Representation of Particles in two-dimensional Thermal Quantum Field Theory

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Contents

1. Introduction

At the end of the 20th century, J. Bros and D. Buchholz proposed an axiomatic program for the study of structural properties of thermal correlation functions in a model independent Wightman-like setting, aiming to show the validity and consistency of the perturbative real- and imaginary-time approaches to thermal quantum field theory [8]. The basic idea in their work is to view the KMS condition and its stronger form, the relativistic KMS condition [10] as the thermal equilibrium analogon to the relativistic spectrum condition of vacuum theories, which allows the study of correlation functions in terms of their analyticity properties.

Among their findings is that there holds a thermal version of the Källén-Lehmann representation, allowing to expand thermal two-point function into free thermal two-point functions of varying masses. This representation, referred to as the *Bros-Buchholz expansion* in this work, provides the basis for a possible answer to the conceptual question of what massive, particle-like excitation in a thermal setting should be, as well as to the subsequent mathematical question of how to infer the particle content of a given thermal theory [7],[8]. In the vacuum case these questions are answered by Wigner's criterion, which identifies particles with irreducible representation of the Poincaré group [2]. Here, the particle content of a given vacuum theory is encoded in discrete mass-shell contributions to the Fourier transformed two-point function.
A result by H. Narnhofer, M. Requardt and W. Thirring states that an adaption of the Wigner criterion for thermal theories, ascribing particles to dispersion laws defined by discrete contributions to the Fourier transformed two-point function, is only meaningful in the case of trivial interaction, due to the dissipative effects of an omnipresent thermal background [5]. A particle criterion proposed by J. Bros and D. Buchholz solves these difficulties by taking into account the anticipated damping effects of a thermal background on the propagation of particle-like excitations. Such a criterion is not only interesting from a conceptual point of view, but may also have concrete applications. Under "normal" conditions quarks are only found in spatially confined pairs or triplets; single quarks alone in space cannot exists. As a consequence, there is no a priori (Wigner) concept of quarks as massive particles. However, experimental evidence suggests the existence of a deconfinement phase, in which quarks and gluons are no longer in a bound state [18]. This phase only emerges at high temperatures and the proposed criterion may give an interpretation of quarks as massive particles in such high energy scenarios.

Though the *Bros-Buchholz criterion* is well motivated, no fully interacting theory exist in four space-time dimensions to which it could be applied [13]. However, the authors showed that the leading order contribution to two-point functions at asymptotic times stems from the presence of massive particles and exploited this insight to compute how such particles would propagate through a thermal background for a class of effective models with polynomial interaction [9]. Their findings exhibit the features one might expect from particles traveling through a dissipative background.

In recent times, C. Gérard and C. Jäkel constructed a class of full fledged thermal theories for polynomial interactions in two space-time dimensions [11] and it is of interest to test the formalism of J. Bros and D. Buchholz in these theories. In order to do this it is necessary to extend their formalism to the said number of dimensions. This is the subject of the work at hand. In addition to the proof of the two-dimensional Bros-Buchholz expansion, this thesis contains a study of asymptotic dynamics of effective theories in analogy to [9] which may serve as a starting point for comparing the dissipative propagation in effective and fully developed theories. In the following an outline of the thesis is given.

Chapter 2 is devoted to setting a notational convention and stating some well-known facts on mathematical objects used throughout this thesis. In addition, an overview of the used quantum field theoretic framework is given.
Chapter 3 contains the proof of the Bros-Buchholz expansion in two space-time dimensions. Some properties of the "expansion coefficients" are studied and the Bros-Buchholz particle criterion is stated and motivated.
Chapter 4 is dedicated to the study of time-asymptotics of thermal correlation functions. Their asymptotic structure is captured in an algebra which admits KMS states for arbitrary positive temperatures.
Chapter 5 makes use of the condition of *asymptotic compatibility* to single out those KMS states which fulfill the field equation of an underlying effective model in an asymptotic sense. In turn, this helps infer the effects of a thermal background on the propagation of massive particles in effective models.
Chapter 6 concludes the discussion of effective theories by translating the established results into the language of retarded, advanced, time-ordered and anti-time-ordered propagators used in other approaches to thermal quantum field theory.
Chapter 7 returns to original motivation for extending the formalism by J. Bros and D. Buchholz to two dimensions, namely its application to the models of C. Gérard and C. Jäkel, and offers a sketchy outline of how this application might be performed.
Appendices A-C cover some of the mathematical tools used throughout this work and provide proofs deemed too lengthy or uninstructive to be placed in the main text.
Appendix D discusses in detail the quantum field theoretical framework employed in this thesis.

2. Preparation

2.1. Preliminaries

The goal of this section is to introduce a variety of mathematical objects and to set a notational convention to be used throughout this thesis. Fundamental facts about these objects are stated here so that they may later be employed in a non-digressive manner.

2.1.1. Units

Natural (Planck) units are used. Of relevance are $c = 1$, $\hbar = 1$ and $k_B = 1$, where $c, \hbar$ and k_B denote the speed of light, reduced Planck constant and Boltzmann constant respectively.

2.1.2. Minkowski Space

The $(n+1)$-dimensional **Minkowski space** $\mathbb{M}^{n+1}$ is $\mathbb{R}^{n+1}$ endowed a non-degenerate symmetric bilinear form η of signature $(1, n)$.
By Sylvester's Inertia Theorem there exist bases $(e_j)_{j=0,\dots,n}$ of $\mathbb{R}^{n+1}$ such that

$$\eta(e_j, e_k) = \begin{cases} 0, & \text{otherwise} \\ 1, & j = k = 0 \\ -1, & j = k > 0\ . \end{cases}$$

A choice of such a basis is called a **Lorentz frame**. With respect to a fixed Lorentz frame, an element in $\mathbb{M}^{n+1}$ is written as

$$x = (\underline{x}, \mathbf{x})\,,$$

where

$$\underline{x} = \eta(x, e_0) \quad \text{and} \quad \mathbf{x} = \sum_{j=1}^{n} \mathbf{x}_j e_j, \quad \mathbf{x}_i = -\eta(x, e_i)\,.$$

$\underline{x}$ is called the **time component** and $\mathbf{x}$ the **space component** of x. With this convention it is

$$\eta(x, y) = \underline{x}\,\underline{y} - \mathbf{x} \cdot \mathbf{y} = \underline{x}\,\underline{y} - \sum_{j=1}^{n} \mathbf{x}_j\, \mathbf{y}_j\,.$$

$\eta(x,x)$ is called **Lorentz square** of $x \in \mathbb{M}^{n+1}$ and will be denoted by x^2 to lighten the notation. Similarly, xy is written instead of $\eta(x,y)$ when there is little danger of confusion. A vector $x \in \mathbb{M}^{n+1}$ is called

$$\begin{aligned} \textbf{timelike} &\quad\Leftrightarrow\quad x^2 > 0\,, \\ \textbf{spacelike} &\quad\Leftrightarrow\quad x^2 < 0\,, \\ \textbf{lightlike} &\quad\Leftrightarrow\quad x^2 = 0\,. \end{aligned}$$

Two vectors $x, y \in \mathbb{M}^{n+1}$ are called **spacelike separated** $:\Leftrightarrow$
$x - y$ is spacelike.
Two subsets $X, Y \subset \mathbb{M}^{n+1}$ are called **spacelike separated** $:\Leftrightarrow$
$\forall\, x \in X, y \in Y : x - y$ is spacelike.
The notation $x \bowtie y$, $X \bowtie Y$ is used to denote spacelike separation.
The cones V^+, V^- in $\mathbb{M}^{n+1}$ defined by

$$\begin{aligned} V^+ &:= \{\, x \in \mathbb{M}^{n+1} \,|\, x \,\text{is timelike,}\ \underline{x} > 0 \,\}\,, \\ V^- &:= \{\, x \in \mathbb{M}^{n+1} \,|\, x \,\text{is timelike,}\ \underline{x} < 0 \,\} \end{aligned}$$

are called **forward** and **backward light-cone** respectively.

Throughout this work n denotes the number of spatial dimensions in physical contexts.

2.1.3. The Poincaré Group

Definition 2.1. The $(n+1)$-dimensional **Lorentz group** $\mathscr{L}_{n+1}$ is the subgroup of $\mathrm{Gl}_{n+1}(\mathbb{R})$ leaving the bilinear form η invariant:

$$\forall\, \Lambda \in \mathscr{L}_{n+1},\ x, y \in \mathbb{M}^{n+1}\ :\ \eta(\Lambda x, \Lambda y) = \eta(x,y)\,.$$

Elements of the Lorentz group have determinant ± 1. The subgroup of the Lorentz group of elements with determinant $+1$ and leaving the sign of the time component unchanged is called **proper orthochronous Lorentz group** and is denoted by $\mathscr{L}^{+,\uparrow}_{n+1}$.
In a given Lorentz frame, the 2-dimensional Lorentz group is generanted by spatial reflections $(\underline{x}, \mathbf{x}) \mapsto (\underline{x}, -\mathbf{x})$, temporal reflections $(\underline{x}, \mathbf{x}) \mapsto (-\underline{x}, \mathbf{x})$ and boosts of rapidity $\theta \in \mathbb{R}$

$$\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}.$$

The 2-dimensional proper orthochronous Lorentz group only consists of boosts.

Definition 2.2. The $(n+1)$-dimensional **Poincaré group** $\mathscr{P}_{n+1} = \mathscr{L}_{n+1} \ltimes \mathbb{R}^{n+1}$ is a semi-direct product of the $(n+1)$-dimensional Lorentz group and $\mathbb{R}^{n+1}$ defined by its action on $\mathbb{M}^{n+1}$:

$$\forall\, (\Lambda, a) \in \mathscr{P}_{n+1},\ x \in \mathbb{M}^{n+1}\ :\ (\Lambda, a) \cdot x := \Lambda x + a\,.$$

Analogously, the $(n+1)$-dimensional proper orthochronous Lorentz group is defined to be $\mathscr{P}^{+,\uparrow}_{n+1} = \mathscr{L}^{+,\uparrow}_{n+1} \ltimes \mathbb{R}^{n+1}$.

2.1.4. Schwartz Space

Definition 2.3. The **Schwartz space** on $\mathbb{R}^n$ is defined to be

$$\mathcal{S}(\mathbb{R}^n) := \{ \varphi \in \mathcal{C}^\infty(\mathbb{R}^n) \,|\, \forall\, \alpha, \beta \in \mathbb{N}_0^n : ||\varphi||_{\alpha,\beta} < \infty \} \,,$$

where $||\varphi||_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} \left| x^\alpha (\partial^\beta \varphi)(x) \right|$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \dots \partial_n^{\beta_n}$ for multiindices $\alpha = (\alpha_1, \dots \alpha_n)$, $\beta = (\beta_1, \dots \beta_n)$ are the **Schwartz semi-norms**. $\mathcal{C}^\infty(\mathbb{R}^n)$ denotes the space of smooth, complex valued function on $\mathbb{R}^n$.

Equipped with the initial toplogy with respect to the family of semi-norms $(||\cdot||_{\alpha,\beta})_{\alpha,\beta\in\mathbb{N}_0^n}$, $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet Space.
$\mathcal{S}(\mathbb{R}^n)$ is closed under addition, pointwise multiplication and convolution: for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ the maps

$$\begin{aligned} x \mapsto (\varphi + \psi)(x) &:= \varphi(x) + \psi(x) \,, \\ x \mapsto (\varphi \cdot \psi)(x) &:= \varphi(x) \cdot \psi(x) \,, \\ x \mapsto (\varphi * \psi)(x) &:= \int_{\mathbb{R}^n} \mathrm{d}y \, \varphi(y) \psi(x-y) \,, \\ x \mapsto \check{\varphi}(x) &:= \varphi(-x) \end{aligned}$$

lie also in $\mathcal{S}(\mathbb{R}^n)$. Elements of $\mathcal{S}(\mathbb{R}^n)$ are called **Schwartz functions**. The closed subspace of real valued Schwartz functions is denoted by $\mathcal{S}_\mathbb{R}(\mathbb{R}^n)$.

2.1.5. Test Functions

Definition 2.4. The **space of test functions** on $\mathbb{R}^n$ is defined to be

$$\mathcal{D}(\mathbb{R}^n) := \{ \varphi \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \mathrm{supp}(\varphi) \text{ is compact} \}$$

equipped with the topology of uniform convergence of all derivatives on compact sets. This topology is the initial topology with respect to the family of semi-norms $(||\cdot||_{\alpha,C})_{\alpha,C}$, where $C \subset \mathbb{R}^n$ compact, $\alpha \in \mathbb{N}_0^n$ and

$$||\varphi||_{\alpha,C} := \sup_{x\in C} |\partial^\alpha \varphi| \,.$$

$\mathcal{D}(\mathbb{R}^n)$ is a dense subspace of $\mathcal{S}(\mathbb{R}^n)$ and the inclusion is continuous.

2.1.6. Distributions

Definition 2.5. The **space of distributions** on $\mathbb{R}^n$ is definied to be to be continuous dual of $\mathcal{D}(\mathbb{R}^n)$:

$$\mathcal{D}'(\mathbb{R}^n) := \{ T \in \hom_\mathbb{C}(\mathcal{D}(\mathbb{R}^n), \mathbb{C}) \,|\, T \text{ is continuous} \} \,.$$

$\mathcal{D}'(\mathbb{R}^n)$ is equipped with the weak-$*$ topolgy.

For the value of a distribution T on a test function φ the notation

$$< T, \varphi >:= T(\varphi)$$

is used.

Definition 2.6. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ **vanishes** on $U \subset \mathbb{R}^n :\Leftrightarrow$
$\forall\, \varphi \in \mathcal{D}(\mathbb{R}^n) \,:\, \mathrm{supp}(\varphi) \subset U \;\Rightarrow\; < T, \varphi >= 0$.

Definition 2.7. The **support** $\mathrm{supp}(T)$ of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is defined to be the complement in $\mathbb{R}^n$ of the largest open subset on which T vanishes.

Definition 2.8. A distribution $T \in \mathcal{D}'(\mathbb{R})$ is called **even** iff for all $\varphi \in \mathcal{D}(\mathbb{R})$ one has

$$< T, \varphi >=< T, \varphi_{\text{even}} > \;.$$

Accordingly $T \in \mathcal{D}'(\mathbb{R})$ is called **odd** iff there holds for all $\varphi \in \mathcal{D}(\mathbb{R})$

$$< T, \varphi >=< T, \varphi_{\text{odd}} > \;.$$

φ_{even} and φ_{odd} denote the even/odd parts of φ which are defined as

$$\varphi_{\text{even/odd}} = \frac{\varphi \pm \check{\varphi}}{2} \,. \tag{2.1}$$

Definition 2.9. The **space of tempered distributions** on $\mathbb{R}^n$ is definied to be to be continuous dual of $\mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{S}'(\mathbb{R}^n) := \{\, T \in \hom_{\mathbb{C}}(\mathcal{S}(\mathbb{R}^n), \mathbb{C}) \,|\, T \text{ is continuous} \,\} \,.$$

$\mathcal{S}'(\mathbb{R}^n)$ is also equipped with the weak-$*$ topolgy. By duality the inclusion of $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$ is continuous. Also $\mathcal{S}'(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$.
A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is called **tempered** iff it extends to a tempered distribution. Such extensions are unique.

Schwartz functions include naturally into their continuous dual: Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ define $T_\varphi \in \mathcal{S}'(\mathbb{R}^n)$ by

$$< T_\varphi, \psi >:= \int_{\mathbb{R}^n} \mathrm{d}x\, \varphi(x)\psi(x) \,.$$

The map $\varphi \mapsto T_\varphi$ is linear, continuous and has dense image. Polynomially bounded continuous functions can be embedded into tempered distributions in the same manner. When there is no danger of confusion, φ is used to both denote the function and the corresponding distribution T_φ.

Tempered distributions can be convoluted with Schwartz functions: For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ define

$$< \varphi * T, \psi >:=< T, \check{\varphi} * \psi > \;. \tag{2.2}$$

This definition is compatible with the natural inclusion of Schwartz functions into tempered distributions in the sense that $\varphi * T_\psi = T_{\varphi*\psi}$ holds for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore $\varphi * T$ is given by the natural inclusion of the polynomially bounded smooth function $x \mapsto < T, \check{\varphi}_x >$, where $\check{\varphi}_x(y) = \varphi(x-y)$.

It is a useful fact that all maps in the following sequence are injective, linear, continuous maps with dense image:

$$\mathcal{D}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)\,.$$

Theorem 2.1 (Structure Theorem for Tempered Distributions)**.** *Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then there exist finitely many multiindices $\alpha \in \mathbb{N}^n$ and polynomially bounded continuous functions T_α such that*

$$\forall\, \varphi \in \mathcal{S}(\mathbb{R}^n)\ :< T, \varphi >= \sum_{\substack{\alpha\\ \text{finite}}} \int \mathrm{d}x\, T_\alpha(x)(\partial^\alpha\varphi)(x)\,.$$

Conversely, if T_α are finitely many polynomially bounded continuous functions, then

$$\varphi \mapsto \sum_{\substack{\alpha\\ \text{finite}}} \int \mathrm{d}x\, T_\alpha(x)(\partial^\alpha\varphi)(x) \tag{2.3}$$

defines a tempered distribution.

A proof along with other distribution-theoretic facts stated here can be found in [14].

2.1.7. Fourier Transform

Definition 2.10. For $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$ define the **Fourier transform** $\hat{\varphi}$ of φ:

$$\hat{\varphi}(p) := \int_{\mathbb{R}^{n+1}} \mathrm{d}x\, \varphi(x) e^{ixp} = \int_{\mathbb{R}} \mathrm{d}\underline{x} \int_{\mathbb{R}^n} \mathrm{d}\mathbf{x}\, \varphi(\underline{x}, \mathbf{x}) e^{i\underline{x}\underline{p}} e^{-i\mathbf{x}\mathbf{p}}\,,$$

where $p = (\underline{p}, \mathbf{p})$.

The Fourier transform is a linear homeomorphism

$$\mathcal{F}\ :\ \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^{n+1}), \quad \varphi \mapsto \hat{\varphi}$$

with inverse given by

$$\begin{aligned}\mathcal{F}^{-1}\ :\ \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^{n+1}), \quad \varphi \mapsto \hat{\varphi}^{-1}\,,\\ \hat{\varphi}^{-1}(x) = (2\pi)^{-(n+1)} \int_{\mathbb{R}^{n+1}} \mathrm{d}p\, \varphi(p) e^{-ixp}\,.\end{aligned}$$

The Fourier transform extends to a unique linear homeomorphism of tempered distributions compatible with the natural inclusion $\mathcal{S} \to \mathcal{S}'$

$$\mathcal{F}\ :\ \mathcal{S}'(\mathbb{R}^{n+1}) \to \mathcal{S}'(\mathbb{R}^{n+1}), \quad T \mapsto \hat{T}$$

given by $< \widehat{T}, \varphi >=< T, \widehat{\varphi} >$.

Partial Fourier transforms with respect to time-energy variable pair or space-momentum variable pairs will be denoted by $\widehat{\ }^{\underline{x},\underline{p}}$ and $\widehat{\ }^{\mathbf{x},\mathbf{p}}$ respectively.

The Fourier transform of the convolution of two Schwartz functions is the product of the Fourier transforms of these functions:

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}\,.$$

Since the natural inclusion of Schwartz functions is compatible with the Fourier transform and convolution it also is

$$\widehat{f * T} = \widehat{f} \cdot \widehat{T}\,.$$

2.1.8. Bessel Functions

Definition 2.11. Bessel functions are solutions to the Bessel differential equation

$$z^2 \cdot f^{(2)}(z) + z \cdot f^{(1)}(z) + (z^2 - k^2)f(z) = 0\,.$$

As a second order ODE, the Bessel differential equation has two linearly independent solutions. A common choice for a pair of such solutions is J_k and Y_k, which are called Bessel functions of the first and second kind respectively. In the following only J_k is of interest. For $k \in \mathbb{N}_0$ one defines J_k as follows (cf. [17]):

$$J_k(z) := \sum_{l=0}^{\infty} \frac{(-1)^l}{(k+l)!l!} \left(\frac{z}{2}\right)^{k+2l} .$$

The series converges absolutely on $\mathbb{C}$. For negative $k \in \mathbb{Z}$ set $J_k := (-1)^k J_{-k}$. All J_k are entire functions.

The following integral representation holds for $k \in \mathbb{N}_0$:

$$J_k(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^k \int_p \mathrm{d}\zeta\, e^{\zeta - \frac{z^2}{4\zeta}}\, \zeta^{-(k+1)}\,,$$

where p is any closed piecewise continuously differentiable path in $\mathbb{C}\backslash\{0\}$ homotopic to $[0, 2\pi] \ni t \mapsto e^{it}$. Compatibility to the defining representation of J_k can be seen expanding the integrand into its Laurent series, computing the -1^{st} coefficient and applying the residue theorem.

2.2. Framework

For the purpose of this work a generalized Wightman framework for real scalar quantum fields according to [6] will be adopted. This framework takes the $*$-algebra $\mathcal{A}$ generated by symbols $\phi(f)$, $f \in \mathcal{S}(\mathbb{R}^{n+1})$, only subject to $\phi(f)^* = \phi(\overline{f})$ and the locality relation $f \bowtie g \Rightarrow [\phi(f), \phi(g)] = 0$, as a starting point. The algebra $\mathcal{A}$ is equipped with a natural automorphic action of the Poincaré group $\mathscr{P}_{n+1}$: for $\Lambda \in \mathscr{P}_{n+1}$ the map $\alpha_\Lambda : \phi(f) \mapsto \phi(f_\Lambda), \;\; f_\Lambda(x) := f(\Lambda^{-1} \cdot x)$ extends to an automorphism of $\mathcal{A}$.

An $(n+1)$-dimensional **quantum field theory** is given by a state ω on $\mathcal{A}$, which imposes further relations on the field encoded in the kernel of the GNS representation associated to ω. States are required to satisfy a regularity condition, which ensures that the maps $f_1, \cdots, f_m \mapsto \omega(\phi(f_1) \cdots \phi(f_m))$ extend to tempered distributions $\mathcal{W}_m \in \mathcal{S}'(\mathbb{R}^{m \cdot (n+1)})$, giving access to a range of distribution-theoretic methods for the analysis of quantum field theories. The tempered distributions $\mathcal{W}_m$ are called **correlation functions**, **m-point functions** or **Wightman functions**. These distributions form a family $(\mathcal{W}_M)_{m \in \mathbb{N}}$, which contains all the information of the associated quantum field theory.

States ω on $\mathcal{A}$ may have the property to be invariant under a subgroup $\mathscr{G}$ of $\mathscr{P}_{n+1}$. For the corresponding theories, $\mathscr{G}$ then is unitarily implemented in the GNS representation associated to ω. In this case, one parameter subgroups of $\mathscr{G}$ correspond to self-adjoint operators on the GNS Hilbert spaces by Stone's theorem.

Of particular interest for the studies in this work are vacuum states and states of (global) thermal equilibrium which are characterized by the relativistic spectrum condition and the (relativistic) KMS condition respectively.

An overview of this settings, including detailed accounts of the various notions introduces above, can be found in appendix D. If not stated otherwise, $n = 1$ is assumed.

3. General Representation of Thermal two-point Functions

As part of their program to study model independent structural properties of thermal correlation functions, J. Bros and D. Buchholz found that there exists an integral representation of thermal two-point function analogous to the Källen-Lehmann representation in vacuum theories (cf. [8]). This Bros-Buchholz representation is based on a continuous expansion of a commutator function C in terms of free mass m commutator functions C_m. The “expansion coefficients” are given in terms of a distribution $D(m, \mathbf{x})$, which weights the contributions C_m and modifies their $\mathbf{x}$-dependence. Using that a thermal two-point function W can be essentially recovered from the commutator function, it can be show that the former can be expanded into free mass m thermal two-point functions $W_{\beta,m}$ with the same “expansion coefficient” D. In light of the expected abatement of two-point correlations increasing with spatial distance due to the interaction with a thermal background, D is called **damping factor**. It describes how local excitations over distance dissipate into a thermal background.

Is this chapter the validity of the Bros-Buchholz representation in two dimensions is proven using a top down approach. The damping factor is first defined and then shown to exhibit the desired properties.

For the remainder of this chapter let ω be a regular state on the two-dimensional field algebra $\mathcal{A}$ satisfying the following properties:

- ω satisfies the relativistic KMS condition (cf. D.19) at inverse temperature β.
- ω is homogeneous and invariant under spatial reflection.
- ω is time-clustering (cf. D.18).

Homogeneity implies the existence of the reduced two-point function W and the commutator function C. By definition, C is an odd distribution, and invariance under spatial reflections implies that C is even in the spatial- and odd in the time-variable. It assumed that ω has vanishing one-point functions.

From a mathematical viewpoint, all that is required to define the damping factor D is that the commutator function C is a tempered distribution with support in $\overline{V^+} \cup \overline{V^-}$. To show that the representation formula for the commutator function (equation 3.2) holds, C is required to be odd in the time variable. For the Bros-Buchholz expansion of

W, the KMS condition, time-clustering and vanishing one-point functions are required. The implications of the relativistic KMS conditon are examined separately.

3.1. The Damping Factor

Proposition 3.1. *The map D given by*

$$D : \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}_{\text{even}}(\mathbb{R}) \to \mathbb{C}\,,$$
$$< D, \varphi \otimes \psi >:=< i\underline{x}C, (\underline{x}, \mathbf{x}) \mapsto \varphi(\mathbf{x})\mathcal{T}\psi(\sqrt{x^2}) >$$

defines a distribution (i.e. is continuous), which is even in φ. $\mathcal{T}$ denotes the Hankel transform (cf. appendix B).

Proof. The definition of D requires clarification as $\sqrt{x^2}$ is only defined on $\overline{V^+} \cup \overline{V^-}$. The map $\mathcal{T}\psi$ is an even Schwartz function which implies that there exists a Schwartz function $\sigma \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{T}\psi(s) = \sigma(s^2)$ for all $s \in \mathbb{R}$. It follows that $(\underline{x}, \mathbf{x}) \mapsto \varphi(\mathbf{x})\mathcal{T}\psi(\sqrt{x^2})$ can be extended to the Schwartz function $(\underline{x}, \mathbf{x}) \mapsto \varphi(\mathbf{x})\sigma(x^2)$. Causality implies that the support of C is contained in $\overline{V^+} \cup \overline{V^-}$. Splitting C into two parts with support in the forward and backward light-cone respectively, an application of the Bros-Epstein-Glaser lemma (cf. [4]) yields that D is independent of the choice of such an extension: there exist continuous functions c_k on $\mathbb{R}^2$, $k \in \mathbb{N}_0^2$ with support in the union of the closure of the light-cones such that for all $\Phi \in \mathcal{S}(\mathbb{R}^2)$ it is

$$< C, \Phi >= \sum_{\substack{k \in \mathbb{N}_0^2 \\ \text{finite}}} \int \mathrm{d}x\, c_k(x) \partial^k \Phi(x)\,, \tag{3.1}$$

where $\partial^k = \partial_{\underline{x}}^{k_1} \partial_{\mathbf{x}}^{k_2}$. With some care and exploiting the support properties of the c_k, one can estimate $| < C, (\underline{x}, \mathbf{x}) \mapsto \underline{x}\varphi(\mathbf{x})\sigma(x^2)\underline{x} > |$ in terms of Schwartz semi-norms of φ and $\mathcal{T}\psi$ using relation 3.1. These estimates involve two arguments:

- For small arguments, the derivatives of $s \mapsto \mathcal{T}\psi(s^{\frac{1}{2}})$ can be estimated in terms of suprema of moduli of higher derivatives. This involves Taylor expanding the even function (all odd derivatives vanish at 0) $\mathcal{T}\psi$ in $s^{\frac{1}{2}}$ about the origin and estimating the Lagrange remainder term for arguments close to zero (e.g. in $[0, 1]$).

  For large arguments, derivatives of $s \mapsto \mathcal{T}\psi(s^{\frac{1}{2}})$ can be estimated in terms of derivatives of $\mathcal{T}\psi$, as the additional inverse powers of $s^{-\frac{1}{2}}$ due to differentiation of $s \mapsto s^{\frac{1}{2}}$ accelerate the decay for large arguments.

- The second argument concerns the mixing of variables $(\underline{x}, \mathbf{x}) \mapsto (\underline{x}, \underline{x}^2 - \mathbf{x}^2)$. The general strategy for estimating the integrals in equation 3.1 works as follows: extract sufficiently high powers of $\frac{1}{1+\underline{x}^2}$ and $\frac{1}{1+\mathbf{x}^2}$ from $\partial^k\Phi$ to a) suppress the polynomial growth of the c_k and, b) make the integral converge all by itself so that the supremum of the appropriate derivative of Φ multiplied by the respective inverse powers of $\frac{1}{1+\underline{x}^2}$ and $\frac{1}{1+\mathbf{x}^2}$ can be pulled out of the integral giving the desired Schwarz semi-norm estimate.

In the case at hand, this procedure works with slight modifications. Derivatives of $(\underline{x}, \underline{x}^2 - \mathbf{x}^2)$ generate polynomials which can be controlled by powers of $\frac{1}{1+\underline{x}^2}$ and $\frac{1}{1+\mathbf{x}^2}$. It remains to estimate appropriate products of powers of $\frac{1}{1+\mathbf{x}^2}$ and $\frac{1}{1+(\underline{x}^2-\mathbf{x}^2)}$ such that their moduli are less than prescribed powers of $\frac{1}{1+\underline{x}^2}$ and $\frac{1}{1+\mathbf{x}^2}$. This can be achieved by splitting the light-cones into two types of regions. The first kind is a sharp cone in which the modulus of the time variable is at least twice that of the spacial variable. Here powers of $\frac{1}{1+(\underline{x}^2-\mathbf{x}^2)}$ can be estimated in terms of prescribed powers of $\frac{1}{1+\underline{x}^2}$. The other type of region are complements of the sharp cones within their respective light-cones. In this case powers of $\frac{1}{1+\mathbf{x}^2}$ can be converted into powers of $\frac{1}{1+\underline{x}^2}$.

Along with the continuity of of $\mathcal{T} : \mathcal{S}_{\text{even}}(\mathbb{R}) \to \mathcal{S}_{\text{even}}(\mathbb{R})$ this proves that D is continuous.

Remark: By the Hahn-Banach extension theorem, it is possible to extend D continuously to $\mathcal{S}(\mathbb{R}^2)$. This extension, however, may be non-unique.

To prove that D is even in the spatial argument note that C is even in the spatial argument. This is a consequence of C being odd altogether and odd in the time variable. It follows

$$\begin{aligned} & < D, \check{\varphi} \otimes \psi > \\ = & < i\underline{x}C, (\underline{x}, \mathbf{x}) \mapsto \check{\varphi}(\mathbf{x}) \mathcal{T}\psi(\sqrt{\underline{x}^2 - \mathbf{x}^2}) > \\ = & < i\underline{x}C, (\underline{x}, \mathbf{x}) \mapsto \check{\varphi}(\mathbf{x}) \mathcal{T}\psi(\sqrt{\underline{x}^2 - (-\mathbf{x})^2}) > \\ = & < i\underline{x}C, (\underline{x}, \mathbf{x}) \mapsto \varphi(\mathbf{x}) \mathcal{T}\psi(\sqrt{\underline{x}^2 - \mathbf{x}^2}) > \\ = & < D, \varphi \otimes \psi > \ . \end{aligned}$$

$\square$

Definition 3.1. The tempered distribution $D \in (\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}_{\text{even}}(\mathbb{R}))'$ is called the **damping factor**.

Theorem 3.2. *The damping factor D satisfies for $\phi, \varphi \in \mathcal{S}(\mathbb{R})$*

$$< C, \phi \otimes \varphi > = < D, (\mathbf{x}, m) \mapsto \varphi(\mathbf{x}) \int \mathrm{d}\underline{x}\, C_m(\underline{x}, \mathbf{x}) \phi(\underline{x}) > ,$$

where C_m is the free mass m commutator function.

Proof. The proof of this theorem is quite technical. Essentially, it relies on establishing the asserted equality for a class of Schwartz functions $\omega \mapsto \omega e^{-\lambda\omega^2}$, $\lambda \in \mathbb{C}_+$ and applying Corollary A.4 to extend it all of $\mathcal{S}(\mathbb{R})$ as in [8]. In the following, extensive use of the results in appendix A is made.
For $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ define

$$\begin{aligned} < \rho, \varphi \otimes \psi > \ &:= \ \frac{1}{\pi} < D, \varphi \otimes (\psi \circ \mathrm{id}^2) > , \\ < D^\varphi, \psi > \ &:= \ < D, \varphi \otimes \psi > , \\ < \rho^\varphi, \psi > \ &:= \ < \rho, \varphi \otimes \psi > \ . \end{aligned}$$

Since $\psi \circ \mathrm{id}^2$ is always even, ρ extends uniquely to a tempered distribution on $\mathbb{R}^2$. If $\mathrm{supp}(\psi) \subset]-\infty, 0[$ then $\psi \circ \mathrm{id}^2 = 0$, so ρ^φ has support in $[0, \infty[$. This allows to take the Laplace transform $\mathcal{L}\rho^\varphi$. For $\lambda \in \mathbb{C}_+$ define

$$\Phi^\varphi(\lambda) := \mathcal{L}\rho^{\varphi\prime}(\lambda) = \lambda \cdot \mathcal{L}\rho^\varphi(\lambda)\,.$$

Claim: It is

$$\Phi^\varphi(\lambda) = \frac{i}{2\pi} < \underline{x}C, x \mapsto \exp\left(\frac{-x^2}{4\lambda}\right)\varphi(\underline{x}) > \ .$$

Proof: Let $f_\lambda \in \mathcal{S}(\mathbb{R})$ such that $f_\lambda|_{[0,\infty[}(s) = e^{-\lambda s}$. Compute

$$\begin{aligned}
& \Phi^\varphi(\lambda) \\
= \ & \lambda < \rho^\varphi, s \mapsto f_\lambda(s) > \\
= \ & \frac{\lambda}{\pi} < D^\varphi, m \mapsto e^{-\lambda m^2} > \\
= \ & \frac{\lambda}{\pi} < i\underline{x}C, x \mapsto \varphi(\mathbf{x})\mathcal{T}(m \mapsto e^{-\lambda m^2})(\sqrt{x^2}) > \\
= \ & \frac{\lambda i}{\pi} < \underline{x}C, x \mapsto \varphi(\mathbf{x})\frac{1}{2\pi}\varpi \circ \mathcal{F} \circ \iota(m \mapsto e^{-\lambda m^2})(\sqrt{x^2}) > \\
= \ & \frac{\lambda i}{2\pi^2} < \underline{x}C, x \mapsto \varphi(\mathbf{x})\mathcal{F}((a,b) \mapsto e^{-\lambda(a^2+b^2)})(\sqrt{x^2}, 0) > \\
= \ & \frac{\lambda i}{2\pi^2} < \underline{x}C, x \mapsto \varphi(\mathbf{x})\frac{\pi}{\lambda}\exp\left(\frac{-x^2}{4\lambda}\right) > \\
= \ & \frac{i}{2\pi} < \underline{x}C, x \mapsto \exp\left(\frac{-x^2}{4\lambda}\right)\varphi(\underline{x}) > \ .
\end{aligned}$$

$\exp\left(\frac{-x^2}{4\lambda}\right)$ is to be understood restricted to $\overline{V^+} \cup \overline{V^-}$ and smoothly extended to a function h_λ on all of $\mathbb{R}^2$ such that $x \mapsto h_\lambda(x)\varphi(\underline{x})$ is a Schwartz function. This is possible by a mixing of variables argument analogous to the the one found in the proof of 3.1. The identity $\mathcal{T} = \varpi \circ \mathcal{F} \circ \iota$ can be found in appendix B.

