bobu}}_\alpha(y, \underline{k}) = \lim_{\epsilon \to 0+} \frac{2^{1-\beta_0}}{(2\pi)^{\mathrm{d}/2}\Gamma(\beta_0)} \int \frac{\mathrm{d}^{\mathrm{d}} x\, |\underline{k}|^{\beta_0}}{\epsilon^{\mathrm{d}/2+1} I_{\beta_0}(|\underline{k}|\epsilon)} \tilde{K}^{\mathrm{bu}}_\alpha(y, (\underline{x}, \epsilon)) e^{i\underline{x}\underline{k}}.$$

In coordinate space, this becomes after some straightforward simplifications

$$\begin{aligned}\tilde{K}^{\mathrm{bobu}}_\alpha(y, \underline{x}) =& \lim_{\epsilon \to 0+} \frac{2\beta_0}{\epsilon^{\frac{\mathrm{d}}{2}+\beta_0+1}} \tilde{K}^{\mathrm{bu}}_\alpha(y, (\underline{x}, \epsilon)) \\ =& \frac{y^0}{\Gamma(\beta_0)\pi^{\frac{\mathrm{d}}{2}}\alpha} \left(\frac{y^0}{4\alpha}\right)^{\frac{\mathrm{d}}{2}+\beta_0} e^{-\frac{(\underline{x}-\underline{y})^2 + (y^0)^2}{4\alpha}}.\end{aligned} \tag{6.2-34}$$

After integration of α, this reduces to the usual bulk-to-boundary propagator (6.1-21), for the un-rescaled fields. In Fourier space, the respective kernel is

$$\tilde{K}^{\mathrm{bobu}}_\alpha((y^0, \underline{k}), \underline{k}') = \delta^{(\mathrm{d})}(\underline{k} - \underline{k}') \frac{(y^0)^{\frac{\mathrm{d}}{2}+1}}{\Gamma(\beta_0)\alpha} \left(\frac{y^0}{4\alpha}\right)^{\beta_0} e^{-\alpha \underline{k}^2 - \frac{(y^0)^2}{4\alpha}}. \tag{6.2-35}$$

Again, the field theoretic prescription is obtained by substituting $\beta_0 \mapsto -\beta_0$. After integration of α, this reduces to the usual bulk-to-boundary propagator (6.1-22), for the un-rescaled fields.

[7] It is worthwhile to note that the integrand $\mathcal{I}$ of the last representation fulfills the differential equation $\tilde{\alpha}\partial_{\tilde{\alpha}}\mathcal{I} + \mathcal{I} + (m^2 - \Box^g)\mathcal{I} = 0$ in the sense that this equation vanishes if integrated over $\tilde{\alpha}$. The Schwinger representation which will formally reproduce directly the kernel $\mathcal{I}$ is [49, 5.55]

$$\Pi^{-1} = \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha} I_{\beta_0}\left(\frac{1}{2\alpha}\right) K_{\sqrt{\beta_0^2 - \Pi}}\left(\frac{1}{2\alpha}\right) = \int_0^\infty \frac{\mathrm{d}\tilde{\alpha}}{\tilde{\alpha}} I_{\beta_0}(\tilde{\alpha}) K_{\sqrt{\beta_0^2 - \Pi}}(\tilde{\alpha}).$$

It is not completely independent of the mass used, and of the dimensionality of the problem (so its meaning is still unclear).

Appendix A

Conformal Propagators and D'EPP Formula

A.1 Conformal Propagators

We are discussing the Schwinger parametrisation of conformal propagators

$$G_\Delta(x-y) = \frac{\hbar}{|x-y|^{2\Delta}}. \tag{A.1-1}$$

The scaling behaviour will be contained solely in a τ-dependent prefactor. Introducing a Schwinger-like representation in coordinate space

$$\frac{1}{(x^2)^\Delta} = \frac{1}{\Gamma(\Delta)}\int_0^\infty \mathrm{d}\alpha\, \alpha^{\Delta-1}e^{-\alpha x^2}, \qquad \Re\Delta > 0, \tag{A.1-2}$$

we can compute the Fourier transform as

$$\begin{aligned}\int \mathrm{d}^d x\, e^{-iq\cdot x}\frac{1}{(x^2)^\Delta} =& \frac{1}{\Gamma(\Delta)}\int \mathrm{d}^d x\, e^{-iq\cdot x}\int_0^\infty \mathrm{d}\alpha\, \alpha^{\Delta-1}e^{-\alpha x^2} \\ =& \frac{\pi^{\frac{\mathrm{d}}{2}}}{\Gamma(\Delta)}\int_0^\infty \mathrm{d}\alpha\, \alpha^{\Delta-\frac{\mathrm{d}}{2}-1}e^{-\frac{q^2}{4\alpha}}\end{aligned}$$

by completing the square. Substituting $\alpha \to (4\tau)^{-1}$, we get the usual Schwinger parametrisation

$$G_\Delta(q) = \hbar\,\frac{2^{d-2\Delta}\pi^{\frac{d}{2}}}{\Gamma(\Delta)}\int_0^\infty \mathrm{d}\tau\, \tau^{\frac{d}{2}-\Delta-1}e^{-\tau q^2}. \tag{A.1-3}$$

This representation is special insofar as the exponential part takes exactly the form of a massless propagator. The only modification is the power of τ in the Schwinger kernel. If $\Re\Delta < \frac{\mathrm{d}}{2}$, we can evaluate the integral explicitly to obtain

$$G_\Delta(q) = \hbar\frac{2^{d-2\Delta}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)}|q|^{2\Delta-d}. \tag{A.1-4}$$

Note that even when Δ is not within the bounds indicated, we may by analytic continuation reach almost every complex Δ.

Formally, the coordinate space δ-distribution is then represented by

$$\delta^{(\mathrm{d})}(\underline{x}) = \frac{\Gamma(\frac{\mathrm{d}}{2})}{\hbar\pi^{\frac{\mathrm{d}}{2}}\Gamma(0)} G_{\frac{\mathrm{d}}{2}}(\underline{x}).$$

A.2 Composition of Conformal Propagators

We have seen that conformal propagators can be defined by Schwinger parametrisation if the conformal dimension fulfills the condition $0 < \Re\Delta < \frac{\mathrm{d}}{2}$. Once the propagators are given in Schwinger parametrised form, they may be switched parallel or serial, and we would like to find out under which conditions these compositions make sense. So let $G_\Delta(k)$ be the conformal propagator (A.1-3) in Schwinger parametrised form (and we *not* demand that these formulas can actually be integrated; so we are allowed arbitrary Δ).

When the actual computations are performed, we find that by parallel switching,

$$G_\Delta(x) G_{\Delta'}(x) = G_{\Delta+\Delta'}(x)$$

For "serial connection", we find

$$G_\Delta(k) G_{\Delta'}(k) = \hbar\pi^{\frac{\mathrm{d}}{2}} \frac{\Gamma(\frac{\mathrm{d}}{2}-\Delta)\Gamma(\frac{\mathrm{d}}{2}-\Delta')\Gamma(\Delta+\Delta'-\frac{\mathrm{d}}{2})}{\Gamma(\Delta)\Gamma(\Delta')\Gamma(\mathrm{d}-\Delta-\Delta')} G_{\Delta+\Delta'-\frac{\mathrm{d}}{2}}(k)$$

In particular, since $\Delta = \frac{\mathrm{d}}{2}$ signifies the "δ-propagator", we find that the inverse conformal propagator $(G_\Delta)^*$ in principle should have dimension $\Delta^* = d - \Delta$; the precise normalisation for the inverse propagator is

$$(G_\Delta)^*(k) = \frac{\Gamma(\Delta)\Gamma(\mathrm{d}-\Delta)}{\hbar^2\pi^{\mathrm{d}}\Gamma(\frac{\mathrm{d}}{2}-\Delta)\Gamma(\Delta-\frac{\mathrm{d}}{2})} G_{\mathrm{d}-\Delta}(k).$$

The serial composition of a conformal propagator with its inverse in the Schwinger domain is however never well defined as the necessary conditions $\Re\Delta < \frac{\mathrm{d}}{2}$, $\Re\Delta^* = \mathrm{d} - \Re\Delta < \frac{\mathrm{d}}{2}$ can never be satisfied simultaneously.