To make the following computation more accessible, introduce the even Fourier pair

$$\begin{aligned}
g_\lambda(\mathbf{x}) &= (4\pi\lambda)^{-\frac{1}{2}}\exp\left(-\frac{\mathbf{x}^2}{4\lambda}\right)\,, \\
\widehat{g}_\lambda(\mathbf{u}) &= e^{-\lambda\mathbf{u}^2}\,.
\end{aligned}$$

It is

$$\begin{aligned}
& < \widehat{C}, p \mapsto \underline{p}e^{-\lambda\underline{p}^2}\varphi(\mathbf{p}) > \\
= \ & < C, x \mapsto \frac{i\underline{x}}{2\lambda}\left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}}\exp\left(\frac{-\underline{x}^2}{4\lambda}\right)\widehat{\varphi}(\mathbf{x}) > \\
= \ & \left(\frac{\pi}{\lambda}\right)^{\frac{3}{2}}\frac{i}{2\pi} < \underline{x}C, \exp\left(\frac{-x^2}{4\lambda}\right)\exp\left(\frac{-\mathbf{x}^2}{4\lambda}\right)\widehat{\varphi}(\mathbf{x}) >
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \left(\frac{\pi}{\lambda}\right) \Phi^{g_\lambda \widehat{\varphi}}(\lambda) \\
&= 2\pi \left(\frac{\pi}{\lambda}\right) \lambda \mathcal{L} \rho^{g_\lambda \widehat{\varphi}}(\lambda) \\
&= 2\pi < D^{g_\lambda \widehat{\varphi}}, m \mapsto e^{-\lambda m^2} > \\
&= 2\pi < D, (\mathbf{x}, m) \mapsto g_\lambda \widehat{\varphi}(\mathbf{x}) e^{-\lambda m^2} > \\
&= 2\pi < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{g_\lambda \widehat{\varphi}}^{-1}{}^{\mathbf{x},\mathbf{p}}(\mathbf{p}) e^{-\lambda m^2} > \\
&= (2\pi)^2 < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{g_\lambda} * \varphi(\mathbf{p}) e^{-\lambda m^2} > \\
&= (2\pi)^2 < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \int \mathrm{d}\mathbf{u}\, \varphi(\mathbf{u}) e^{-\lambda(\mathbf{p}-\mathbf{u})^2} e^{\lambda m^2} > \\
&= (2\pi)^2 < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \int \mathrm{d}\mathbf{u}\, \varphi(\mathbf{u}) e^{-\lambda[(\mathbf{p}-\mathbf{u})^2 + m^2]} > \ .
\end{aligned}
$$

The next step is to connect the function $\mathbf{p} \mapsto e^{-\lambda(\mathbf{p}^2+m^2)}$ to the free mass m commutator function C_m. Define

$$< \widehat{C}_m^\phi, \varphi > := < \widehat{C}_m, \phi \otimes \varphi > \ .$$

Note that for $\phi \in \mathcal{S}(\mathbb{R})$ the distribution $\widehat{C}_m^\phi$ is given by the canonical inclusion of the Schwartz function $\mathbf{p} \mapsto 2\pi \phi_{\mathrm{odd}}((\mathbf{p}^2 + m^2)^{\frac{1}{2}}) \cdot (\mathbf{p}^2 + m^2)^{-\frac{1}{2}}$ into $\mathcal{S}'(\mathbb{R})$:

$$
\begin{aligned}
& < \widehat{C}_m^\phi, \varphi > \\
&= < \widehat{C}_m, \phi \otimes \varphi > \\
&= 2\pi \int \mathrm{d}\mathbf{p} \int \mathrm{d}\underline{p}\, \varepsilon(\underline{p}) \delta(p^2 - m^2) \phi(\underline{p}) \varphi(\mathbf{p}) \\
&= 2\pi \int \mathrm{d}\mathbf{p}\, \varphi(\mathbf{p}) \int \mathrm{d}\underline{p}\, \frac{\delta(\underline{p} - \sqrt{\mathbf{p}^2 + m^2}) - \delta(\underline{p} + \sqrt{\mathbf{p}^2 + m^2})}{2\sqrt{\mathbf{p}^2 + m^2}} \phi(\underline{p}) \\
&= 2\pi \int \mathrm{d}\mathbf{p}\, \varphi(\mathbf{p}) \frac{\phi_{\mathrm{odd}}(\sqrt{\mathbf{p}^2 + m^2})}{\sqrt{\mathbf{p}^2 + m^2}} \ .
\end{aligned}
$$

Denote this Schwartz function also by $\widehat{C}_m^\phi$. It follows

$$\widehat{C}_m^{\underline{p} \mapsto \underline{p} e^{-\lambda \underline{p}^2}}(\mathbf{p}) = 2\pi e^{-\lambda(\mathbf{p}^2 + m^2)} \ ,$$

which, inserted into the previous expression for $< \widehat{C}, p \mapsto \underline{p} e^{-\lambda \underline{p}^2} \varphi(\mathbf{p}) >$, gives

$$
\begin{aligned}
& < \widehat{C}, p \mapsto \underline{p} e^{-\lambda \underline{p}^2} \varphi(\mathbf{p}) > \\
&= 2\pi < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{C}_m^{\underline{p} \mapsto \underline{p} e^{-\lambda \underline{p}^2}} * \varphi(\mathbf{p}) > \ .
\end{aligned}
$$

The key argument now is that for fixed $\varphi \in \mathcal{S}(\mathbb{R})$ both sides define odd tempered distributions

$$
\begin{aligned}
\phi &\mapsto < \widehat{C}, p \mapsto \phi(\underline{p}) \varphi(\mathbf{p}) > , \\
\phi &\mapsto 2\pi < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{C}_m^\phi * \varphi(\mathbf{p}) > ,
\end{aligned}
$$

which agree for all $\underline{p} \mapsto \phi_\lambda(\underline{p}) := \underline{p}e^{-\lambda \underline{p}^2}$, $\lambda \in \mathbb{C}_+$. It follows from the result obtained in corollary A.4 that they are identical, i.e.

$$< \widehat{C}, \phi \otimes \varphi >= 2\pi < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{C}^{\phi}_m * \varphi(\mathbf{p}) > \tag{3.2}$$

for all $\phi, \varphi \in \mathcal{S}(\mathbb{R})$.
The final step consists in taking the inverse Fourier transform and simplifying the obtained expression:

$$\begin{aligned}
& < C, \phi \otimes \varphi > \\
= \;& < \widehat{C}, \widehat{\phi}^{-1} \otimes \widehat{\varphi}^{-1} > \\
= \;& 2\pi < \widehat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{C}^{\widehat{\phi}^{-1}}_m * \widehat{\varphi}^{-1}(\mathbf{p}) > \\
= \;& 2\pi < D, (\mathbf{x}, m) \mapsto (\mathcal{F}_{\mathbf{x},\mathbf{p}} \widehat{C}^{\widehat{\phi}^{-1}}_m)(\mathbf{x}) \cdot \varphi(\mathbf{x}) > \; .
\end{aligned}$$

If $(\sigma_n)_{n \in \mathbb{N}}$ is a sequence which converges weakly to $\delta_{\mathbf{x}}$, $< \delta_{\mathbf{x}}, \varphi >= \varphi(\mathbf{x})$ in $\mathcal{S}'(\mathbb{R})$, then it is

$$\begin{aligned}
& \mathcal{F}_{\mathbf{x},\mathbf{p}} \widehat{C}^{\widehat{\phi}^{-1}}_m(\mathbf{x}) \\
= \;& \lim_n < \mathcal{F}_{\mathbf{x},\mathbf{p}} \widehat{C}^{\widehat{\phi}^{-1}}_m, \sigma_n > \\
= \;& \lim_n < \widehat{C}^{\widehat{\phi}^{-1}}_m, \widehat{\sigma}_n > \\
= \;& \lim_n < C_m, \widehat{\phi}^{-1} \otimes \widehat{\sigma}_n > \\
= \;& \frac{1}{2\pi} \lim_n < C_m, \phi \otimes \check{\sigma}_n > \\
= \;& \frac{1}{2\pi} \lim_n < C^{\phi}_m, \sigma_n > ,
\end{aligned}$$

where the last step involves exploiting that C_m is even in the spatial argument. It follows that C^{ϕ}_m is also given by the canonical inclusion of a Schwartz function (denoted by $\mathbf{x} \mapsto C^{\phi}_m(\mathbf{x})$) into $\mathcal{S}'(\mathbb{R})$ and that

$$\mathcal{F}_{\mathbf{x},\mathbf{p}} \widehat{C}^{\widehat{\phi}^{-1}}_m(\mathbf{x}) = \frac{1}{2\pi} C^{\phi}_m(\mathbf{x}) \, .$$

Inserting this into the above expression for $< C, \phi \otimes \varphi >$ yields

$$< C, \phi \otimes \varphi >=< D, (\mathbf{x}, m) \mapsto C^{\phi}_m(\mathbf{x}) \varphi(\mathbf{x}) > \; .$$

Noting that $C^{\phi}_m(\mathbf{x}) = \int \mathrm{d}\underline{x}\, C_m(\underline{x}, \mathbf{x}) \phi(\underline{x})$ in the heuristic notation used to state the theorem concludes the proof.

□

Corollary 3.3 (Bros-Buchholz Representation)**.** *The damping factor D satisfies for $\phi, \varphi \in \mathcal{S}(\mathbb{R})$:*

$$< W, \phi \otimes \varphi >=< D, (\mathbf{x}, m) \mapsto \varphi(\mathbf{x}) \int \mathrm{d}\underline{x}\, W_{\beta,m}(\underline{x}, \mathbf{x}) \phi(\underline{x}) > ,$$

where $W_{\beta,m}$ is the reduced two-point function of free mass m β-KMS state (cf. equation D.7).

Proof. The time-clustering property and vanishing one-point functions ensure that

$$\widehat{W} = \frac{1}{1 - e^{-\beta \underline{p}}} \hat{C}$$

is an unambiguous relation between tempered distributions (cf. derivation of equation D.6). The same relation holds between the free mass m reduced two-point and commutator function.

Reconsider equation 3.2

$$< \hat{C}, \phi \otimes \varphi >= 2\pi < \hat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \hat{C}_m^{\phi} * \varphi(\mathbf{p}) >$$

taken from the proof of theorem 3.2. In light of the above arguments it is possible to divide ϕ by $(1 - e^{-\beta \underline{p}})$ to obtain

$$< \widehat{W}, \phi \otimes \varphi >= 2\pi < \hat{D}^{\mathbf{x},\mathbf{p}}, (\mathbf{p}, m) \mapsto \widehat{W}_{\beta,m}^{\phi} * \varphi(\mathbf{p}) > \ .$$

Noting that $\widehat{W}_{\beta,m}^{\phi}$ is a Schwartz function in $\mathbf{p}$, one can proceed as in the proof of theorem 3.2 from equation 3.2 onward to show

$$< W, \phi \otimes \varphi >=< D, (\mathbf{x}, m) \mapsto \varphi(\mathbf{x}) W_{\beta,m}^{\phi}(\mathbf{x}) > \ .$$

□

3.2. Properties of the Damping Factor

The KMS and the relativistic KMS condition imply certain regularity and analyticity properties of the reduced two-point function W. These properties and their implications for the damping factor D are studied in the first part of this section. In the second part an estimate giving an upper bound on how far the damping factor $D(\mathbf{x}, m)$ is from being a measure in m is stated.

3.2.1. Regularity and Analyticity in $\mathbf{x}$

Proposition 3.4. *Let ω_β be a β-KMS state on $\mathcal{A}$. Let $f = \underline{f} \otimes \mathbf{f}, g = \underline{g} \otimes \mathbf{g} \in \mathcal{S}(\mathbb{R}^2)$, where the underlined functions denote time components and the boldface functions spatial ones. Let $F_{f,g}$ be the function holomorphic on $\mathcal{S}_\beta$ for $a = \phi(f)$, $b = \phi(g)$ as defined in the KMS condition. The limit*

$$\lim_{\underline{f},\underline{g} \to \delta} F_{f,g}(\underline{z})$$

exists for $z_0 \in \mathcal{S}_\beta$ and there defines a holomorphic function denoted $F_{\mathbf{f},\mathbf{g}}$.

Proof. For $\underline{x} \in \mathbb{R}$ it is

$$\begin{aligned} F_{f,g}(\underline{x}) &= \omega(\phi(f)\phi(g_{\underline{x}})) \\ &= < W, \underline{f} * \breve{\underline{g}}_{\underline{x}} \otimes \mathbf{f} * \breve{\mathbf{g}} > \\ &= (2\pi)^2 < \widehat{W}, \check{\hat{\underline{f}}} \cdot \hat{\underline{g}}_{\underline{x}} \otimes \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}} > \\ &= (2\pi)^2 < \widehat{W}, p \mapsto \check{\hat{\underline{f}}} \cdot \hat{\underline{g}}(\underline{p}) e^{i\underline{xp}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) > \ . \end{aligned}$$

By the KMS condition this extends to $\underline{x} + i\underline{y}$ for $0 \leqslant \underline{y} \leqslant \beta$. In particular it is

$$F_{f,g}(\underline{x} + i\beta) = (2\pi)^2 < \widehat{W}, p \mapsto \check{\hat{\underline{f}}} \cdot \hat{\underline{g}}(\underline{p}) e^{i\underline{xp}} e^{-\beta\underline{p}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) > \ < \infty \,. \tag{3.3}$$

For general $0 < \underline{y} < \beta$ it is

$$F_{f,g}(\underline{x} + i\underline{y}) = (2\pi)^2 < \widehat{W}, p \mapsto \check{\hat{\underline{f}}} \cdot \hat{\underline{g}}(\underline{p}) e^{i\underline{xp}} e^{(\beta-\underline{y})\underline{p}} e^{-\beta\underline{p}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) > \ .$$

Note that this implies that the tempered distribution Q given by

$$h \mapsto < Q, h > := < \widehat{W}, p \mapsto h\underline{p}\, e^{\underline{xp}} e^{(\beta-\underline{y})} e^{-\beta\underline{p}} \cdot \hat{\mathbf{f}}\check{\hat{\mathbf{g}}}(\mathbf{p}) >$$

decays exponentially for large $|\underline{p}|$: for $\underline{p} \to -\infty$, the function $\underline{p} \mapsto e^{(\beta-\underline{y})\underline{p}}$ decays exponentially and $e^{-\beta\underline{p}}\widehat{W}$ is tempered by the KMS condition (cp. equation 3.3). Similarly, for $\underline{p} \to \infty$, the function $\underline{p} \mapsto e^{(\beta-\underline{y})\underline{p}} e^{-\beta\underline{p}} = e^{-\underline{yp}}$ decays exponentially and $\widehat{W}$ is tempered. One can find a nowhere vanishing Schwartz function $r \in \mathcal{S}(\mathbb{R})$, dominated by $\min\{e^{-\underline{yp}}, e^{(\beta-\underline{y})\underline{p}}\}$, such that $\frac{Q}{r}$ is still tempered. This allows to use theorem C.3 to prove that $\lim_{\underline{f},\underline{g}\to\delta} F_{f,g}(\underline{z})$ exists for $\underline{z} \in \mathcal{S}_\beta$ and is given by

$$\lim_{\underline{f},\underline{g}\to\delta} F_{f,g}(\underline{z}) = (2\pi)^2 < \widehat{W}, p \mapsto e^{i\underline{zp}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) > \ .$$

An application of Morera's theorem shows that $\underline{z} \mapsto \lim_{\underline{f},\underline{g}\to\delta} F_{f,g}(\underline{z}) =: F_{\mathbf{f},\mathbf{g}}(\underline{z})$ is holomorphic in $\mathcal{S}_\beta$. □

The domain of analyticity of $F_{\mathbf{f},\mathbf{g}}$ can be further extended in two ways:

- Suppose $\mathbf{f}, \mathbf{g}$ have compact support and $\underline{x} \in \mathbb{R}$ is such that $(\{\underline{x}\} \times \mathrm{supp}(\mathbf{g}) - \{0\} \times \mathrm{supp}(\mathbf{f})) \cap (\overline{V^+} \cup \overline{V^-}) = \varnothing$. As a consequence of locality it is

$$< \widehat{W}, p \mapsto e^{i\underline{xp}} e^{-\underline{yp}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) > = < \widehat{W}, p \mapsto e^{-i\underline{xp}} e^{\underline{yp}} \cdot \hat{\mathbf{f}} \cdot \check{\hat{\mathbf{g}}}(\mathbf{p}) > , \tag{3.4}$$

  which means the decay of $\widehat{W}$ must be sufficiently rapid to compensate the exponential growth of both $\underline{p} \mapsto e^{\underline{yp}}$ and $\underline{p} \mapsto e^{-\underline{yp}}$. The limit $\underline{y} \to 0$ exists. This allows to extend $F_{\mathbf{f},\mathbf{g}}$ to the domain $\{\underline{z} \in \mathbb{C} \mid -\beta < \underline{y} < \beta\} \backslash \{\underline{z} \in \mathbb{C} \,|\, \underline{y} = 0, (\{\underline{x}\} \times \mathrm{supp}(\mathbf{g}) - \{0\} \times \mathrm{supp}(\mathbf{f})) \cap (\overline{V^+} \cup \overline{V^-}) \neq \varnothing\}$.

- As a direct consequence of the KMS condition it is

$$\begin{aligned}
&\lim_{\underline{y}\to\beta} < \widehat{W}, p \mapsto e^{i\underline{xp}} e^{-\underline{yp}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) > \\
= \; &\lim_{\underline{y}\to 0} < \widehat{W}, p \mapsto e^{-i\underline{xp}} e^{\underline{yp}} \cdot \hat{\mathbf{f}} \cdot \check{\hat{\mathbf{g}}}(\mathbf{p}) > ,
\end{aligned}$$

  which again exists if $(\{\underline{x}\} \times \mathrm{supp}(\underline{f}) - \{0\} \times \mathrm{supp}(\underline{f})) \cap (\overline{V^+} \cup \overline{V^-}) = \varnothing$ and by equation 3.4 coincided with $\lim_{\underline{y}\to 0} < \widehat{W}, p \mapsto e^{i\underline{xp}} e^{-\underline{yp}} \cdot \check{\hat{\mathbf{f}}} \cdot \hat{\mathbf{g}}(\mathbf{p}) >$. This allows to extend $F_{\mathbf{f},\mathbf{g}}$ as an $i\beta$ periodic function to $\mathbb{C} \setminus \{\, \underline{z} \in \mathbb{C} \,|\, \underline{y} \in \beta\mathbb{Z},\ (\{\underline{x}\} \times \mathrm{supp}(\mathbf{g}) - \{0\} \times \mathrm{supp}(\mathbf{f})) \cap (\overline{V^+} \cup \overline{V^-}) \neq \varnothing \,\}$.

The relativistic KMS condition allows to further refine the above results.

Proposition 3.5. *Let ω_β be a relativistic β-KMS state on $\mathcal{A}$. Let $f = \underline{f} \otimes \mathbf{f}, g = \underline{g} \otimes \mathbf{g} \in \mathcal{S}(\mathbb{R}^2)$, where the underlined functions denote time components and boldface functions spatial ones. Let $F_{f,g}$ be the function holomorphic on $\mathcal{R}_{\beta e}$ for $a = \phi(f)$, $b = \phi(g)$ as defined in the relativistic KMS condition. The limit*

$$\lim_{f,g\to\delta} F_{f,g}(z)$$

exists for $z \in \mathcal{R}_{\beta e}$ and there defines a holomorphic function denoted F.

The proof can be carried out using arguments analogous to those in the (non-relativistic) KMS case, which here also apply to the spatial variable. The function F is given by

$$F(z) = (2\pi)^2 < \widehat{W}, p \mapsto e^{izp} > ,$$

where $e^{izp} = e^{i\eta(p,z)} = e^{i\eta(p,x)} e^{-\eta(y,p)}$ for $z = x + iy$. Note that the above equation means that F is an analytic continuation of W. In particular if F can be extended analytically to a region in $\mathbb{R}^2$, then W is regular in that region.
As before, the domain of analyticity of F can be extended using locality and the KMS condition:

- First extend F as an $i\beta e$ periodic function to $\bigcup_{k\in\mathbb{Z}} \mathcal{R}_{\beta e} + ki\beta e$.

- If x is spacelike, then locality implies that

$$< \widehat{W}, e^{ipx} e^{-yp} > = < \widehat{W}, e^{-ipx} e^{yp} > \; .$$

  As before, this allows to perform the limit $V^+ \ni y \to 0$ and extend F continuously to the edges $\{\, x + ik\beta e \,|\, k \in \mathbb{Z},\, x \,\text{spacelike} \,\}$ connecting the $\mathbb{Z}$ indexed union of translated tubes $\mathcal{R}_{\beta e}$. The continuity of this extension is due to

$$\lim_{\substack{y\to\beta e \\ y\in\beta e + V^-}} < \widehat{W}, e^{ipx} e^{-yp} > = \lim_{\substack{y\to 0 \\ y\in V^+}} < \widehat{W}, e^{-ipx} e^{yp} > ,$$

  which follows directly from the relativistic KMS condition.

- By the edge-of-the-wedge theorem (cf. [1]) there exist open neighborhoods $\mathcal{N}_k \subset \mathbb{C}^2$ of the edges $\{x + ik\beta e \,|\, x \,\text{spacelike}\}$, such that F is holomorphic on the domain $\bigcup_{k\in\mathbb{Z}}(\mathcal{R}_{\beta e} + ki\beta e) \cup \mathcal{N}_k$.

It can be shown that these analyticity properties are sufficient for the distribution W to be regular in the spatial component, i.e. for each $\underline{f} \in \mathcal{S}(\mathbb{R})$ the tempered distribution $\mathbf{f} \mapsto < W^{\underline{f}}, \mathbf{f} > := < W, \underline{f} \otimes \mathbf{f} >$ is given by a polynomially bounded smooth function also denoted $W^{\underline{f}}$. From this it can also be shown, that the damping factor D admits an analytic continuation and hence is regular in the spatial variable (see [7] and [8]).

3.2.2. Mass Dependence of D

In light of the particle criterion stated in section 3.3 it would be satisfying from an interpretational point of view, if D were a measure m, i.e. if D extended continuously from even Schwartz functions to compactly supported continuous functions. The problem in approaching the question whether D is a measure, is that one has little information on the commutator function $\widehat{C}$, except for its support properties and the fact that, by the Bochner-Schwartz theorem, $\widehat{C}$ is the difference of the polynomially bounded positive measure $\widehat{W}$ and $\widecheck{\widehat{W}}$. It is assumed that the underlying state ω satisfies the relativistic KMS condition, so that W, C and D are analytic in $\mathbf{x}$. In light of the latter assumption one has by definition 3.1

$$
\begin{aligned}
& < D, \delta_{\mathbf{y}} \otimes \psi > \\
= \;& i < C, x \mapsto \underline{x}\delta_{\mathbf{y}}(\mathbf{x}) \int_0^\infty \mathrm{d}m\, m\psi(m) J_0(\sqrt{\underline{x}^2 - \mathbf{x}^2}m) > \\
= \;& \frac{i}{(2\pi)^2} < \widehat{C}, p \mapsto e^{i\mathbf{p}\mathbf{y}} \int_{\mathbb{R}\setminus[-|\mathbf{y}|,|\mathbf{y}|]} \mathrm{d}\underline{x}\, e^{-i\underline{x}\underline{p}} \underline{x} \int_0^\infty \mathrm{d}m\, m\psi(m) J_0(\sqrt{\underline{x}^2 - \mathbf{y}^2}m) > \;.
\end{aligned}
$$

The factor $\underline{x}$ disallows performing the Fourier transform inside the m-integral in a reasonable way. However, using the relation $J_k'(x) = J_{k-1}(x) - \frac{k}{x}J_k(x)$, which can be derived from the integral representation of J_k in 2.1.8, one obtains the following identity by integration by parts:

$$
\int_0^\infty \mathrm{d}m\, m J_0(am)\psi(m) = -\frac{1}{a}\int_0^\infty \mathrm{d}m\, m J_1(am)\psi'(m)\,.
$$

Insertion into the above expression for $< D, \delta_{\mathbf{y}} \otimes \psi >$ yields

$$
\frac{-i}{(2\pi)^2} < \widehat{C}, p \mapsto e^{i\mathbf{p}\mathbf{y}} \int_{\mathbb{R}\setminus[-|\mathbf{y}|,|\mathbf{y}|]} \mathrm{d}\underline{x}\, e^{-i\underline{x}\underline{p}} \frac{\underline{x}}{\sqrt{\underline{x}^2 - \mathbf{y}^2}} J_1(\sqrt{\underline{x}^2 - \mathbf{y}^2}m) \int_0^\infty \mathrm{d}m\, m\psi'(m) > \;.
$$

To estimate this in terms of ψ, additional information on $\widehat{C}$ is needed. It is assumed here that, as in the free case, $\frac{|\widehat{C}|}{|\underline{p}|^\delta}$ is finite for $0 < \delta \leqslant 1$. Unfortunately the expression

$$
\int_{\mathbb{R}\setminus[-|\mathbf{y}|,|\mathbf{y}|]} \mathrm{d}\underline{x}\, e^{-i\underline{x}\underline{p}} \frac{\underline{x}}{\sqrt{\underline{x}^2 - \mathbf{y}^2}} J_1(\sqrt{\underline{x}^2 - \mathbf{y}^2}m)
$$

is difficult to handle and its behavior in $\underline{p}$ and m is not obvious. However, as $\sqrt{\underline{x}^2 - \mathbf{y}^2}$ asymptotically behaves like $|\underline{x}|$ and the integrand remains bounded for $|\underline{x}|$ close to $|\mathbf{y}|$, one may hope that the $\underline{p}$ and m dependence is similar to that in the computationally more convenient case of $\mathbf{y} = 0$. Here the above integral can be evaluated in a distributional sense (cf. [15]):

$$-2i(m^2 - \underline{p}^2)^{-\frac{1}{2}} \cdot \frac{\underline{p}}{m}\theta(|m| - |\underline{p}|)\,.$$

Substituting $m \to m|\underline{p}|$ in the m integral yields

$$\begin{aligned} &< D, \delta_0 \otimes \psi > \\ = \quad & \frac{-2}{(2\pi)^2} < \widehat{C}, \varepsilon(\underline{p}) \int_1^\infty \mathrm{d}m\, |\underline{p}|\psi'(m|\underline{p}|)\,(m^2-1)^{-\frac{1}{2}} > \\ = \quad & \frac{-2}{(2\pi)^2} < \frac{\widehat{C}}{|\underline{p}|^\delta}, \varepsilon(\underline{p}) \int_1^\infty \mathrm{d}m\, \frac{|\underline{p}|^{1+\delta}}{(|\underline{p}|m)^{1+\delta}}\,(m^2-1)^{-\frac{1}{2}} > \cdot (|\underline{p}|m)^{1+\delta}\psi'(m|\underline{p}|)\,. \end{aligned}$$

In this form one can make use of the assumption that $\frac{|\widehat{C}|}{|\underline{p}|^\delta}$ is a finite, positive measure to estimate

$$\begin{aligned} &| < D, \delta_0 \otimes \psi > | \\ \leqslant \quad & \frac{2}{(2\pi)^2} \left| < \frac{|\widehat{C}|}{|\underline{p}|^\delta}, 1 > \right| \cdot \int_1^\infty \mathrm{d}m\, \frac{1}{m^{1+\delta}}(m^2-1)^{-\frac{1}{2}} \cdot \sup_{x \in \mathbb{R}_+} \left| x^{1+\delta}\psi'(x) \right| \\ =: \quad & A_{C,\delta} \cdot \sup_{x \in \mathbb{R}_+} \left| x^{1+\delta}\psi'(x) \right|\,. \end{aligned}$$

This estimate is a bit disappointing, as it involves the first derivative of ψ. An explicit computation of D in the free mass M case reveals that $D = \delta(m - M)$, as one should expect from the representation formula given in theorem 3.2. Here one can of course estimate $| < D, \delta_{\mathbf{y}} \otimes \psi > | \leqslant \sup_{x \in \mathbb{R}_+} |\psi(x)|$, but in the explicit computation, one has to exploit the specific structure of the free commutator function.

3.3. Particle Interpretation

In vacuum theories, Wigner's criterion provides a concept of particles [2]. According to this criterion, a vacuum theory describes particles if the unitary representation of the proper orthochronous Poincaré group contains an irreducible one with discrete weight. This definition is in part motivated by the fact that the joint spectrum of the self-adjoint generators of the space-time translations in an irreducible representation is confined to a hyperboloid $\{\, p \in \mathbb{R}^2 \,|\, p^2 = m^2 \,\}$, where the parameter m is to be interpreted as a rest mass. Irreducible parts of the representation of space-time symmetries carrying discrete weight manifest themselves as contributions of the form $\theta(\underline{p})\delta(p^2 - m^2)$ to the Fourier transformed, reduced two-point function $\widehat{W}$.
An analogous criterion in theories induced by homogeneous thermal states is not sensible. This due to a theorem by H. Narnhofer, M. Requardt and W. Thirring which states

that entities corresponding to singular contributions to $\widehat{W}$ do not scatter [5]. As a consequence, such entities described by sharp dispersion laws do not couple to a thermal background and only provide a suitable notion of massive particles in the case of trivial interaction. However, another criterion has been proposed by J. Bros and D. Buchholz in [7] and [8].

Criterion 3.2. *A theory induced by a homogeneous KMS state invariant under spatial reflections and satisfying the time-clustering property is said to describe particles of mass M if the damping factor $D(x, m)$ contains a discrete contribution of the form $\delta(m - M)D_{\mathrm{d}}(\mathbf{x})$.*

The Bros-Buchholz criterion is motivated by several heuristic considerations.

- In the free case, the two-point function $W_{\beta,m}$ can be interpreted in terms of propagation of particles/holes. The positive/negative frequency parts of $W_{\beta,m}(x)$ correspond to the probability amplitude of particles/holes created the field ϕ acting at 0 to be found at x ($W_{\beta,m}$ coincides with the time-ordered propagator T outside of$\overline{V^-}$, see chapter 6). In the presence of interaction, particles/holes of mass m can collide with other constituents of the thermal background resulting in excitations which are expected to primarily contribute to higher correlation functions. Hence the spatial part of the interacting two-point function W_β is subject to additional abating effects, which result in a decrease of propagation amplitude. This can be modeled by the relation
$$W_\beta(x) = D(\mathbf{x})W_{\beta,m}(x)\,,$$
where the damping factor D accounts for the additional propagation hindering effects.

- The Bros-Buchholz representation theorem states that W_β can be represented in the form
$$W_\beta(x) = \int_0^\infty \mathrm{d}m\, D(\mathbf{x}, m)W_{\beta,m}(x)\,.$$
The manifest interpretation of $D(\mathbf{x}, m)$ is that of the above damping factor for possible particles of mass m. If $D(\mathbf{x}, m)$ vanishes for a certain M then the corresponding theory would be void of particles of mass M. However not all M for which $D(\mathbf{x}, m)$ does not vanish should correspond to masses of particles present in that theory. Suppose $D(\mathbf{x}, m)$ is continuous in a mass interval, then for each M in that interval $D(\mathbf{x}, M)W_{\beta,M}(x)$ only contributes infinitesimally to $W_\beta(x)$. Omitting countably many of such contributions leaves $W_\beta(x)$ unchanged, which is not an expected characteristic of particle contributions. Discrete parts $\delta(m - M)D_{\mathrm{d}}(\mathbf{x})$ of $D(x, m)$ contribute non-negligibly to W_β and behave as outlined in the previous point. It appears natural to identify particles of mass M with such contributions.

- As shown in chapter 4, discrete contributions to $D(x, m)$ of the form $\delta(m - M)D_{\mathrm{d}}(\mathbf{x})$ give rise to leading order contributions in $W_\beta(\underline{x}, \mathbf{x})$ for asymptotic $|\underline{x}|$. This further corroborates the interpretation of contributions $W_\beta(\underline{x}, \mathbf{x})$ being due to particles, as they correspond to the most stable possible excitations of the thermal background.

Adopting the Bros-Buchholz criterion, determining the particle content of a thermal theory is a matter of computing the damping factor by means of formula 3.1 and identifying the singular contributions.

4. Asymptotic Fields

In this chapter, it is shown in analogy to [9] that, suppressing low energy contributions, the leading part in thermal two-point functions for asymptotic times are due to discrete contributions to the damping factor. Under the assumption that no collective memory effects manifest themselves in higher correlation functions, the asymptotically dominant contributions to the latter are shown to exhibit the structure of a quasifree state, which can be realized as an actual KMS state on an **algebra of asymptotic fields**.

4.1. Analysis of Asymptotic Thermal Correlation Functions

Definition 4.1. For $\underline{g} \in \mathcal{S}(\mathbb{R})$ define the **time-regularized fields**

$$\phi_{\underline{g}}(f) := \phi(g *_{\underline{x}} f)\,,$$

where $*_{\underline{x}}$ denotes convolution in the time variable. In a more heuristic notation, this reads

$$\phi_{\underline{g}}(x) = \int \mathrm{d}\underline{y} g(\underline{y} - \underline{x})\, \phi(\underline{y}, \mathbf{x})\,.$$

It is noteworthy that if ω is a regular homogeneous relativistic β-KMS state, then the limit

$$\lim_{\substack{f_1 \to \delta_{x_1} \\ f_2 \to \delta_{x_2}}} \omega(\phi_{\underline{g}_1}(f_1)\phi_{\underline{g}_2}(f_2))$$

exists. This can be seen as follows: it is

$$\omega(\phi_{\underline{g}_1}(f_1)\phi_{\underline{g}_2}(f_2)) =< W, \underline{g}_1 *_{\underline{x}} f_1 * \widecheck{\underline{g}_2 *_{\underline{x}} f_2} >=< (\check{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W, \check{f}_1 * f_2 > \ .$$

Since the relativistic KMS condition implies regularity of W in the spatial variable, the time-regularized distribution $(\check{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W$ is regular. The corresponding polynomially bounded, smooth function will also be denoted $(\check{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W$.

The time-regularization of the fields allows to to formalize the suppression of low energy contributions to the two-point function. Suppose the Schwartz functions $\underline{g}_1, \underline{g}_2$ have Fourier transforms which vanish at the origin. Then $\mathcal{F}((\check{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W) = \check{\hat{\underline{g}}}_1 \hat{\underline{g}}_2 \cdot \widehat{W}$. As $\widehat{W}$ is a measure, multiplication by $\underline{p} \mapsto \check{\hat{\underline{g}}}_1 \hat{\underline{g}}_2(\underline{p})$ suppresses low energy (small $|\underline{p}|$) contributions to $\widehat{W}$.

Lemma 4.1. *Let ω be a regular homogeneous β-KMS state invariant under spatial reflection and W the corresponding reduced two-point function. It then is*

$$< (\breve{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W, \underline{f} \otimes \mathbf{f} >=< D, (\mathbf{x}, m) \mapsto \mathbf{f}(\mathbf{x}) \int \mathrm{d}\underline{x}\, (\breve{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W_{\beta,m}(\underline{x}, \mathbf{x}) \underline{f}(\underline{x}) > .$$

Proof. The statement is a straightforward consequence of corollary 3.3 using $(\breve{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W^{\underline{f}} = W^{(\underline{g}_1 * \breve{\underline{g}}_2) * \underline{f}}$. □

Theorem 4.2. *Let $\mu \in \mathbb{N}_0$ and $\underline{h} \in \mathcal{S}(\mathbb{R})$ such that $\widehat{h}$ has a double zero at the origin. Then there exist*

- $K_\pm : \mathbb{R} \to \mathbb{C}$ *rapidly decreasing, i.e.* $\forall\, N \in \mathbb{N}_0 : \sup_{m \in \mathbb{R}} |K_\pm(m)(1+m)^N| < \infty$,
- $r : [0,\infty[\times([0,\infty[\times\mathbb{R}) \to \mathbb{C}$ *such that* $\forall\, K \subset \mathbb{R}$ *compact,* $N \in \mathbb{N}_0\, \exists\, C_{N,K}\, \forall\, m \in [0,\infty[,\ \underline{x} \in \mathbb{R} : \sup_{\mathbf{x} \subset K} |r(m,x)| < C_{N,K} |\underline{x}|^{-\frac{\mu+1}{2}-\delta}(1+m)^{-N}$, *where* $0 < \delta < \frac{1}{2}$,

with the property that for $m \in [0,\infty[, \underline{x} \in [0,\infty[, \mathbf{x} \in \mathbb{R}$ *it is*

$$\underline{h} *_{\underline{x}} \partial^\mu_{\mathbf{x}} W_{\beta,m}(x) = \underline{x}^{-\frac{\mu+1}{2}} \sum_{\sigma=\pm} e^{-\sigma i m \underline{x}} K_\sigma(m) + r(m,x) .$$

An analogous result holds for $\underline{x} \in]-\infty, 0]$.

Proof. Cf. Appendix C. □

The statement of the theorem is that for asymptotic times and suppressing low energy contributions, $W_{\beta,m}(x)$ behaves like $|\underline{x}|^{-\frac{1}{2}}$ for $\mathbf{x}$ varying in compact sets. With the help of the Bros-Buchholz representation, it can be shown that a general thermal W behaves similarly and that the leading order asymptotic contributions are due to discrete parts of the damping factor D.

For the remainder of this chapter, assume that ω is a homogeneous time-clustering relativistic β-KMS state and that the damping factor is of the form

$$D(\mathbf{x}, m) = \delta(m - M) D_{\mathrm{d}}(\mathbf{x}) + D_{\mathrm{ac}}(\mathbf{x}, m) \tag{4.1}$$

for fixed $M > 0$. Both the discrete part D_{d} of D and D_{ac} are analytic in $\mathbf{x}$ as a consequence of the relativistic KMS condition and D_{ac} is assumed to be Lebesgue absolutely continuous. As a consequence of D being tempered, D_{ac} and D_{d} are polynomially bounded in $\mathbf{x}$ and D_{ac} is polynomially bounded in m.

Define (in the sense of distributions in $\underline{x}$):

$$W_{\mathrm{d}}(x) := D_{\mathrm{d}}(\mathbf{x}) W_{M,\beta}(x) .$$

W_{d} is the part of the reduced two-point function corresponding to the discrete contribution $\delta(m - M) D_{\mathrm{d}}$ to the damping factor.

Claim 4.3. *For $K \subset \mathbb{R}$ compact and $\underline{g}_1, \underline{g}_2 \in \mathcal{S}(\mathbb{R})$, $\hat{\underline{g}}_1(0) = \hat{\underline{g}}_1(0) = 0$ it is*

$$\lim_{\underline{x}\to\infty} |\underline{x}|^{\frac{1}{2}} |(\check{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W(x) - (\check{\underline{g}}_1 * \underline{g}_2) *_{\underline{x}} W_{\mathrm{d}}(x)| = 0$$

uniformly for $\mathbf{x} \in K$.

Proof. Define $\underline{h} := \check{\underline{g}}_1 * \underline{g}_2$. Using the Bros-Buchholz representation, it is in light of 4.1 and 4.2

$$\begin{aligned}
& \lim_{\underline{x}\to\infty} |\underline{x}|^{\frac{1}{2}} |\underline{h} *_{\underline{x}} W(x) - \underline{h} *_{\underline{x}} W_{\mathrm{d}}(x)| \\
= \; & \lim_{\underline{x}\to\infty} |\underline{x}|^{\frac{1}{2}} \left| \int_0^\infty \mathrm{d}m \, D_{\mathrm{ac}}(\mathbf{x}, m) \underline{h} *_{\underline{x}} W_{\beta,m}(x) \right| \\
= \; & \lim_{\underline{x}\to\infty} |\underline{x}|^{\frac{1}{2}} \left| \int_0^\infty \mathrm{d}m \, D_{\mathrm{ac}}(\mathbf{x}, m) (\underline{x}^{-\frac{1}{2}} \sum_{\sigma=\pm} e^{-\sigma i m \underline{x}} K_\sigma(m) + r(m,x)) \right| \\
\leqslant \; & \sum_{\sigma=\pm} \lim_{\underline{x}\to\infty} \left| \int_0^\infty \mathrm{d}m \, D_{\mathrm{ac}}(\mathbf{x}, m) e^{-\sigma i m \underline{x}} K_\sigma(m) \right| \\
& + \lim_{\underline{x}\to\infty} |\underline{x}|^{\frac{1}{2}} \left| \int_0^\infty \mathrm{d}m \, D_{\mathrm{ac}}(\mathbf{x}, m) r(m,x) \right| .
\end{aligned}$$

By the Riemann-Lebesgue lemma, the first term vanishes uniformly for $\mathbf{x} \in K$, as $\sup_{\mathbf{x}\in K} |D_{\mathrm{ac}}(\mathbf{x}, m)| k_\sigma(m)$ is integrable. To see that the second term vanishes uniformly for $\mathbf{x} \in K$, note that D_{ac} is polynomially bounded, so there exists $N \in \mathbb{N}_0$ such that $\sup_{\mathbf{x}\in K} |D_{\mathrm{ac}}(\mathbf{x}, m)|(1+m)^{-N}$ is integrable. But for this N there exists $C_{N,K}$ such that $\sup_{\mathbf{x}\subset K} |r(m,x)| < C_{N,K} |\underline{x}|^{-\frac{1}{2}-\delta}(1+m)^{-N}$, so that one can estimate

$$\begin{aligned}
& \sup_{\mathbf{x}\in K} |\underline{x}|^{\frac{1}{2}} \left| \int_0^\infty \mathrm{d}m \, D_{\mathrm{ac}}(\mathbf{x}, m) r(m,x) \right| \\
\leqslant \; & |\underline{x}|^{-\delta} \int_0^\infty \mathrm{d}m \sup_{\mathbf{x}\in K} |D_{\mathrm{ac}}(\mathbf{x}, m)|(1+m)^{-N} ,
\end{aligned}$$

which vanishes as $\underline{x} \to \infty$. □

This shows that $\underline{h} *_{\underline{x}} W(x)$ behaves like $\underline{h} *_{\underline{x}} W_{\mathrm{d}}(x)$ for asymptotic $\underline{x}$, i.e. suppressing low energy contributions, the asymptotically leading part of the reduced two-point function corresponds to discrete contributions in the damping factor. As such discrete contributions are associated to particle content of the theory, it is natural to interpret the asymptotically dominant part of $W(x)$ to be due to the exchange of particles between two points in space-time separated by x.

As a general representation of higher correlation functions in thermal states has not been developed, no analogous statement for the asymptotic behavior of general m-point functions can be derived. The further analysis relies on the assumption, that the asymptotically dominant contributions to higher m-point functions with suppressed low energy

contributions again stem from the exchange of constituent particles between space-time positions at which field measurements are performed and do not encode collective memory effects decaying only slowly in time. In mathematical terms this is formulated as follows:
For $m \in \mathbb{N}$ let $\vartheta := \min\{|\underline{x}_j - \underline{x}_k| \,|\, 1 \leqslant j,j \leqslant m,\, j \neq k\}$ denote the minimal time separation between points $x_1, \cdots, x_m$ in space-time. For the (regular) m-point functions of fields regularized in time by Schwartz functions whose Fourier transform vanish at 0, it is assumed that

$$\lim_{\vartheta \to \infty} \vartheta^{\frac{m-1}{2} - \delta} \left(\bigotimes_{j=1}^{m} \underline{\breve{g}}_j \right) *_{\underline{x}} \mathcal{W}_m^T(x_1, \cdots, x_m) = 0$$

for all $\delta > 0$. Here $*_{\underline{x}}$ denotes convolution in all time variables and T truncation. This mathematical formulation of the physical situation described above may require some explanation. First note that for $f_1, \cdots, f_m \in \mathcal{S}(\mathbb{R}^2)$ it is $\omega(\prod_{j=1}^m \phi_{g_j}(f_j)) =< \mathcal{W}_m, \bigotimes_{j=1}^m \underline{g}_j *_{\underline{x}} f_j >=< \bigotimes_{j=1}^m \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_m, \bigotimes_{j=1}^m f_j >$, so the expression $\left(\bigotimes_{j=1}^m \underline{\breve{g}}_j\right) *_{\underline{x}} \mathcal{W}_m^T(x_1, \cdots, x_m)$ describes the correlated part of the time-regularized m-point function. The above assumption means that this quantity decays at least like $\vartheta^{\frac{m-1}{2}}$ for large time separation of the $x_1, \cdots, x_m$, which is the rate expected from correlations due to particle exchange. This assumption has the following implications:

Lemma 4.4. *Assuming vanishing one-point functions it is (using the same notation as in section D.4 and definition D.10)*

$$\lim_{\vartheta \to \infty} \vartheta^{\frac{m}{4}} \left| \bigotimes_{j=1}^{m} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T} \in \mathcal{P}_2} \prod_{S \in \mathfrak{T}} \Big(\bigotimes_{\substack{j \in S \\ \text{ordered}}} \underline{\breve{g}}_j \Big) *_{\underline{x}} \mathcal{W}_{|S|}(\hat{x}_1^S,\cdots,\hat{x}_m^S) \right| = 0\,,$$

where $\hat{x}_j^S$ denotes omission if $j \notin S$.

Proof. By the assumption of vanishing one-point functions it is

$$\underline{\breve{g}} *_{\underline{x}} \mathcal{W}_1^T(x) = \underline{\breve{g}} *_{\underline{x}} \mathcal{W}_1(x) = 0$$

and

$$\begin{aligned} & \underline{\breve{g}}_1 \otimes \underline{\breve{g}}_2 *_{\underline{x}} \mathcal{W}_2^T(x_1, x_2) \\ = \; & \underline{\breve{g}}_1 \otimes \underline{\breve{g}}_2 *_{\underline{x}} \mathcal{W}_2(x_1, x_2) - \underline{\breve{g}}_1 *_{\underline{x}} \mathcal{W}_1(x_1) \cdot \underline{\breve{g}}_2 *_{\underline{x}} \mathcal{W}_1(x_2) \\ = \; & \underline{\breve{g}}_1 \otimes \underline{\breve{g}}_2 *_{\underline{x}} \mathcal{W}_2(x_1, x_2)\,. \end{aligned}$$

Let $\mathcal{P}_1$ denote the set of partitions of $M_m = \{1, \cdots, m\}$ containing a set with a single element, i.e. $\mathfrak{T} \in \mathcal{P}_1 \Leftrightarrow \exists\, S \in \mathfrak{T} \,:\, |S| = 1$. It is

$$
\begin{aligned}
& \bigotimes_{j=1}^{m} \breve{g}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) \\
& - \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{S\in\mathfrak{T}} \bigotimes_{\substack{j\in S\\ \text{ordered}}} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_{|S|}(\widehat{x}_1^S,\cdots,\widehat{x}_m^S) \\
= & \bigotimes_{j=1}^{m} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) \\
& - \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{S\in\mathfrak{T}} \bigotimes_{\substack{j\in S\\ \text{ordered}}} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}^T_{|S|}(\widehat{x}_1^S,\cdots,\widehat{x}_m^S) \\
& - \sum_{\mathfrak{T}\in\mathcal{P}_1} \prod_{S\in\mathfrak{T}} \bigotimes_{\substack{j\in S\\ \text{ordered}}} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}^T_{|S|}(\widehat{x}_1^S,\cdots,\widehat{x}_m^S) \\
= & \sum_{\mathfrak{T}\in\mathcal{P}\setminus(\mathcal{P}_1\cup\mathcal{P}_2)} \prod_{S\in\mathfrak{T}} \bigotimes_{\substack{j\in S\\ \text{ordered}}} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}^T_{|S|}(\widehat{x}_1^S,\cdots,\widehat{x}_m^S)\,.
\end{aligned}
$$

As $\mathfrak{T} \in \mathcal{P}\setminus(\mathcal{P}_1 \cup \mathcal{P}_2)$ contains no sets containing just one element and is not entirely composed of sets containing two elements, $\mathfrak{T}$ contains at most $\frac{m-1}{2}$ elements. Let each $S_i \in \mathfrak{T}$ contain m_i elements, $1 \leqslant i \leqslant |\mathfrak{T}|$. Since $\mathfrak{T}$ is a partition, it is $\sum_{i=1}^{l} m_i = m$. This leads to the estimate

$$
\frac{m}{4} < \frac{m+1}{4} = \frac{1}{2}\left(m - \frac{m-1}{2}\right) \leqslant \frac{1}{2}(m - |\mathfrak{T}|) = \sum_{i=1}^{|\mathfrak{T}|} \frac{m_i - 1}{2}\,.
$$

The sharpness of this inequality implies that there exists $\delta_i > 0$, $1 \leqslant i \leqslant m$ such that

$$
\frac{m}{4} < \sum_{i=1}^{|\mathfrak{T}|} \left(\frac{m_i - 1}{2} - \delta_i\right)\,.
$$

It follows that for $\vartheta > 1$ one has

$$
\vartheta^{\frac{m}{4}} < \vartheta^{\sum_{i=1}^{|\mathfrak{T}|}\left(\frac{m_i-1}{2}-\delta_i\right)} = \prod_{i=1}^{|\mathfrak{T}|} \vartheta^{\frac{m_i-1}{2}-\delta_i}\,,
$$

which finally implies

$$
\begin{aligned}
& \lim_{\vartheta\to\infty} \vartheta^{\frac{m}{4}} \Bigg| \bigotimes_{j=1}^{m} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) \\
& \qquad - \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{S\in\mathfrak{T}} \bigotimes_{\substack{j\in S\\ \text{ordered}}} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_{|S|}(\widehat{x}_1^S,\cdots,\widehat{x}_m^S) \Bigg| \\
= & \lim_{\vartheta\to\infty} \vartheta^{\frac{m}{4}} \Bigg| \sum_{\mathfrak{T}\in\mathcal{P}\setminus(\mathcal{P}_1\cup\mathcal{P}_2)} \prod_{S\in\mathfrak{T}} \bigotimes_{\substack{j\in S\\ \text{ordered}}} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}^T_{|S|}(\widehat{x}_1^S,\cdots,\widehat{x}_m^S) \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sum_{\mathfrak{T}\in\mathcal{P}\setminus(\mathcal{P}_1\cup\mathcal{P}_2)} \lim_{\vartheta\to\infty}\left|\vartheta^{\frac{m}{4}}\prod_{i=1}^{|\mathfrak{T}|}\bigotimes_{\substack{j\in S_i\\\text{ordered}}}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}^T_{|S_i|}(\widehat{x}_1^{S_i},\cdots,\widehat{x}_m^{S_i})\right| \\
&\leqslant \sum_{\mathfrak{T}\in\mathcal{P}\setminus(\mathcal{P}_1\cup\mathcal{P}_2)} \left|\prod_{i=1}^{|\mathfrak{T}|}\underbrace{\lim_{\vartheta\to\infty}\vartheta^{\frac{m_i-1}{2}-\delta_i}\bigotimes_{\substack{j\in S_i\\\text{ordered}}}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}^T_{|S_i|}(\widehat{x}_1^{S_i},\cdots,\widehat{x}_m^{S_i})}_{=0}\right| \\
&= 0\,.
\end{aligned}
$$

□

The statement of this lemma can be rewritten as

$$
\lim_{\vartheta\to\infty}\vartheta^{\frac{m}{4}}\left|\bigotimes_{j=1}^{m}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2}\prod_{\substack{i=1\\S_i=(j_i,k_i)}}^{|\mathfrak{T}|}(\breve{\underline{g}}_{j_i} * \underline{g}_{k_i}) *_{\underline{x}} W(x_{j_i}-x_{k_i})\right| = 0\,.
$$

Combined with the result that for asymptotic times the $(\breve{\underline{g}}_{j_i} * \underline{g}_{k_i}) *_{\underline{x}} W$ behaves like $(\breve{\underline{g}}_{j_i} * \underline{g}_{k_i}) *_{\underline{x}} W_{\mathrm{d}}$, this can be used to show that ω is asymptotically quasifree in the following sense:

Theorem 4.5.

$$
\lim_{\vartheta\to\infty}\vartheta^{\frac{m}{4}}\left|\bigotimes_{j=1}^{m}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2}\prod_{\substack{i=1\\S_i=(j_i,k_i)}}^{|\mathfrak{T}|}(\breve{\underline{g}}_{j_i} * \underline{g}_{k_i}) *_{\underline{x}} W_{\mathrm{d}}(x_{j_i}-x_{k_i})\right| = 0\,.
$$

Proof. For odd m this can be seen from lemma 4.4 and noting that $\mathcal{P}_2$ is empty for sets containing an odd number of elements. For even m and $\mathfrak{T}\in\mathcal{P}_2$ it is $|\mathfrak{T}| = \frac{m}{2}$. In this case define $J := \{1,\cdots,\frac{m}{2}\}$, $\underline{h}_{jk} := \breve{\underline{g}}_j * \underline{g}_k$ and compute

$$
\begin{aligned}
&\lim_{\vartheta\to\infty}\vartheta^{\frac{m}{4}}\left|\bigotimes_{j=1}^{m}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2}\prod_{\substack{i=1\\S_i=(j_i,k_i)}}^{|\mathfrak{T}|}\underline{h}_{j_ik_i} *_{\underline{x}} W_{\mathrm{d}}(x_{j_i}-x_{k_i})\right| \\
=\;&\lim_{\vartheta\to\infty}\vartheta^{\frac{m}{4}}\Bigg|\bigotimes_{j=1}^{m}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2}\prod_{\substack{i=1\\S_i=(j_i,k_i)}}^{|\mathfrak{T}|} \\
&\Big[\underbrace{\underline{h}_{j_ik_i} *_{\underline{x}} W(x_{j_i}-x_{k_i})}_{:=a_i} + (\underbrace{\underline{h}_{j_ik_i} *_{\underline{x}} W_{\mathrm{d}}(x_{j_i}-x_{k_i}) - \underline{h}_{j_ik_i} *_{\underline{x}} W(x_{j_i}-x_{k_i})}_{:=b_i})\Big]\Bigg| \\
=\;&\lim_{\vartheta\to\infty}\vartheta^{\frac{m}{4}}\left|\bigotimes_{j=1}^{m}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2}\sum_{I\subset J}\left[\prod_{i\in I}a_i\cdot\prod_{i\in J\setminus I}b_i\right]\right| \\
=\;&\lim_{\vartheta\to\infty}\vartheta^{\frac{m}{4}}\left|\bigotimes_{j=1}^{m}\breve{\underline{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2}\left(\prod_{i\in J}a_i + \sum_{I\subsetneq J}\left[\prod_{i\in I}a_i\cdot\prod_{i\in J\setminus I}b_i\right]\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \underbrace{\lim_{\vartheta\to\infty} \vartheta^{\frac{m}{4}} \left| \bigotimes_{j=1}^{m} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{i\in J} a_i \right|}_{=0} \\
&\quad + \sum_{\mathfrak{T}\subset\mathcal{P}_2} \sum_{I\subsetneq J} \left(\prod_{i\in I} \underbrace{\sup_{\vartheta} \vartheta^{\frac{1}{2}} |a_i|}_{<\infty} \right) \left(\prod_{i\in J\setminus I} \underbrace{\lim_{\vartheta\to\infty} \vartheta^{\frac{1}{2}} |b_i|}_{=0} \right) \\
&= 0 \,.
\end{aligned}
$$

The first term vanishes by lemma 4.4, $\vartheta^{\frac{1}{2}}|a_i|$ is bounded as a consequence of theorem 4.2 and $\lim_{\vartheta\to\infty}\vartheta^{\frac{1}{2}}|b_i| = 0$ by virtue of claim 4.3. □

4.2. Algebra of Asymptotic Fields

It has been seen in the preceding section that m-point functions of thermal states with suppressed low energy contributions behave, for large time separation of the measurements, like sums of products of those contributions to two-point functions stemming from discrete parts of the damping factor. This sections aims to capture that asymptotic structure algebraically by introducing an algebra of asymptotic fields, suitable to model asymptotic dynamics for effective models.

The definition of the algebra of asymptotic fields takes some preparation. Let $\mathcal{A}_M$ be the free mass M field algebra (see definition D.14). In order to lighten the notation, an abstract formulation of $\mathcal{A}_M$ is given. This formulation is equivalent to the represented field algebra $\pi(\mathcal{A})$, where π is the GNS representation corresponding to the free mass m vacuum state on $\mathcal{A}$.

Definition 4.2 (Abstract mass M free field algebra)**.** The **abstract mass M free field algebra** in two dimensions, denoted $\mathcal{A}_M$, is given by the unital $*$-algebra generated by symbols $\phi(f)$ for $f \in \mathcal{S}(\mathbb{R}^2)$, subject to relations

$$
\begin{aligned}
\phi(\lambda_1 f_1 + \lambda_2 f_2) &= \lambda_1\phi(f_1) + \lambda_2\phi(f_2)\,, \\
\phi(f)^* &= \phi(\overline{f})\,, \\
[\phi(f_1), \phi(f_2)] &= < C_M, f_1 * \check{f}_2 > \mathbf{1}
\end{aligned}
$$

for $f, f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$, $\lambda_1, \lambda_2 \in \mathbb{C}$. C_M denotes the mass M free commutator function, which is given by

$$
< C_M, f_1 * \check{f}_2 >= (2\pi)^{-1} \int \mathrm{d}p\, \varepsilon(\underline{p})\delta(p^2 - m^2)\, \check{\hat{f}}_1 \cdot \hat{f}_2(p)\,.
$$

Definition 4.3 (Extension algebra)**.** The **extension algebra** $\mathcal{Z}$ is the free unital $*$-

algebra generated by symbols $Z(\mathbf{f})$ for $\mathbf{f} \in \mathcal{S}(\mathbb{R})$ subject to the relations

$$\begin{aligned}
Z(\lambda_1 \mathbf{f}_1 + \lambda_1 \mathbf{f}_1) &= \lambda_1 Z(\mathbf{f}_1) + \lambda_2 Z(\mathbf{f}_2)\,, \\
Z(\mathbf{f})^* &= Z(\check{\overline{\mathbf{f}}})\,, \\
Z(\check{\mathbf{f}}) &= Z(\mathbf{f})\,, \\
[Z(\mathbf{f}_1), Z(\mathbf{f}_2)] &= 0
\end{aligned}$$

for $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2 \in \mathcal{S}(\mathbb{R})$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Definition 4.4 (Algebra of asymptotic fields). The **algebra of asymptotic fields** $\mathcal{A}_0$ is the free unital $*$-algebra generated by symbols $Z_0(\mathbf{f})$ for $\mathbf{f} \in \mathcal{S}(\mathbb{R})$ and $\phi_0(f)$ for $f \in \mathcal{S}(\mathbb{R}^2)$ subject to the following relations

$$\begin{aligned}
\phi_0(\lambda_1 f_1 + \lambda_2 f_2) &= \lambda_1 \phi_0(f_1) + \lambda_2 \phi_0(f_2)\,, \\
Z_0(\lambda_1 \mathbf{f}_1 + \lambda_1 \mathbf{f}_1) &= \lambda_1 Z_0(\mathbf{f}_1) + \lambda_2 Z_0(\mathbf{f}_2)\,, \\
\phi_0(f)^* &= \phi_0(\overline{f})\,, \\
Z_0(\mathbf{f})^* &= Z_0(\check{\overline{\mathbf{f}}})\,, \\
Z_0(\check{\mathbf{f}}) &= Z_0(\mathbf{f})\,, \\
[\phi_0(f_1), \phi_0(f_2)] &= Z_0(f_1 \vartriangle f_2)\,, \\
[Z_0(\mathbf{f}_1), Z_0(\mathbf{f}_2)] &= 0\,, \\
[Z_0(\mathbf{f}), \phi_0(f)] &= 0
\end{aligned}$$

for $f, f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$, $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2 \in \mathcal{S}(\mathbb{R})$, $\lambda_1, \lambda_2 \in \mathbb{C}$. The bilinear map $\vartriangle\colon \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R})$ is defined as follows:

$$\begin{aligned}
&f_1 \vartriangle f_2(\mathbf{x}) := (2\pi)^{-2} \times \\
&\mathcal{F}^{-1}_{\mathbf{x},\mathbf{p}}\left(\mathbf{p} \mapsto \frac{\mathcal{F}_{\underline{x},\underline{p}}(f_1 * \check{f}_2)(-\sqrt{\mathbf{p}^2 + M^2}, \mathbf{x}) - \mathcal{F}_{\underline{x},\underline{p}}(f_1 * \check{f}_2)(\sqrt{\mathbf{p}^2 + M^2}, \mathbf{x})}{2\sqrt{\mathbf{p}^2 + M^2}}\right)(\mathbf{x})\,.
\end{aligned}$$

Heuristically this reads $f_1 \vartriangle f_2(\mathbf{x}) = \int \mathrm{d}\underline{x}\, C_M(x) f_1 * \check{f}_2(x)$. From the rigorous expression it can be inferred that $f_1 \vartriangle f_2$ is a test function.

Note that $f_1 \vartriangle f_2$ satisfies $\widecheck{f_1 \vartriangle f_2} = -f_2 \vartriangle f_1$ and $\overline{f_1 \vartriangle f_2} = -\overline{f_1} \vartriangle \overline{f_2}$. It can be checked that the defining relations for $\mathcal{A}_0$ are consistent in the sense that the sum of the two-sided ideals they generate is not too large and $\mathcal{A}_0$ is not trivial. In particular it is

$$Z_0(f_1 \vartriangle f_2)^* = [\phi_0(f_1), \phi_0(f_2)]^* = [\phi_0(\overline{f_2}), \phi_0(\overline{f_1})] = Z_0(-\overline{f_2 \vartriangle f_1}) = Z_0(\widecheck{\overline{f_2 \vartriangle f_1}})\,,$$

$$Z_0(\widecheck{f_1 \vartriangle f_2}) = -Z_0(f_2 \vartriangle f_1) = -[\phi_0(f_2), \phi_0(f_1)] = [\phi_0(f_1), \phi_0(f_2)] = Z_0(f_1 \vartriangle f_2)\,,$$

$$\begin{aligned}
Z_0(\mathbf{f}) Z_0(f_1 \vartriangle f_2) &= Z_0(\mathbf{f})[\phi_0(f_1), \phi_0(f_2)] \\
&= [\phi_0(f_1), \phi_0(f_2)] Z_0(\mathbf{f}) = Z_0(f_1 \vartriangle f_2) Z_0(\mathbf{f})\,.
\end{aligned}$$

The algebra of asymptotic fields $\mathcal{A}_0$ can be viewed as a central extension of $\mathcal{A}$ by $\mathcal{Z}$ is the following way:
The map $Z(\mathbf{f}) \mapsto Z_0(\mathbf{f}) - \int \mathrm{d}\mathbf{x}\, \mathbf{f}(\mathbf{x})\mathbf{1}$ extends to an injective $*$-algebra homomorphism

$$\iota \,:\, \mathcal{Z} \to \mathcal{A}_0 \,.$$

Similarly the map $Z_0(\mathbf{f}) \mapsto \int \mathrm{d}\mathbf{x}\, \mathbf{f}(\mathbf{x})\mathbf{1}$, $\phi_0(f) \mapsto \phi(f)$ extends to a surjective $*$-algebra homomorphism

$$\pi \,:\, \mathcal{A}_0 \to \mathcal{Z}_0 \,.$$

It is straightforward to check that ι and π map the defining ideals in $\mathcal{Z}, \mathcal{A}_0$ respectively, to zero and thus are well-definied. The algebras fit into a sequence

$$\mathcal{Z} \xrightarrow{\iota} \mathcal{A}_0 \xrightarrow{\pi} \mathcal{A}$$

and it can be checked that $\mathrm{ima}(\iota)$ generates $\ker(\pi)$. If $\mathcal{I}$ denotes the ideal in $\mathcal{A}_0$ generated by $\mathrm{ima}(\iota)$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}_0/\mathcal{I}$. Furthermore $\mathrm{ima}(\iota)$ is central in $\mathcal{A}_0$.

Space-time translations act automorphically on $\mathcal{A}_0$. For $a \in \mathbb{R}^2$ define α_a on the generators of $\mathcal{A}_0$ by

$$\begin{aligned} \alpha_a(\phi_0(f)) &:= \phi_0(f_a)\,, \\ \alpha_a(Z_0(\mathbf{f})) &:= Z_0(\mathbf{f})\,. \end{aligned}$$

This map extends to a homomorphism on the free algebra generated by $\phi_0(f), Z_0(\mathbf{f})$, preserves the defining ideals of $\mathcal{A}_0$ and hence defines an endomorphism of $\mathcal{A}_0$. As $\alpha_a \circ \alpha_b = \alpha_{a+b}$, α is a representation of $\mathbb{R}^2$. In particular each α_a is an automorphism of $\mathcal{A}_0$.
The action of $\mathbb{R}^2$ on $\mathcal{A}_0$ is compatible with that on $\mathcal{A}_M$ with $\mathbb{R}^2$ seen as a subgroup of the Poincaré group, i.e. π intertwines the actions of $\mathbb{R}^2$: $\alpha_a(\pi(\phi_0(f))) = \pi(\alpha_a(\phi_0(f)))$. In this sense, the action of $\mathbb{R}^2$ on $\mathcal{A}_0$ extends that on $\mathcal{A}_M$.

4.2.1. KMS States on the Algebra of Asymptotic Fields

In section D.8, KMS states are defined only on the field algebra $\mathcal{A}$. As the definition makes no specific reference to generators of $\mathcal{A}$ and only requires an action of $\mathbb{R}$, a generalization of the KMS condition to $\mathcal{A}_0$ is at hand. The same is true for the definition of time-clustering. The definition of regularity has to be extended in the sense that words in Z_0 and ϕ_0 extend to tempered distributions in regular states.