A.3 D'EPP Relation

We prove a relation introduced first by D'Eramo, Parisi and Peliti [25]. It describes the transformation of a star graph of conformal propagators into a triangle graph by integrating out the central vertex. In the electric circuit analogy, this is the star-δ-transform [17]. Consider the star graph defined by the equation

$$G(x_1, x_2, x_3) = \int \mathrm{d}^d u \, |x_1 - u|^{-2\Delta_1} |x_2 - u|^{-2\Delta_2} |x_3 - u|^{-2\Delta_3},$$

with $\Delta_1+\Delta_2+\Delta_3=\Delta$. Performing a Fourier transform, this becomes with (A.1-3)

$$G(k_1,k_2,k_3)=(2\pi)^{-\frac{d}{2}}\delta^{(d)}(k_1+k_2+k_3)\prod_{j=1}^{3}\frac{2^{d-2\Delta_j}\pi^{\frac{d}{2}}}{\Gamma(\Delta_j)}\int_0^\infty \mathrm{d}\alpha_j\,\alpha_j^{\frac{d}{2}-\Delta_j-1}e^{-\alpha_j k_j^2}.$$

Using momentum conservation, $k_1^2=-k_1(k_2+k_3)$ etc.; then the exponential becomes

$$(\alpha_1+\alpha_2)k_1k_2+(\alpha_1+\alpha_3)k_1k_3+(\alpha_2+\alpha_3)k_2k_3.$$

This is already the correct form for the exponent of a triangle graph. However, we still need to get the correct prefactors for the momenta. Going from a star to a delta network means to substitute

$$\tau_j=\frac{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}{\alpha_j},$$

where τ_j is the resistance opposite to node j. The inverse transformation is

$$\alpha_j=\frac{\tau_1\tau_2\tau_3}{\tau_j(\tau_1+\tau_2+\tau_3)}.$$

For the Jacobian of the transformation we get

$$\mathrm{d}\alpha_1\;\mathrm{d}\alpha_2\;\mathrm{d}\alpha_3\;=\frac{\tau_1\tau_2\tau_3}{(\tau_1+\tau_2+\tau_3)^3}\,\mathrm{d}\tau_1\;\mathrm{d}\tau_2\;\mathrm{d}\tau_3\,,$$

or

$$\mathrm{d}\tau_1\;\mathrm{d}\tau_2\;\mathrm{d}\tau_3\;=\frac{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^3}{\alpha_1^2\alpha_2^2\alpha_3^2}\,\mathrm{d}\alpha_1\;\mathrm{d}\alpha_2\;\mathrm{d}\alpha_3\,.$$

Performing all the necessary substitutions, the amplitude reads

$$\begin{aligned}G(k_1,k_2,k_3)=&(2\pi)^{-\frac{d}{2}}\delta^{(d)}(k_1+k_2+k_3)\left(\prod_{j=1}^{3}\frac{2^{d-2\Delta_j}\pi^{\frac{d}{2}}}{\Gamma(\Delta_j)}\int_0^\infty \mathrm{d}\tau_j\,\tau_j^{d-\Delta+\Delta_j-1}\right)(\tau_1+\tau_2+\tau_3)^{\Delta-\frac{3d}{2}}\\ &e^{\frac{\tau_3(\tau_1+\tau_2)k_1\cdot k_2+\tau_2(\tau_1+\tau_3)k_1\cdot k_3+\tau_1(\tau_2+\tau_3)k_2\cdot k_3}{\tau_1+\tau_2+\tau_3}}.\end{aligned}$$

We see now that under the condition $\Delta=d$, we can indeed reach a Schwinger parametrised triangle graph formula; substituting back the conformal propagator formula (A.1-3), we obtain finally in coordinate space

$$G(x_1,x_2,x_3)=\pi^{\frac{d}{2}}\frac{\Gamma(\Delta_{23})\Gamma(\Delta_{13})\Gamma(\Delta_{12})}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}\frac{1}{|x_1-x_2|^{2\Delta_{12}}}\frac{1}{|x_1-x_3|^{2\Delta_{13}}}\frac{1}{|x_2-x_3|^{2\Delta_{23}}},$$

with $\Delta_{12}=\frac{d}{2}-\Delta_3$ resp. $\Delta_3=\Delta_{13}+\Delta_{23}$ etc. They obey $\Delta_{12}+\Delta_{13}+\Delta_{23}=\frac{d}{2}$. Note that in coordinate space, the formula is valid only in the range $0<\Re\Delta_j<\frac{d}{2}$. It can be analytically continued to disallowed scaling dimensions.

Appendix B

Vertex Integration in EAdS

In this appendix, we compute general EAdS-integrals needed in section 3.4.5 for the EAdS-presentation of three-point functions of currents. The summations in this appendix have been performed with the symbolic computation package Maple 10 (Waterloo Software).

B.1 Non-conformally Covariant Integrals

In the first section, we compute the elementary integrals

$$\mathcal{I} = \int \frac{\mathrm{d}z^0\ \mathrm{d}^{\mathrm{d}}z}{(z^0)^{\mathrm{d}+1}} \prod_{k=1}^{3} (I_{x_k z})^{i_k} (J_{v_k z})^{j_k} - \text{traces}, \tag{B.1-1}$$

where $i_k \in \mathbb{R}_+$, $j_k \in \mathbb{N}_0$ are arbitrary powers, $\underline{x}_k \in \mathbb{R}^{\mathrm{d}}$ are points on the conformal boundary of EAdS, $\underline{v}_k \in T_{\underline{x}_k}$ are tangent vectors at these points, and the symbols

$$\begin{aligned} I_{x_k z} &= -\frac{1}{(\tilde{x}_k, \tilde{z})} = \frac{2z^0}{(\underline{x}_k - \underline{z})^2 + (z^0)^2} \\ J_{v_k z} &= \frac{(\tilde{v}_k, \tilde{z})}{(\tilde{x}_k, \tilde{z})} = -\frac{\underline{v}_k \cdot (\underline{z} - \underline{x}_k)}{z^0} \frac{2z^0}{(\underline{x}_k - \underline{z})^2 + (z^0)^2} \\ K_{v_j v_k} &= (\tilde{v}_j, \tilde{v}_k) - \frac{(\tilde{v}_j, \tilde{x}_k)(\tilde{x}_j, \tilde{v}_k)}{(\tilde{x}_j, \tilde{x}_k)} \end{aligned} \tag{B.1-2}$$

are the bulk/boundary invariants which we introduce in (3.4-116). Subtraction of traces is with respect to the tangent vectors $\underline{v}_k$.

Methodically, we will resort to Schwinger parametrisation, as a technical tool only.

The amplitude can be formally written as

$$\begin{aligned}
\mathcal{I} =& \int \frac{\mathrm{d}z^0\ \mathrm{d}^{\mathrm{d}} z}{(z^0)^{\mathrm{d}+1}} \prod_{k=1}^{3} \frac{(-1)^{j_k}}{\Gamma(i_k+j_k)}\, \partial_{\beta_k}^{j_k} \int_0^\infty \mathrm{d}\gamma_k\, \gamma_k^{i_k-1} \exp\left(\gamma_k(\tilde{x}_k,\tilde{z}) + \beta_k\gamma_k(\tilde{v},\tilde{z})\right)\Big|_{\beta\equiv 0} - \text{traces}\\
=& \int \frac{\mathrm{d}z^0\ \mathrm{d}^{\mathrm{d}} z}{(z^0)^{\mathrm{d}+1}} \prod_{k=1}^{3} \frac{(-1)^{j_k}}{\Gamma(i_k+j_k)} (z^0)^{i_k} \partial_{\beta_k}^{j_k} \int_0^\infty \mathrm{d}\gamma_k\, \gamma_k^{i_k-1}\\
& \exp\left(-\gamma_k \frac{(\underline{x}_k - \underline{z})^2 + (z^0)^2}{2} + \beta_k\gamma_k \underline{v}\cdot(\underline{z}-\underline{x})\right)\Big|_{\beta\equiv 0} - \text{traces.}
\end{aligned}$$

The z-integrations can be exchanged with the γ-integrations and done, resulting in

$$\begin{aligned}
\mathcal{I} =& 2^{\frac{i_1+i_2+i_3}{2}-1} \pi^{\frac{\mathrm{d}}{2}} \Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right) \int_0^\infty \mathrm{d}^3\gamma\, G^{-\frac{i_1+i_2+i_3}{2}} \left(\prod_{k=1}^{3} \frac{(-1)^{j_k}}{\Gamma(i_k+j_k)} \partial_{\beta_k}^{j_k} \gamma_k^{i_k-1}\right)\\
& \exp\left(\frac{\gamma_1\gamma_2(\tilde{x}_1,\tilde{x}_2)}{G} + \beta_1 \frac{\gamma_1\gamma_2(\tilde{v}_1,\tilde{x}_2)}{G} + \beta_1 \frac{\gamma_1\gamma_3(\tilde{v}_1,\tilde{x}_3)}{G} + \text{cycl. perms.}\right.\\
& \left. + \frac{(\beta_1\gamma_1\underline{v}_1 + \beta_2\gamma_2\underline{v}_2 + \beta_3\gamma_3\underline{v}_3)^2}{2G}\right)\Big|_{\beta\equiv 0} - \text{traces,}
\end{aligned}$$

with

$$G = \gamma_1 + \gamma_2 + \gamma_3.$$

This was the main computation; the rest is the organisation of the prefactors. As usual, the subtraction of traces will completely remove the $\underline{v}_j^2$-terms (however this is not sufficient for a complete removal of trace terms).