The set of possible β-KMS states on $\mathcal{A}_0$ can be computed by means of standard methods: Let ω be a regular, time-clustering β-KMS state. Since $\forall\, t \in \mathbb{R} \,:\, f_t \vartriangle g_t = f \vartriangle g$, it is $\alpha_t([\phi_0(f_1), \phi_0(f_2)]) = [\phi_0(f_1), \phi_0(f_2)]$. Assuming vanishing one-point functions, time-clustering then implies for $f_1, f_2, g_1, \cdots g_k \in \mathcal{S}(\mathbb{R}^2)$

$$\begin{aligned} 0 &= \lim_{t\to\pm\infty}\Bigg[\omega\Bigg(\prod_{l=1}^{k}\phi_0(g_k)\cdot\alpha_t([\phi_0(f_1),\phi_0(f_2)])\Bigg) \\ &\qquad -\omega\Bigg(\prod_{l=1}^{k}\phi_0(g_k)\Bigg)\omega([\phi_0(f_1),\phi_0(f_2)])\Bigg] \\ &= \Bigg[\omega\Bigg(\prod_{l=1}^{k}\phi_0(g_k)\cdot[\phi_0(f_1),\phi_0(f_2)]\Bigg) \\ &\qquad -\omega\Bigg(\prod_{l=1}^{k}\phi_0(g_k)\Bigg)\omega([\phi_0(f_1),\phi_0(f_2)])\Bigg], \end{aligned}$$

i.e.

$$\omega\Bigg(\prod_{l=1}^{k}\phi_0(g_k)\cdot[\phi_0(f_1),\phi_0(f_2)]\Bigg) = \omega\Bigg(\prod_{l=1}^{k}\phi_0(g_k)\Bigg)\cdot\omega([\phi_0(f_1),\phi_0(f_2)])\,. \tag{4.2}$$

Corollary 4.6. *For $f, g_1, \cdots, g_k \in \mathcal{S}(\mathbb{R})$ it is*

$$\omega\Bigg(\Bigg[\prod_{l=1}^{k}\phi_0(g_k),\phi_0(f)\Bigg]\Bigg) = \sum_{l=1}^{k}\omega\Bigg(\prod_{\substack{j=1\\ j\neq l}}^{k}\phi_0(g_k)\Bigg)\cdot\omega(Z_0(g_l\vartriangle f))\,.$$

Proof. Since for $1 \leqslant l \leqslant k$ the commutators $[\phi_0(g_l), \phi_0(f)] = Z_0(g_l \vartriangle f)$ are central in $\mathcal{A}_0$, the statement is derived by considering $\omega(\phi_0(g_1)\cdots\phi_0(g_k)\phi_0(f))$, permuting $\phi_0(f)$ to the left and applying relation 4.2 to the commutator terms. $\square$

Set $a := \prod_{l=1}^{k}\phi_0(g_l)$ and $b := \phi_0(f)$. Let $F_{a,b}$ be the function $t \mapsto \omega(a\,\alpha_t(b))$ extended to the strip $\mathcal{S}_\beta$ using the KMS condition. It is

$$\begin{aligned} F_{a,b}(t) &= <\mathcal{W}_{k+1}, g_1\otimes\cdots\otimes g_k\otimes f_t> \\ &= <\widehat{\mathcal{W}}^{-1}_{k+1}, \widehat{g}_1\otimes\cdots\otimes\widehat{g}_k\otimes e^{it\underline{p}}\widehat{f}> \end{aligned}$$

and by the KMS condition

$$\begin{aligned} F_{a,b}(t+i\beta) &= <\mathcal{W}_{k+1}, f_t\otimes g_1\otimes\cdots\otimes g_k> \\ &= <\widehat{\mathcal{W}}^{-1}_{k+1}, \widehat{g}_1\otimes\cdots\otimes\widehat{g}_k\otimes e^{it\underline{p}}e^{-\beta\underline{p}}\widehat{f}>\ . \end{aligned}$$

Define the tempered distribution $\mathcal{C}_{k,1}$ by

$$<\mathcal{C}_{k,1}, g_1\otimes\cdots\otimes g_k\otimes f> := \omega\Bigg(\Bigg[\prod_{l=1}^{k}\phi_0(g_l),\phi_0(f)\Bigg]\Bigg)\,.$$

In light of the preceding computations it is

$$\begin{aligned}
& < \widehat{\mathcal{C}}_{k,1}^{-1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \\
= \;& < \mathcal{C}_{k,1}, g_1 \otimes \cdots \otimes g_k \otimes f > \\
= \;& F_{a,b}(0) - F_{a,b}(i\beta) \\
= \;& < \widehat{\mathcal{W}}_{k+1}^{-1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes (1 - e^{-\beta \underline{p}}) \widehat{f} > \\
= \;& < (1 - e^{-\beta \underline{p}}) \widehat{\mathcal{W}}_{k+1}^{-1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \ .
\end{aligned}$$

As a consequence of time-clustering, $\widehat{\mathcal{W}}_{k+1}^{-1}$ does not have a discrete contribution in the $(k+1)^{\text{th}}$ energy variable $\underline{p}$:

$$\begin{aligned}
0 \;=\;& \lim_{t \to \pm\infty} \omega\Big(\prod_{l=1}^{k} \phi_0(g_l) \cdot \alpha_t(\phi_0(f)) \Big) \\
=\;& \lim_{t \to \pm\infty} < e^{it\underline{p}} \widehat{\mathcal{W}}_{k+1}^{-1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \ .
\end{aligned}$$

This would not be possible in the presence of a discrete contribution at $\underline{p} = 0$.

The relation $\widehat{\mathcal{C}}_{k,1}^{-1} = (1 - e^{-\beta \underline{p}}) \widehat{\mathcal{W}}_{k+1}^{-1}$ can hence be writte as $\widehat{\mathcal{W}}_{k+1}^{-1} = (1 - e^{-\beta \underline{p}})^{-1} \widehat{\mathcal{C}}_{k,1}^{-1}$ in an unambiguous way. Using a somewhat sloppy notation, compute

$$\begin{aligned}
& \omega\Big(\prod_{l=1}^{k} \phi_0(g_l) \cdot \phi_0(f) \Big) \\
=\;& < \widehat{\mathcal{W}}_{k+1}^{-1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \\
=\;& < (1 - e^{-\beta \underline{p}})^{-1} \widehat{\mathcal{C}}_{k,1}^{-1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \\
=\;& < \mathcal{C}_{k,1}, g_1 \otimes \cdots \otimes g_k \otimes \mathcal{F}^{-1}_{\underline{x},\underline{p}}((1 - e^{-\beta \underline{p}})^{-1} \widehat{f}) > \\
=\;& \omega\Big(\Big[\prod_{l=1}^{k} \phi_0(g_l), \phi_0(\mathcal{F}^{-1}_{\underline{x},\underline{p}}((1 - e^{-\beta \underline{p}})^{-1} \widehat{f})) \Big] \Big) \\
=\;& \sum_{l=1}^{k} \omega\Big(\prod_{\substack{j=1 \\ j \neq l}}^{k} \phi_0(g_k) \Big) \cdot \omega(Z_0(g_l \vartriangle \mathcal{F}^{-1}_{\underline{x},\underline{p}}((1 - e^{-\beta \underline{p}})^{-1} \widehat{f}))) \\
=\;& \sum_{l=1}^{k} \omega\Big(\prod_{\substack{j=1 \\ j \neq l}}^{k} \phi_0(g_k) \Big) \cdot \omega(Z_0(g_l \vartriangle_\beta f)) \,,
\end{aligned}$$

where $\vartriangle_\beta$ is given by $f_1 \vartriangle_\beta f_2(\mathbf{x}) = \int \mathrm{d}\underline{x}\, W_{M,\beta}(x) f_1 * \check{f}_2(x)$ for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$. A more rigorous, but lengthy expression can be given for $f_1 \vartriangle_\beta f_2 \in \mathcal{S}(\mathbb{R})$ in analogy to $f_1 \vartriangle f_2$.

A combinatorial argument now shows that for $f_1, \cdots, f_m \in \mathcal{S}(\mathbb{R}^2)$ one has

$$\omega\left(\prod_{j=1}^{m} \phi_0(f_l)\right) = \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{\substack{(s_1,s_2)\in\mathfrak{T} \\ s_1<s_2}} \omega(Z_0(f_{s_1} \vartriangle_\beta f_{s_2}))\,. \tag{4.3}$$

It follows that time-clustering β-KMS states on $\mathcal{A}_0$ are homogeneous and quasifree with two-point functions given by

$$\omega(\phi_0(f)\phi_0(g)) = \omega(Z_0(f \vartriangle_\beta g))\,. \tag{4.4}$$

The set of KMS states for a given inverse temperature β is degenerate with a residual degree of freedoms in the choice of $\omega(Z_0(\mathbf{f}))$. For prescribed $\omega(Z_0(\mathbf{f}))$ it has to be checked, whether ω defines an actual state as the positivity condition need not be automatically satisfied.

Now consider the following comparison between KMS states on $\mathcal{A}_0$ and the asymptotic time structure of correlation functions in general thermal states with suppressed low energy contributions:
In the sense of distributions equation 4.3 reads

$$\mathcal{W}_{0,m}(x_1, \cdots, x_m) = \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{\substack{i=1 \\ S_i=(j_i,k_i)}}^{|\mathfrak{T}|} \omega(Z_0(\mathbf{x}_{j_i} - \mathbf{x}_{k_i}))W_{m,\beta}(x_{j_i} - x_{k_i})\,.$$

Recall the result previously obtained for time-regularized correlation functions

$$\lim_{\vartheta\to\infty} \vartheta^{\frac{m}{4}} \left| \bigotimes_{j=1}^{m} \underline{\breve{g}}_j *_{\underline{x}} \mathcal{W}_m(x_1,\cdots,x_m) - \sum_{\mathfrak{T}\in\mathcal{P}_2} \prod_{\substack{i=1 \\ S_i=(j_i,k_i)}}^{|\mathfrak{T}|} (\underline{\breve{g}}_{j_i} * \underline{g}_{k_i}) *_{\underline{x}} W_{\mathrm{d}}(x_{j_i} - x_{k_i}) \right| = 0$$

with $W_{\mathrm{d}}(x) = D_{\mathrm{d}}(\mathbf{x})W_{M,\beta}(x)$. Time-regularizing the asymptotic fields, it becomes apparent that, identifying the tempered distributions $\omega(Z_0(\mathbf{x}))$ and $D_{\mathrm{d}}(\mathbf{x})$, the algebra $\mathcal{A}_0$ captures the asymptotic time structure of general thermal correlation functions. In the following, the normalization $\omega(Z_0(0)) = D_{\mathrm{d}}(0) = 1$ is assumed.

4.2.2. Vacuum State on the Algebra of Asymptotic Fields

By similar methods, vacuum states on $\mathcal{A}_0$ can be computed. It turns out to be essentially unique. Let ω be a vacuum state on $\mathcal{A}_0$. As a consequence of the relativistic spectrum condition it is

$$\begin{aligned} & < \widehat{\mathcal{C}}^{-1}_{k,1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \\ = \; & < \mathcal{C}_{k,1}, g_1 \otimes \cdots \otimes g_k \otimes f > \\ = \; & < \theta(\underline{p})\widehat{\mathcal{W}}^{-1}_{k+1}, \widehat{g}_1 \otimes \cdots \otimes \widehat{g}_k \otimes \widehat{f} > \;. \end{aligned}$$

With this modification in mind and using spatial- instead of time-clustering, the method for computing KMS states on $\mathcal{A}_0$ can be applied in the vacuum case. It turns out that vacuum states on $\mathcal{A}_0$ are quasifree with two-point function

$$\omega(\phi_0(f)\phi_0(g)) = \omega(Z_0(f \vartriangle_\infty g)),$$

where $f \vartriangle_\infty g$ is formally given by $f \vartriangle_\infty g(\mathbf{x}) = \int \mathrm{d}\underline{x}\, W_{M,\infty}(x) f * \widehat{g}(x)$. In the vacuum case, Poincaré invariance further restricts the possibilities for $\omega(Z_0(f \vartriangle_\infty g))$ to multiples of $\int \mathrm{d}x\, W_{M,\infty}(x) f * \widehat{g}(x)$. With a suitable normalization it is $Z_0(\mathbf{x}) = 1$ and

$$\omega(Z_0(f \vartriangle_\infty g)) =< W_{M,\infty}, f * \widehat{g} >,$$

or equivalently in heuristic notation

$$W_0(x) = Z_0(\mathbf{x}) W_{M,\infty}(x) = W_{M,\infty}(x).$$

In short, identifying $\phi_0(f) \in \mathcal{A}_0$ with $\phi(f) \in \mathcal{A}$, the vacuum state on $\mathcal{A}_0$ coincides with the free mass M vacuum. This is not surprising: The identification $\omega(Z_0(\mathbf{x})) \leftrightarrow D_{\mathrm{d}}(\mathbf{x})$ leads to the expectation that the residual degree of freedom in the choice of $\omega(Z_0(\mathbf{f}))$ in KMS states on $\mathcal{A}_0$ does not persist in the vacuum case, as the vacuum does not admit a non-trivial damping factor.

5. Asymptotic Dynamics

Assuming the original field obeyes a dynamical law given by a field equation of the form $(\Box + M^2)\phi + P(\phi) = 0$ for some polynomial P in the sense of an effective theory, the condition of **asymptotic compatibility** imposes restrictions on the admissible KMS states on the algebra of asymptotic fields $\mathcal{A}_0$. As in the four-dimensional case (cf. [9]), these restrictions are sufficiently strong to recover the discrete parts of the damping factor and thus to identify the particle content of such an effective theory.

5.1. Normal-Ordering

The formulation of a field equation $(\Box + M^2)\phi + P(\phi) = 0$ requires a notion of powers of a field. For the purposes of describing the asymptotic dynamics of a thermal theory satisfying the assumptions made in 4.1, such a notion of powers is needed for the field ϕ_0. The"naive" power $\phi_0(f)^m$ turns out to be too singular in KMS states on $\mathcal{A}_0$, when considering to the limit $f \to \delta_x$ required to formulate the condition of asymptotic compatibility. A way out lies in subtracting from $\phi_0(f)^m$ a suitable combination of vacuum expectation values corresponding to singular parts in $\phi_0(f)$ for increasingly localized f. This procedure, called **normal-ordering**, is defined as follows:

Definition 5.1. Let κ be a regular, quasifree state on $\mathcal{A}_0$. For $f \in \mathcal{S}(\mathbb{R}^2)$, **normal-ordering** is a map from $\{\, \phi_0(f)^m \,|\, m \in \mathbb{N}_0 \,\}$ to $\mathcal{A}_0$ given by

$$N_\kappa(\phi_0(f)^m) := \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \binom{m}{2k} \kappa(\phi_0(f)^{2k}) \phi_0(f)^{m-2k} \,,$$

where $\lfloor \frac{m}{2} \rfloor$ is the largest integer less or equal than $\frac{m}{2}$. N_κ extends linearly to the space $\mathrm{span}(\{\phi_0(f)^m \,|\, m \in \mathbb{N}_0\})$. $N_\kappa(\phi_0(f)^m)$ is called $m^{\mathbf{th}}$ **normal-ordered power of** $\phi_0(f)$ **with respect to** κ.

Lemma 5.1. *Let κ, ω be quasifree states on $\mathcal{A}_0$ and $f_1, \cdots, f_{m_1}, \cdots, f_{m_2}, f \in \mathcal{S}(\mathbb{R}^2)$.*

Define $M := \{1, \cdots, m_1 + m_2\}$ *and* $K_f := \omega(\phi_0(f)\phi_0(f)) - \kappa(\phi_0(f)\phi_0(f))$. *It is*

$$\begin{aligned}
&\omega\Big(\prod_{k=1}^{m_1}\phi_0(f_k)\cdot N_\kappa(\phi_0(f)^m)\cdot\prod_{k=m_1+1}^{m_2}\phi_0(f_k)\Big)\\
=&\sum_{\substack{m-(m_1+m_2)\\ \leqslant 2k\leqslant m}}\frac{m!}{2^k k!}K_f^k\sum_{\substack{S\subset M\\ |S|=m-2k}}\omega\Big(\prod_{\substack{p\in M\setminus S\\ \text{ordered}}}\phi_0(f_p)\Big)\\
&\times\prod_{\substack{p\in S\\ p\leqslant m_1}}\omega(\phi_0(f_p)\phi_0(f))\cdot\prod_{\substack{p\in S\\ p> m_1}}\omega(\phi_0(f)\phi_0(f_p))\,.
\end{aligned}$$

Proof. The statement can be proven by combinatorial methods exploiting the quasifreeness of ω and κ. However, the combinatorial arguments are difficult to present in a comprehensible manner. The proof given in appendix C.4 makes use of generating functionals for normal-ordered powers. □

Definition 5.2. Let κ, ω be two regular, quasifree states on $\mathcal{A}_0$. ω is called **κ-regular** :⇔ The following limits exist for all $g \in \mathcal{S}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$:

$$\begin{array}{ll}
\lim_{f\to\delta_x} & \omega(\phi_0(f)\phi_0(f)) - \kappa(\phi_0(f)\phi_0(f))\\
\lim_{f\to\delta_x} & \omega(\phi_0(f)\phi_0(g))\\
\lim_{f\to\delta_x} & \omega(\phi_0(g)\phi_0(f))\,.
\end{array}$$

Lemma 5.2. *Let κ be a regular quasifree and ω a regular, κ-regular, quasifree state on $\mathcal{A}$. If π_ω denotes the GNS representation on $\mathcal{H}$ with invariant dense domain $\mathcal{D}$ and inner product $\langle\cdot|\cdot\rangle$, then for $x \in \mathbb{R}^2$*

$$\lim_{f\to\delta_x}\langle a|\pi_\omega(N_\kappa(\phi_0(f)^m))\cdot b\rangle,\quad a,b\in\mathcal{D}$$

defines a sesquilinear form on $\mathcal{D}\times\mathcal{D}$.

Proof. Let $a, b \in \mathcal{D}$. $\langle a, \pi_\omega(N_\kappa(\phi_0(f)^m))\cdot b\rangle$ can be decomposed into summands $\omega(\cdots)$ as in lemma 5.1. The κ-regularity of ω then guarantees the existence of the $f \to \delta_x$ limit. Sequlinearity follows from the sesquilinearity of $\langle\cdot|\cdot\rangle$. □

In the following all normal-orderings are with respect to the vacuum sate $\omega_{0,\infty}$ in $\mathcal{A}_0$. The index in $N_{\omega_{0,\infty}}$ is omitted.

5.2. Asymptotic Field Equation

Now suppose $P = \sum_{m=0}^{l} c_m X^m$ is the polynomial interaction term of a modified Klein-Gordon equation $(\Box + M^2)\phi + P(\phi) = 0$ satisfied by original field ϕ in the GNS representation π of a relativistic KMS state as in 4.1. This is to say that given a suitable notion of powers of fields on $\pi(\mathcal{A})$, it is $\pi(\phi((\Box + M^2)f)) + \pi(P(\phi(f))) = 0$ for any $f \in \mathcal{S}(\mathbb{R}^2)$. In fact, no such thermal state or notion of powers of fields need exist. The original fields

satisfying a field equation serves only as a starting point for heuristic considerations and it suffices if the field equation is satisfied in the sense of an effective theory.
The asymptotic field ϕ_0 cannot satisfy the field equation exactly in any finite temperature KMS state on $\mathcal{A}_0$, as the kernels $\ker(\pi)$ of the corresponding GNS representation do not contain the ideal generated by $\phi_0((\Box + M^2)f) + \sum_{m=0}^{l} c_m N(\phi_0(f)^m)$, unless $P = 0$. However, it is reasonable to assume that ϕ_0 satisfies the field equation in an asymptotic sense in suitable KMS states on $\mathcal{A}_0$. This leads to the notion of asymptotic compatibility. The exact formulation requires some preparation.

The field equation map

$$\begin{aligned} L \,:\, & \mathcal{S}(\mathbb{R}^2) \to \mathcal{A}_0\,, \\ & f \mapsto L(f) := \phi_0((\Box + M^2)f) + \sum_{m=0}^{l} c_m N(\phi_0(f)^m) \end{aligned}$$

can be used to measure how badly ϕ_0 fails to satisfy the field equation in a given β-KMS state $\omega_{0,\beta}$ on $\mathcal{A}_0$. Let $\pi_{0,\beta}$ be the GNS representation of $\mathcal{A}_0$ induced by $\omega_{0,\beta}$ on the Hilbert space $\mathcal{H}$ with dense invariant subspace $\mathcal{D}$ and inner product $\langle\cdot|\cdot\rangle$. If $\pi_{0,\beta} \circ L$ was the zero map, then ϕ_0 would satisfy the field equation in $\omega_{0,\beta}$. The condition $\pi_{0,\beta} \circ L = 0$ is equivalent to $\forall\, a, b \in \mathcal{D},\ f \in \mathcal{S}(\mathbb{R}^2)\ :\ \langle a|\pi \circ L(f)b\rangle = 0$, which in turn means that $\forall\, f_1, \cdots, f_{m_1}, \cdots, f_{m_2}, f \in \mathcal{S}(\mathbb{R}^2)\ :$

$$\omega_{0,\beta}(\phi_0(f_1)\cdots\phi_0(f_{m_1})L(f)\phi_0(f_{m_1+1})\cdots\phi_0(f_{m_2})) = 0\,.$$

This motivates the use $\omega_{0,\beta}(\phi_0(f_1)\cdots\phi_0(f_{m_1})L(f)\phi_0(f_{m_1+1})\cdots\phi_0(f_{m_2}))$ as a basis for formulating an asymptotic field equation.

Lemma 5.3. *Let $\omega_{0,\infty}$ be the vacuum state and $\omega_{0,\beta}$ a β-KMS state on $\mathcal{A}_0$ such that the tempered distribution $\mathbf{f} \mapsto \omega_{0,\beta}(Z_0(\mathbf{f}))$ is given by a smooth function $\omega_{0,\beta}(Z_0(\mathbf{x})) =: D_{\mathrm{d}}(\mathbf{x})$ with $D_{\mathrm{d}}(0) = 1$. Then $\omega_{0,\beta}$ is $\omega_{0,\infty}$-regular.*

Proof. Because of temperedness, D_{d} is polynomially bounded and hence a multiplier in $\mathcal{S}'(\mathbb{R}^2)$. Using the results on β-KMS states on $\mathcal{A}_0$ it is

$$\begin{aligned} & \omega_{0,\beta}(\phi_0(f)\phi_0(f)) \\ = \ & \omega_{0,\beta}(Z(f \vartriangle_\beta f)) \\ = \ & \int \mathrm{d}\mathbf{x}\, D_{\mathrm{d}}(\mathbf{x}) f \vartriangle_\beta f(\mathbf{x}) \\ = \ & < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} >\,, \end{aligned}$$

were $W_{M,\beta}$ is the free, reduced mass M β-KMS two-point function. It follows that

$$\begin{aligned} & \omega_{0,\beta}(\phi_0(f)\phi_0(f)) - \omega_{0,\infty}(\phi_0(f)\phi_0(f)) \\ = \ & < W_{0,\beta}, f * \check{f} > - < W_{0,\infty}, f * \check{f} > \\ = \ & < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\beta}, f * \check{f} > \\ = \ & (< W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} >) \\ & +(< W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\beta}, f * \check{f} >)\,. \end{aligned}$$

In the limit $f \to \delta_x$ the first summand in the latter expression converges to a constant only depending on M and β and the second vanishes as shown below. Convergence is to be understood in a weak sense and the computations can be made rigorous using arguments similar to those in theorem C.3.

First summand: Using that $D_{\mathrm{d}} \cdot f * \check{f}$ and consequently $\mathcal{F}^{-1}(D_{\mathrm{d}} \cdot f * \check{f})$ are even in $\underline{x}, \underline{p}$ respectively, it is in slightly sloppy notation

$$\begin{aligned} & < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\infty}, D_{\mathrm{d}} \cdot f * \check{f} > \\ = \; & < \widehat{W}_{M,\beta} - \widehat{W}_{M,\infty}, \mathcal{F}^{-1}(D_{\mathrm{d}} \cdot f * \check{f}) > \\ = \; & 2\pi < \frac{\varepsilon(\underline{p})\delta(p^2 - M^2)}{1 - e^{-\beta \underline{p}}} - \theta(\underline{p})\delta(p^2 - M^2), \mathcal{F}^{-1}(D_{\mathrm{d}} \cdot f * \check{f}) > \\ = \; & 2\pi < \frac{2\theta(\underline{p})}{e^{\beta \underline{p}} - 1}\delta(p^2 - M^2), \mathcal{F}^{-1}(D_{\mathrm{d}} \cdot f * \check{f}) > \; . \end{aligned}$$

For $f \to \delta_x$, $f * \check{f}$ converges to δ. As $D_{\mathrm{d}}(0) = 1$, $\mathcal{F}^{-1}(D_{\mathrm{d}} \cdot f * \check{f})$ then converges to the constant function $(2\pi)^{-2}$. Consequently

$$\begin{aligned} & \lim_{f \to \delta_x} < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\infty}, D_{\mathrm{d}} \cdot f * \check{f} > \\ &= \lim_{f \to \delta_x} (2\pi) \int \mathrm{d}\mathbf{p} \, \frac{\mathcal{F}^{-1}(D_{\mathrm{d}} \cdot f * \check{f})(\sqrt{\mathbf{p}^2 + M^2})}{\sqrt{\mathbf{p}^2 + M^2} \cdot (e^{\beta\sqrt{\mathbf{p}^2 + M^2}} - 1)} \\ &= (2\pi)^{-1} \int \mathrm{d}\mathbf{p} \, \frac{1}{\sqrt{\mathbf{p}^2 + M^2}} \frac{1}{(e^{\beta\sqrt{\mathbf{p}^2 + M^2}} - 1)} \\ &=: K_\beta \, . \end{aligned} \tag{5.1}$$

Second summand: Similarly one gets

$$\begin{aligned} & < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\beta}, f * \check{f} > \\ = \; & < \widehat{W}_{M,\beta}, \mathcal{F}^{-1}((D_{\mathrm{d}} - 1) \cdot f * \check{f}) > \; . \end{aligned}$$

Using again that $D_{\mathrm{d}}(0) = 1$, the expression $\mathcal{F}^{-1}((D_{\mathrm{d}} - 1) \cdot f * \check{f})$ vanishes as $f \to \delta_x$, which implies

$$\lim_{f \to \delta_x} < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{f} > - < W_{M,\beta}, f * \check{f} > = 0 \, .$$

Altogether it is

$$\lim_{f \to \delta_x} \omega_{0,\beta}(\phi_0(f)\phi_0(f)) - \omega_{0,\infty}(\phi_0(f)\phi_0(f)) = K_\beta \, .$$

It remains to show that $\lim_{f \to \delta_x} \omega_{0,\beta}(\phi_0(f)\phi_0(g))$ and $\lim_{f \to \delta_x} \omega_{0,\beta}(\phi_0(g)\phi_0(f))$ exist. Since $\omega_{0,\beta}(\phi_0(f)\phi_0(g)) = < W_{M,\beta}, D_{\mathrm{d}} \cdot f * \check{g} >$ and $f * \check{g} \to \check{g}_x$ as $f \to \delta_x$ it is

$$\lim_{f \to \delta_x} \omega_{0,\beta}(\phi_0(f)\phi_0(g)) = < W_{M,\beta}, D_{\mathrm{d}} \cdot \check{g}_x > \; .$$

Interchanging f, g, the same argument shows that

$$\lim_{f\to\delta_x} \omega_{0,\beta}(\phi_0(g)\phi_0(f)) =< W_{M,\beta}, D_{\mathrm{d}} \cdot g_{-x} > \ .$$

$\square$

Lemma 5.4. *Let $\omega_{0,\beta}$ be a β-KMS state on $\mathcal{A}_0$ such that the tempered distribution $\mathbf{f} \mapsto \omega_{0,\beta}(Z_0(\mathbf{f}))$ is given by a smooth function $\omega_{0,\beta}(Z_0(\mathbf{x})) =: D_{\mathrm{d}}(\mathbf{x})$ with $D_{\mathrm{d}}(0) = 1$. If $\pi_{0,\beta}$ is the GNS representation on the Hilbert space $\mathcal{H}$ with dense invariant domain $\mathcal{D}$, associated to $\omega_{0,\beta}$, then $\pi_{0,\beta} \circ \phi_0((\square + M^2)f)$ admits the limit $f \to \delta_x$ in the sense of a sesquilinear form on $\mathcal{D} \times \mathcal{D}$.*

Proof. Let $f_1, \cdots, f_{m_1}, \cdots, f_{m_2}, f \in \mathcal{S}(\mathbb{R}^2)$. It is by quasifreeness of $\omega_{0,\beta}$:

$$\begin{aligned} &\omega_{0,\beta}\Big(\prod_{k=1}^{m_1} \phi_0(f_k) \cdot \phi_0((\square + M^2)f) \cdot \prod_{k=m_1+1}^{m_2} \phi_0(f_k)\Big) \\ = &\sum_{j=1}^{m_1} \omega_{0,\beta}(\phi_0(f_j)\phi_0((\square + M^2)f)) \cdot \omega_{0,\beta}\Big(\prod_{\substack{k=1\\k\neq j}}^{m_2} \phi_0(f_k)\Big) \\ + &\sum_{j=m_1+1}^{m_2} \omega_{0,\beta}(\phi_0((\square + M^2)f)\phi_0(f_j)) \cdot \omega_{0,\beta}\Big(\prod_{\substack{k=1\\k\neq j}}^{m_2} \phi_0(f_k)\Big) . \end{aligned}$$

This shows that it is sufficient to prove that the limit $f \to \delta_x$ exists in the two-point functions

$$\omega_{0,\beta}(\phi_0(g)\phi_0((\square + M^2)f))$$

and

$$\omega_{0,\beta}(\phi_0((\square + M^2)f)\phi_0(g)) .$$

It is

$$\begin{aligned} &\omega_{0,\beta}(\phi_0(g)\phi_0((\square + M^2)f)) \\ = \ &< D_{\mathrm{d}} \cdot W_{M,\beta}, g * ((\square + M^2)\check{f}) > \\ = \ &< D_{\mathrm{d}} \cdot W_{M,\beta}, (\square + M^2)g * \check{f} > \\ = \ &< (\square + M^2)\, D_{\mathrm{d}} \cdot W_{M,\beta}, g * \check{f} > \ . \end{aligned}$$

In this form it is obvious that the limit $f \to \delta_x$ exists and that

$$\lim_{f\to\delta_x} \omega_{0,\beta}(\phi_0(g)\phi_0((\square + M^2)f)) =< (\square + M^2)\, D_{\mathrm{d}} \cdot W_{M,\beta}, g_{-x} > \ .$$

Similarly it is

$$\lim_{f\to\delta_x} \omega_{0,\beta}(\phi_0((\square + M^2)f)\phi_0(g)) =< (\square + M^2)\, D_{\mathrm{d}} \cdot W_{M,\beta}, \check{g}_x > \ .$$

$\square$

It follows from lemmata 5.2 and 5.4 that $\pi_{0,\beta} \circ L(f)$ admits the limit $f \to \delta_x$ for $x \in \mathbb{R}^2$ in the sense of sesquilinear forms on $\mathcal{D} \times \mathcal{D}$. The resulting sesquilinear form, denoted $\pi_{0,\beta} \circ L(x)$ can be time-regularized by a Schwartz function $\underline{g} \in \mathcal{S}(\mathbb{R})$, whose Fourier transform vanishes at zero, resulting in the sesquilinear form $\pi_{0,\beta} \circ L_{\underline{g}}(x)$:

$$\begin{aligned} &\omega_{0,\beta}(\phi_0(f_1)\cdots\phi_0(f_{m_1})L_{\underline{g}}(x)\phi_0(f_{m_1+1})\cdots\phi_0(f_{m_2})) \\ := &\int \mathrm{d}\underline{x}\, \underline{g}(\underline{x})\omega_{0,\beta}(\phi_0(f_1)\cdots\phi_0(f_{m_1})L(x)\phi_0(f_{m_1+1})\cdots\phi_0(f_{m_2}))\,. \end{aligned}$$

It seems natural to assume that if ϕ_0 is to satisfy the field equation L in $\omega_{0,\beta}$ in an asymptotic sense, $\pi_{0,\beta} \circ L_{\underline{g}}(x)$ should decay more rapidly than its components, namely the field $\pi_{0,\beta}(\phi_0(x))$ and its normal-ordered powers. The decay is to be understood in the sense of decay of the respective sesquilinear form applied to arbitrary $a, b \in \mathcal{D}$. For the field ϕ_0 to share the asymptotic time behavior in thermal states with the original time-regularized field $\phi_{\underline{g}}$, it also needs to be time-regularized by a Schwartz function $\underline{g} \in \mathcal{S}(\mathbb{R})$, whose Fourier transform vanishes at zero. These considerations allow to formulate an asymptotic field equation:

Definition 5.3. Let $\omega_{0,\beta}$ be a β-KMS state on $\mathcal{A}_0$ such that the tempered distribution $\mathbf{f} \mapsto \omega_{0,\beta}(Z_0(\mathbf{f}))$ is given a by smooth function $\omega_{0,\beta}(Z_0(\mathbf{x})) =: D_{\mathrm{d}}(\mathbf{x})$ with $D_{\mathrm{d}}(0) = 1$. Let $\pi_{0,\beta}$ be the GNS representation associated to $\omega_{0,\beta}$. The field $\pi_{0,\beta} \circ \phi_0$ is said to comply with the **asymptotic field equation** $:\Leftrightarrow$
For all $f_1, \cdots, f_{m_1}, \cdots, f_{m_2} \in \mathcal{S}(\mathbb{R}^2)$, $\underline{g}_1, \cdots, \underline{g}_{m_1}, \cdots, \underline{g}_{m_2}, g \in \mathcal{S}(\mathbb{R})$ with $\hat{\underline{g}}_i(0) = \hat{\underline{g}} = 0$ and $m \in \mathbb{N}_0$:

$$\omega_{0,\beta}(\phi_{0,\underline{g}_1}(f_1)\cdots\phi_{0,\underline{g}_{m_1}}(f_{m_1})L_{\underline{g}}(x)\phi_{0,\underline{g}_{m_1+1}}(f_{m_1+1})\cdots\phi_{0,\underline{g}_{m_2}}(f_{m_2}))$$

decays more rapidly in $\underline{x}$ than

$$\omega_{0,\beta}(\phi_{0,\underline{g}_1}(f_1)\cdots\phi_{0,\underline{g}_{m_1}}(f_{m_1})N_{\underline{g}}(\phi_0(x)^m)\phi_{0,\underline{g}_{m_1+1}}(f_{m_1+1})\cdots\phi_{0,\underline{g}_{m_2}}(f_{m_2}))\,, \tag{5.2}$$

where $N_{\underline{g}}(\phi_0(x)^m)$ is the sesquilinear form associated to the m^{th} normal-ordered power of $\phi_0(f)$ in the limit $f \to \delta$ time-regularized by $\underline{g}$. If $\pi_{0,\beta} \circ \phi_0$ complies with the asymptotic field equation, $\omega_{0,\beta}$ is said to be **asymptotically compatible** with L.

By determining the asymptotic behavior of 5.2, the asymptotic field equation can be formulated in a more convenient way.

Lemma 5.5. *In the situation of definition 5.3, the leading order in the $\underline{x}$-asymptotic behavior of 5.2 is at most $\sim |\underline{x}|^{-\frac{1}{2}}$. The cases of slowest possible rate of decay in $\underline{x}$ exhibit a $\sim |\underline{x}|^{-\frac{1}{2}}$ asymptotic behavior.*

Proof. It follows from lemma 5.1 and the proof of 5.3, that

$$\begin{aligned}
&\omega_{0,\beta}(\phi_{0,\underline{g}_1}(f_1)\cdots\phi_{0,\underline{g}_{m_1}}(f_{m_1})N_{\underline{g}}(\phi_0(x)^m)\phi_{0,\underline{g}_{m_1+1}}(f_{m_1+1})\cdots\phi_{0,\underline{g}_{m_2}}(f_{m_2}))\\
&= \sum_{\substack{m-(m_1+m_2)\\ \leqslant 2k\leqslant m}} \frac{m!}{2^k k!}K_\beta^k \sum_{\substack{S\subset M\\ |S|=m-2k}} \omega_{0,\beta}\Bigg(\prod_{\substack{p\in M\setminus S\\ \text{ordered}}} \phi_{0,\underline{g}_p}(f_p)\Bigg)\\
&\quad\times\int \mathrm{d}\underline{y}\, \underline{g}(\underline{y}) \prod_{\substack{p\in S\\ p\leqslant m_1}} \omega_{0,\beta}(\phi_{0,\underline{g}_p}(f_p)\phi_0(\underline{x}+\underline{y},\mathbf{x}))\cdot \prod_{\substack{p\in S\\ p> m_1}} \omega_{0,\beta}(\phi_0(\underline{x}+\underline{y},\mathbf{x})\phi_{0,\underline{g}_p}(f_p))\,,
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
\omega_{0,\beta}(\phi_{0,\underline{g}_p}(f_p)\phi_0(\underline{x}+\underline{y},\mathbf{x})) &= <\underline{g}_p *_{\underline{x}} W_{M,\beta}, D_{\mathrm{d}}\cdot \check{f}_{p,(\underline{x}+\underline{y},\mathbf{x})}>\,,\\
\omega_{0,\beta}(\phi_0(\underline{x}+\underline{y},\mathbf{x})\phi_{0,\underline{g}_p}(f_p)) &= <\check{\underline{g}}_p *_{\underline{x}} W_{M,\beta}, D_{\mathrm{d}}\cdot f_{p,-(\underline{x}+\underline{y},\mathbf{x})}>\;.
\end{aligned}$$

By the asymptotic properties of the free reduced mass M β-KMS two-point function $W_{M,\beta}$, each such factor vanishes for asymptotic $\underline{x}$. Consequently the time-regularized product of these factors can be estimated from above by applying the time-regularization to only one factor. Time-regularization of one such factor with $\underline{g}$ gives

$$< (\underline{g}_p * \check{\underline{g}}) *_{\underline{x}} W_{M,\beta}, D_{\mathrm{d}}\cdot \check{f}_{p,x} > \tag{5.4}$$

and

$$< (\underline{g} * \check{\underline{g}}_p) *_{\underline{x}} W_{M,\beta}, D_{\mathrm{d}}\cdot f_{p,-x} > \tag{5.5}$$

respectively. As the appearance of the time independent D_{d} does not impact the time asymptotic behavior of $W_{M,\beta}$, these terms exhibit a $\sim|\underline{x}|^{-\frac{1}{2}}$ asymptotic behavior (cp. theorem 4.2). The remaining non-time-regularized factors still vanish for asymptotic $\underline{x}$, though their rate of decay is possibly slower than $\sim|\underline{x}|^{-\frac{1}{2}}$, as low energy contributions are not as strongly suppressed.

This can be used to determine the asymptotic behavior of each summand in $\sum_{\substack{S\subset M\\ |S|=m-2k}}$ appearing in 5.3:

- If $|S|=0$, the corresponding summand contains no factor depending on $\underline{y}$. As $\widehat{g}(0)=0$, such a summand vanishes as a result of time-regularization.
- If $|S|=1$, then the summand contains precisely one factor of the form 5.4 or 5.5 and consequently exhibits a $\sim|\underline{x}|^{-\frac{1}{2}}$ asymptotic behavior.
- If $|S|>1$, then the summand can estimated from above by one factor of the form 5.4 or 5.5 and additional factors vanishing for large $|\underline{x}|$. Such summands decay faster than $|\underline{x}|^{-\frac{1}{2}}$.

$\square$

Corollary 5.6. *A state $\omega_{0,\beta}$ as in definition 5.3 is asymptotically compatible with a field equation $L \Leftrightarrow$ for all $f_1, \cdots, f_{m_1}, \cdots, f_{m_2} \in \mathcal{S}(\mathbb{R}^2)$, $\underline{g}_1, \cdots, \underline{g}_{m_1}, \cdots, \underline{g}_{m_2}, g \in \mathcal{S}(\mathbb{R})$ with $\widehat{\underline{g}}_i(0) = \widehat{\underline{g}} = 0$:*

$$\lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \omega_{0,\beta}\Bigg(\prod_{k=1}^{m_1} \phi_{0,\underline{g}_k}(f_k) \cdot L_{\underline{g}}(x) \cdot \prod_{k=m_1+1}^{m_2} \phi_{0,\underline{g}_k}(f_k)\Bigg) = 0\,.$$

This reformulation of the condition of asymptotic compatibility has an interesting consequence: it implies that in a asymptotically compatible thermal state on $\mathcal{A}_0$, D_{d} satisfies a differential equation, which essentially fixes the state.

Theorem 5.7. *Let $\omega_{0,\beta}$ be a β-KMS state on $\mathcal{A}_0$ such that the tempered distribution $\mathbf{f} \mapsto \omega_{0,\beta}(Z_0(\mathbf{f}))$ is given by a smooth function $\omega_{0,\beta}(Z_0(\mathbf{x})) =: D_{\mathrm{d}}(\mathbf{x})$ with $D_{\mathrm{d}}(0) = 1$. Then if $\omega_{0,\beta}$ is asymptotically compatible with a field equation L, D_{d} satisfies the differential equation*

$$\Big(\partial^2 - \sum_{\substack{k\in\mathbb{N}_0 \\ 2k+1\leqslant l}} \tfrac{(2k+1)!}{2^k k!} K_\beta^k c_{2m+1}\Big) D_{\mathrm{d}} = 0\,.$$

Proof. Let $f_1, \cdots, f_{m_1}, \cdots, f_{m_2} \in \mathcal{S}(\mathbb{R}^2)$, $\underline{g}_1, \cdots, \underline{g}_{m_1}, \cdots, \underline{g}_{m_2}, g \in \mathcal{S}(\mathbb{R})$ with the property $\widehat{\underline{g}}_i(0) = \widehat{\underline{g}} = 0$. Consider

$$\begin{aligned}
&\omega_{0,\beta}\Bigg(\prod_{k=1}^{m_1} \phi_{0,\underline{g}_k}(f_k) \cdot L_{\underline{g}}(x) \cdot \prod_{k=m_1+1}^{m_2} \phi_{0,\underline{g}_k}(f_k)\Bigg) \\
= \;& \underbrace{\omega_{0,\beta}\Bigg(\prod_{k=1}^{m_1} \phi_{0,\underline{g}_k}(f_k) \cdot (\Box + M^2)\phi_{0,\underline{g}}(x) \cdot \prod_{k=m_1+1}^{m_2} \phi_{0,\underline{g}_k}(f_k)\Bigg)}_{:=A(x)} \\
+ \;& \sum_{m=0}^{l} c_m \cdot \underbrace{\omega_{0,\beta}\Bigg(\prod_{k=1}^{m_1} \phi_{0,\underline{g}_k}(f_k) \cdot N_{\underline{g}}(\phi_0(x)^m) \cdot \prod_{k=m_1+1}^{m_2} \phi_{0,\underline{g}_k}(f_k)\Bigg)}_{:=B_m(x)}\,.
\end{aligned}$$

Making use of the computations in the proof of lemma 5.4 it is

$$\begin{aligned}
&A(x) \\
= \;& \sum_{j=1}^{m_1} < (\check{\underline{g}}_j * \underline{g}) *_{\underline{y}} (\Box + M^2) D_{\mathrm{d}} \cdot W_{M,\beta}, f_{j,-x} > \cdot\, \omega_{0,\beta}\Bigg(\prod_{\substack{k=1\\k\neq j}}^{m_2} \phi_{0,\underline{g}_k}(f_k)\Bigg) \\
+ & \sum_{j=m_1+1}^{m_2} < (\underline{g}_j * \check{\underline{g}}) *_{\underline{y}} (\Box + M^2) D_{\mathrm{d}} \cdot W_{M,\beta}, \check{f}_{j,x} > \cdot\, \omega_{0,\beta}\Bigg(\prod_{\substack{k=1\\k\neq j}}^{m_2} \phi_{0,\underline{g}_k}(f_k)\Bigg)\,.
\end{aligned}$$

Note that since $W_{M,\beta}$ satisfies the mass M Klein-Gordon equation, it is

$$(\Box + M^2) D_{\mathrm{d}} \cdot W_{M,\beta} = (-\partial_{\mathbf{y}}^2 D_{\mathrm{d}}) W_{M,\beta} - 2(\partial_{\mathbf{y}} D_{\mathrm{d}})(\partial_{\mathbf{y}} W_{M,\beta})\,,$$

where the first term decays like $|\underline{x}|^{-\frac{1}{2}}$ and the second like $|\underline{x}|^{-1}$ when time-regularized by $\breve{\underline{g}}_j * \underline{g}$ or $\underline{g}_j * \breve{\underline{g}}$, as a result of theorem 4.2. Consequently, it is

$$\begin{aligned}
&\lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} A(x) \\
= \;& \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \sum_{j=1}^{m_1} < (\breve{\underline{g}}_j * \underline{g}) *_{\underline{y}} W_{M,\beta}, -(\partial_{\mathbf{y}}^2 D_{\mathrm{d}}) f_{j,-x} > \cdot\, \omega_{0,\beta}\Big(\prod_{\substack{p=1\\ p\neq j}}^{m_2} \phi_{0,\underline{g}_p}(f_p) \Big) \\
+ \;& \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \sum_{j=m_1+1}^{m_2} < (\underline{g}_j * \breve{\underline{g}}) *_{\underline{y}} \cdot W_{M,\beta}, -(\partial_{\mathbf{y}}^2 D_{\mathrm{d}}) \breve{f}_{j,x} > \cdot\, \omega_{0,\beta}\Big(\prod_{\substack{p=1\\ p\neq j}}^{m_2} \phi_{0,\underline{g}_p}(f_p) \Big) .
\end{aligned}$$

Now recall (proof of lemma 5.5) that

$$\begin{aligned}
&B_m(x) \\
= \;& \sum_{\substack{m-(m_1+m_2)\\ \leqslant 2k \leqslant m}} \frac{m!}{2^k k!} K_\beta^k \sum_{\substack{S\subset M\\ |S|=m-2k}} \omega_{0,\beta}\Big(\prod_{\substack{p\in M\setminus S\\ \text{ordered}}} \phi_{0,\underline{g}_p}(f_p) \Big) \\
& \times \int \mathrm{d}\underline{y}\, \underline{g}(\underline{y}) \prod_{\substack{p\in S\\ p\leqslant m_1}} \omega_{0,\beta}(\phi_{0,\underline{g}_p}(f_p)\phi_0(\underline{x}+\underline{y},\mathbf{x})) \cdot \prod_{\substack{p\in S\\ p> m_1}} \omega_{0,\beta}(\phi_0(\underline{x}+\underline{y},\mathbf{x})\phi_{0,\underline{g}_p}(f_p)) ,
\end{aligned}$$

where the summands with $m-2k = |S| = 1$ give rise to the leading order $|\underline{x}|^{-\frac{1}{2}}$ behavior. Further using results from the proof of lemma 5.5 it follows that

$$\begin{aligned}
&\lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} B_m(x) \\
= \;& \sum_{\substack{m-(m_1+m_2)\\ \leqslant 2k \leqslant m}} \frac{m!}{2^k k!} K_\beta^k \cdot \delta_{m-2k,1} \;\times \\
& \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \sum_{j=1}^{m_1} < (\breve{\underline{g}}_j * \underline{g}) *_{\underline{y}} W_{M,\beta}, D_{\mathrm{d}} \cdot f_{j,-x} > \cdot\, \omega_{0,\beta}\Big(\prod_{\substack{p=1\\ p\neq j}}^{m_2} \phi_{0,\underline{g}_p}(f_p) \Big) \\
+ \;& \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \sum_{j=m_1+1}^{m_2} < (\underline{g}_j * \breve{\underline{g}}) *_{\underline{y}} \cdot W_{M,\beta}, D_{\mathrm{d}} \cdot \breve{f}_{j,x} > \cdot\, \omega_{0,\beta}\Big(\prod_{\substack{p=1\\ p\neq j}}^{m_2} \phi_{0,\underline{g}_p}(f_p) \Big) .
\end{aligned}$$

Combining the obtained expressions for $\lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} A(x)$ and $\lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} B_m(x)$, the asymptotic field equation can be expressed as follows:

$$
\begin{aligned}
0 &= \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \omega_{0,\beta}\left(\prod_{k=1}^{m_1} \phi_{0,\underline{g}_k}(f_k) \cdot L_{\underline{g}}(x) \cdot \prod_{k=m_1+1}^{m_2} \phi_{0,\underline{g}_k}(f_k)\right) \\
&= \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \left(A(x) + \sum_{m=0}^{l} c_m \cdot B_m(x)\right) \\
&= \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \sum_{j=1}^{m_1} < (\breve{\underline{g}}_j * \underline{g}) *_{\underline{y}} W_{M,\beta}, f_{j,-x} \cdot \left(-\partial_{\mathbf{y}}^2 + \sum_{\substack{k\in\mathbb{N}_0\\ 2k+1\leqslant l}} \tfrac{(2k+1)!}{2^k k!} K_\beta^k c_{2k+1}\right) D_{\mathrm{d}} > \\
&\quad \times \omega_{0,\beta}\left(\prod_{\substack{p=1\\ p\neq j}}^{m_2} \phi_{0,\underline{g}_p}(f_p)\right) \\
&+ \lim_{|\underline{x}|\to\infty} |\underline{x}|^{\frac{1}{2}} \sum_{j=m_1+1}^{m_2} < (\underline{g}_j * \breve{\underline{g}}) *_{\underline{y}} \cdot W_{M,\beta}, \breve{f}_{j,x} \cdot \left(-\partial_{\mathbf{y}}^2 + \sum_{\substack{k\in\mathbb{N}_0\\ 2k+1\leqslant l}} \tfrac{(2k+1)!}{2^k k!} K_\beta^k c_{2k+1}\right) D_{\mathrm{d}} > \\
&\quad \times \omega_{0,\beta}\left(\prod_{\substack{p=1\\ p\neq j}}^{m_2} \phi_{0,\underline{g}_p}(f_p)\right),
\end{aligned}
$$

which is satisfied iff

$$\left(\partial^2 - \sum_{\substack{k\in\mathbb{N}_0\\ 2k+1\leqslant l}} \tfrac{(2k+1)!}{2^k k!} K_\beta^k c_{2k+1}\right) D_{\mathrm{d}} = 0\,.$$

□

The constant $\xi_{P,\beta} := \sum_{\substack{k\in\mathbb{N}_0\\ 2k+1\leqslant l}} \frac{(2k+1)!}{2^k k!} K_\beta^k c_{2k+1}$ depends only on the interaction polynomial P and the inverse temperature β. In light of the identification of D_{d} with the discrete part of the damping factor, D_{d} is even. Along with the normalization condition $D_{\mathrm{d}}(0) = 1$ and the assumed smoothness of D_{d}, this guarantees that the differential equation $D_{\mathrm{d}}'' - \xi_{P,\beta} D_{\mathrm{d}} = 0$ has unique solutions:

- If $\xi_{P,\beta} < 0$, then $D_{\mathrm{d}}(\mathbf{x}) = \cos(\sqrt{-\xi_{P,\beta}}\mathbf{x})$.
- If $\xi_{P,\beta} = 0$, then $D_{\mathrm{d}}(\mathbf{x}) = 1$.
- If $\xi_{P,\beta} > 0$, then $D_{\mathrm{d}}(\mathbf{x}) = \cosh(\sqrt{\xi_{P,\beta}}\mathbf{x})$.

In the latter case D_{d} is not tempered, which means that the identification with the discrete part of a damping factor cannot be valid. Under the assumptions made on D_{d}, the method of asymptotic analysis is not suitable to detect the particle content of a thermal theory with interaction P at temperatures such that $\xi_{P,\beta} > 0$. If one is willing to consider non-relativistic KMS states, the smoothness condition on D_{d} can be eased and solutions like $D_{\mathrm{d}}(\mathbf{x}) = e^{-\xi_{P,\beta}|\mathbf{x}|}$ become admissible for $\xi_{P,\beta} > 0$. This ansatz is not pursued further in this thesis.

For $\xi_{P,\beta} = 0$, the asymptotically compatible β-KMS state behaves like the free β-KMS state in the sense that their reduced two-point functions define the same tempered distribution. It is noteworthy that P need not be trivial for this case to occur at special temperatures β. In this scenario attractive and repulsive influences of constituents of the thermal background on particles would statistically. The interaction would only manifest itself in quickly decaying contributions to correlation functions due to unstable excitations.

In the $\xi_{P,\beta} < 0$ case, $\omega_{\beta,0}$ with $D_{\mathrm{d}}(\mathbf{x}) = \cos(\sqrt{|\xi_{P,\beta}|}\mathbf{x})$ defines in fact a state. It suffices to check positivity:

$$\begin{aligned}
& \omega_{0,\beta}(\phi_0(f)^* \phi_0(f)) \\
= \; & < W_{M,\beta}, D_{\mathrm{d}} \cdot \overline{f} * \check{f} > \\
= \; & < W_{M,\beta}, x \mapsto \cos(\sqrt{|\xi_{P,\beta}|}\mathbf{x}) \cdot \overline{f} * \check{f}(x) > \\
= \; & < W_{M,\beta}, x \mapsto \overline{f_c} * \check{f_c}(x) + \overline{f_s} * \check{f_s}(x) > \\
= \; & < W_{M,\beta}, \overline{f_c} * \check{f_c} > + < W_{M,\beta}, \overline{f_s} * \check{f_s} > ,
\end{aligned}$$

where $f_s(x) := f(x)\sin(\sqrt{|\xi_{P,\beta}|}\mathbf{x})$ and $f_c(x) := f(x)\cos(\sqrt{|\xi_{P,\beta}|}\mathbf{x})$. The used identity $\cos(\sqrt{|\xi_{P,\beta}|}\mathbf{x}) \cdot \overline{f} * \check{f}(x) = \overline{f_c} * \check{f_c}(x) + \overline{f_s} * \check{f_s}(x)$ is a consequence of angle sum identities for sin and cos. $< W_{M,\beta}, \overline{f_c} * \check{f_c} > 0$ and $< W_{M,\beta}, \overline{f_s} * \check{f_s} >$ are non-negative, as $W_{M,\beta}$ is of positive type. The damping factor being spatially periodic is an indication for $W_{0,\beta}$ exhibiting a standing wave character. It is also apparent that no interaction of the considered form essentially dampens particle propagation for asymptotic times: At certain distances (namely integer multiples of $\frac{2\pi}{\sqrt{|\xi_{P,\beta}|}}$) the propagation amplitudes are as if they were in the absence of interaction. Computing the Fourier transform of $W_{0,\beta}$ one obtains (using $\mathcal{F}(\cos)(\mathbf{p}) = \pi(\delta(1-\mathbf{p}) + \delta(1+\mathbf{p}))$):

$$\widehat{W}_{0,\beta}(p) = 2\pi^2 \frac{\varepsilon(\underline{p})}{1-e^{-\beta \underline{p}}} \left[\delta(\underline{p}^2 - (\mathbf{p} - \sqrt{|\xi_{P,\beta}|})^2 - M^2) + \delta(\underline{p}^2 - (\mathbf{p} + \sqrt{|\xi_{P,\beta}|})^2 - M^2) \right]. \quad (5.6)$$

This expression defines a sharp dispersion law

$$\underline{p}(\mathbf{p}) = \pm_1 \sqrt{\left(\mathbf{p} \pm_2 \sqrt{|\xi_{P,\beta}|}\right)^2 + M^2} \, .$$

The choice of sign in $\pm_1$ also appears in the free, undamped case and corresponds to particles/holes. The second choice of sign in $\pm_2$ is due to the periodic damping and can be ascribed to incoming/outgoing waves responsible for the standing wave character of $W_{0,\beta}$.

The singular nature of $\widehat{W}_{0,\beta}$ does not stand in contradiction to the Narnhofer-Requardt-Thirring theorem: the contribution $W_{\mathrm{d}} = D_{\mathrm{d}} W_{M,\beta}$ to the full two-point function of the original theory due to the exchange of stable particles of mass M - here identified with $W_{0,\beta}$ - already subsumes the collective effects of the interaction with the thermal background into the damping factor D_d. The statement of the theorem is only that this subsumption of interaction possibilities is complete and no "residual interaction" is encoded in the asymptotic time structure of the truncated m-point functions.

The occurrence of the incoming/outgoing wave structure can be attributed to scattering of a wave created by ϕ at 0 off a complex vector potential A. For given D_d the vector potential ansatz

$$\left[\partial_{\underline{x}}^2 - (\partial_\mathbf{x} - A(\mathbf{x}))^2 + M^2\right] D_\mathrm{d}(\mathbf{x}) W_{M,\beta}(\underline{x}, \mathbf{x}) = 0 \tag{5.7}$$

has the solution $A(\mathbf{x}) = \partial_\mathbf{x} D_\mathrm{d}(\mathbf{x})/D_\mathrm{d}(\mathbf{x})$. In the case at hand ($D_\mathrm{d}(\mathbf{x}) = \cos(\sqrt{|\xi_{P,\beta}|}\mathbf{x})$) it is

$$A(\mathbf{x}) = -|\xi_{P,\beta}| \tan(\sqrt{|\xi_{P,\beta}|}\, \mathbf{x}) .$$

The ansatz in equation 5.7 corresponds to the Klein-Gordon equation for a free charged particle in the presence of a magnetic field with A as its vector potential. However, the magnetic analogy introduces interpretational difficulties: in the present case of one spatial dimension, no magnetic fields exist. In addition, $A(\mathbf{x})$ is real, resulting in a non-hermitian canonical momentum.
A different interpretation presents itself when expanding the term in equation 5.7 containing the vector potential:

$$(\partial_\mathbf{x} - A(\mathbf{x}))^2 = \partial_\mathbf{x}^2 + A(\mathbf{x})^2 - (A(\mathbf{x})\partial_\mathbf{x} + \partial_\mathbf{x} A(\mathbf{x})) .$$

$A(\mathbf{x})^2$ then plays the role of a regular potential and $(A(\mathbf{x})\partial_\mathbf{x} + \partial_\mathbf{x} A(\mathbf{x}))$ that of an anti-hermitian, velocity-dependent potential. The anti-hermitian potential is related to dissipative effects in the following way:
One can associate to the "particle part" $D_\mathrm{d} W_{M,\beta}$ of the full two-point function ($D_\mathrm{d} W_{M,\beta}$ is due to the discrete part of the damping factor, which, according to the Bros-Buchholz criterion, stems from stable massive particles) a "particle Hamiltonian" corresponding to $i\partial_t$ in equation 5.7. This "particle Hamiltonian" describes the non-unitary time evolution of massive particles in the theory. Mathematically, the non-unitarity of this time evolution is due to the anti-hermitian term in the potential. The physical interpretation is that massive particles are subjected to dissipative effects of the thermal background and an energy transfer between the particle and the background takes place in the associated collision processes. As the resulting excitations of the thermal background are expected to only contribute to the continuous part of the two-point function or to higher correlation functions, this exchange of energy with the thermal background will be reflected in the "particle part" of the two-point function. The full Hamiltonian takes both sides of the process - particle excitations and the thermal background - into account and the time evolution of the entire system is of course unitary.
The notable periodicity of A can be interpreted in terms of a mean particle distribution. In a statistical average, constituent particles of the thermal background are spaced equidistantly giving rise to the periodic vector potential A with poles at the expected particle positions. In this picture $|\xi_{P,\beta}|^{-\frac{1}{2}}$ can be given meaning as a mean free path

length. The fact that the damping factor does not essentially decay in $\mathbf{x}$ is expected to be a feature of particles not having additional spatial directions in which they can disperse.

In light of the difficulties plaguing the $\xi_{P,\beta} > 0$ case, it comes as a relief that the restriction $\xi \leqslant 0$ is still capable of describing a range of physically interesting situations. To illustrate this, consider an effective, stable ϕ^6 theory, where the leading order coefficient c_5 of the interaction polynomial P appearing in the field equation $(\Box + M^2)\phi + P(\phi) = 0$ is positive. In this case $\xi_{P,\beta} = \sum_{\substack{k\in\mathbb{N}_0 \\ 2k+1\leqslant l}} \frac{(2k+1)!}{2^k k!} K_\beta^k c_{2k+1}$ takes the form

$$\xi_{P,\beta} = 3c_1 K_\beta + 15c_3 K_\beta^2 + 21c_5 K_\beta^3 \,. \tag{5.8}$$

As $\xi_{P,\beta}$ is linear in P, one can divide by the positive leading order coefficient without changing the sign $\varepsilon(\xi_{P,\beta})$ of $\xi_{P,\beta}$, i.e. $21c_5 = 1$ can be assumed without loss of generality. Modifying c_1, c_3 by their respective combinatorial factor to obtain new constants d_1, d_3, equation 5.9 can be written as

$$\xi_{P,\beta} = d_1 K_\beta + d_3 K_\beta^2 + K_\beta^3 \,. \tag{5.9}$$

Recall that K_β was given by (cp. equation 5.1)

$$K_\beta = \frac{1}{2\pi} \int \mathrm{d}\mathbf{p} \, \frac{1}{\sqrt{\mathbf{p}^2 + M^2}} \frac{1}{e^{\beta\sqrt{\mathbf{p}^2+M^2}} - 1} \,.$$

For all $\beta \in]0,\infty[$ the integrand is positive and becomes monotonously larger for increasing $T = \beta^{-1}$. By the Lebesgue dominated convergence theorem, K_β vanishes in the limit $T \to 0$. In addition, $T \mapsto K_\beta$ is unbounded, continuous and consequently extends to a continuous, monotonously increasing bijection of $[0,\infty[$ onto itself. Excluding the $T = 0$ case where $\xi_{P,\beta} \leqslant 0$ is trivially satisfied and $W_{0,\beta}$ becomes the free mass M vacuum two-point function, the condition $\xi_{P,\beta} \leqslant 0$ can be phrased as $d_1 + d_3 K_\beta + K_\beta^2 \leqslant 0$, which is equivalent to

$$\frac{d_3^2}{4} \geqslant d_1 \quad \wedge \quad K_\beta \leqslant \sqrt{\frac{d_3^2}{4} - d_1} - \frac{d_3}{2} \quad \wedge \quad K_\beta \geqslant -\sqrt{\frac{d_3^2}{4} - d_1} - \frac{d_3}{2} \,.$$

- For $d_1 < 0$ and arbitrary d_3, this condition is satisfied for small temperatures up to a boundary temperature T_b determined by the $K_{1/T_b} = \sqrt{\frac{d_3^2}{4} - d_1} - \frac{d_3}{2}$.
- For $d_1 \geqslant 0$ and $d_3 \leqslant -2d_1$ the condition is satisfied for temperatures $T \in [T_{b-}, T_{b+}]$, where the boundary temperatures $T_{b\pm}$ are determined by

$$K_{1/T_{b+}} = \sqrt{\frac{d_3^2}{4} - d_1} - \frac{d_3}{2} \quad \text{and} \quad K_{1/T_{b-}} = -\sqrt{\frac{d_3^2}{4} - d_1} - \frac{d_3}{2} \,.$$

- In all other cases the condition $\xi_{P,\beta} \leqslant 0$ is not met for any temperature.

Part of this result qualitatively generalizes to arbitrary interactions: as long as c_1 (or the lowest non-vanishing odd coefficient) is negative, the condition $\xi_{P,\beta}$ is always satisfied for sufficiently low temperatures.

6. Asymptotic Green's Functions

The approaches of real time and imaginary time formalism to thermal quantum field theory rely on the study of certain functions, namely the retarded, advanced, time-ordered and anti-time-ordered propagators. It is the goal of this chapter to translate the results established for asymptotic thermal two-point functions into this framework by computing these functions. Computations are performed with a KMS state $\omega_{0,\beta}$ on the algebra of asymptotic fields in mind.

6.1. General Computation

By locality the commutator function $C = W - \breve{W}$ has support in $\overline{V}^+ \cup \overline{V}^-$. The advanced and retarded propagators A, R are obtained by splitting the commutator function C into parts with support in $\overline{V}^-$ and $\overline{V}^+$ respectively such that

$$iC = R - A\,.$$

In the cases of interest this splitting can be performed unambiguously by setting

$$\begin{aligned} R(x) &= i\theta(\underline{x})C(x)\,, \\ A(x) &= -i\theta(-\underline{x})C(x)\,. \end{aligned}$$

These products of distributions are to be understood in the following sense: In the cases considered, C becomes a smooth, polynomially bounded function $\underline{x} \mapsto C^{\mathbf{f}}(\underline{x})$ if smeared with a Schwartz function $\mathbf{f} \in \mathcal{S}(\mathbb{R})$ in the spatial argument. The product $x \mapsto \theta(\underline{x})C^{\mathbf{f}}(\underline{x})$ can then be integrated with a Schwartz function $\underline{f} \in \mathcal{S}(\mathbb{R})$ defining $< \theta \cdot C, \underline{f} \otimes \mathbf{f} >$.

The support properties of R and A imply that their Fourier transforms $\widehat{R}, \widehat{A}$ are distributional boundary values of holomorphic functions on the respective tube domains $\mathcal{R}_\pm := \{z \in \mathbb{C}^2 \mid \mathrm{Im}(z) \in V^\pm\}$ (cf. [4]). These holomorphic functions are also denoted $\widehat{R}, \widehat{A}$.

Lemma 6.1. *If $\widehat{C}$ vanishes on a open set $\mathcal{E} \subset \mathbb{R}^2$, then there exists an open set $\mathcal{D} \subset \mathbb{C}^2$ containing $\mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{E}$ and a holomorphic function*

$$\widehat{G} : \mathcal{D} \to \mathbb{C}\,,$$

such that $\widehat{G}|_{\mathcal{R}_+} = \widehat{R}$ and $\widehat{G}|_{\mathcal{R}_-} = \widehat{A}$.

Proof. $\widehat{C}|_{\mathcal{E}} = 0$ means that $\widehat{R}|_{\mathcal{E}} = \widehat{A}|_{\mathcal{E}}$. Hence, for any test function f with support in $\mathcal{E}$ it is

$$\lim_{\substack{\varepsilon\searrow 0\\ \varepsilon\in V^+}} \int \mathrm{d}p\, \widehat{R}(p+i\varepsilon) f(p) = \lim_{\substack{\varepsilon\nearrow 0\\ \varepsilon\in V^-}} \int \mathrm{d}p\, \widehat{A}(p+i\varepsilon) f(p)\,.$$

The statement of the lemma follows from an application of the edge-of-the-wedge theorem. □

Definition 6.1. The holomorphic function $\widehat{G}$ from the preceding lemma is called **Green's function**.

As the Green's function allows to recover the commutator function via

$$\begin{aligned} i\widehat{C}(p) = \widehat{R}(p) - \widehat{A}(p) &= \lim_{\substack{\varepsilon\searrow 0\\ \varepsilon\in V^+}} \widehat{R}(p+i\varepsilon) - \lim_{\substack{\varepsilon\nearrow 0\\ \varepsilon\in V^-}} \widehat{A}(p+i\varepsilon) \\ &= \lim_{\substack{\varepsilon\searrow 0\\ \varepsilon\in V^+}} \widehat{G}(p+i\varepsilon) - \lim_{\substack{\varepsilon\nearrow 0\\ \varepsilon\in V^-}} \widehat{G}(p+i\varepsilon)\,, \end{aligned}$$

and in time-clustering, translation invariant thermal states, the two-point function can be computed form the commutator function, the Green's function contains complete information about two-point correlations of theories induced by such states. If the state in question is quasifree, the entire theory can be recovered from the Green's function.

The **time-ordered** and **anti-time-ordered propagators** are defined as the following distributions:

$$\begin{aligned} T &:= W - iA = \breve{W} - iR\,, \\ S &:= W + iR = \breve{W} + iA\,. \end{aligned}$$

6.2. Concrete Computation

Using the general results of the previous section, advanced, retarded, time-ordered and anti-time-ordered propagators are computed for asymptotic thermal fields in the case of "ood coupling", where the two-point function is $W_{0,\beta}(x) = \cos(\sqrt{|\xi|}\mathbf{x}) \cdot W_{M,\beta}(x)$. The Fourier transform of $W_{0,\beta}$ is given by equation 5.6, which can be used to compute the Fourier transform of the commutator function:

$$\widehat{C}_{0,\beta}(p) = 2\pi^2 \varepsilon(\underline{p})[\delta(\underline{p}^2 - (\mathbf{p} - \sqrt{|\xi|})^2 - M^2) + \delta(\underline{p}^2 - (\mathbf{p} + \sqrt{|\xi|})^2 - M^2)]\,.$$

The support of $\widehat{C}_{0,\beta}$ is contained in $\{p \in \mathbb{R}^2 \,|\, \underline{p}^2 \geqslant M^2\}$, so for $\mathcal{E} := \{p \in \mathbb{R}^2 \,|\, \underline{p}^2 < M^2\}$ ones has $\widehat{C}_{0,\beta}|_{\mathcal{E}} = 0$ and lemma 6.1 applies, which guarantees the existence of a Green's Function as defined in the previous section.