Note that the exponent can be written in a very compact notation: Denoting by γ a vector in $\mathbb{R}^3$ with entries γ_k, and by W the symmetric 3×3-matrix with entries

$$\begin{aligned}
W_{jk} =& (\tilde{x}_j + \beta_j\tilde{v}_j,\ \tilde{x}_k + \beta_k\tilde{v}_k) \qquad (j\neq k),\\
W_{jj} =& 0,
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{I} =& 2^{\frac{i_1+i_2+i_3}{2}-1} \pi^{\frac{\mathrm{d}}{2}} \Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right) \int_0^\infty \mathrm{d}^3\gamma\, G^{-\frac{i_1+i_2+i_3}{2}}\\
& \left(\prod_{k=1}^{3} \frac{(-1)^{j_k}}{\Gamma(i_k+j_k)} \partial_{\beta_k}^{j_k} \gamma_k^{i_k-1}\right) \exp \frac{\gamma^{\mathrm{tr}} W \gamma}{2G}\Big|_{\beta\equiv 0} - \text{traces.}
\end{aligned} \tag{B.1-3}$$

The diagonal terms of W vanish since $(\tilde{x}_k,\tilde{x}_k) = (\tilde{x}_k,\tilde{v}_k) = 0$, and $(\tilde{v}_k,\tilde{v}_k) = \underline{v}_k\cdot\underline{v}_k$ is removed by the subtraction of traces. We perform the variable change

$$\delta_k = \frac{\gamma_k}{\sqrt{G}},$$

with the Jacobian

$$\mathrm{d}^3\gamma\ = 2(\delta_1+\delta_2+\delta_3)^3\,\mathrm{d}^3\delta\,,$$

and obtain

$$\mathfrak{I} = 2^{\frac{i_1+i_2+i_3}{2}}\pi^{\frac{\mathrm{d}}{2}}\Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right)\int_0^\infty \mathrm{d}^3\delta\,\left(\prod_{k=1}^3 \frac{(-1)^{j_k}}{\Gamma(i_k+j_k)}\partial_{\beta_k}^{j_k}\delta_k^{i_k-1}\right)\exp\frac{\delta^{\mathrm{tr}}W\delta}{2}\bigg|_{\beta\equiv 0} - \text{traces.}$$

We compute the action of the β-derivatives. A quick combinatorial argument shows that

$$\begin{aligned}
&\left(\prod_{k=1}^3 \partial_{\beta_k}^{j_k}\right)\exp\frac{\delta^{\mathrm{tr}}W\delta}{2}\bigg|_{\beta_k=0}\\
=&\sum_{n_{12}}\sum_{n_{23}}\sum_{n_{31}}\ \sum_{m_{12}}\sum_{m_{23}}\sum_{m_{31}}\\
&\frac{1}{n_{12}!n_{23}!n_{31}!}\ \frac{j_1!}{m_{12}!(j_1-n_{12}-n_{31}-m_{12})!}\\
&\frac{j_2!}{m_{23}!(j_2-n_{23}-n_{12}-m_{23})!}\frac{j_3!}{m_{31}!(j_3-n_{31}-n_{23}-m_{31})!}\\
&(\tilde v_1,\tilde v_2)^{n_{12}}(\tilde v_2,\tilde v_3)^{n_{23}}(\tilde v_3,\tilde v_1)^{n_{31}}\ (\tilde v_1,\tilde x_2)^{m_{12}}(\tilde v_2,\tilde x_3)^{m_{23}}(\tilde v_3,\tilde x_1)^{m_{31}}\\
&(\tilde v_1,\tilde x_3)^{j_1-n_{12}-n_{31}-m_{12}}(\tilde v_2,\tilde x_1)^{j_2-n_{23}-n_{12}-m_{23}}(\tilde v_3,\tilde x_2)^{j_3-n_{31}-n_{23}-m_{31}}\\
&\delta_1^{j_1+j_2-n_{23}-n_{12}-m_{23}+m_{31}}\delta_2^{j_2+j_3-n_{31}-n_{23}-m_{31}+m_{12}}\delta_3^{j_3+j_1-n_{12}-n_{31}-m_{12}+m_{23}}\exp\frac{\delta^{\mathrm{tr}}W^0\delta}{2}.
\end{aligned}$$

The sums are supposed to cover the whole range of integers; practically, the summation is limited to those values of the counting parameters for which the factorials are finite (with the rule $n! = \Gamma(n+1) = \infty$ if $n < 0$). We have abbreviated $W^0 = W|_{\beta\equiv 0}$. Now we can perform the δ-integrals; they are of the type

$$\begin{aligned}
&\int_0^\infty \mathrm{d}^3\delta\,\left(\prod_{k=1}^3 \delta_k^{d_k-1}\right)\exp\frac{\delta^{\mathrm{tr}}W^0\delta}{2}\\
=&\frac{\Gamma\left(\frac{d_1+d_2-d_3}{2}\right)\Gamma\left(\frac{d_2+d_3-d_1}{2}\right)\Gamma\left(\frac{d_3+d_1-d_2}{2}\right)}{2}(-W^0_{12})^{\frac{d_3-d_1-d_2}{2}}(-W^0_{23})^{\frac{d_1-d_2-d_3}{2}}(-W^0_{31})^{\frac{d_2-d_3-d_1}{2}}.
\end{aligned}$$

and can be solved in terms of simple algebraic functions because W^0 does not have diagonal terms (otherwise, the degree of complexity increases greatly).

Combining all terms and making the obvious substitutions of the invariants $J_{v_jx_k}$

and $I_{x_j x_k}$, we obtain

$$
\begin{aligned}
\mathcal{J} =& 2^{\frac{i_1+i_2+i_3}{2}-1}\pi^{\frac{\mathrm{d}}{2}}\Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right)\left(\prod_{k=1}^{3}\frac{1}{\Gamma(i_k+j_k)}\right)\\
&\sum_{n_{12}}\sum_{n_{23}}\sum_{n_{31}}\sum_{m_{12}}\sum_{m_{23}}\sum_{m_{31}}\\
&\frac{1}{n_{12}!n_{23}!n_{31}!}\frac{j_1!}{m_{12}!(j_1-n_{12}-n_{31}-m_{12})!}\\
&\frac{j_2!}{m_{23}!(j_2-n_{23}-n_{12}-m_{23})!}\frac{j_3!}{m_{31}!(j_3-n_{31}-n_{23}-m_{31})!}\\
&\Gamma\left(\frac{i_1+i_2-i_3}{2}+j_2-n_{23}-m_{23}+m_{12}\right)\Gamma\left(\frac{i_2+i_3-i_1}{2}+j_3-n_{31}-m_{31}+m_{23}\right)\\
&\Gamma\left(\frac{i_3+i_1-i_2}{2}+j_1-n_{12}-m_{12}+m_{31}\right)\\
&(I_{x_1x_2})^{\frac{i_1+i_2-i_3}{2}+n_{12}}(I_{x_2x_3})^{\frac{i_2+i_3-i_1}{2}+n_{23}}(I_{x_3x_1})^{\frac{i_3+i_1-i_2}{2}+n_{31}}\\
&(\tilde{v}_1,\tilde{v}_2)^{n_{12}}(\tilde{v}_2,\tilde{v}_3)^{n_{23}}(\tilde{v}_3,\tilde{v}_1)^{n_{31}}\;(J_{v_1x_2})^{m_{12}}(J_{v_2x_3})^{m_{23}}(J_{v_3x_1})^{m_{31}}\\
&(J_{v_1x_3})^{j_1-n_{12}-n_{31}-m_{12}}(J_{v_2x_1})^{j_2-n_{23}-n_{12}-m_{23}}(J_{v_3x_2})^{j_3-n_{31}-n_{23}-m_{31}}-\text{traces.}
\end{aligned}
$$

This is not yet the preferred version, since we still have the scalar products $(\tilde{v}_j,\tilde{v}_k)$ between boundary tangent vectors which are non-invariant under $\tilde{v}_k \mapsto \tilde{v}_k + \mathbb{R}\tilde{x}_k$. These should be substituted by the invariants $K_{v_jv_k}$, by means of the equality

$$(\tilde{v}_j,\tilde{v}_k) = K_{v_jv_k} - \frac{J_{v_jx_k}J_{v_kx_j}}{I_{x_jx_k}}.$$

After expanding the powers of these by the binomial formula, reshuffling the summations, and summing up what can be summed, we get the final result

$$\begin{aligned}
&\int \frac{\mathrm{d}z^0\ \mathrm{d}^\mathrm{d} z}{(z^0)^{\mathrm{d}+1}} \prod_{k=1}^{3} I_{x_k z}^{i_k} J_{v_k z}^{j_k} - \text{traces} \qquad \text{(B.1-4)}\\
=&2^{\frac{i_1+i_2+i_3}{2}-1} \pi^{\frac{\mathrm{d}}{2}} \Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right)\left(\prod_{k=1}^{3}\frac{1}{\Gamma(i_k+j_k)}\right)\\
&\sum_{p_{12}}\sum_{p_{23}}\sum_{p_{31}}\sum_{n_{12}}\sum_{n_{23}}\sum_{n_{31}} \frac{1}{n_{12}!p_{12}!n_{23}!p_{23}!n_{31}!p_{31}!}\\
&\frac{j_1!}{(j_1-p_{12}-p_{31}-n_{12})!}\frac{j_2!}{(j_2-p_{23}-p_{12}-n_{23})!}\frac{j_3!}{(j_3-p_{31}-p_{23}-n_{31})!}\\
&\Gamma\left(\frac{i_1+i_2-i_3}{2}+j_2-p_{23}-n_{23}\right)\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)_{n_{12}}\\
&\Gamma\left(\frac{i_2+i_3-i_1}{2}+j_3-p_{31}-n_{31}\right)\left(\frac{i_2+i_3-i_1}{2}+p_{23}\right)_{n_{23}}\\
&\Gamma\left(\frac{i_3+i_1-i_2}{2}+j_1-p_{12}-n_{12}\right)\left(\frac{i_3+i_1-i_2}{2}+p_{31}\right)_{n_{31}}\\
&(I_{x_1x_2})^{\frac{i_1+i_2-i_3}{2}}(I_{x_2x_3})^{\frac{i_2+i_3-i_1}{2}}(I_{x_3x_1})^{\frac{i_3+i_1-i_2}{2}}(I_{x_1x_2}K_{v_1v_2})^{p_{12}}(I_{x_2x_3}K_{v_2v_3})^{p_{23}}(I_{x_3x_1}K_{v_3v_1})^{p_{31}}\\
&(J_{v_1x_2})^{n_{12}}(J_{v_1x_3})^{j_1-p_{12}-p_{31}-n_{12}}\ (J_{v_2x_3})^{n_{23}}(J_{v_2x_1})^{j_2-p_{23}-p_{12}-n_{23}}\\
&(J_{v_3x_1})^{n_{31}}(J_{v_3x_2})^{j_3-p_{31}-p_{23}-n_{31}} - \text{traces.}
\end{aligned}$$

The summations run over all values which are admitted by the factorials in the denominator. The mirror symmetry (123-132) is difficult to see on the Γ-factors in the middle, but we have checked that it is indeed there.