The Green function $\widehat{G}_{0,\beta}$ can be computed by means of a comparison to the free mass M case. Note that in the sense of distributions it is $C_{0,\beta}(x) = \cos(\sqrt{|\xi|})\mathbf{x})C_M(x)$. If $iC_M = R_M - A_M$, then

$$\begin{aligned} R_{0,\beta}(x) &= \cos(\sqrt{|\xi|})\mathbf{x})R_M(x)\,, \\ A_{0,\beta}(x) &= \cos(\sqrt{|\xi|})\mathbf{x})A_M(x)\,. \end{aligned}$$

It is known [6] that the free mass M advanced and retarded propagators have the limit representation (as distributions)

$$\begin{aligned} \widehat{R}_M(p) &= \lim_{\substack{\varepsilon\searrow 0\\ \varepsilon\in V+}} \frac{1}{(p+i\varepsilon)^2 - M^2}\,, \\ \widehat{A}_M(p) &= \lim_{\substack{\varepsilon\nearrow 0\\ \varepsilon\in V-}} \frac{1}{(p+i\varepsilon)^2 - M^2}\,. \end{aligned}$$

Multiplication by $\cos(\sqrt{|\xi|}\mathbf{x})$ acts as a sum of translations of $\mathbf{p}$ by $\pm\sqrt{|\xi|}$ in momentum space. It follows that

$$\begin{aligned} \widehat{R}_{0,\beta}(p) &= \lim_{\substack{\varepsilon\searrow 0\\ \varepsilon\in V+}} \left[\frac{1}{(p+(0,\sqrt{|\xi|})+i\varepsilon)^2 - M^2} + \frac{1}{(p-(0,\sqrt{|\xi|})+i\varepsilon)^2 - M^2}\right]\,, \\ \widehat{A}_{0,\beta}(p) &= \lim_{\substack{\varepsilon\nearrow 0\\ \varepsilon\in V-}} \left[\frac{1}{(p+(0,\sqrt{|\xi|})+i\varepsilon)^2 - M^2} + \frac{1}{(p-(0,\sqrt{|\xi|})+i\varepsilon)^2 - M^2}\right]\,. \end{aligned}$$

In this form it is obvious that $\widehat{R}_{0,\beta}$ and $\widehat{A}_{0,\beta}$ are distributional boundary values of the holomorphic function

$$\widehat{G}_{0,\beta}(k) = \frac{1}{k_+^2 - M^2} + \frac{1}{k_-^2 - M^2}\,,$$

where $k_\pm = p_\pm + is$ and $p_\pm = p \pm (\sqrt{|\xi|}, 0)$. The following lemma shows that the domain of holomorphy contains $\mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{E}$ and is in fact the Green's function.

Lemma 6.2. *If $k \in \mathcal{R}_+ \cup \mathcal{R}_- \cup \mathcal{E}$, then $k_+^2 - M^2 \neq 0$ and $k_-^2 - M^2 \neq 0$.*

Proof. First note that for $k = p + is$ it is

$$\begin{aligned} & k_\pm^2 - M^2 = 0 \\ \Leftrightarrow\ & p_\pm^2 - s^2 - M^2 + 2isp_\pm = 0 \\ \Leftrightarrow\ & p_\pm = s^2 + M^2 \quad \wedge \quad sp_\pm = 0\,. \end{aligned}$$

Now let $k \in \mathcal{R}_+$. As $s \in V^+$ it is $\underline{s} > |\mathbf{s}| \geqslant 0$. Assume that $sp_\pm = 0$.

$$sp_\pm = 0 \Rightarrow \underline{p}\underline{s} = \mathbf{p}_\pm\mathbf{s} \Rightarrow \underline{p} = \mathbf{p}_\pm \underbrace{\frac{\underline{s}}{\mathbf{s}}}_{|\cdot|<1}\,.$$

Using this, one obtains the estimate

$$p_\pm^2 = \underline{p}^2 - \mathbf{p}_\pm^2 = \mathbf{p}_\pm^2 \left(\left(\frac{\underline{s}}{\mathbf{s}} \right)^2 - 1 \right) \leqslant 0 < s^2 < s^2 + M^2 \,,$$

so $p_\pm \neq 0$ and consequently $k_\pm^2 - M^2 \neq 0$.

In the case of $k \in \mathcal{R}_-$, it is $-\underline{s} > |\mathbf{s}| \geqslant 0$ and again $\left(\frac{\underline{s}}{\mathbf{s}}\right)^2 < 1$. The above argument thus also holds in this case and it is $k_\pm^2 - M^2 \neq 0$.

Finally, for $k \in \mathcal{E}$ it is $s = 0$, so

$$p_\pm^2 = \underline{p}^2 - \mathbf{p}^2 < M^2 - \mathbf{p}^2 \leqslant M^2 = s^2 + M^2$$

and $p_\pm^2 \neq s^2 + M^2$ which implies $k_\pm^2 - M^2 \neq 0$. □

The time-ordered and anti-time-ordered propagators can be computed from $\widehat{G}_{0,\beta}$ in a straightforward manner:

$$\begin{aligned}
\widehat{T}_{0,\beta}(p) &= \widehat{W}_{0,\beta}(p) - i\widehat{A}_{0,\beta}(p) \\
&= \frac{\widehat{C}_{0,\beta}(p)}{1 - e^{-\beta\underline{p}}} - i\widehat{A}_{0,\beta}(p) \\
&= i\frac{\widehat{A}_{0,\beta}(p) - \widehat{R}_{0,\beta}(p)}{1 - e^{-\beta\underline{p}}} - i\widehat{A}_{0,\beta}(p) \\
&= -i\left[\frac{\widehat{A}_{0,\beta}(p)}{1 - e^{\beta\underline{p}}} + \frac{\widehat{R}_{0,\beta}(p)}{1 - e^{-\beta\underline{p}}}\right] \\
&= -i \lim_{\substack{\varepsilon\searrow 0 \\ \varepsilon\in V^+}} \left[\frac{\widehat{G}_{0,\beta}(p - i\varepsilon)}{1 - e^{\beta\underline{p}}} + \frac{\widehat{G}_{0,\beta}(p + i\varepsilon)}{1 - e^{-\beta\underline{p}}}\right] .
\end{aligned}$$

Similarly, the anti-time-ordered propagator is given by

$$\begin{aligned}
\widehat{S}_{0,\beta}(p) &= i\left[\frac{\widehat{A}_{0,\beta}(p)}{1 - e^{-\beta\underline{p}}} + \frac{\widehat{R}_{0,\beta}(p)}{1 - e^{\beta\underline{p}}}\right] \\
&= i \lim_{\substack{\varepsilon\searrow 0 \\ \varepsilon\in V^+}} \left[\frac{\widehat{G}_{0,\beta}(p - i\varepsilon)}{1 - e^{-\beta\underline{p}}} + \frac{\widehat{G}_{0,\beta}(p + i\varepsilon)}{1 - e^{\beta\underline{p}}}\right] .
\end{aligned}$$

7. Summary and Outlook

At the core of this work is the Bros-Buchholz expansion in two-dimensional thermal quantum field theory. A large class of thermal two-point functions can be expanded in terms of free thermal two-point functions of varying masses, weighted by a damping factor. This representation gives rise to a natural criterion for a thermal theory to describe massive particles and provides the basis for the analysis of asymptotic dynamics in effective theories. The effective theories need not be full fledged quantum field theories, only the heuristic input that the fields in question in some sense satisfy a field equation with polynomial interaction is needed. These heuristic considerations manifest themselves in the condition of asymptotic compatibility, which allows to compute the contributions to the damping factor responsible for the leading order of the two-point function at asymptotic times. However, there is no guarantee that these "andiate" contributions, computed on the basis of heuristic assumptions, do in fact match their rigorous counterparts in full theories with polynomial interaction.
The expansion of thermal two-point function and the analysis of asymptotic dynamics in four-dimensional quantum field theory is a feat that has been accomplished by J. Bros and D. Buchholz in 1992 ([7], [8], [9]). The motivation for extending the four-dimensional formalism to two dimensions is found in the fact that, to date, no rigorous non-trivial four-dimensional quantum field theory exists. This is not the case in two dimensions. In 2004 C. Gérard and C. Jäkel rigorously constructed thermal quantum field theories with polynomial interaction ([11]). They later showed that two-point functions of these theories satisfy the relativistic KMS condition, making them good candidates for the application of the methods established in this work ([12]). In particular, two questions arise naturally.

- In theory, the damping factor D can be computed for two-point functions in such theories via the formula given in proposition 3.1. Does the damping factor contain discrete parts of the form $\delta(m - M)D_M$ and, consequently, the corresponding theory describe particles according to the proposed criterion?

- If the first question has a positive answer, the formalism of asymptotic analysis can be applied using the field equation of the full theory and the resulting "candidate" D_M can be compared to the corresponding discrete contributions to D. Is the method of asymptotic dynamics capable of describing the full particle content of the rigorous theories?

A starting point for the computation of D is the two-point function of a full theory provided in [12]:

$$\underline{g}_1 * \check{\underline{g}}_2 *_{\underline{x}} W_\beta(\underline{x}, \mathbf{x}) = < \pi_\beta(\phi(\underline{g}_1))\Omega_\beta | e^{i\underline{x}L_\beta} \alpha_{\mathbf{x}}(\pi_\beta(\phi(\underline{g}_2)))\Omega_\beta > ,$$

Here π_β is the GNS representation of the underlying state ω_β, Ω_β the GNS vector and $\underline{g}_1, \underline{g}_2 \in \mathcal{D}_{\mathbb{R}}(\mathbb{R})$. $\pi_\beta(\phi(\underline{g}))$ has to be understood as the limit $\lim_{\mathbf{g}\to\delta} \pi_\beta(\phi(\underline{g} \otimes \mathbf{g}))$, which already defines an operator on the dense invariant domain $\mathcal{D}_\beta$, as ω_β satisfies the relativistic KMS condition. The dynamical information (the field equation) is contained in the **Liouvillian** L_β, which generates time translations and is constructed using Euclidian methods. The challenge in computing D lies in a) finding an accessible expression for L_β and b) using the definition of the damping factor to obtain a useful expression for D.

A. Distributional Laplace Transform

Proposition A.1. *Let $T \in \mathcal{S}'(\mathbb{R})$ with* $\operatorname{supp}(T) \subset [0, \infty[$. *For $\lambda \in \mathbb{C}_+$ let $f_\lambda \in \mathcal{S}(\mathbb{R})$ such that $f_\lambda|_{[0,\infty[}(s) = e^{-\lambda s}$. Then the map*

$$\begin{aligned} \mathcal{L}T : \mathbb{C}_+ &\to \mathbb{C}\,, \\ \lambda &\mapsto \; < T, f_\lambda > \end{aligned}$$

is well-defined, holomorphic and polynomially bounded on the shifted half planes $\mathbb{C}_\gamma := \{\, \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \gamma \,\}$, $\gamma > 0$.

Proof. $\mathcal{L}T$ is well-definied: for $a \in \mathbb{R}$, let $\tau_a : x \mapsto x + a$ be the translation by a on $\mathbb{R}$. For $\varphi \in \mathcal{S}(\mathbb{R})$ the map

$$\begin{aligned} \mathbb{R} &\to \mathcal{S}(\mathbb{R})\,, \\ a &\mapsto \varphi \circ \tau_a \end{aligned}$$

is continuous. If $f_\lambda, g_\lambda \in \mathcal{S}(\mathbb{R})$ have the property $f_\lambda|_{[0,\infty[}(s) = g_\lambda|_{[0,\infty[}(s) = e^{-\lambda s}$, then $(f_\lambda - g_\lambda) \circ \tau_a$ has support in $]-\infty, -a]$. The support properties of T ensure that for all $a > 0$ it is $< T, (f_\lambda - g_\lambda) \circ \tau_a >= 0$. Hence

$$< T, (f_\lambda - g_\lambda) >= \lim_{a \searrow 0} < T, (f_\lambda - g_\lambda) \circ \tau_a >= 0\,.$$

This shows that $\mathcal{L}T$ is well-defined.

Since $]0, \infty[$ is an open convex cone in $\mathbb{R}$, by the Bros-Epstein-Glaser lemma can be applied and T has the form

$$\forall\, \varphi \in \mathcal{S}(\mathbb{R}) \; :< T, \varphi >= \sum_{\substack{k \in \mathbb{N}_0 \\ \text{finite}}} \int \mathrm{d}s\, T_k(s) (\partial^k \varphi)(s)\,,$$

where the T_α have support in $[0, \infty[$ and are polynomially bounded. Hence it is

$$\mathcal{L}T(\lambda) = \sum_{\substack{k \in \mathbb{N}_0 \\ \text{finite}}} \int_0^\infty \mathrm{d}s\, T_k(s) (-\lambda)^k e^{-\lambda s}\,.$$

In order to prove that $\mathcal{L}T$ is holomorphic on $\mathbb{C}_+$, it is sufficient to show that all line integrals over closed paths p in $\mathbb{C}_+$ vanish (Morera's theorem). As suprema in λ of the integrands $|T_k(s)(-\lambda)^k e^{-\lambda s}|$ over compact sets $K \subset \mathbb{C}_+$ remain integrable with respect to s, Fubini's theorem allows the interchange of line- and ds-integration. Since for each

s the integrands are holomorphic in λ, the line integrals vanish. This proves the holomorphy of $\mathcal{L}T$ on $\mathbb{C}_+$.

Polynomial boundedness of $\mathcal{L}T$ in each $\mathbb{C}_\gamma$ is a consequence of the polynomial boundedness of the continuous functions T_k: there exist constants C_k and exponents m_k such that $\forall\, s \in \mathbb{R} \,:\, T_k(s) < C_k s^{m_k}$. m_k-fold integration by parts yields

$$\left| \int_0^\infty \mathrm{d}s\, T_k(s)(-\lambda)^k e^{-\lambda s} \right| < \int_0^\infty \mathrm{d}s\, C_k s^{m_k} |\lambda|^k e^{-s \operatorname{Re}\lambda} = C_k \frac{|\lambda|^k}{\operatorname{Re}\lambda^{m_k+1}} \,.$$

$\square$

Definition A.1. Let $T \in \mathcal{S}'(\mathbb{R})$ with $\operatorname{supp}(T) \subset [0, \infty[$. The holomorphic map $\mathcal{L}T : \mathbb{C}_+ \to \mathbb{C}$ is called the (distributional) **Laplace transform** of T.

Theorem A.2. *Let* $(\mathbb{C}_+)$ *denote the set of holomorphic functions on* $\mathbb{C}_+$ *which are polynomially bounded on each* $\mathbb{C}_\gamma$, $\gamma > 0$ *and let* $\mathcal{S}'([0, \infty[)$ *denote the set of tempered distributions on* $\mathbb{R}$ *with support in* $[0, \infty[$. *The map*

$$\begin{array}{rcl} \mathcal{L} \,:\, \mathcal{S}'([0,\infty[) & \to & \mathscr{H}_{\mathrm{pb}}(\mathbb{C}_+)\,, \\ T & \mapsto & \mathcal{L}T \end{array}$$

is injective.

Proof. There is a map

$$\mathcal{L}^{-1} \,:\, \mathscr{H}_{\mathrm{pb}}(\mathbb{C}_+) \to \mathcal{D}'([0, \infty[)$$

such that $\mathcal{L}^{-1} \circ \mathcal{L}$ is the inclusion map $\mathcal{S}'([0, \infty[) \to \mathcal{D}'([0, \infty[)$, where $\mathcal{D}'([0, \infty[)$ denotes the set of distributions on $\mathbb{R}$ with support in $[0, \infty[$. $\mathcal{L}^{-1}$ is given by

$$< \mathcal{L}^{-1}h, \varphi > := \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{d}\zeta\, h(\zeta) \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta} \varphi(s)$$

for $\gamma > 0$, $h \in \mathscr{H}_{\mathrm{pb}}(\mathbb{C}_+)$ and $\varphi \in \mathcal{D}(\mathbb{R})$.
Write $\zeta = \gamma + i\nu$ and define $\varphi_\gamma := e^{\gamma s}\varphi(s) \in \mathcal{D}(\mathbb{R})$. φ_γ depends continuously on φ and it is

$$< \mathcal{L}^{-1}h, \varphi > = \int_{-\infty}^{\infty} \mathrm{d}\nu\, h(\gamma + i\nu) \widehat{\varphi_\gamma}^{-1}(\nu) \,.$$

Since $\widehat{\varphi_\gamma}^{-1}$ is a Schwartz function, which depends continuously on φ, and $\nu \mapsto h(\gamma + i\nu)$ is continuous and polynomially bounded, this defines a distribution.

An application of Cauchy's integral theorem shows that for $h \in \mathscr{H}_{\mathrm{pb}}(\mathbb{C}_+)$, the distribution $\mathcal{L}^{-1}h$ has support in $[0, \infty[$: for $R > 0$, consider the line integral of the holomorphic function $\zeta \mapsto h(\zeta) \int_{-\infty}^{0} \mathrm{d}s\, e^{s\zeta}\varphi(s)$ along the closed path given by the line $[\gamma - iR, \gamma + iR]$ connected to a semicircle of radius R extending into $\mathbb{C}_\gamma$. By Cauchy's integral theorem it is zero. In the limit $R \to \infty$ the line contribution converges to

$\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta\, h(\zeta) \int_{-\infty}^{0} \mathrm{d}s\, e^{s\zeta}\varphi(s)$, while the semi-circle contribution vanishes by an application of the Lebesgue dominated convergence theorem. Hence, $\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta\, h(\zeta) \int_{-\infty}^{0} \mathrm{d}s\, e^{s\zeta}\varphi(s) = 0$ and

$$< \mathcal{L}^{-1}h, \varphi >:= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta\, h(\zeta) \int_{0}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s)\,.$$

In this form, it is obvious that $\mathcal{L}^{-1}h$ has support in $[0,\infty[$. Also, by a similar application of Cauchy's integral theorem, it can be shown that $\mathcal{L}^{-1}h$ is independent of the choice of $\gamma > 0$.

To see that $T|_{\mathcal{D}(\mathbb{R})} = \mathcal{L}^{-1}(\mathcal{L}T)$ compute for $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\begin{aligned}
& < \mathcal{L}^{-1}(LT), \varphi > \\
= \;& \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta \; < T, f_\zeta > \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s) \\
= \;& \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta \sum_k \int_0^{\infty} \mathrm{d}t\, T_k(t)(\partial_t^k f_\zeta)(t) \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s) \\
= \;& \frac{1}{2\pi i} \sum_k \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta \int_0^{\infty} \mathrm{d}t\, T_k(t)(-\zeta)^k e^{-t\zeta} \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s) \\
= \;& \sum_k \int_{-\infty}^{\infty} \mathrm{d}\nu \int_0^{\infty} \mathrm{d}t\, T_k(t) e^{-t\gamma} e^{-it\nu}(-1)^k(\gamma + i\nu)^k \widehat{\varphi_\gamma}^{-1}(\nu)\,.
\end{aligned}$$

The integrand is integrable with respect to $\mathrm{d}(\nu, t)$ since $\widehat{\varphi_\gamma}^{-1}$ is a Schwartz function and T_k is polynomially bounded. Hence, by Fubini's theorem, integration order can be interchanged:

$$\begin{aligned}
& < \mathcal{L}^{-1}(LT), \varphi > \\
= \;& \sum_k \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\nu\, T_k(t) e^{-t\gamma} e^{-it\nu}(-1)^k(\gamma + i\nu)^k \widehat{\varphi_\gamma}^{-1}(\nu) \\
= \;& \sum_k \int_0^{\infty} \mathrm{d}t\, T_k(t)\, \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta\, (-\zeta)^k e^{-t\zeta} \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s) \\
= \;& < T, t \mapsto \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta\, e^{-t\zeta} \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s) > \\
= \;& < T, t \mapsto \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}\zeta \int_{-\infty}^{\infty} \mathrm{d}s\, e^{s\zeta}\varphi(s+t) > \\
\overset{\gamma\to 0}{=} \;& < T, t \mapsto \frac{1}{2\pi} \int_{\infty}^{\infty} \mathrm{d}\nu \int_{-\infty}^{\infty} \mathrm{d}s\, e^{is\nu}\sigma_t(s) > \\
= \;& < T, t \mapsto \int_{\infty}^{\infty} \mathrm{d}\nu\, \widehat{\sigma_t}^{-1}(\nu) > \\
= \;& < T, t \mapsto \sigma_t(0) > \\
= \;& < T, \varphi >\,,
\end{aligned}$$

where $\sigma_t(s) := \varphi(s+t)$ has been used. It follows that $T|\mathcal{D}(\mathbb{R})$ and therefore T itself (by continuity) can be recovered from $\mathcal{L}T$. This proves the injectivity of $\mathcal{L}$. $\square$

Theorem A.3. *Let $T \in \mathcal{S}'(\mathbb{R})$. The distribution $T \circ \frac{1}{2}$ defined by*

$$< T \circ \tfrac{1}{2}, \varphi > := < T, s \mapsto s\varphi(s^2) >$$

is tempered and has support in $[0, \infty[$. Furthermore the restriction of the map $T \mapsto T \circ \frac{1}{2}$ to odd distributions is injective.

Proof. $\varphi \mapsto \mathrm{id}\,\varphi \circ {}^2$ defines a continuous map $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$. Hence, $T \circ \frac{1}{2}$ is tempered. If $\mathrm{supp}(\varphi) \subset]-\infty, 0[$ then $\mathrm{id}\,\varphi \circ {}^2$ vanishes, so $\mathrm{supp}(T) \subset [0, \infty[$. Now let T be odd. Given $\varphi \in \mathcal{S}(\mathbb{R})$ define for $s > 0$:

$$\psi(s) := \frac{\varphi_{\mathrm{odd}}(s^{\frac{1}{2}})}{s^{\frac{1}{2}}}\,.$$

Taylor expanding φ_{odd} about 0, it can be seen that ψ can be extended to a Schwartz function on $\mathbb{R}$. Note that $\mathrm{id}\,\psi \circ {}^2 = \varphi_{\mathrm{odd}}$. Therefore it is

$$< T \circ \tfrac{1}{2}, \psi > = < T, \varphi_{\mathrm{odd}} > = < T, \varphi >\,,$$

which proves the injectivity of $T \mapsto T \circ \frac{1}{2}$ on odd distributions. $\square$

Corollary A.4. *Let $T \in \mathcal{S}'(\mathbb{R})$ be odd. The function*

$$\lambda \mapsto < T, s \mapsto s e^{-\lambda s^2} >$$

is holomorphic on $\mathbb{C}_+$ and allows to recover T.

Proof. It is $< T, s \mapsto s e^{-\lambda s^2} > = < T \circ \frac{1}{2}, f_\lambda > = \mathcal{L}(T \circ \frac{1}{2})(\lambda)$, where $f_\lambda \in \mathcal{S}(\mathbb{R})$, such that $f_\lambda|_{[0,\infty[} = e^{-\lambda s}$. Being tempered, $T \circ \frac{1}{2}$ can be recovered from $\mathcal{L}(T \circ \frac{1}{2})$, and since T is odd it can be recovered from $T \circ \frac{1}{2}$. $\square$

B. Hankel Transform

Proposition B.1. *Let $\varphi \in \mathcal{S}_{\text{even}}(\mathbb{R})$. The function*

$$\begin{array}{rcl} \mathcal{T}\varphi : \mathbb{R} & \to & \mathbb{C}\,, \\ x & \mapsto & \mathcal{T}\varphi(x) := \displaystyle\int_0^\infty \mathrm{d}m\, m\varphi(m) J_0(mx) \end{array}$$

lies again $\mathcal{S}(\mathbb{R})$ and the map

$$\begin{array}{rcl} \mathcal{T} : \mathcal{S}_{\text{even}}(\mathbb{R}) & \to & \mathcal{S}_{\text{even}}(\mathbb{R}) \\ \varphi & \mapsto & \mathcal{T}\varphi \end{array}$$

is continuous in the topology of $\mathcal{S}(\mathbb{R})$.

Proof. Let $\varphi \in \mathcal{S}_{\text{even}}(\mathbb{R})$. There exists a Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\varphi(x) = \psi(x^2)$. This can be seen as follows: for $s > 0$ define $\psi(s) := \varphi(\sqrt{s})$. Noting that all odd derivatives of even functions vanish at the origin, it can be seen by Taylor expanding φ in $\sqrt{s}$ about 0, that ψ can be extended to a Schwartz function.
This will help to prove that even Schwartz functions in one variable are in one-to-one correspondence with rotationally symmetric Schwartz functions in two variables. Let $\mathcal{S}_{\text{rs}}(\mathbb{R}^2)$ denote the space of the latter. Define the map

$$\varpi : \mathcal{S}_{\text{rs}}(\mathbb{R}^2) \to \mathcal{S}_{\text{even}}(\mathbb{R}),\ \varpi\Phi(x) := \Phi(x\cos\vartheta, x\sin\vartheta)$$

for any $\vartheta \in \mathbb{R}$. Because of the rotational invariance of Φ, this is well-defined and even (consider $\vartheta = 0$ and $\vartheta = \pi$). Since the Schwartz semi-norms of $\varpi\Phi$ can be easily estimated in terms of corresponding semi-norms of Φ, the map ϖ is continuous. Now define

$$\iota : \mathcal{S}_{\text{even}}(\mathbb{R}) \to \mathcal{S}_{\text{rs}}(\mathbb{R}^2),\ \iota\varphi(x, y) := \varphi(\sqrt{x^2 + y^2})\,.$$

Using the above result on even Schwartz functions and writing $\varphi(\sqrt{x^2 + y^2}) = \psi(x^2 + y^2)$, where $\psi \in \mathcal{S}(\mathbb{R})$, it becomes apparent that $\iota\varphi \in \mathcal{S}_{\text{rs}}(\mathbb{R}^2)$.
Note that ι and ϖ are inverse to each other and that $\mathcal{S}_{\text{even}}(\mathbb{R})$ and $\mathcal{S}_{\text{rs}}(\mathbb{R}^2)$ are Fréchet spaces as closed subspaces of Fréchet spaces. As ϖ is a continuous linear bijection between Fréchet spaces, it is also open by the open mapping theorem ([3]). Hence ι is continuous.
Note that the Fourier transform $\mathcal{F}$ is continuous and maps rotationally invariant Schwartz

functions into themselves. Showing that $2\pi\mathcal{T} = \varpi \circ \mathcal{F} \circ \iota$ hence concludes the proof.

$$\begin{aligned}
& \varpi \circ \mathcal{F} \circ \iota\varphi(s) \\
= \; & \widehat{\iota\varphi}(s,0) \\
= \; & \int \mathrm{d}x \int \mathrm{d}y \, e^{-ixs} \varphi(\sqrt{x^2+y^2}) \\
= \; & \int_0^\infty \mathrm{d}m \int_0^{2\pi} \mathrm{d}\vartheta \, m \, e^{-ism\cos\vartheta} \varphi(m) \\
= \; & 2\pi \int_0^\infty \mathrm{d}m \, m\varphi(m) \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\vartheta \, e^{ism\sin\vartheta} \\
= \; & 2\pi \int_0^\infty \mathrm{d}m \, m\varphi(m) J_0(ms) \\
= \; & 2\pi\mathcal{T}\varphi(s) \,.
\end{aligned}$$

The identity $J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\vartheta \, e^{ism\sin\vartheta}$ can be derived from the integral representation of Bessel functions provided in 2.1.8 by choosing the path $p(\vartheta) := \frac{z}{2}e^{i\vartheta}$, $\vartheta \in [0, 2\pi]$. $\mathcal{T}\varphi$ is even since J_0 is. $\square$

Definition B.1. Let $\varphi \in \mathcal{S}_{\text{even}}(\mathbb{R})$. The map $\mathcal{T}\varphi \in \mathcal{S}_{\text{even}}(\mathbb{R})$ is called the **Hankel transform** of φ.

C. Various Proofs

C.1. Translationally Invariant Distributions

Proposition C.1. *Let $T \in \mathcal{S}'(\mathbb{R}^{2n})$ such that for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$: $< T, \psi_a \otimes \varphi_a > = < T, \psi \otimes \varphi >$, where $\psi_a(x) = \psi(x-a)$. Then there exists $\widetilde{T} \in \mathcal{S}'(\mathbb{R}^n)$ such that for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ it is $< T, \psi \otimes \varphi > = < \widetilde{T}, \psi * \check{\varphi} >$.*

This proposition can be considered a distributional version of the following lemma:

Lemma C.2. *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a function such that for all $x, y, a \in \mathbb{R}^n$ it is $f(x+a, y+a) = f(x,y)$. Then there exists $\widetilde{f} : \mathbb{R}^n \to \mathbb{C}$ such that for all $x, y \in \mathbb{R}^n$ it is $f(x,y) = \widetilde{f}(x-y)$.*

Proof of the Lemma. $\mathbb{R}^n$ acts by translation on both factors on $\mathbb{R}^n \times \mathbb{R}^n$. Let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n / \mathbb{R}^n$ the canonical map to the orbit space. A function f satisfying the requirements in the lemma is invariant under that action and hence there exists a function $\widetilde{f}$ on the orbit space satisfying $\widetilde{f} \circ \pi = f$. The orbit space $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n / \mathbb{R}^n$ stands in bijection to $\mathbb{R}^n$ by mapping the equivalence class $(x+a, y+a)$ to $x-y$. Identifying the two sets concludes the proof. □

Proof of the Proposition. By the structure theorem for tempered distributions (2.1), there exist finitely many continuous, polynomially bounded functions $c_{k_1,k_2} : \mathbb{R}^{2n} \to \mathbb{C}$, such that for $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ it is

$$< T, \psi \otimes \varphi > = \sum_{(k_1,k_2)} \int \mathrm{d}x \mathrm{d}y \, c_{k_1,k_2}(x,y) (\partial_x^{k_1} \psi)(x) (\partial_y^{k_2} \varphi)(y) \,.$$

Since $< T, \psi_a \otimes \varphi_a > = < T, \psi \otimes \varphi >$ for all $a \in \mathbb{R}^n$, it is, again for all $a \in \mathbb{R}^n$,

$$< T, \psi \otimes \varphi > = \sum_{(k_1,k_2)} \int \mathrm{d}x \mathrm{d}y \, c_{k_1,k_2}(x+a, y+a) (\partial_x^{k_1} \psi)(x) (\partial_y^{k_2} \varphi)(y) \,.$$

As this is true for arbitrary $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, it can be shown that for all $x, y, a \in \mathbb{R}^n$, $k_1, k_2 \in \mathbb{N}_0$ there holds

$$c_{k_1,k_2}(x+a, y+a) = c_{k_1,k_2}(x,y) \,.$$

The preceding lemma implies the existence of $\widetilde{c}_{k_1,k_2}$, such that $c_{k_1,k_2}(x,y) = \widetilde{c}_{k_1,k_2}(x-y)$ and the $\widetilde{c}_{k_1,k_2}$ are continuous and polynomially bounded. Define for $\psi \in \mathcal{S}(\mathbb{R}^n)$:

$$< \widetilde{T}, \psi > := \sum_{(k_1,k_2)} \int \mathrm{d}x \, \widetilde{c}_{k_1,k_2}(x) (\partial_x^{k_1+k_2} \psi)(x) \,.$$

By the structure theorem for tempered distributions, $\widetilde{T}$ is tempered and it is

$$\begin{aligned}
< T, \psi \otimes \varphi > &= \sum_{(k_1,k_2)} \int \mathrm{d}x\mathrm{d}y\, \widetilde{c}_{k_1,k_2}(x-y)(\partial_x^{k_1}\psi)(x)(\partial_y^{k_2}\varphi)(y) \\
&= \sum_{(k_1,k_2)} \int \mathrm{d}x\mathrm{d}y\, \widetilde{c}_{k_1,k_2}(x)(\partial_x^{k_1}\psi)(x+y)(\partial_y^{k_2}\varphi)(y) \\
&= \sum_{(k_1,k_2)} \int \mathrm{d}x\, \widetilde{c}_{k_1,k_2}(x) \int \mathrm{d}y\, (\partial_x^{k_1}\psi)(x+y)(\partial_y^{k_2}\varphi)(y) \\
&= \sum_{(k_1,k_2)} \int \mathrm{d}x\, \widetilde{c}_{k_1,k_2}(x)(\partial_x^{k_1}\psi) * (\partial_y^{k_2}\check{\varphi})(x) \\
&= \sum_{(k_1,k_2)} \int \mathrm{d}x\, \widetilde{c}_{k_1,k_2}(x)\partial_x^{k_1+k_2}\psi * \check{\varphi}(x) \\
&= < \widetilde{T}, \psi * \check{\varphi}(x) > \ .
\end{aligned}$$

□

C.2. A Regularity Theorem for Distributions

Intuitively, the regularity of an eligible object, e.g. a function or a distribution, is encoded in the decay property of its Fourier transform. The following theorem uses correspondence to establish a sufficient condition for a tempered distributions to be given by a function.

Theorem C.3. *Let $T \in \mathcal{S}'(\mathbb{R})$ be a tempered distribution, such that its Fourier transform $\widehat{T}$ has the following decay property: there exists a nowhere vanishing Schwartz function $g \in \mathcal{S}(\mathbb{R})$, such that $S := \frac{\widehat{T}}{g}$ is again a tempered distribution. Then for any sequence φ_n in $\mathcal{S}(\mathbb{R})$ converging weakly to δ in $\mathcal{S}'(\mathbb{R})$, the limit $\lim_n < T, \varphi_n >$ exists and is given by*

$$\lim_n < T, \varphi_n > = < S, g > = < \widehat{T}, 1 > \ .$$

Proof. Define $\phi_n := \widehat{\varphi}_n$. The Fourier transform is a homeomorphism of $\mathcal{S}'(\mathbb{R})$, so the weak convergence $\varphi_n \to \delta$ is equivalent to the weak convergence $\phi_n \to \widehat{\delta} = 1$ and

$$\lim_n < T, \varphi_n > = \lim_n < \widehat{T}, \phi_n > = \lim_n < S, g\phi_n > \ .$$

By the structure theorem for tempered distributions (theorem 2.1), there exist continuous, polynomially bounded functions s_k, such that for all $f \in \mathcal{S}(\mathbb{R})$ it is

$$< S, f > = \sum_{\substack{k \\ \text{finite}}} \int \mathrm{d}x\, s_k(x) \cdot (\partial_x^k f)(x) = \sum_{\substack{k \\ \text{finite}}} < s_k, f^{(k)} > \ .$$

Using the Leibniz rule one obtails

$$
\begin{aligned}
< S, g\phi_n > &= \sum_{\substack{k\\\text{finite}}} \int \mathrm{d}x\, s_k(x) \cdot (\partial_x^k g\phi_n)(x) \\
&= \sum_{\substack{k\\\text{finite}}} \sum_{l=0}^{k} \binom{k}{l} \int \mathrm{d}x\, s_k \phi_n^{(l)}(x) \cdot g^{(k-l)}(x) \\
&= \sum_{\substack{k\\\text{finite}}} \sum_{l=0}^{k} \binom{k}{l} < s_k \phi_n^{(l)}, g^{(k-l)} > \ .
\end{aligned}
$$

Since differentiation and multiplication by polynomially bounded, continuous functions are continuous maps of $\mathcal{S}'(\mathbb{R})$ onto itself, the tempered distributions $s_k \phi_n^{(l)}$ converge weakly to $s_k 1^{(l)}$ in the limit $n \to \infty$. Noting that $1^{(l)} = 0$ for $l > 0$ yields

$$
\begin{aligned}
\lim_n < S, g\phi_n > &= \sum_{\substack{k\\\text{finite}}} \sum_{l=0}^{k} \binom{k}{l} < s_k 1^{(l)}, g^{(k-l)} > \\
&= \sum_{\substack{k\\\text{finite}}} < s_k, g^{(k)} > \\
&= < S, g > ,
\end{aligned}
$$

which concludes the proof. □

C.3. Asymptotic Behavior of $\partial^\mu_{\mathbf{x}} W_{M,\beta}$

The study of the asymptotic structure of thermal correlation function for large times mainly relies on the analysis of $\partial^\mu_{\mathbf{x}} W_{M,\beta}$, where $W_{M,\beta}$ is the free mass M β-KMS reduced two-point function. Due to its technical and unenlightening nature, this analysis is performed here rather than in the main text. The analysis is largely based on the outline given in [9].