B.2 Conformally Covariant Integrals

We now consider the conformally covariant integrals

$$\mathcal{I}' = \int \frac{\mathrm{d}z^0\ \mathrm{d}^\mathrm{d} z}{(z^0)^{\mathrm{d}+1}} \prod_{k=1}^{3}(I_{x_k z})^{i_k}\prod_{p\neq q}(J_{v_p z}-J_{v_p x_q})^{j_{pq}} - \text{traces.} \qquad \text{(B.2-5)}$$

These can be obtained from (B.1-4) by manipulation of the summations. We keep the convention that sums run over all integers, and the factorials of negative argument are formally infinite and limit the domain of summation. The idea is to expand the powers j_{pq} by the binomial formula, introducing new summation variables k_{pq}; the summands are then of the type covered in the first section and can be done by (B.1-4).

After some translation of the summations, we get a form which we will see mirrors already the invariant character of the total amplitude:

$$
\begin{aligned}
\mathcal{J}' =& 2^{\frac{i_1+i_2+i_3}{2}-1}\pi^{\frac{\mathrm{d}}{2}}\Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right)\coprod_{p\neq q}\left[\sum_{k_{pq}}\binom{j_{pq}}{k_{pq}}(-1)^{j_{pq}-k_{pq}}\right] \\
& \sum_{p_{12}}\sum_{p_{23}}\sum_{p_{31}}\sum_{n_{12}}\sum_{n_{23}}\sum_{n_{31}}\frac{1}{(n_{12}-j_{12}+k_{12})!p_{12}!(n_{23}-j_{23}+k_{23})!p_{23}!(n_{31}-j_{31}+k_{31})!p_{31}!} \\
& \frac{1}{\Gamma(k_{12}+k_{13}+i_1)\Gamma(k_{23}+k_{21}+i_2)\Gamma(k_{31}+k_{32}+i_3)} \\
& \frac{(k_{12}+k_{13})!}{(j_{12}+k_{13}-p_{12}-p_{31}-n_{12})!}\frac{(k_{23}+k_{21})!}{(j_{23}+k_{21}-p_{23}-p_{12}-n_{23})!}\frac{(k_{31}+k_{32})!}{(j_{31}+k_{32}-p_{31}-p_{23}-n_{31})!} \\
& \Gamma\left(\frac{i_3+i_1-i_2}{2}+j_{12}+k_{13}-p_{12}-n_{12}\right)\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)_{n_{12}-j_{12}+k_{12}} \\
& \Gamma\left(\frac{i_1+i_2-i_3}{2}+j_{23}+k_{21}-p_{23}-n_{23}\right)\left(\frac{i_2+i_3-i_1}{2}+p_{23}\right)_{n_{23}-j_{23}+k_{23}} \\
& \Gamma\left(\frac{i_2+i_3-i_1}{2}+j_{31}+k_{32}-p_{31}-n_{31}\right)\left(\frac{i_3+i_1-i_2}{2}+p_{31}\right)_{n_{31}-j_{31}+k_{31}} \\
& (I_{x_1x_2})^{\frac{i_1+i_2-i_3}{2}}(I_{x_2x_3})^{\frac{i_2+i_3-i_1}{2}}(I_{x_3x_1})^{\frac{i_3+i_1-i_2}{2}}(I_{x_1x_2}K_{v_1v_2})^{p_{12}}(I_{x_2x_3}K_{v_2v_3})^{p_{23}}(I_{x_3x_1}K_{v_3v_1})^{p_{31}} \\
& (J_{v_1x_2})^{n_{12}}(J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}-n_{12}}\,(J_{v_2x_3})^{n_{23}}(J_{v_2x_1})^{j_{23}+j_{21}-p_{23}-p_{12}-n_{23}} \\
& (J_{v_3x_1})^{n_{31}}(J_{v_3x_2})^{j_{31}+j_{32}-p_{31}-p_{23}-n_{31}} - \text{traces.}
\end{aligned}
$$

The summations over k_{pq} are not interferring with the invariants any more; they are concerning just the prefactors, and we will see what we can do to sum them up efficiently. This expression falls apart into three independent blocks, the first block eg containing the relevant summations

$$
\begin{aligned}
B_1 =& \sum_{k_{12}}\sum_{k_{13}}\sum_{n_{12}}\binom{j_{12}}{k_{12}}\binom{j_{13}}{k_{13}}(-1)^{j_{12}-k_{12}+j_{13}-k_{13}}\frac{1}{(n_{12}-j_{12}+k_{12})!p_{12}!}. \\
& \frac{1}{\Gamma(k_{12}+k_{13}+i_1)}\frac{(k_{12}+k_{13})!}{(j_{12}+k_{13}-p_{12}-p_{31}-n_{12})!} \\
& \Gamma\left(\frac{i_3+i_1-i_2}{2}+j_{12}+k_{13}-p_{12}-n_{12}\right)\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)_{n_{12}-j_{12}+k_{12}} \\
& (J_{v_1x_2})^{n_{12}}(J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}-n_{12}}
\end{aligned}
\tag{B.2-6}
$$

and the other blocks the respective cyclic permutations of indices $1 \to 2 \to 3 \to 1$. The sums k_{12} and k_{13} are hard and we failed to sum them symbolically or via computer algebra; however, we can knock them out them by a trick. We know that since the total expression is conformally invariant by construction, the J-invariants must combine into powers

$$(J_{v_1x_2}-J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}}. \tag{B.2-7}$$

The n_{12}-summation is just the binomial expansion of these, with the corresponding binomial coefficients

$$(-1)^{j_{12}+j_{13}-p_{12}-p_{31}-n_{12}}\binom{j_{12}+j_{13}-p_{12}-p_{31}}{n_{12}}.$$

If we divide out these binomial coefficients from the summands of B_1 and at the same time substitute (B.2-7) for the invariants J appearing in B_1, then the remaining sum must be independent of n_{12} (although it appears in several places):

$$\begin{aligned}B_1 = &\sum_{k_{12}}\sum_{k_{13}}\binom{j_{12}}{k_{12}}\binom{j_{13}}{k_{13}}(-1)^{p_{12}-k_{12}+p_{31}-k_{13}+n_{12}}\\ &\binom{j_{12}+j_{13}-p_{12}-p_{31}}{n_{12}}^{-1}\frac{1}{(n_{12}-j_{12}+k_{12})!p_{12}!}\\ &\frac{1}{\Gamma(k_{12}+k_{13}+i_1)}\frac{(k_{12}+k_{13})!}{(j_{12}+k_{13}-p_{12}-p_{31}-n_{12})!}\\ &\Gamma\left(\frac{i_3+i_1-i_2}{2}+j_{12}+k_{13}-p_{12}-n_{12}\right)\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)_{n_{12}-j_{12}+k_{12}}\\ &(J_{v_1x_2}-J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}}.\end{aligned}$$

The judicious choice $n_{12}=j_{12}+j_{13}-p_{12}-p_{31}\geq 0$ now kills the k_{13}-summation:

$$\begin{aligned}B_1 = &\sum_{k_{12}}\sum_{k_{13}}\binom{j_{12}}{k_{12}}\binom{j_{13}}{k_{13}}(-1)^{-k_{12}-k_{13}+j_{12}+j_{13}}\\ &\frac{1}{(j_{13}-p_{12}-p_{31}+k_{12})!p_{12}!}\frac{1}{\Gamma(k_{12}+k_{13}+i_1)}\frac{(k_{12}+k_{13})!}{(k_{13}-j_{13})!}\\ &\Gamma\left(\frac{i_3+i_1-i_2}{2}+k_{13}-j_{13}+p_{31}\right)\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)_{j_{13}-p_{12}-p_{31}+k_{12}}\\ &(J_{v_1x_2}-J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}}\\ = &\sum_{k_{12}}\binom{j_{12}}{k_{12}}(-1)^{-k_{12}+j_{12}}\frac{1}{(j_{13}-p_{12}-p_{31}+k_{12})!p_{12}!}\frac{(k_{12}+j_{13})!}{\Gamma(k_{12}+j_{13}+i_1)}\\ &\Gamma\left(\frac{i_3+i_1-i_2}{2}+p_{31}\right)\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)_{j_{13}-p_{12}-p_{31}+k_{12}}\\ &(J_{v_1x_2}-J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}},\end{aligned}$$

since only $k_{13}=j_{13}$ contributes to the sum (all other contributions are suppressed by the factor $\frac{1}{(k_{13}-j_{13})!}$). That the pochhammer symbol might have a negative index is no problem, as we may represent it by

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}, \qquad n\in\mathbb{Z},$$

$\frac{i_1+i_2-i_3}{2}$ is supposed to be arbitrary real, and therefore, we may exclude the possibility that the Γ-function in the enumerator will encounter a pole when counting downwards (we take the correlation to be analytic in the exponents i_1, i_2, i_3).
The sum k_{12} can be evaluated in terms of a hypergeometric function

$$
\begin{aligned}
B_1 =& \frac{(j_{12}+j_{13})!}{p_{12}!(j_{12}+j_{13}-p_{12}-p_{31})!} \frac{\Gamma\left(\frac{i_3+i_1-i_2}{2}+p_{31}\right)\left(\frac{i_2+i_1-i_3}{2}+p_{12}\right)_{j_{12}+j_{13}-p_{12}-p_{31}}}{\Gamma(i_1+j_{12}+j_{13})} \\
& {}_3\mathrm{F}_2\Big(-j_{12}, 1-i_1-j_{12}-j_{13}, -j_{12}-j_{13}+p_{12}+p_{31}; -j_{12}-j_{13}, 1-\tfrac{i_2+i_1-i_3}{2}-j_{12}-j_{13}+p_{31}; 1\Big) \\
& (J_{v_1x_2}-J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}}, \qquad \text{(B.2-8)}
\end{aligned}
$$

however, there is no simple expression for this hypergeometric function; it is just an abbreviation for the (finite) sum indicated. The total conformally invariant amplitudes is

$$
\begin{aligned}
&\int \frac{\mathrm{d}z^0\ \mathrm{d}^{\mathrm{d}} z}{(z^0)^{\mathrm{d}+1}} \prod_{k=1}^{3} I^{i_k}_{x_k z} \prod_{p\neq q} (J_{v_p z}-J_{v_p x_q})^{j_{pq}} - \text{traces} \qquad \text{(B.2-9)} \\
=&2^{\frac{i_1+i_2+i_3}{2}-1}\pi^{\frac{\mathrm{d}}{2}}\Gamma\left(\frac{i_1+i_2+i_3-\mathrm{d}}{2}\right)\sum_{p_{12}}\sum_{p_{23}}\sum_{p_{31}} \\
& \frac{(j_{12}+j_{13})!}{p_{12}!(j_{12}+j_{13}-p_{12}-p_{31})!} \frac{\Gamma\left(\frac{i_3+i_1-i_2}{2}+p_{31}\right)\left(\frac{i_2+i_1-i_3}{2}+p_{12}\right)_{j_{12}+j_{13}-p_{12}-p_{31}}}{\Gamma(i_1+j_{12}+j_{13})} \\
& \frac{(j_{23}+j_{21})!}{p_{23}!(j_{23}+j_{21}-p_{23}-p_{12})!} \frac{\Gamma\left(\frac{i_1+i_2-i_3}{2}+p_{12}\right)\left(\frac{i_3+i_2-i_1}{2}+p_{23}\right)_{j_{23}+j_{21}-p_{23}-p_{12}}}{\Gamma(i_2+j_{23}+j_{21})} \\
& \frac{(j_{31}+j_{32})!}{p_{31}!(j_{31}+j_{32}-p_{31}-p_{23})!} \frac{\Gamma\left(\frac{i_2+i_3-i_1}{2}+p_{23}\right)\left(\frac{i_1+i_3-i_2}{2}+p_{31}\right)_{j_{31}+j_{32}-p_{31}-p_{23}}}{\Gamma(i_3+j_{31}+j_{32})} \\
& {}_3\mathrm{F}_2\Big(-j_{12}, 1-i_1-j_{12}-j_{13}, -j_{12}-j_{13}+p_{12}+p_{31}; -j_{12}-j_{13}, 1-\tfrac{i_2+i_1-i_3}{2}-j_{12}-j_{13}+p_{31}; 1\Big) \\
& {}_3\mathrm{F}_2\Big(-j_{23}, 1-i_2-j_{23}-j_{21}, -j_{23}-j_{21}+p_{23}+p_{12}; -j_{23}-j_{21}, 1-\tfrac{i_3+i_2-i_1}{2}-j_{23}-j_{21}+p_{12}; 1\Big) \\
& {}_3\mathrm{F}_2\Big(-j_{31}, 1-i_3-j_{31}-j_{32}, -j_{31}-j_{32}+p_{31}+p_{23}; -j_{31}-j_{32}, 1-\tfrac{i_1+i_3-i_2}{2}-j_{31}-j_{32}+p_{23}; 1\Big) \\
& (I_{x_1x_2})^{\frac{i_1+i_2-i_3}{2}}(I_{x_2x_3})^{\frac{i_2+i_3-i_1}{2}}(I_{x_3x_1})^{\frac{i_3+i_1-i_2}{2}}(I_{x_1x_2}K_{v_1v_2})^{p_{12}}(I_{x_2x_3}K_{v_2v_3})^{p_{23}}(I_{x_3x_1}K_{v_3v_1})^{p_{31}} \\
& (J_{v_1x_2}-J_{v_1x_3})^{j_{12}+j_{13}-p_{12}-p_{31}}(J_{v_2x_3}-J_{v_2x_1})^{j_{23}+j_{21}-p_{23}-p_{12}}(J_{v_3x_1}-J_{v_3x_2})^{j_{31}+j_{32}-p_{31}-p_{23}} - \text{traces}.
\end{aligned}
$$

An application of this formula is given in section 3.4.5.

Appendix C

Some Integrals of Bessel functions

C.1 Coordinate Space Representation of Bulk-To-Boundary Vectors

The coordinate space representation of $e_{\underline{k},\beta_0}$ is obtained by performing a Fourier transform in the index variable $\underline{k}$,

$$
\begin{aligned}
e_{\underline{y},\beta_0}(x) =&(2\pi)^{-\mathrm{d}/2}\int \mathrm{d}^{\mathrm{d}}k\, e^{-i\underline{k}\underline{y}} e_{\underline{k},\beta_0}(x) \\
=&\frac{C_{\beta_0}(x^0)^{\mathrm{d}/2}}{(2\pi)^{\mathrm{d}/2}}\int \mathrm{d}^{\mathrm{d}}k\, e^{i\underline{k}(\underline{x}-\underline{y})} K_{\beta_0}(|\underline{k}|x^0) \\
=&C_{\beta_0}(x^0)^{\mathrm{d}/2}|\underline{x}-\underline{y}|^{1-\mathrm{d}/2}\int \mathrm{d}k\, k^{\mathrm{d}/2}\; J_{\mathrm{d}/2-1}(|\underline{x}-\underline{y}|k)\; K_{\beta_0}(kx^0) \\
=&C_{\beta_0}\frac{\Gamma(\frac{\mathrm{d}+\beta_0}{2})\Gamma(\frac{\mathrm{d}-\beta_0}{2})}{2^{1-\mathrm{d}/2}\Gamma(\frac{\mathrm{d}}{2})}(x^0)^{-\mathrm{d}/2}\,{}_2\mathrm{F}_1\left(\frac{\mathrm{d}+\beta_0}{2},\frac{\mathrm{d}-\beta_0}{2};\;\frac{\mathrm{d}}{2};\;-\frac{|\underline{x}-\underline{y}|^2}{(x^0)^2}\right).
\end{aligned}
\tag{C.1-1}
$$

C.2 Bulk-to-bulk Propagator and Heat Kernel

We want to study the integral

$$
\begin{aligned}
K_\alpha^{\mathrm{bu}}(x,y) =&(x^0y^0)^{\mathrm{d}/2}\int \mathrm{d}^{\mathrm{d}}k\int_0^\infty \mathrm{d}\beta\, e^{-(\beta_0^2+\beta^2)\alpha}\; C_{i\beta}^2\; e^{i(\underline{x}-\underline{y})\underline{k}}\; K_{i\beta}(|\underline{k}|x^0)\; K_{i\beta}(|\underline{k}|y^0) \\
=&\frac{2(x^0y^0)^{\mathrm{d}/2}}{(2\pi)^{\mathrm{d}}\pi^2}e^{-\beta_0^2\alpha}\int \mathrm{d}^{\mathrm{d}}k\int_0^\infty \mathrm{d}\beta\,\beta\sinh\pi\beta e^{-\beta^2\alpha}\; e^{i(\underline{x}-\underline{y})\underline{k}}\; K_{i\beta}(|\underline{k}|x^0)\; K_{i\beta}(|\underline{k}|y^0).
\end{aligned}
$$

We use the helpful representation [37, 7.14.2(60)]

$$
K_\mu(z)K_\mu(Z) = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\;\cosh\mu t\; K_0[(z^2+Z^2+2zZ\cosh t)^{1/2}] \qquad \Re z>0,\; \Re Z>0.
\tag{C.2-2}
$$

This yields

$$K_\alpha^{\mathrm{bu}}(x,y) = \frac{(x^0y^0)^{\mathrm{d}/2}}{(2\pi)^{\mathrm{d}}\pi^2} e^{-\beta_0^2\alpha} \int \mathrm{d}^{\mathrm{d}}k \int_0^\infty \mathrm{d}\beta\, \beta \sinh \pi\beta e^{-\beta^2\alpha}\, e^{i(\underline{x}-\underline{y})\underline{k}} \int_{-\infty}^{\infty} \mathrm{d}t\, \cos\beta t\, K_0[|\underline{k}|((x^0)^2+(y^0)^2+2x^0y^0\cosh t)^{1/2}].$$