Let $\underline{h} \in \mathcal{S}(\mathbb{R})$ be such that the Fourier transform of $\underline{h}$ has a double zero at the origin (i.e. $\lim_{\underline{x}\to 0} \underline{x}^{-2} h(\underline{x}) < \infty$). In the main text this $\underline{h}$ arises by convolution of two Schwartz functions whose Fourier transforms vanish at the origin. These Schwartz function serve the purpose of reflecting the suppression of low energy contributions, which would otherwise give rise to an unwanted leading order contributions to the two-point function.

Compute for $\underline{x} > 0$:

$$
\begin{aligned}
& \underline{h} *_{\underline{x}} \partial^\mu_{\mathbf{x}} W_{M,\beta}(x) \\
= \; & \mathcal{F}^{-1}(\underline{\widehat{h}} \cdot \widehat{\partial_{\mathbf{x}^\mu} W}_{M,\beta})(x) \\
= \; & \frac{(-i)^\mu}{2\pi} \int \mathrm{d}p \, \frac{\varepsilon(\underline{p})\delta(p^2 - M^2)}{1 - e^{-\beta \underline{p}}} \underline{\widehat{h}}(p) \mathbf{p}^\mu e^{-ipx} \\
= \; & \sum_{\sigma = \pm 1} \sigma \frac{(-i)^\mu}{2\pi} \int \frac{\mathrm{d}\mathbf{p}}{2\omega(\mathbf{p})} (1 - e^{-\sigma\beta\omega(\mathbf{p})})^{-1} \underline{\widehat{h}}(\sigma\omega(p)) \mathbf{p}^\mu e^{-i\sigma \underline{x}\omega(\mathbf{p})} e^{i\mathbf{x}\mathbf{p}} \\
= \; & \sum_{\sigma = \pm 1} \sigma \frac{(-i)^\mu}{2\pi} \int_0^\infty \frac{\mathrm{d}\mathbf{p}}{2\omega(\mathbf{p})} (1 - e^{-\sigma\beta\omega(\mathbf{p})})^{-1} \underline{\widehat{h}}(\sigma\omega(p)) \mathbf{p}^\mu e^{-i\sigma \underline{x}\omega(\mathbf{p})} e^{i\mathbf{x}\mathbf{p}} \\
+ \; & \sum_{\sigma = \pm 1} \sigma \frac{(-i)^\mu}{2\pi} \int_0^\infty \frac{\mathrm{d}\mathbf{p}}{2\omega(\mathbf{p})} (1 - e^{-\sigma\beta\omega(\mathbf{p})})^{-1} \underline{\widehat{h}}(\sigma\omega(p)) (-\mathbf{p})^\mu e^{-i\sigma \underline{x}\omega(\mathbf{p})} e^{-i\mathbf{x}\mathbf{p}} ,
\end{aligned}
$$

with $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}$. Substitute $w = \underline{x} \cdot \left(\sqrt{\underline{p}^2 + M^2} - M^2\right)$ to get

$$
\begin{aligned}
& \sum_{\sigma = \pm} \sigma \frac{(-i)^\mu}{2\pi} \int_0^\infty \mathrm{d}w \, \frac{1}{\underline{x}} \left(\frac{w}{\underline{x}}\right)^{\frac{\mu-1}{2}} \left(\frac{w}{\underline{x}} + 2M\right)^{\frac{\mu-1}{2}} \underline{\widehat{h}} \left(\sigma \left(\frac{w}{\underline{x}} + M\right)\right) e^{-i\sigma\underline{x}\left(\frac{w}{\underline{x}} + M\right)} \\
& \times \left(1 - e^{-\sigma\beta\left(\frac{w}{\underline{x}} + M\right)}\right)^{-1} \left[e^{i\mathbf{x}\sqrt{\left(\frac{w}{\underline{x}}\right)^2 + 2M\left(\frac{w}{\underline{x}}\right)}} + (-1)^\mu e^{-i\mathbf{x}\sqrt{\left(\frac{w}{\underline{x}}\right)^2 + 2M\left(\frac{w}{\underline{x}}\right)}} \right] \\
= \; & \sum_{\sigma = \pm} \frac{(-i)^\mu}{2\pi} e^{-i\sigma\underline{x}M} \underline{x}^{-\frac{\mu+1}{2}} \int_0^\infty \mathrm{d}w \, e^{-i\sigma w} w^{\frac{\mu-1}{2}} k_\sigma \left(\frac{w}{\underline{x}}; m\right) ,
\end{aligned}
$$

where

$$
\begin{aligned}
k_\sigma(v; M) \; &= \; \sigma(v + 2M)^{\frac{\mu-1}{2}} (1 - e^{-\sigma\beta(v+M)})^{-1} \underline{\widehat{h}}(\sigma(v + M)) \\
&\times \; \left[e^{i\mathbf{x}\sqrt{v^2+2Mv}} + (-1)^\mu e^{-i\mathbf{x}\sqrt{v^2+2Mv}} \right] .
\end{aligned}
$$

Define:

- $A := (v + 2M)^{\frac{\mu-1}{2}}$,
- $B := (1 - e^{-\sigma\beta(v+M)})^{-1} \underline{\widehat{h}}(\sigma(v + M))$,
- $C := e^{i\mathbf{x}\sqrt{v^2+2Mv}} + (-1)^\mu e^{-i\mathbf{x}\sqrt{v^2+2Mv}}$.

With these definitions it is $k_\sigma(v; M) = \sigma A \cdot B \cdot C$. Note that B is a Schwartz function in $v + M$ which vanishes at zero. This can be used to show that k_σ is bounded and rapidly decreasing in both v and M varying in $[0,\infty[$:

$$\begin{aligned} |k_\sigma(v;M)| &= |A|\cdot|B|\cdot|C| \\ &\leqslant 2|A|\cdot|B| \\ &\leqslant (v+M)(v+2M)^{\frac{\mu-1}{2}}\cdot\frac{|B|}{v+M} \\ &\leqslant (v+M)^{\frac{\mu+1}{2}}\cdot\left|\frac{B}{v+M}\right| . \end{aligned}$$

Since $B/(v+M)$ is a Schwartz function in $v+M$ and $(v+M)^{\frac{\mu+1}{2}}$ grows at most polynomially, it follows that for any $N\in\mathbb{N}$ and suitable constants C_N it is

$$|k_\sigma(v;M)|\leqslant C_N(1+(v+M)^{-N} .$$

Definition C.1. Let $U\subset\mathbb{R}^m$ be open. On the set of functions $U\to\mathbb{R}$ define the following transitive relation

$$f\lesssim g :\Leftrightarrow \exists\, C>0\,\forall\, x\in\mathbb{R}^m\,:\, f(x)\leqslant Cg(x)$$

for $f,g:U\to\mathbb{R}$.

This definition serves to lighten the notation on the estimates to follow.

Lemma C.4. *For* $0<\delta<\frac{1}{2}$ *and* $m,N\in\mathbb{N}$ *it is*

$$\sup_{v>0}\left|v^{m-\delta}\partial_v^m k_\sigma(v;M)\right|\lesssim(1+M)^{-N} .$$

Proof. The only property of B used for this proof is that $B/(v+M)$ is a Schwartz function in $v+M$. Hence the case $\sigma=\pm$ can be treated simultaneously.

Note that derivates $\partial_v^m k_\sigma$ are sums of products of $\partial_v^{m_A}A$, $\partial_v^{m_B}B$ and $\partial_v^{m_C}C$, where $m_A+m_B+m_C=m$. The individual factors can be estimated as follows:

- Estimates for A: It is

$$|\partial_v^{m_A}A|\lesssim(v+2M)^{\frac{\mu+1}{2}}(v+2M)^{-(m_A+1)}\lesssim(v+M)^{\frac{\mu+1}{2}}(v+M)^{-(m_A+1)} .$$

- Estimates for B: As $B/(v+M)$ is a Schwartz function in $v+M$, one gets for $\beta(v+M)\leqslant 1$:

$$|\partial_v^{m_B}B|\lesssim(v+M)^{\max\{0,1-m_B\}} .$$

  For generic $v+M$ and any $N\in\mathbb{N}$ it is

$$|\partial_v^{m_B}B|\lesssim(1+(v+M))^{-N} .$$

- Estimates for C: From the definition of C, it is immediately obvious that

$$|\partial_v^{m_C}| \leqslant 2 \left|\partial_v^{m_C} e^{i\mathbf{x}\sqrt{v^2+2Mv}}\right| .$$

Define $g(v) = e^{f(v)}$ with $f(v) = i\mathbf{x}\sqrt{v^2+2Mv}$. One shows by induction, that derivates $g^{(l)}$, $l > 0$ are a product of g itself and a finite linear combination of products of the form $\prod\limits_{\substack{i\\ \text{finite}}} f^{(k_i)}$ where $\sum\limits_{\substack{i\\ \text{finite}}} k_i = l$.

Also by induction, derivates $f^{(k)}$ can be shown to be linear combinations of $(v + M)^{k-2j}(v^2+2Mv)^{-(k-k-\frac{1}{2})}$ where $0 \leqslant 2j \leqslant k$. This allows the estimate

$$|f^{(k)}| \lesssim \sum_{0\leqslant 2j\leqslant k} (v+M)^{\frac{1}{2}-k} v^{-(k-j-\frac{1}{2})} ,$$

which in turn can be used to show that

$$\prod_{\substack{i\\ \text{finite}}} f^{(k_i)} \lesssim (v+M)^{\frac{1}{2}} v^{\frac{1}{2}-l} .$$

Using $|g(v)| = 1$ and the above arguments, this gives the following estimates for derivatives of C:

$$\begin{aligned} |\partial_v^{m_C} C| &\lesssim 1, && \text{if}\, m_C = 0\,, \\ |\partial_v^{m_C} C| &\lesssim (v+M)^{\frac{1}{2}} v^{\frac{1}{2}-m_C}, && \text{if}\, m_C > 0\,. \end{aligned}$$

Let $0 < \delta < \frac{1}{2}$. For $0 < \beta(v+M) \leqslant 1$, combining the above estimates gives

$$\begin{aligned} &|v^{m-\delta}\partial_v^m k_\sigma(v;M)| \\ \lesssim &\sum_{m_A+m_B+m_C=m} |v^{n-\delta}\partial_v^{m_A} A| \cdot |\partial_v^{m_B} B| \cdot |\partial_v^{m_C} C| \\ \lesssim &\sum_{\substack{m_A+m_B+m_C=m\\ m_C=0}} v^{m-\delta}(v+M)^{\frac{\mu+1}{2}}(v+M)^{-(m_A+1)}(v+M)^{\max\{0,1-m_B\}} \\ + &\sum_{\substack{m_A+m_B+m_C=m\\ m_C\neq 0}} v^{m-\delta}(v+M)^{\frac{\mu+1}{2}}(v+M)^{-(m_A+1)}(v+M)^{\max\{0,1-m_B\}}(v+M)^{\frac{1}{2}}v^{\frac{1}{2}-m_C} \\ \lesssim &\sum_{\substack{m_A+m_B+m_C=m\\ m_C=0}} (v+M)^{\frac{\mu}{2}+(\frac{1}{2}-\delta)+(m-m_A+1+\max\{0,1-m_B\})} \\ + &\sum_{\substack{m_A+m_B+m_C=m\\ m_C\neq 0}} v^{\frac{1}{2}-\delta}(v+M)^{\frac{\mu}{2}+(m-m_A-m_C+\max\{0,1-m_B\})} \\ \lesssim &\;\; 1\,. \end{aligned}$$

Similarly one obtains for $\beta(v+M) \geqslant 1$:

$$|v^{m-\delta}\partial_v^m k_\sigma(v;M)| \lesssim (1+(v+M))^{-N} \lesssim (1+M)^{-N}\,.$$

Along with the boundedness for small $(v+M)$, this proves the lemma. Note that the estimates for C, and therefore also for k_σ, hold uniformly for $\mathbf{x}$ varying in a compact set.

□

Corollary C.5. *The following chain of inequalities holds for $0 < \delta < \frac{1}{2}$ and $N \in \mathbb{N}$:*

$$|k_\sigma(v;M) - k_\sigma(0;M)| \leqslant \delta v^\delta \sup_{u>0} |u^{1-\delta}\partial_u k_\sigma(u;M)| \lesssim v^\delta (1+M)^{-N}\,.$$

Proof. The second inequality follows from lemma C.4. The first inequality can be proven as follows:
Define $t \mapsto g(t) = (tv)^{\delta^{-1}}$ and $t \mapsto h_M(t) = k_\sigma(g(t);M)$. In the case $v = 0$ the inequality in question is trivial. For $v > 0$ the map g is invertible and it is $g^{-1}(u) = v^{-1}u^\delta$. Also note that $g'(t) = \delta^{-1}v^{\delta^{-1}}t^{\delta^{-1}-1}$. Set $t_v = v^{\delta-1}$ and $t_0 = 0$. It is $g(t_v) = v$ and $g(t_0) = 0$. By the mean value theorem it is

$$|h_M(t_v) - h_M(t_0)| = |t_v - t_0| \cdot \sup_{t \in]t_0,t_v[} |h'_M(t)|\,.$$

The left hand side is nothing but $|k_\sigma(v;M) - k_\sigma(0;M)|$. For the right hand side compute

$$\begin{aligned}
|t_v - t_0| \cdot \sup_{t_0<t<t_v} |h'_M(t)| &\leqslant v^{\delta-1} \sup_{t\in]t_0,t_v[} |g'(t)\cdot(\partial_1 k_\sigma)(g(t);M)| \\
&= v^{\delta-1}\delta^{-1}v^{\delta^{-1}} \sup_{u\in]g(t_0),g(t_v)[} |g^{-1}(u)^{\delta^{-1}-1}\partial_u k_\sigma(u;M)| \\
&= \delta^{-1}v^\delta \sup_{u\in]0,v[} |u^{1-\delta}\partial_u k_\sigma(u;M)| \\
&\leqslant \delta^{-1}v^\delta \sup_{u>0} |u^{1-\delta}\partial_u k_\sigma(u;M)|\,.
\end{aligned}$$

□

Recall that for $\underline{x} > 0$ the following equation holds:

$$\underline{h} *_{\underline{x}} \partial_{\mathbf{x}}^{\mu} W_{M,\beta}(x) = \sum_{\sigma=\pm} \frac{(-i)^\mu}{2\pi} e^{-i\sigma\underline{x}M}\,\underline{x}^{-\frac{\mu+1}{2}} \int_0^\infty \mathrm{d}w\, e^{-i\sigma w} w^{\frac{\mu-1}{2}} k_\sigma\left(\frac{w}{\underline{x}};m\right)\,.$$

Lemma C.4 and its corollary are used to identify the asymptotically leading contribution in $\underline{x}$ from the integral appearing in the above relation. Define

$$k_\sigma^{(m)}(v;M) := \partial_v^m[k_\sigma(v;M) - k_\sigma(0;M)]$$

and split the the integral in question as follows

$$\begin{aligned}
&\int_0^\infty \mathrm{d}w\, e^{-i\sigma w} w^{\frac{\mu-1}{2}} k_\sigma\left(\frac{w}{\underline{x}};M\right)\\
=\;& \lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}} k_\sigma\left(\frac{w}{\underline{x}};M\right)\\
=\;& k_\sigma(0;M)\underbrace{\lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}}}_{\text{First limit}}\\
+\;& \underbrace{\lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}} k_\sigma^{(0)}\left(\frac{w}{\underline{x}};M\right)}_{\text{Second limit}}\,.
\end{aligned}$$

The first equality is due to the Lebesgue dominated convergence theorem. The splitting of the limit is justified below by showing that the individual limits exist. To ease the notation a bit, set $a = i\sigma + \varepsilon$.

First limit:
This part is independent of $\underline{x}$ and gives rise to the asymptotically leading contribution to $\underline{h} *_{\underline{x}} \partial^\mu_{\mathbf{x}} W_{M,\beta}$. The integral $\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}}$ can be computed explicitly: Set $z = \frac{\mu+1}{2}$. It is

$$\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}} = \int_0^\infty \mathrm{d}t\, e^{-at} t^{z-1} = \mathcal{M}(t \mapsto e^{-at})(z) = a^{-z}\Gamma(z)\,,$$

where $\mathcal{M}$ denotes the Mellin transform and Γ the Gamma function. The identity $\mathcal{M}(t \mapsto e^{-at})(z) = a^{-z}\Gamma(z)$ for $\mathrm{Re}(a) > 0$, $\mathrm{Re}(z) > 0$ can be found in integral tables (e.g. [15]). In the limit $\varepsilon \searrow 0$ this becomes

$$\lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}} = e^{-\sigma\pi\frac{\mu+1}{4}}\Gamma\left(\frac{\mu+1}{2}\right)\,.$$

Second limit:
The goal of this analysis is to show that the second limit decays in $\underline{x}$ and gives rise to asymptotically suppressed contribution to $\underline{h} *_{\underline{x}} W_{M,\beta}$. This requires some work. First define

$$u_\mu(w) := w^{\frac{\mu-1}{2}} k_\sigma^{(0)}\left(\frac{w}{\underline{x}};M\right)\,,$$

so that $\int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} w^{\frac{\mu-1}{2}} k_\sigma^{(0)}\left(\frac{w}{\underline{x}};m\right) = \int_0^\infty \mathrm{d}w\, e^{(-i\sigma-\varepsilon)w} u_\mu(w)$.

Lemma C.6. *For $\nu \in \mathbb{N}_0$ it is*

$$u_\mu^\nu(w) = w^{\frac{\mu-1}{2}-\nu}\sum_{l=0}^{\nu}\binom{\nu}{l}\left(\frac{w}{\underline{x}}\right)^l k_\sigma^{(l)}\left(\frac{w}{\underline{x}};M\right)\prod_{j=0}^{\nu-l-1}\left(\frac{\mu-1}{2}-j\right)\,.$$

Proof. The statement follows from induction on ν and repeated application of the chain rule. □

Lemma C.7. *For $l \in \mathbb{N}_0$ and $0 < \delta < \frac{1}{2}$ it is*

$$\lim_{v \searrow 0} v^{l-\delta} k_\sigma^{(l)}(v; M) = 0 \,.$$

Proof. Choose $\delta < \rho < \frac{1}{2}$. For $l > 0$, it is $\sup_{v>0} \left| v^{l-\rho} k_\sigma^{(l)}(v; M) \right| < \infty$ by lemma C.4 and thus

$$\lim_{v \searrow 0} \left| v^{l-\delta} k_\sigma^{(l)}(v; M) \right| = \lim_{v \searrow 0} v^{\rho-\delta} \left| v^{l-\rho} k_\sigma^{(l)}(v; M) \right| = 0 \,.$$

For $l = 0$, corollary C.5 gives $\left| k_\sigma^{(0)}(v; M) \right| \lesssim v^\rho$. Consequently it is $\left| v^{-\delta} k_\sigma^{(0)}(v; M) \right| \lesssim v^{\rho-\delta}$ and

$$\lim_{v \searrow 0} \left| v^{-\delta} k_\sigma^{(0)}(v; M) \right| = 0 \,.$$

□

Corollary C.8. *For $\kappa \in \mathbb{N}_0$ and $0 < \delta < \frac{1}{2}$ it is*

$$\lim_{w \searrow 0} w^{\kappa - \frac{\mu-1}{2} - \delta} u_\mu^{(\kappa)}(w) = 0 \,.$$

Proof. This follows immediately from the previous two lemmata:

$$\begin{aligned} & \lim_{w \searrow 0} w^{\kappa - \frac{\mu-1}{2} - \delta} u_\mu^{(\kappa)}(w) \\ = \;& \sum_{l=0}^{\kappa} \binom{\kappa}{l} \underline{x}^{-\rho} \underbrace{\lim_{w \searrow 0} \left(\frac{w}{\underline{x}} \right)^{l-\rho} k_\sigma^{(l)} \left(\frac{w}{\underline{x}}; M \right)}_{=0} \prod_{j=0}^{\kappa-l-1} \left(\frac{\mu-1}{2} - j \right) \\ = \;& 0 \,. \end{aligned}$$

□

Claim C.9. *For integer $\frac{\mu-1}{2} < \eta \leqslant \frac{\mu+1}{2}$ it is*

$$\int_0^\infty \mathrm{d}w \, e^{-aw} u_\mu^{(0)}(w) = a^{-\eta} \int_0^\infty \mathrm{d}w \, a^{-aw} u_\mu^{(\eta)}(w) \,.$$

Proof. First show that for integer $0 \leqslant \kappa < \eta$ it is $\lim_{w \searrow 0} u_\mu^{(\kappa)}(w) = 0$. Let $0 < \delta < \frac{1}{2}$. As $\kappa < \eta$ and $\eta \leqslant \frac{\mu+1}{2}$, it is $\delta + \frac{\mu-1}{2} - \kappa > 0$. It follows that

$$\lim_{w \searrow 0} u_\mu^{(\kappa)}(w) = 0 = \lim_{w \searrow 0} \underbrace{w^{\delta + \frac{\mu-1}{2} - \kappa}}_{\to 0} \cdot \underbrace{w^{\kappa - \frac{\mu-1}{2} - \delta} u_\mu^{(\kappa)}(w)}_{\to 0 \text{ by preceding corollary}} = 0 \,.$$

The claim is then proven by η-fold partial integration. The boundary terms vanish due to the small argument behavior of $u_\mu^{(\kappa)}$ established above. □

Claim C.10. *For integer* $\frac{\mu+1}{2} < \nu \leqslant \frac{\mu+3}{2}$ *it is*

$$\int_0^\infty \mathrm{d}w\, e^{-aw} u_\mu^{(0)}(w) = a^{-\nu} \int_0^\infty \mathrm{d}w\, (e^{-aw} - 1) u_\mu^{(\nu)}(w)\,.$$

Proof. Since $\frac{\mu-1}{2} < \nu - 1 \leqslant \frac{\mu+1}{2}$ it is

$$\int_0^\infty \mathrm{d}w\, e^{-aw} u_\mu^{(0)}(w) = a^{-(\nu-1)} \int_0^\infty \mathrm{d}w\, a^{-aw} u_\mu^{(\nu-1)}(w)$$

by the previous result. Further analyzing the integral on the right hand side yields

$$\begin{aligned}
&\int_0^\infty \mathrm{d}w\, a^{-aw} u_\mu^{(\nu-1)}(w) \\
=\;& \lim_{\zeta\searrow 0} \int_\zeta^\infty \mathrm{d}w\, a^{-aw} u_\mu^{(\nu-1)}(w) \\
=\;& \lim_{\zeta\searrow 0} \left[-\frac{1}{a} e^{-aw} u_\mu^{(\nu-1)}(w) \Big|_\zeta^\infty + \frac{1}{a} \int_\zeta^\infty \mathrm{d}w\, e^{-aw} u_\mu^{(\nu)}(w) \right] \\
=\;& \frac{1}{a} \lim_{\zeta\searrow 0} \Bigg[e^{-a\zeta} u_\mu^{(\nu-1)}(\zeta) + \overbrace{\underbrace{\int_\zeta^\infty \mathrm{d}w\, u_\mu^{(\nu)}(w)}_{-u_\mu^{(\nu-1)}(\zeta)} - \int_\zeta^\infty \mathrm{d}w\, u_\mu^{(\nu)}(w)}^{0} + \int_\zeta^\infty \mathrm{d}w\, e^{-aw} u_\mu^{(\nu)}(w) \Bigg] \\
=\;& \frac{1}{a} \lim_{\zeta\searrow 0} \left[(e^{-a\zeta} - 1) u_\mu^{(\nu-1)}(\zeta) + \int_\zeta^\infty \mathrm{d}w\, (e^{-aw} - 1) u_\mu^{(\nu)}(w) \right] \\
=\;& \frac{1}{a} \lim_{\zeta\searrow 0} \frac{(e^{-a\zeta} - 1)}{\zeta} \cdot \underbrace{\lim_{\zeta\searrow 0} \zeta u_\mu^{(\nu-1)}(\zeta)}_{=0 \text{ by C.8}} + \frac{1}{a} \int_0^\infty \mathrm{d}w\, (e^{-aw} - 1) u_\mu^{(\nu)}(w) \\
=\;& \frac{1}{a} \int_0^\infty \mathrm{d}w\, (e^{-aw} - 1) u_\mu^{(\nu)}(w)\,.
\end{aligned}$$

It follows immediately that

$$\int_0^\infty \mathrm{d}w\, e^{-aw} u_\mu^{(0)}(w) = a^{-\nu} \int_0^\infty \mathrm{d}w\, (e^{-aw} - 1) u_\mu^{(\nu)}(w)$$

□

All the previous work goes into the following estimate, which establishes that the second limit in fact decays in $\underline{x}$ and is rapidly decreasing in M. Assuming $0 < \delta < \frac{1}{2}$ and $N \in \mathbb{N}_0$ it is

$$
\begin{aligned}
&\left|\int_0^\infty \mathrm{d}w\, e^{-aw} u_\mu^{(0)}(w)\right| \\
=\;& |a^{-\nu}| \left|\int_0^\infty \mathrm{d}w\, (e^{-aw}-1) u_\mu^{(\nu)}(w)\right| \\
=\;& |a^{-\nu}| \left|\int_0^\infty \mathrm{d}w\, (e^{-aw}-1) w^{\frac{\mu-1}{2}-\nu} \sum_{l=0}^{\nu} \binom{\nu}{l} \left(\frac{w}{\underline{x}}\right)^l k_\sigma^{(l)}\left(\frac{w}{\underline{x}}; M\right) \prod_{j=0}^{\nu-l-1} \left(\frac{\mu-1}{2}-j\right)\right| \\
\lesssim\;& \int_0^\infty \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta} |\underline{x}|^{-\delta} \sum_{l=0}^{\nu} \left| \left(\frac{w}{\underline{x}}\right)^{l-\delta} k_\sigma^{(l)}\left(\frac{w}{\underline{x}}; M\right)\right| \\
\lesssim\;& |\underline{x}|^{-\delta} (1+M)^{-N} \int_0^\infty \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta} .
\end{aligned}
$$

It remains to show that the integral $\int_0^\infty \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta}$ and its limit $\varepsilon \searrow 0$ exit. To see this, note that as $\frac{\mu+1}{2} < \nu \leqslant \frac{\mu+3}{2}$ it is $-2 < \frac{\mu-1}{2} - \nu + \delta < -1$. It is convenient to split the integral into two parts, which can be analyzed with relative ease:

$$
\int_0^\infty \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta} = \int_0^1 \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta} + \int_1^\infty \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta} .
$$

For the first part, note that for $\varepsilon < 1$ and $w \geqslant 0$ the term $|e^{-aw}-1|$ can be estimated as follows:

$$
\begin{aligned}
|e^{-aw}-1| &= |e^{-(\sigma+\varepsilon)w}-1| \\
&= \left|\sum_{l=0}^{\infty} \frac{-1^l}{l!} (i\sigma+\varepsilon)^l w^l\right| \\
&= w \left|\sum_{l=0}^{\infty} \frac{-1^{l+1}}{(l+1)!} (i\sigma+\varepsilon)^{l+1} w^l\right| \\
&\leqslant w \sum_{l=0}^{\infty} \frac{1}{(l+1)!} 2^{l+1} w^l \\
&\leqslant w \sup_{w\in[0,1]} \sum_{l=0}^{\infty} \frac{1}{(l+1)!} 2^{l+1} w^l .
\end{aligned}
$$

Since $w^{\frac{\mu+1}{2}-\nu+\delta} \in L_1([0,1])$, an application of the Lebesgue dominated convergence theorem yields

$$
\lim_{\varepsilon\searrow 0} \int_0^1 \mathrm{d}w\, |e^{-aw}-1| w^{\frac{\mu-1}{2}-\nu+\delta} \lesssim \int_0^1 \mathrm{d}w\, w^{\frac{\mu+1}{2}-\nu+\delta} < \infty .
$$

For the second part, also assume $\varepsilon < 1$ and estimate

$$
|e^{-aw}-1| = |e^{-(\sigma+\varepsilon)w}-1| \leqslant 2 .
$$

Using $w^{\frac{\mu-1}{2}-\nu+\delta} \in L_1([1,\infty])$, the Lebesgue dominated convergence theorem gives

$$\lim_{\varepsilon\searrow 0}\int_1^\infty \mathrm{d}w\,|e^{-aw}-1|w^{\frac{\mu-1}{2}-\nu+\delta} \lesssim \int_1^\infty \mathrm{d}w\,w^{\frac{\mu-1}{2}-\nu+\delta} < \infty\,.$$

This proves $\lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\,|e^{-aw}-1|w^{\frac{\mu-1}{2}-\nu+\delta} < \infty$ which concludes the discussion of the second limit.

The preceding results can be summarized into the following theorem:

Theorem C.11. *Let $\mu \in \mathbb{N}_0$ and $\underline{h} \in \mathcal{S}(\mathbb{R})$ such that $\widehat{h}$ has a double zero at the origin. Then there exist*

- $K_\pm : \mathbb{R} \to \mathbb{C}$ *rapidly decreasing, i.e.* $\forall\, N \in \mathbb{N}_0 : \sup_{M\in\mathbb{R}} |K_\pm(M)(1+M)^N| < \infty$,
- $r : [0,\infty[\times([0,\infty[\times\mathbb{R}) \to \mathbb{C}$ *such that* $\forall\, K \subset \mathbb{R}$ *compact,* $N \in \mathbb{N}_0 \exists\, C_{N,K} \forall M \in [0,\infty[, \underline{x} \in \mathbb{R} : \sup_{\mathbf{x}\subset K} |r(M,x)| < C_{N,K}|\underline{x}|^{-\frac{\mu+1}{2}-\delta}(1+m)^{-N}$, *where* $0 < \delta < \frac{1}{2}$,

with the property that for $M \in [0,\infty[, \underline{x} \in [0,\infty[, \mathbf{x} \in \mathbb{R}$ it is

$$\underline{h} *_{\underline{x}} \partial_{\mathbf{x}}^\mu W_{\beta,M}(x) = \underline{x}^{-\frac{\mu+1}{2}} \sum_{\sigma=\pm} e^{-\sigma i M \underline{x}} K_\sigma(M) + r(M,x)\,.$$

An analogous result holds for $\underline{x} \in]-\infty, 0]$.

Proof. Define:

$$K_\pm(M) := \frac{(-1)^\mu}{2\pi} k_\pm(0;M) \lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\, e^{-(\pm 1+\varepsilon)w} w^{\frac{\mu-1}{2}}\,,$$

$$r(M,x) := \sum_{\sigma=\pm} \frac{(-1)^\mu}{2\pi} e^{-\sigma M \underline{x}} \underline{x}^{-\frac{\mu+1}{2}} \lim_{\varepsilon\searrow 0}\int_0^\infty \mathrm{d}w\, e^{-(\pm 1+\varepsilon)w} w^{\frac{\mu-1}{2}} k_\sigma^{(0)}\left(\frac{w}{\underline{x}}; M\right)\,.$$

The computations leading up to this theorem prove that $K_\pm$ and r have the desired properties. In the case of $\underline{x} < 0$ substitute $w = -\underline{x}\left(\sqrt{\underline{p}^2+M^2} - M^2\right)$ instead of $w = \underline{x}\left(\sqrt{\underline{p}^2+M^2} - M^2\right)$ in the computation of $\underline{h} *_{\underline{x}} \partial_{\mathbf{x}}^\mu W_{\beta,M}(x)$. The rest of the analysis can the be performed analogously. □

C.4. Combinatorics of Normal Ordering

It is the aim of this section to prove the lemma 5.1, giving an explicit expression for expectation values of normal-ordered powers of ϕ_0 in arbitrary quasifree states. The lemma is restated here for convenience.

Claim C.12. *Let κ, ω be quasifree states on $\mathcal{A}_0$ and $g_1, \cdots, g_{m_1}, \cdots, g_{m_2}, f \in \mathcal{S}(\mathbb{R}^2)$. Define $M := \{1, \cdots, m_1 + m_2\}$ and $K_f := \omega(\phi_0(f)\phi_0(f)) - \kappa(\phi_0(f)\phi_0(f))$. It is*

$$
\begin{aligned}
&\omega\Big(\prod_{k=1}^{m_1} \phi_0(g_k) \cdot N_\kappa(\phi_0(f)^m) \cdot \prod_{k=m_1+1}^{m_2} \phi_0(g_k)\Big) \\
= &\sum_{\substack{m-(m_1+m_2)\\ \leqslant 2k \leqslant m}} \frac{m!}{2^k k!} K_f^k \sum_{\substack{S \subset M \\ |S|=m-2k}} \omega\Big(\prod_{\substack{p \in M\setminus S\\ \text{ordered}}} \phi_0(g_p)\Big) \\
&\times \prod_{\substack{p\in S\\ p\leqslant m_1}} \omega(\phi_0(g_p)\phi_0(f)) \cdot \prod_{\substack{p\in S\\ p> m_1}} \omega(\phi_0(f)\phi_0(g_p)) \,.
\end{aligned}
$$

The idea of the proof is to proceed to generating functionals of normal-ordered powers which can be expressed in terms of Weyl operators. It turns out that expectation values of products of Weyl operators and their normal-ordered equivalent can be easily computed, bypassing combinatorial difficulties. The expectation values of products of field operators and a ordered power can then be recovered by differentiation.