Using the integrals

$$\int_0^\infty \mathrm{d}\beta\, \beta \sinh \pi\beta e^{-\beta^2\alpha} \cos\beta t = \frac{i\sqrt{\pi}}{8\alpha^{3/2}} \left((t-i\pi)e^{-\frac{(t-i\pi)^2}{4\alpha}} - c.c. \right) \tag{C.2-3}$$

and

$$\begin{aligned} \frac{1}{(2\pi)^{\mathrm{d}/2}} \int \mathrm{d}^{\mathrm{d}}k\, e^{i\underline{z}\underline{k}} K_0(|\underline{k}|a) =& |\underline{z}|^{1-\mathrm{d}/2} \int_0^\infty \mathrm{d}k\, k^{\mathrm{d}/2} J_{\mathrm{d}/2-1}(|\underline{z}|k) K_0(ka) \\ =& \frac{2^{\mathrm{d}/2-1}\Gamma(\frac{\mathrm{d}}{2})}{(|\underline{z}|^2+a^2)^{\mathrm{d}/2}} \end{aligned} \tag{C.2-4}$$

[49, 6.576 7], obtain

$$\begin{aligned} K_\alpha^{\mathrm{bu}}(x,y) =& \frac{(x^0y^0)^{\mathrm{d}/2}}{(2\pi)^{\mathrm{d}/2}\pi^2} e^{-\alpha\beta_0^2} \int_{-\infty}^\infty \mathrm{d}t\, \frac{i\sqrt{\pi}}{8\alpha^{3/2}} \left((t-i\pi)e^{-\frac{(t-i\pi)^2}{4\alpha}} - c.c. \right) \\ & \frac{2^{\mathrm{d}/2-1}\Gamma(\frac{\mathrm{d}}{2})}{(|\underline{x}-\underline{y}|^2+(x^0)^2+(y^0)^2+2x^0y^0\cosh t)^{\mathrm{d}/2}} \\ =& \frac{\Gamma(\frac{\mathrm{d}}{2})i}{16\,\pi^{(\mathrm{d}+3)/2}\alpha^{3/2}} e^{-\beta_0^2\alpha} \int_{C_1} \mathrm{d}t\, \frac{te^{-\frac{t^2}{4\alpha}}}{(\sigma(x,y)+2-2\cosh t)^{\mathrm{d}/2}}, \end{aligned}$$

where the contour C_1 runs from $-\infty - i\pi$ to $\infty - i\pi$ and from $\infty + i\pi$ back to $-\infty + i\pi$. The value of the integral is determined by the singularities and cuts on the strip enclosed by C_1.

In even dimensions, the function in the denominator is analytic in t. There are singularities determined by the equation

$$\cosh t = 1 + \frac{\sigma(x,y)}{2} > 1$$

at $t_0 = \operatorname{arccosh}(1+\frac{\sigma(x,y)}{2})$ on the positive axis and at $-t_0$ on the negative axis. In odd dimensions, there are two cuts $[t_0,\infty[$ and $]-\infty,-t_0]$. If we introduce the contour C_2 running from $\infty + i\pi$ to $i\pi$ to $-i\pi$ to $\infty - i\pi$, then $C_1 = C_2 \cup -C_2$. However, the integral is symmetric with respect to the contour, and therefore integration along C_2 and $-C_2$ yields the same value. We restrict therefore to the contour C_2; and pushing this contour close to the real axis, we get the contour C_ϵ running from $\infty + i\epsilon$ to $i\epsilon$ to $-i\epsilon$ to $\infty - i\epsilon$.

For even dimensions, we are almost there. Integrating by parts and substituting $u = \cosh t$,

$$K^{\mathrm{bu}}_\alpha(x,y) = \frac{\Gamma(\frac{\mathrm{d}}{2}+1)i}{2^{\mathrm{d}/2+2}\,\pi^{(\mathrm{d}+3)/2}\alpha^{1/2}} e^{-\beta_0^2\alpha} \int_{C_\epsilon} \mathrm{d}t\, \frac{(\sinh t)e^{-\frac{t^2}{4\alpha}}}{((1+\frac{\sigma(x,y)}{2}) - \cosh t)^{\mathrm{d}/2+1}} \tag{C.2-5}$$

$$= \frac{\Gamma(\frac{\mathrm{d}}{2}+1)i}{2^{\mathrm{d}/2+2}\,\pi^{(\mathrm{d}+3)/2}\alpha^{1/2}} e^{-\beta_0^2\alpha} \int_{\cosh C_\epsilon} \mathrm{d}u\, \frac{e^{-\frac{(\operatorname{arccosh} u)^2}{4\alpha}}}{((1+\frac{\sigma(x,y)}{2}) - u)^{\mathrm{d}/2+1}}. \tag{C.2-6}$$

There is a single pole enclosed in the contour; it is the pole at $u = 1 + \frac{\sigma(x,y)}{2}$, and it has order $\mathrm{d}/2+1$. Using Cauchy's formula,

$$K^{\mathrm{bu}}_\alpha(x,y) = \frac{e^{-\beta_0^2\alpha}}{2^{\mathrm{d}/2+1}\,\pi^{(\mathrm{d}+1)/2}\alpha^{1/2}} \left(-\frac{\partial}{\partial u}\right)^{\frac{\mathrm{d}}{2}} e^{-\frac{(\operatorname{arccosh} u)^2}{4\alpha}}\Big|_{u=1+\frac{\sigma(x,y)}{2}}$$

$$= \frac{e^{-\beta_0^2\alpha}}{2\,\pi^{(\mathrm{d}+1)/2}\alpha^{1/2}} \left(-\frac{\partial}{\partial \sigma}\right)^{\frac{\mathrm{d}}{2}} e^{-\frac{(\operatorname{arccosh}(1+\frac{\sigma(x,y)}{2}))^2}{4\alpha}}. \tag{C.2-7}$$

As a series,

$$\left(\operatorname{arccosh}(1+\frac{\sigma}{2})\right)^2 \approx \sigma - \frac{\sigma^2}{12} + \frac{\sigma^3}{90} - \frac{\sigma^4}{560} + \frac{\sigma^5}{3150} - \dots$$

For small α, $K^{\mathrm{bu}}_\alpha(x,y)$ is supported significantly only in the region of small σ; in this case, as the series expansion shows, we get a Gaussian profile as in flat Euclidean space (with $\sigma \sim r^2$ taking the role of the distance squared). In particular, for $\mathrm{d} = 0$ we get the result from the real line computation after reparametrising the x^0-coordinate as $\ln x^0 = u$.

The kernel of the propagator is obtained by integrating

$$G_{\mathrm{bu}}(x,y) = \hbar \int_0^\infty \mathrm{d}\alpha\, K^{\mathrm{bu}}_\alpha(x,y).$$

In the even dimensional case, we can determine the propagator directly using

$$\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^{1/2}} e^{-m^2\alpha - \frac{t^2}{4\alpha}} = \frac{\sqrt{\pi}e^{-mt}}{m} \qquad (|\Im t| < \Re t),$$

whence

$$G_{\mathrm{bu}}(x,y) = \frac{\hbar}{2\,\pi^{\mathrm{d}/2}\,\beta_0} \left(-\frac{\partial}{\partial \sigma}\right)^{\frac{\mathrm{d}}{2}} e^{-\beta_0 \operatorname{arccosh}(1+\frac{\sigma(x,y)}{2})}.$$

In the general case, we integrate the contour integral representation (C.2-5), as it is valid for all dimensions. The propagator then appears as

$$G_{\mathrm{bu}}(x,y) = \hbar \frac{\Gamma(\frac{\mathrm{d}}{2}+1)i}{2^{\mathrm{d}/2+2}\,\pi^{\mathrm{d}/2+1}\,\beta_0} \int_{C_\epsilon} \mathrm{d}t\, \frac{(\sinh t)e^{-\beta_0 t}}{((1+\frac{\sigma(x,y)}{2}) - \cosh t)^{\mathrm{d}/2+1}}.$$

This integral can be done analytically. We push out the contour again to C_2. The vertical contribution from $i\pi$ to $-i\pi$ can be written

$$-i\int_0^\pi \mathrm{d}t\,\frac{\cos(\beta_0-1)t-\cos(\beta_0+1)t}{((1+\frac{\sigma(x,y)}{2})-\cosh t)^{\mathrm{d}/2+1}}.$$

The horizontal contributions add up to

$$-i\sin\pi\beta_0\int_0^\infty \mathrm{d}t\,\frac{e^{-(\beta_0-1)t}-e^{-(\beta_0+1)t}}{((1+\frac{\sigma(x,y)}{2})+\cosh t)^{\mathrm{d}/2+1}}.$$

According to [49, 8.713 1]

$$\int_0^\pi \mathrm{d}t\,\frac{\cos\nu t}{(z-\cos t)^{\mathrm{d}/2+1}}-\sin\pi\nu\int_0^\infty \mathrm{d}t\,\frac{e^{-\nu t}}{(z+\cosh t)^{\mathrm{d}/2+1}}=\frac{\sqrt{2\pi}e^{-i\pi\frac{\mathrm{d}+1}{2}}Q^{\frac{\mathrm{d}+1}{2}}_{\nu-\frac{1}{2}}(z)}{\Gamma\left(\frac{\mathrm{d}}{2}+1\right)(z^2-1)^{\frac{\mathrm{d}+1}{4}}},$$

integration yields associated Legendre functions and the result is ($\sigma\equiv\sigma(x,y)$)

$$\begin{aligned}G_{\mathrm{bu}}(x,y)=&\hbar\frac{e^{-i\pi\frac{\mathrm{d}+1}{2}}}{2\,\pi^{\frac{\mathrm{d}+1}{2}}\,\beta_0[\sigma(\sigma+4)]^{\frac{\mathrm{d}+1}{4}}}\left[Q^{\frac{\mathrm{d}+1}{2}}_{\beta_0-\frac{3}{2}}(1+\frac{\sigma}{2})-Q^{\frac{\mathrm{d}+1}{2}}_{\beta_0+\frac{1}{2}}(1+\frac{\sigma}{2})\right]\\ =&\hbar\frac{e^{-i\pi\frac{\mathrm{d}-1}{2}}}{2\,\pi^{\frac{\mathrm{d}+1}{2}}\,[\sigma(\sigma+4)]^{\frac{\mathrm{d}-1}{4}}}Q^{\frac{\mathrm{d}-1}{2}}_{\beta_0-\frac{1}{2}}(1+\frac{\sigma}{2}) \qquad \text{(C.2-8)}\\ =&\hbar\frac{\Gamma(\beta_0+\frac{\mathrm{d}}{2})}{\Gamma(\beta_0+1-\frac{\mathrm{d}}{2})}\frac{e^{i\pi\frac{\mathrm{d}-1}{2}}}{2\,\pi^{\frac{\mathrm{d}+1}{2}}\,[\sigma(\sigma+4)]^{\frac{\mathrm{d}-1}{4}}}Q^{\frac{1-\mathrm{d}}{2}}_{\beta_0-\frac{1}{2}}(1+\frac{\sigma}{2}).\end{aligned}$$

C.3 Further Relevant Integrals

The following integrals will be needed throughout the text:

$$\begin{aligned}&\int_0^\infty\frac{\mathrm{d}\tau}{\tau}\tau^\mu e^{X\tau}K_{\pm\nu}(\tau) \qquad (X\le -1,\ \mu-|\nu|>0)\\ &=\frac{\sqrt{\pi}}{\sqrt{2}(1-X)^{\mu-\frac{1}{2}}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+\frac{1}{2})}\,{}_2\mathrm{F}_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\ \frac{1}{2}+\mu;\ \frac{1+X}{2}\right), \qquad \text{(C.3-9)}\end{aligned}$$

using [49, 6.621 3 and 9.131 1]. In particular, this implies a representation

$$\begin{aligned}\left(X^2-1\right)^{\frac{\mu}{2}}P^\mu_{\nu-\frac{1}{2}}(-X)=&\frac{(1-X)^\mu}{\Gamma(1-\mu)}\,{}_2\mathrm{F}_1\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;\ 1-\mu;\ \frac{X+1}{2}\right)\\ =&\sqrt{\frac{2}{\pi}}\frac{1}{\Gamma(\frac{1}{2}-\mu+\nu)\Gamma(\frac{1}{2}-\mu-\nu)}\int_0^\infty\frac{\mathrm{d}\tau}{\tau}\tau^{\frac{1}{2}-\mu}e^{X\tau}K_{\pm\nu}(\tau)\end{aligned}$$

(C.3-10)

for associated Legendre functions (using [49, 8.702]). Similarly, from [49, 6.622 3, 8.736 4 and 8.773 2], for $\frac{1}{2} - \mu + \nu > 0$,

$$\sqrt{\frac{\pi}{2}} \frac{1}{\Gamma(\frac{1}{2}+\nu-\mu)} \int_0^\infty \frac{\mathrm{d}\tau}{\tau} \tau^{\frac{1}{2}-\mu} e^{X\tau} I_\nu(\tau) = \frac{e^{-\mu\pi i}}{\Gamma(\frac{1}{2}+\nu+\mu)} \left(X^2-1\right)^{\frac{\mu}{2}} Q^\mu_{\nu-\frac{1}{2}}(-X)$$
$$= \frac{\Gamma(\mu)}{2\,\Gamma(\frac{1}{2}+\nu+\mu)} (1-X)^\mu {}_2\mathrm{F}_1\Big(\frac{1}{2}+\nu, \frac{1}{2}-\nu;\ 1-\mu; \frac{X+1}{2}\Big)$$
$$+ \frac{\Gamma(-\mu)}{2^{\mu+1}\,\Gamma(\frac{1}{2}+\nu-\mu)} (X^2-1)^\mu {}_2\mathrm{F}_1\Big(\frac{1}{2}+\mu-\nu, \frac{1}{2}+\mu+\nu;\ 1+\mu; \frac{X+1}{2}\Big). \quad \text{(C.3-11)}$$

For the hypergeometric functions to be expandable in a series around $X = -1$, we must have $\mu < 1$.

We discuss the validity of this formula: The generic case is $\nu \notin -\mathbb{N}$ (so $I_\nu(\tau) \sim \tau^\nu$). The Legendre function $Q^\mu_{\nu-\frac{1}{2}}$ diverges with a simple pole if the parameters approach $\mu + \nu - \frac{1}{2} \in -\mathbb{N}$; however this is balanced by the Γ-function on the right-hand-side which diverges in this limit as well, so that the quotient is finite. Similarly, the integral on the left-hand side diverges (ie, the analytic continuation in the parameters of the integral diverges) if $\frac{1}{2}+\nu-\mu \in -\mathbb{N}_0$, but this is balanced by the Γ-function on the left-hand side which diverges in the same parametric domain, so that the limit is finite when one approaches one of these points.

Finally, there are the special points where $\nu \in -\mathbb{N}$, and $I_\nu(\tau) \sim \tau^{-\nu}$. Then, the condition for the integral on the LHS to diverge is $\frac{1}{2} - \nu - \mu \in -\mathbb{N}_0$. This implies that $\frac{1}{2} + \nu - \mu \in -\mathbb{N}_0$ as well, so the Γ-function on the left-hand side still diverges and the limit is finite. The only case where both sides of the equation vanish is when $\nu \in -\mathbb{N}$, the integral is convergent ($\frac{1}{2} - \nu - \mu \notin -\mathbb{N}_0$), but the Γ-function diverges ($\frac{1}{2}+\nu-\mu \in -\mathbb{N}_0$). In this case, it is easy to prove that $\mu+\nu+\frac{1}{2} \in -\mathbb{N}_0$, so that the Γ-function on the right-hand-side diverges. In parallel, the Legendre function on the right-hand-side diverges *logarithimcally*, so that it is dominated by the Γ-function, and the right hand side will vanish as well. So these cases have to be excluded from the above formula (the formula is correct, but useless).

C.4 An Orthonormal Basis for the EAdS Klein-Gordon Operator

We need to check the orthogonality and normalisation of the eigenfunctions

$$e_{\underline{k},\beta}(x) = C_\beta\, e^{i\underline{xk}}\, (x^0)^{\mathrm{d}/2}\, K_\beta(|\underline{k}|x^0) \qquad \text{if } \frac{\mathrm{d}^2}{4} + m^2 - \lambda > 0$$
$$e_{\underline{k},i\beta}(x) = C_{i\beta}\, e^{i\underline{xk}}\, (x^0)^{\mathrm{d}/2}\, K_{i\beta}(|\underline{k}|x^0) \qquad \text{if } \frac{\mathrm{d}^2}{4} + m^2 - \lambda \le 0$$

of the Klein-Gordon form (6.1-6). For the vectors $e_{\underline{k},i\beta}$,

$$\begin{aligned}\langle e_{\underline{k},i\beta}, e_{\underline{k}',i\beta'}\rangle_m =&|C_{i\beta}|^2 \int \mathrm{d}^{\mathrm{d}+1}x\,(x^0)^{1-2\beta_0}\,\partial_\mu\left[(x^0)^{\beta_0}\,e^{i\underline{kx}}\,K_{i\beta}(|\underline{k}|x^0)\right]^*\cdot\partial_\mu\left[(x^0)^{\beta_0}\,e^{i\underline{k}'\underline{x}}\,K_{i\beta'}(|\underline{k}'|x^0)\right]\\ =&|C_{i\beta}|^2(2\pi)^{\mathrm{d}}\delta^{(\mathrm{d})}(\underline{k}-\underline{k}')\int \mathrm{d}x^0\left\{x^0K_{-i\beta}(x^0)K_{i\beta'}(x^0)+\right.\\ &\left.(x^0)^{1-2\beta_0}\,\partial_0\left[(x^0)^{\beta_0}\,K_{-i\beta}(x^0)\right]\cdot\partial_0\left[(x^0)^{\beta_0}\,K_{i\beta'}(x^0)\right]\right\}.\end{aligned} \tag{C.4-12}$$

The integral can be evaluated if we make some basic assumptions characteristic for the bulk part of the wave functions; this will enable us to perform integration by parts. We study the second summand in the brackets. Ignoring the prefactors and abbreviating $K_\nu(x^0)\equiv K_\nu$, it can be written as

$$\begin{aligned}&\int \mathrm{d}x^0\left\{(x^0)^{1-2\beta_0}\,\partial_0\left[(x^0)^{\beta_0-i\beta}(x^0)^{i\beta}\,K_{-i\beta}\right]\cdot\partial_0\left[(x^0)^{\beta_0+i\beta'}(x^0)^{-i\beta'}\,K_{i\beta'}\right]\right\}\\ =&\int \mathrm{d}x^0\,x^0\,\left[\frac{\beta_0-i\beta}{x^0}K_{-i\beta}-K_{-i\beta+1}\right]\cdot\left[\frac{\beta_0+i\beta'}{x^0}K_{i\beta'}-K_{i\beta'+1}\right]\\ =&(\beta_0-i\beta)(\beta_0+i\beta')\int \mathrm{d}x^0\,\frac{K_{-i\beta}K_{i\beta'}}{x^0}\\ &-\beta_0\int \mathrm{d}x^0\,\left(K_{-i\beta}K_{i\beta'+1}+K_{-i\beta+1}K_{i\beta'}\right)\\ &+\int \mathrm{d}x^0\,\left(x^0K_{-i\beta+1}K_{i\beta'+1}+i\beta K_{-i\beta}K_{i\beta'+1}-i\beta'K_{-i\beta+1}K_{i\beta'}\right).\end{aligned}$$

These integrals can be evaluated using the formula

$$\int_0^\infty \frac{\mathrm{d}z}{z}K_{-i\beta}(z)K_{i\beta'}(z)=\frac{\pi}{2}|\Gamma(i\beta)|^2\delta(\beta-\beta')=\frac{\pi^2}{2\beta\sinh\pi\beta}\delta(\beta-\beta')\qquad \beta,\beta'>0.$$

(This is the well-known kernel of the Kontorovich-Lebedev transform [1]) [36, 95]. Concerning the second integral, we may write it as a total derivative and a Kontorovich-Lebedev remainder,

$$-K_{-i\beta}K_{i\beta'+1}-K_{-i\beta+1}K_{i\beta'}=\partial_0K_{-i\beta}K_{i\beta'}+\frac{i(\beta-\beta')}{x^0}K_{-i\beta}K_{i\beta'}.$$

The last integral can be compounded with the contribution from the transverse derivatives which we left out so far. This gives

$$\begin{aligned}&x^0K_{-i\beta}K_{i\beta'}+x^0K_{-i\beta+1}K_{i\beta'+1}+i\beta K_{-i\beta}K_{i\beta'+1}-i\beta'K_{-i\beta+1}K_{i\beta'}\\ &\qquad=\partial_0\left(-x^0K_{-i\beta}K_{i\beta'+1}+i\beta'K_{-i\beta}K_{i\beta'}\right)+i\beta'\frac{i(\beta-\beta')}{x^0}K_{-i\beta}K_{i\beta'}.\end{aligned}$$

When we compute the total integral, we assume that the Bessel functions are sufficiently smeared in β resp. β' so that $\lim_{x^0\to0+}\int \mathrm{d}\beta\,f(\beta)K_{i\beta}(x^0)=0$ fast enough,

[1] Also called "Kantorovich-Lebedev" transform in the literature.

where β is the smearing function. Then, the integral over the total derivatives vanishes. That this assumption is reasonable will be obvious once we have computed the scalar product in $(\underline{k}, \beta)$-space; the normalisation conditions will be seen to guarantee this assumption .

Employing the Kontorovich-Lebedev kernel, we obtain for the integrations

$$[(\beta_0 - i\beta)(\beta_0 + i\beta') + (i\beta' + \beta_0)(i\beta - i\beta')] \, \frac{\pi^2}{2\beta \sinh \pi\beta} \delta(\beta - \beta').$$

So the *kernel* of the scalar product is

$$\langle e_{\underline{k},i\beta}, e_{\underline{k}',i\beta'} \rangle = |C_{i\beta}|^2 (2\pi)^{\mathrm{d}} \delta^{(\mathrm{d})}(\underline{k} - \underline{k}') \frac{\pi^2 \, \lambda_{i\beta}}{2\beta \sinh \pi\beta} \delta(\beta - \beta'),$$

where we have used $\lambda_{i\beta} \equiv \beta_0^2 + \beta^2$.

It is assumed here implicitely that the "plane waves" in this scalar product are smeared with wave functions $f(\beta, \underline{k})$ which are at least $L^2_{\mathrm{loc}}(\mathrm{AdS})$. We choose therefore a normalisation

$$C_{i\beta} = \left(\frac{2\beta \sinh \pi\beta}{(2\pi)^{\mathrm{d}} \pi^2} \right)^{1/2} .$$

Then, $\langle e_{\underline{k},i\beta}, e_{\underline{k}',i\beta'} \rangle = \lambda_{i\beta} \delta^{(\mathrm{d})}(\underline{k} - \underline{k}') \delta(\beta - \beta')$.

For the eigenvectors $e_{\underline{k},\beta}$, the situation is completely different. For small x^0, $K_\beta(x^0) \sim (x^0)^{-\beta}$. Accordingly, the integrand of the scalar product becomes

$$\begin{aligned} x^0 K_\beta(x^0) K_{\beta'}(x^0) + (x^0)^{1-2\beta_0} \, \partial_0 \left[(x^0)^{\beta_0} \, K_\beta(x^0) \right] \cdot \partial_0 \left[(x^0)^{\beta_0} \, K_{\beta'}(x^0) \right] \\ \sim (x^0)^{1-\beta-\beta'} + (\beta_0 - \beta)(\beta_0 - \beta')(x^0)^{-1-\beta-\beta'}. \end{aligned}$$

One can see that if the second terms does not vanish, the integral in the scalar product must diverge - smearing with respect to β or β' can not cure the problem. So either $\beta = \beta_0$ or $\beta' = \beta_0$; and because of (generalised) normalisability of the eigenvectors, this means that there can be no generalised eigenvector for real $\beta \neq \beta_0$. A bit of analysis yields

$$\begin{aligned} \langle e_{\underline{k},\beta_0}, e_{\underline{k}',\beta_0} \rangle =& |C_{\beta_0}|^2 (2\pi)^{\mathrm{d}} \delta^{(\mathrm{d})}(\underline{k} - \underline{k}') \int \mathrm{d}x^0 \, \Big\{ x^0 K_{\beta_0}(x^0) K_{\beta_0}(x^0) + \\ & (x^0)^{1-2\beta_0} \, \partial_0 \left[(x^0)^{\beta_0} \, K_{\beta_0}(x^0) \right] \cdot \partial_0 \left[(x^0)^{\beta_0} \, K_{\beta_0}(x^0) \right] \Big\} \\ =& |C_{\beta_0}|^2 (2\pi)^{\mathrm{d}} \delta^{(\mathrm{d})}(\underline{k} - \underline{k}') \int \mathrm{d}x^0 \, \Big\{ x^0 K_{\beta_0}(x^0)^2 + x^0 K_{\beta_0 - 1}(x^0)^2 \Big\} \\ =& |C_{\beta_0}|^2 (2\pi)^{\mathrm{d}} \delta^{(\mathrm{d})}(\underline{k} - \underline{k}') \frac{\pi}{2 \sin \beta_0 \pi}, \qquad \text{if } 0 < \beta_0 < 1 \end{aligned}$$

[49, 6.521 3]. Convergence imposes the bounds $-\frac{\mathrm{d}^2}{4} < m^2 < (1 - \frac{\mathrm{d}^2}{4})$ on the range of possible masses. Choosing the normalisation

$$C_{\beta_0} = \left(\frac{2 \sin \beta_0 \pi}{(2\pi)^{\mathrm{d}} \pi} \right)^{1/2} ,$$

the scalar product becomes $\langle e_{\underline{k},\beta_0}, e_{\underline{k}',\beta_0}\rangle = \delta^{(\mathrm{d})}(\underline{k}-\underline{k}')$.

We have yet to check the orthogonality of the imaginary and the real indexed eigenfunctions:

$$\begin{aligned}\langle e_{\underline{k},i\beta}, e_{\underline{k}',\beta_0}\rangle =& C_{i\beta}C_{\beta_0}(2\pi)^{\mathrm{d}}\delta^{(\mathrm{d})}(\underline{k}-\underline{k}')\int \mathrm{d}x^0\, \Big\{x^0 K_{-i\beta}(x^0)K_{\beta_0}(x^0)+\\ &(x^0)^{1-2\beta_0}\,\partial_0\left[(x^0)^{\beta_0}\,K_{-i\beta}(x^0)\right]\cdot\partial_0\left[(x^0)^{\beta_0}\,K_{\beta_0}(x^0)\right]\Big\}\\ =& C_{i\beta}C_{\beta_0}(2\pi)^{\mathrm{d}}\delta^{(\mathrm{d})}(\underline{k}-\underline{k}')\int \mathrm{d}x^0\, \Big\{x^0 K_{-i\beta}(x^0)K_{\beta_0}(x^0)+\\ &-\partial_0\left[(x^0)^{\beta_0}\,K_{-i\beta}(x^0)\right]\cdot(x^0)^{1-\beta_0}\,K_{\beta_0-1}(x^0)\Big\} \qquad = 0\end{aligned}$$

by integration by parts.

Appendix D

Worldgraph Formalism for Feynman Amplitudes

We summarise very briefly the content of a previous publication by the author [53], which is used in section 3.4 to derive the generating formula (3.4-82) for the three-point functions of tensor currents.

A unified treatment of Schwinger parametrised Feynman amplitudes is suggested which addresses vertices of arbitrary order on the same footing as propagators. Contributions from distinct diagrams are organised col- lectively. The scheme is based on the continuous graph Laplacian. The analogy to a classical statistical diffusion system of vector charges on the graph is explored.

Given a Euclidean Feynman graph for particles of mass m, the propagators can be parametrised in wave number space as

$$\frac{1}{k^2+m^2} = \int_0^\infty \mathrm{d}\tau \, e^{-\tau(k^2+m^2)}. \tag{D.0-1}$$

If the "Schwinger parameters" τ associated thus with each propagator are interpreted as "times" (although this is only a formal notion), then one can give a complete statistical interpretation in terms of a classical diffusive system to the Feynman graph amplitudes. This can be understood from the fact that the integral kernel in (D.0-1) is nothing but the d-dimensional diffusion kernel in wave number space, for time τ.

Not only does this allow very simple and highly symmetric generating function arguments like the one we use here, for graphs of arbitrary topology; the similarity to correlations as implied by string theory is stressed in this approach, and the similarity of the Schwinger parameters to the "moduli" of string theory can be driven to an astonishing degree.

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