In the following, let π_ω be the GNS representation of $\mathcal{A}_0$ on the GNS Hilbert space $\mathcal{H}_\omega$ with GNS vector Ω_ω, dense invariant domain $\mathcal{D}_\omega$ and scalar product $\langle\cdot|\cdot\rangle$ of a quasifreestate ω. The notation $< f, g >_\omega := \omega(\phi_0(f)\phi_0(g))$ is used for $f, g \in \mathcal{S}(\mathbb{R}^2)$. Exploiting the quasifreeness of ω, it is straightforward to compute the expectation values of Weyl operators $W_\omega(f)$, $f \in \mathcal{S}_\mathbb{R}(\mathbb{R}^2)$.

Lemma C.13.

Let $f \in \mathcal{S}_\mathbb{R}(\mathbb{R}^2)$. For expectation values of Weyl operators there holds

$$
\langle \Omega_\omega | W_\omega(f) \cdot \Omega_\omega \rangle = e^{-\frac{1}{2}< f,f>_\omega} \,.
$$

Proof. Exploiting the quasifreeness of ω, one obtains for $k \in \mathbb{N}_0$:

$$
\omega(\phi_0(f)^k) = \begin{cases} 0, & \text{otherwise} \\ (k-1)!!\,\omega(\phi_0(f)\phi_0(f))^{k/2}, & k \text{ is even}\,. \end{cases}
$$

Using this, compute

$$
\begin{aligned}
&\langle \Omega_\omega | W_\omega(f) \cdot \Omega_\omega \rangle \\
= &\sum_{k=0}^{\infty} \frac{i^k}{k!} \langle \Omega_\omega | \pi_\omega(\phi_0(f)^k) \cdot \Omega_\omega \rangle \\
= &\sum_{k=0}^{\infty} \frac{i^k}{k!} \omega(\phi_0(f)^k) \\
= &\sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} (2k-1)!! \, < f, f >_\omega^k
\end{aligned}
$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} < f, f >_\omega^k \\ &= e^{-\frac{1}{2} < f,f >_\omega} . \end{aligned}$$

□

Lemma C.14. *For $f_1, \cdots, f_m \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^2)$ it is*

$$\langle \Omega_\omega | \prod_{k=1}^{m} W_\omega(f_k) \cdot \Omega_\omega \rangle = e^{-\sum_{1 \leqslant l < k \leqslant m} < f_l, f_k >_\omega} e^{-\frac{1}{2} \sum_{k=1}^{m} < f_k, f_k >_\omega} .$$

Proof. This is proven by induction on m using the Weyl relations and the preceding lemma. □

This lemma will be used in the following form:

Corollary C.15. *For $g_1, \cdots, g_{m_1}, \cdots, g_{m_2}, f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^2)$ it is*

$$\begin{aligned} & \langle \Omega_\omega | \prod_{k=1}^{m_1} W_\omega(g_k) \cdot W_\omega(f) \cdot \prod_{k=m_1+1}^{m_2} W_\omega(g_k) \cdot \Omega_\omega \rangle \\ = \; & e^{-\frac{1}{2} < f,f >_\omega - \sum_{k=1}^{m_1} < g_k, f >_\omega - \sum_{k=m_1+1}^{m_2} < f, g_k >_\omega} \\ & \times \langle \Omega_\omega | \prod_{k=1}^{m_2} W_\omega(g_k) \cdot \Omega_\omega \rangle . \end{aligned}$$

Proof. The relation in question is verified by expressing the expectation values on both sides as products of exponentials using the previous lemma. □

Now recall the definition of normal-ordered powers of $\phi_0(f)$ with respect to the quasifree state κ:

$$N_\kappa(\phi_0(f)^m) := \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \binom{m}{2k} < f, f >_\kappa^k \phi_0(f)^{m-2} .$$

The next lemma serves to compute a generating functional for normal-ordered powers of $\phi_0(f)$.

Lemma C.16. *On $\mathcal{D}_\omega$ the following identity holds:*

$$N_\kappa(W_\omega(f)) := \sum_{k=\infty}^{m} \frac{i^k}{k!} \pi_\omega(N_\kappa(\phi_0(f)^k)) = e^{\frac{1}{2} < f,f >_\kappa} W_\omega(f) .$$

In particular, $N_\kappa(W_\omega(f))$ extends to all of $\mathcal{H}$ and $t \mapsto N_\kappa(W_\omega(tf))$ is strongly continuous.

Proof. By quasifreeness of κ, it is $\kappa(\phi_0(f)^{2l}) = \frac{(2l)!}{2^l l!} < f, f >_\kappa^k$. Using this, compute on $\mathcal{D}_\omega$:

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{i^k}{k!} \pi_\omega(N_\kappa(\phi_0(f)^k)) \\
= \; &\sum_{k=0}^{\infty} \frac{i^k}{k!} \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^l \binom{k}{2l} \frac{(2l)!}{2^l l!} < f, f >_\kappa^l \; \pi_\omega(\phi_0(f))^{k-2l} \\
= \; &\sum_{k=0}^{\infty} \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{i^k}{k!} (i)^{-2l} \frac{k!}{(2l)!(k-2l)!} \frac{(2l)!}{2^l l!} < f, f >_\kappa^l \; \pi_\omega(\phi_0(f))^{k-2l} \\
= \; &\sum_{k=0}^{\infty} \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{l!} \Big(\frac{1}{2} < f, f >_\kappa \Big)^l \cdot \frac{i^{k-2l}}{(k-2l)!} \pi_\omega(\phi_0(f))^{k-2l} \\
= \; &\sum_{l=0}^{\infty} \frac{1}{l!} \Big(\frac{1}{2} < f, f >_\kappa \Big)^l \cdot \sum_{j=0}^{\infty} \frac{i^j}{j!} \pi_\omega(\phi_0(f))^j \\
= \; &e^{\frac{1}{2} < f, f >_\kappa} W_\omega(f) \, .
\end{aligned}$$

The resummation in the second to the last step is done by setting $j = k - 2l$. □

The lemma allows to express normal-ordered Weyl operators in terms of multiples of Weyl operators. Using the previous corollary, C.15, it is straightforward to compute expectation values of products of normal-ordered and non-normal-ordered Weyl operators:

$$\begin{aligned}
&\langle \Omega_\omega | \prod_{k=1}^{m_1} W_\omega(\mu_k g_k) \cdot N_\kappa(W_\omega(\lambda f)) \cdot \prod_{k=m_1+1}^{m_2} W_\omega(\mu_k g_k) \cdot \Omega_\omega \rangle \\
= \; &e^{-\frac{1}{2}\lambda^2 (\overbrace{< f, f >_\omega - < f, f >_\kappa}^{:= K_f}) - \lambda \sum_{k=1}^{m_1} \mu_k < g_k, f >_\omega - \lambda \sum_{k=m_1+1}^{m_2} \mu_k < f, g_k >_\omega} \\
&\times \langle \Omega_\omega | \prod_{k=1}^{m_2} W_\omega(\mu_k g_k) \cdot \Omega_\omega \rangle \, ,
\end{aligned}$$

where $\mu_1, \cdots, \mu_{m_1}, \cdots, \mu_{m_2}, \lambda \in \mathbb{R}$. This can be used to prove the original claim: $\omega\left(\prod_{k=1}^{m_1} \phi_0(g_k) \cdot N_\kappa(\phi_0() f^m) \prod_{k=m_1+1}^{m_2} \phi_0(g_k)\right)$ can be recovered by differentiation form the above expression.

Proof. Assume $m_1 = m_2$ for computational simplicity. In the case of real valued Schwartz functions, this represents no loss of generality as the relative position of the $W_\omega(\mu_k g_k)$ to $N_\kappa(W_\omega(\lambda f))$ does not matter. The correct generalization for complex valued Schwartz functions and arbitrary m_1, m_2 is evident. It is

$$\begin{aligned}
&\omega\Bigl(\prod_{k=1}^{m_2}\phi_0(g_k)\cdot N_\kappa(\phi_0()f^m)\Bigr)\\
=\ &(-i\partial_\lambda)^m|_{\lambda=0}\prod_{k=1}^{m_2}(-i\partial_{\mu_k})|_{\mu_k=0}\\
&\times\Bigl[\langle\Omega_\omega|\prod_{k=1}^{m_2}W_\omega(\mu_k g_k)\cdot N_\kappa(W_\omega(\lambda f))\cdot\Omega_\omega\rangle\Bigr]\\
=\ &-(i\partial_\lambda)^m|_{\lambda=0}\prod_{k=1}^{m_2}(-i\partial_{\mu_k})|_{\mu_k=0}\\
&\times\Bigl[e^{-\frac{1}{2}\lambda^2K_f-\lambda\sum_{k=1}^{m_2}\mu_k<g_k,f>_\omega}\langle\Omega_\omega|\prod_{k=1}^{m_2}W_\omega(\mu_k g_k)\cdot\Omega_\omega\rangle\Bigr].
\end{aligned}$$

To compute the ∂_{μ_k} derivates, it is useful to prove the following lemma by induction on q: Let $0\leqslant q\leqslant m_2$. Define $M_2 := \{1,\cdots,m_2\}$, $Q := \{1,\cdots,q\}\subset M_2$. It is:

$$\begin{aligned}
&\prod_{k=1}^{q}(-i\partial_{\mu_k})|_{\mu_k=0}\Bigl[e^{-\frac{1}{2}\lambda^2K_f-\lambda\sum_{k=1}^{m_2}\mu_k<g_k,f>_\omega}\langle\Omega_\omega|\prod_{k=1}^{m_2}W_\omega(\mu_k g_k)\cdot\Omega_\omega\rangle\Bigr]\\
=\ &e^{-\frac{1}{2}\lambda^2K_f-\lambda\sum_{k\in M_2\setminus Q}\mu_k<g_k,f>_\omega}\\
\times\ &\sum_{l=1}^{q}(i\lambda)^l\sum_{\substack{S\subset Q\\|S|=l}}\prod_{k\in S}<g_k,f>_\omega\cdot\langle\Omega_\omega|\prod_{k\in Q\setminus S}\pi_\omega(\phi_0(g_k))\cdot\prod_{k\in M_2\setminus Q}W_\omega(\mu_k g_k)\cdot\Omega_\omega\rangle.
\end{aligned}$$

For $q=0$ this is true. Assume the statement is true for q. In order to make the induction step somewhat presentable, use the following abbreviations: $q' := q+1$, $Q' := Q\cup\{q'\}$, $K(\lambda) := -\frac{1}{2}\lambda^2K_f$, $P_k :=< g_k, f>_\omega$, $\pi_k := \pi_\omega(\phi_0(g_k))$, $W_k(\mu_k) := W_\omega(\mu_k g_k)$. It is

$$\begin{aligned}
&\prod_{k=1}^{q+1}-i\partial_{\mu_k}|_{\mu_k=0}\Bigl[e^{K(\lambda)-\lambda\sum_{k=1}^{m_2}\mu_kP_k}\langle\Omega_\omega|\prod_{k=1}^{m_2}W_k(\mu_k)\cdot\Omega_\omega\rangle\Bigr]\\
=\ &-i\partial_{q+1}\Bigl[e^{K(\lambda)-\lambda\sum_{k\in M_2\setminus Q}\mu_kP_k}\\
&\times\sum_{l=1}^{q}(i\lambda)^l\sum_{\substack{S\subset Q\\|S|=l}}\prod_{k\in S}P_k\cdot\langle\Omega_\omega|\prod_{k\in Q\setminus S}\pi_k\cdot\prod_{k\in M_2\setminus Q}W_k(\mu_k)\cdot\Omega_\omega\rangle\Bigr]\\
=\ &e^{K(\lambda)-\lambda\sum_{k\in M_2\setminus Q'}\mu_kP_k}\sum_{l=1}^{q}(i\lambda)^{l+1}\sum_{\substack{S\subset Q\\|S|=l}}P_{q'}\prod_{k\in S}P_k\\
&\times\langle\Omega_\omega|\prod_{k\in Q\setminus S}\pi_k\cdot\prod_{k\in M_2\setminus Q'}W_k(\mu_k)\cdot\Omega_\omega\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\ e^{K(\lambda)-\lambda\sum_{k\in M_2\backslash Q'}\mu_k P_k}\sum_{l=1}^{q}(i\lambda)^l\sum_{\substack{S\subset Q\\|S|=l}}\prod_{k\in S}P_k\\
&\quad\times\langle\,\Omega_\omega|\prod_{k\in Q\backslash S}\pi_k\cdot\pi_{q'}\cdot\prod_{k\in M_2\backslash Q'}W_k(\mu_k)\cdot\Omega_\omega\rangle\\
&=\ e^{K(\lambda)-\lambda\sum_{k\in M_2\backslash Q'}\mu_k P_k}\sum_{l=1}^{q'}(i\lambda)^l\sum_{\substack{S\subset Q'\\|S|=l,q'\in S}}\prod_{k\in S}P_k\\
&\quad\times\langle\Omega_\omega|\prod_{k\in Q'\backslash S}\pi_k\cdot\prod_{k\in M_2\backslash Q'}W_k(\mu_k)\cdot\Omega_\omega\rangle\\
&+\ e^{K(\lambda)-\lambda\sum_{k\in M_2\backslash Q'}\mu_k P_k}\sum_{l=1}^{q'}(i\lambda)^l\sum_{\substack{S\subset Q'\\|S|=l,q'\notin S}}\prod_{k\in S}P_k\\
&\quad\times\langle\Omega_\omega|\prod_{k\in Q'\backslash S}\pi_k\cdot\prod_{k\in M_2\backslash Q'}W_k(\mu_k)\cdot\Omega_\omega\rangle\\
&=\ e^{K(\lambda)-\lambda\sum_{k\in M_2\backslash Q'}\mu_k P_k}\sum_{l=1}^{q'}(i\lambda)^l\sum_{\substack{S\subset Q'\\|S|=l}}\prod_{k\in S}P_k\\
&\quad\times\langle\Omega_\omega|\prod_{k\in Q'\backslash S}\pi_k\cdot\prod_{k\in M_2\backslash Q'}W_k(\mu_k)\cdot\Omega_\omega\rangle\,.
\end{aligned}
$$

This proves the lemma. Setting $q=m_2$ evaluates the ∂_{μ_k} derivatives:

$$
\begin{aligned}
&\prod_{k=1}^{m_2}(-i\partial_{\mu_k})|_{\mu_k=0}\Big[\langle\Omega_\omega|\prod_{k=1}^{m_2}W_\omega(\mu_k g_k)\cdot N_\kappa(W_\omega(\lambda f))\cdot\Omega_\omega\rangle\Big]\\
=\ &\sum_{l=0}^{m_2}(i\lambda)^l\sum_{S\subset M_2}\prod_{k\in S}<g_k,f>_\omega\cdot\omega\Bigg(\prod_{\substack{k\in M_2\backslash S\\\text{ordered}}}\phi_0(g_k)\Bigg)\cdot e^{-\frac{1}{2}\lambda^2 K_f}\,.
\end{aligned}
$$

It remains to compute the ∂_λ^m derivative:

$$
\begin{aligned}
&\omega\Bigg(\prod_{k=1}^{m_2}\phi_0(g_k)\cdot N_\kappa(\phi_0()f^m)\Bigg)\\
=\ &(-i\partial_\lambda)^m|_{\lambda=0}\prod_{k=1}^{m_2}(-i\partial_{\mu_k})|_{\mu_k=0}\Big[\langle\Omega_\omega|\prod_{k=1}^{m_2}W_\omega(\mu_k g_k)\cdot N_\kappa(W_\omega(\lambda f))\cdot\Omega_\omega\rangle\Big]\\
=\ &(-i\partial_\lambda)^m|_{\lambda=0}\Bigg[\sum_{l=0}^{m_2}(i\lambda)^l\sum_{S\subset M_2}\prod_{k\in S}<g_k,f>_\omega\cdot\omega\Bigg(\prod_{\substack{k\in M_2\backslash S\\\text{ordered}}}\phi_0(g_k)\Bigg)\cdot e^{-\frac{1}{2}\lambda^2 K_f}\Bigg]\\
=\ &(-i)^m\sum_{q=0}^{m}\binom{m}{q}\sum_{l=0}^{m_2}\partial_\lambda^{m-q}|_{\lambda=0}\left[(i\lambda)^l\right]\sum_{S\subset M_2}\prod_{k\in S}<g_k,f>_\omega
\end{aligned}
$$

$$\times\,\omega\Bigg(\prod_{\substack{k\in M_2\setminus S\\ \text{ordered}}}\phi_0(g_k)\Bigg)\cdot\partial_\lambda^q|_{\lambda=0}\left[e^{-\frac{1}{2}\lambda^2K_f}\right]$$

$$= (-i)^m\sum_{0\leqslant 2p\leqslant m}\binom{m}{2p}\sum_{l=0}^{m_2}\partial_\lambda^{m-2p}|_{\lambda=0}\left[(i\lambda)^l\right]\sum_{S\subset M_2}\prod_{k\in S}<g_k,f>_\omega$$

$$\times\,\omega\Bigg(\prod_{\substack{k\in M_2\setminus S\\ \text{ordered}}}\phi_0(g_k)\Bigg)\cdot(-K_f)^p\frac{(2p)!}{2^pp!}$$

$$=\sum_{m-m_2\leqslant 2p\leqslant m}\frac{m!}{2^pp!}K_f^p\sum_{\substack{S\subset M_2\\|S|=m-2p}}\prod_{k\in S}<g_k,f>_\omega\cdot\omega\Bigg(\prod_{\substack{k\in M_2\setminus S\\ \text{ordered}}}\phi_0(g_k)\Bigg).$$

For arbitrary m_1 and m_2 and complex valued Schwartz functions, this generalizes to

$$\omega\Bigg(\prod_{k=1}^{m_1}\phi_0(g_k)\cdot N_\kappa(\phi_0()f^m)\cdot\prod_{k=m_1+1}^{m_2}\phi_0(g_k)\Bigg)$$

$$=\sum_{m-m_2\leqslant 2p\leqslant m}\frac{m!}{2^pp!}K_f^p\sum_{\substack{S\subset M_2\\|S|=m-2p}}\prod_{\substack{k\in S\\k\leqslant m_1}}<g_k,f>_\omega\cdot\prod_{\substack{k\in S\\k>m_1}}<f,g_k>_\omega\cdot\omega\Bigg(\prod_{\substack{k\in M_2\setminus S\\ \text{ordered}}}\phi_0(g_k)\Bigg).$$

$\square$

D. Basics of Quantum Field Theory

It is the goal of this appendix to provide a systematic overview of the quantum field theoretical notions used throughout this thesis. The framework presented is based on [6] and considers a single scalar, hermitian field.

D.1. General Considerations and Definitions

For the purpose of stating a framework for quantum field theory, it is convenient to define a quantum field ϕ by means of a class of measuring devices. For a sufficiently nice real valued function f on space-time (assumed to be given by Minkowski space $\mathbb{M}^{n+1}$), $\phi(f)$ is to be thought of as a measurement performed by a device of that class indexed by f.
To justify the heuristic picture of $\phi(f) = \int_{\mathbb{M}^{n+1}} \mathrm{d}x\, f(x)\phi(x)$ being a *weighted average* of measurements performed at points $x \in \mathbb{M}^{n+1}$, the symbols $\phi(f)$ are given the structure of a real vector space by defining ϕ to be linear in f. In particular, the measurement $\phi(f)$ takes place in $\mathrm{supp}(f) \subset \mathbb{M}^{n+1}$.

In a setting where a field is defined in terms of measurement, the information about the state of the a system needs also be phrased in terms of measurement. All information an observer can obtain about the state of a system is given by the value of measurements performed in that state. This motivates the definition of a state ω as a map from the set of possible measurements to real numbers, associating to each observable its expectation value.

To describe correlation measurements, an additional multiplicative structure on the symbols $\phi(f)$, such that one obtains an algebra, is required. This, however, entails some conceptual problems. It's a basic fact of quantum mechanics that no uncorrelated states, i.e. states where all expectation values of products equal the products of the factors' expectation values, exist. This forces the algebra to be non-commutative and having two non-commuting observables brings forth the question of how to measure their product. No satisfactory operative prescription exists, since, in general, expectation values of products depend on their order.

The way out lies in acknowledging that the algebra generated by symbols $\phi(f)$ also contains unobservable elements, which in turn imposes the need for a criterion to identify the observable ones. Allowing the functions f in $\phi(f)$ and states to be complex valued and replacing $\mathbb{R}$-linearity by $\mathbb{C}$-linearity, such a criterion is given by equipping the resulting algebra with the involution $\phi(f)^* = \phi(\overline{f})$ extended to the whole algebra. Observables are then defined to be fixed points of the involution. In particular, this definition includes all $\phi(f)$ for real valued f as observables. Also, if the product of two oberservables a, b is again observable, then $ab = (ab)^* = b^* a^* = ba$, so a, b commute. This allows to define the measurement ab for commuting a, b as performing measurement a and b and multiplying their results.

An involution also allows to introduce a natural notion of positivity: positive elements in the algebra are those, which can be written in the form $a^* a$. A state is required to be non-negative on positive elements. If one also imposes the condition that in every state the empty measurement gives the result 1, states become involutive, i.e. for all a in the algebra one has $\omega(a^*) = \overline{\omega(a)}$. This has two relieving consequences:

- Expectation values of observables are real.
- States are fully determined by their values on observables.

This is conceptually satisfying, as it means that a state cannot contain information intrinsically inaccessible to the observer.

In order to keep the framework as generic as possible, no a priori dynamical relation is imposed on the algebra. However, all physically reasonable theories should disallow effects propagating faster than light. That is to say that all two measurements performed in spacelike separated regions should be uncorrelated, or in mathematical terms that $[\phi(f), \phi(g)] = 0$, if $\operatorname{supp}(f) \bowtie \operatorname{supp}(g)$.

These considerations lead to the following definition:

Definition D.1. The **field algebra** $\mathcal{A}$ of a single hermitian field ϕ is the free unital algebra generated by symbols $\phi(f)$, $f \in \mathcal{S}(\mathbb{R}^{n+1})$, subject to the following relations:

- Linearity:
  $\forall\, f_1, f_2 \in \mathcal{S}(\mathbb{R}^{n+1}),\ \lambda_1, \lambda_2 \in \mathbb{C} \,:\, \phi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \phi(f_1) + \lambda_2 \phi(f_2)$,
- Causality:
  $\forall\, f_1, f_2 \in \mathcal{S}(\mathbb{R}^{n+1}) \,:\, \operatorname{supp}(f_1) \bowtie \operatorname{supp}(f_2) \;\Rightarrow [\phi(f_1), \phi(f_2)] = 0$

and endowed with an involution

- $\phi(f)^* = \phi(\overline{f})$ extended anti-automorphically to $\mathcal{A}$.

Definition D.2. A **state** ω on a unital $*$-algebra is a map $\omega \,:\, \mathcal{A} \to \mathbb{C}$ satisfying the following properties:

- Linearity:
  $\forall\, a_1, a_2 \in \mathcal{A},\ \lambda_1, \lambda_2 \in \mathbb{C} \,:\, \omega(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \omega(a_1) + \lambda_2 \omega(a_2)$.

- Positivity:
  $\forall\, a \in \mathcal{A} \,:\, \omega(a^* a) \geqslant 0.$

- Normalization:
  $\omega(\mathbf{1}) = 1.$

States with the following continuity property exhibit desirable properties

Definition D.3. A state ω on the field algebra $\mathcal{A}$ is called **regular** :$\Leftrightarrow$

$$\forall\, k, m \in \mathbb{N},\ 1 \leqslant k \leqslant m\ f_1, \cdots f_m \in \mathcal{S}(\mathbb{R}^{n+1}) \ :\ f_k \mapsto \omega(\phi(f_1) \cdots \phi(f_m))$$

is a continuous map $\mathcal{S}(\mathbb{R}^{n+1}) \to \mathbb{C}$.

By the nuclear theorem (cf. [1] and references therein), there exisst for each $m \in \mathbb{N}$, a unique tempered distribution $\mathcal{W}_m \in \mathcal{S}'(\mathbb{R}^{m(n+1)})$ such that for $f_1, \cdots, f_m$ it is

$$< \mathcal{W}_m, f_1 \otimes \cdots \otimes f_m >= \omega(\phi(f_1) \cdots \phi(f_m))\,.$$

These tempered distributions are called **correlation functions** or m**-point functions**.

The Poincaré group acts automorphically on $\mathcal{A}$ in a canonical way:
For $(\Lambda, a) \in \mathscr{P}_{n+1}$, define $\alpha_{(\Lambda,a)} \in \mathrm{aut}(\mathcal{A})$ on generators of $\mathcal{A}$ by

$$\alpha_{(\Lambda,a)}(\phi(f)) := \phi(f_{(\Lambda,a)})$$

and extend it as a $*$-homomorphism to all of $\mathcal{A}$, where $f_{(\Lambda,a)}(x) := f((\Lambda, a)^{-1} \cdot x)$. $\alpha_{(\Lambda,a)}$ is well-defined, since it preserves the (two-sided) $*$-ideal generated by $\{\, [\phi(f), \phi(g)] \in \mathcal{A} \,|\, f, g \in \mathcal{S}'(\mathbb{R}^{n+1}),\ f \bowtie g \,\}$.

Within this framework, specific theories are given by representations π of the field algebra $\mathcal{A}$. By the fist isomorphism theorem, the represented algebra $\pi(\mathcal{A})$ is isomorphic to $\mathcal{A}/\ker(\pi)$. In this manner, a representation imposes new relations $\ker(\pi)$ on the field algebra, which may include dynamical laws. A natural way to construct representations from states is given by the **GNS construction**.

D.2. The GNS Construction

The Cauchy-Schwarz inequality is usually stated for spaces equipped with an inner product. In the following a slightly stronger version will be needed.

Lemma D.1. *Let V be a vectorspace over $\mathbb{C}$ equipped with a (possibly indefinite) positive hermitian sesquilinear form $(\cdot|\cdot)$. Then for $a, b \in V$ there holds*

$$|(a|b)|^2 \leqslant (a|a)(b|b)\,.$$

Proof. For $\lambda \in \mathbb{C}$, $a, b \in V$ it is $0 \leqslant (a - \lambda b | a - \lambda b) = (a|a) - 2\operatorname{Re}\lambda(a|b) + |\lambda|^2 (b|b)$. If $(b|b) \neq 0$ set $\lambda := \overline{(a|b)}(b|b)^{-1}$ to get the desired result. If $(b|b) = 0$, consider λ of the form $\lambda = \frac{\alpha}{2}\overline{(a|b)}$ for $\alpha \in \mathbb{R}$. It follows that $\alpha|(a|b)|^2 \leqslant (b|b)$ for all $\alpha \in \mathbb{R}$ which is only possible, if $(a|b) = 0$. □

Lemma D.2. *Let $\mathcal{A}$ be a unital $*$-algebra and ω a state on $\mathcal{A}$. Then ω is involutive.*

Proof. Exploiting the positivity property of ω it is $\omega((a + \mathbf{1})^*(a + \mathbf{1})) - \omega(a^* a) - 1 = \omega(a^*) + \omega(a) \in \mathbb{R}$ and thus

$$\omega(a^*) + \omega(a) = \overline{\omega(a^*)} + \overline{\omega(a)}\,. \tag{D.1}$$

Similarly it is $\omega((a + i\mathbf{1})^*(a + i\mathbf{1})) - \omega(a * a) - 1 = i(\omega(a^*) - \omega(a)) \in bR$ and

$$\omega(a^*) - \omega(a) = -\overline{\omega(a^*)} + \overline{\omega(a)}\,. \tag{D.2}$$

Adding D.1 and D.2 gives $\omega(a^*) = \overline{\omega(a)}$. □

Theorem D.3. *Let $\mathcal{A}$ be a unital $*$-algebra and ω a state on $\mathcal{A}$. Then there exist a Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$, a dense domain $\mathcal{D} \subset \mathcal{H}$, a $*$-representation π of $\mathcal{A}$ as closable unbounded operators on $\mathcal{H}$ with stable domain $\mathcal{D}$ (i.e. $\pi(\mathcal{A})\mathcal{D} \subset \mathcal{D}$) and a cyclic vector $\Omega \in \mathcal{D}$, such that for all $a \in \mathcal{A}$ it is $\omega(a) = \langle \Omega | \pi(a) \Omega \rangle$.*

Proof. On the $\mathbb{C}$-vector-space $\mathcal{A}$ consider the sesquilinear form

$$\begin{aligned} (\cdot|\cdot) : \mathcal{A} \times \mathcal{A} &\to \mathbb{C}\,, \\ (a, b) &\mapsto (a|b) := \omega(a^* b)\,. \end{aligned}$$

For $a \in \mathcal{A}$ it is $(a|a) = \omega(a^* a) \geqslant 0$, i.e. $(\cdot|\cdot)$ is positive. There also holds $\overline{(a|b)} = \overline{\omega(a^* b)} = \omega(b^* a) = (b|a)$, so $(\cdot|\cdot)$ is hermitian. Define the set $\mathcal{I} := \{\, a \in \mathcal{A} \,|\, (a|a) = 0 \,\}$. $\mathcal{I} \subset \mathcal{A}$ is a sub-vector-space: for $a \in \mathcal{I}$ and $\lambda \in \mathbb{C}$ it is $(\lambda a | \lambda a) = |\lambda|^2 (a|a) = 0$ and for $a, b \in \mathcal{I}$, the Cauchy-Schwarz inequality gives $(a|b) = 0$ and hence $(a + b | a + b) = (a|a) + (a|b) + (b|a) + (b|b) = 0$. Now let $b \in \mathcal{A}, a \in \mathcal{I}$. It is $(ba|ba) = \omega((ba)^* ba) = \omega(a^* b^* ba) = \omega((b^* ba)^* a)) = (b^* ba | a) = 0$, again by virtue of the Cauchy-Schwarz inequality. This shows that $\mathcal{I}$ is a left ideal in $\mathcal{A}$. On the vector space $\mathcal{D} := \mathcal{A}/\mathcal{I}$ define

$$\begin{aligned} \langle\cdot|\cdot\rangle : \mathcal{D} \times \mathcal{D} &\to \mathbb{C}\,, \\ (a/\mathcal{I}, b/\mathcal{I}) &\mapsto \langle a/\mathcal{I} | b/\mathcal{I} \rangle := (a|b)\,. \end{aligned}$$

Since for $a, b \in \mathcal{A}$ and $a', b' \in \mathcal{I}$ it is $(a + a' | b + b') = (a|b) + (a|b') + (a'|b) + (a'|b') = (a|b)$ (Cauchy-Schwarz), $\langle\cdot|\cdot\rangle$ is well-defined. It inherits positivity, hermicity and sesquilinearity from $(\cdot|\cdot)$ and is in addition positive definite: Suppose $\langle a/\mathcal{I} | a/\mathcal{I} \rangle = 0$. Then it is $(a|a) = 0$, so $a \in \mathcal{I}$ and $a/\mathcal{I} = 0$. In conclusion, $(\mathcal{D}, \langle\cdot|\cdot\rangle)$ is a pre-Hilbert space, which can be completed to a Hilbert space $\mathcal{H}$.

$\mathcal{A}$ acts on $\mathcal{H}$ as unbounded operators with common domain $\mathcal{D}$ in the following way: For $a \in \mathcal{A}, b/\mathcal{I} \in \mathcal{D}$ define $\pi(a) \cdot b/\mathcal{I} := (ab)/\mathcal{I}$. Note that $\pi(a) \cdot b/\mathcal{I} \in \mathcal{D}$, so $\pi(a) \in \hom_{\mathbb{C}}(\mathcal{D}, \mathcal{D})$. It can be readily seen that

$$\begin{aligned} \pi : \mathcal{A} &\to \hom_{\mathbb{C}}(\mathcal{D}, \mathcal{D}), \\ a &\mapsto \pi(a) \end{aligned}$$

is a homomorphism of associative algebras.

For $b/\mathcal{I}$, $c/\mathcal{I} \in \mathcal{D}$, $a \in \mathcal{A}$ it is $\langle b/\mathcal{I}|\pi(a) \cdot c/\mathcal{I}\rangle = \langle b/\mathcal{I}|(ac)/\mathcal{I}\rangle = \omega(b^*ac) = \omega((a^*b)^*c) = \langle (a^*b)/\mathcal{I}|c/\mathcal{I}\rangle = \langle \pi(a^*) \cdot b/\mathcal{I}|c/\mathcal{I}\rangle$. It follows that $\mathcal{D}$ lies in the domain of $\pi(a)^*$ and that $\pi(a)^*|_{\mathcal{D}} = \pi(a^*)$. In particular the domain of $\pi(a)^*$ is dense in $\mathcal{H}$, so $\pi(a)$ is closable. Understanding $\pi(a)^*$ as $\pi(a)^*|_{\mathcal{D}}$, π is a $*$-homomorphism.

Lastly define $\Omega := \mathbf{1}/\mathcal{I} \in \mathcal{D}$. Apparently $\pi(\mathcal{A}) \cdot \Omega = \mathcal{D}$, so Ω is cyclic. Furthermore it is $\langle \Omega|\pi(a)\Omega\rangle = \langle \mathbf{1}/\mathcal{I}|a/\mathcal{I}\rangle = (\mathbf{1}|a) = \omega(\mathbf{1}^*a) = \omega(a)$.

□

Given a state ω on $\mathcal{A}$, it is possible to look for states that are "not too different" from ω by constructing states on the represented algebra $\pi_\omega(\mathcal{A})$, where π_ω is the GNS representation of $\mathcal{A}$ on the GNS Hilbert space $\mathcal{H}_\omega$. States ρ on $\pi_\omega(\mathcal{A})$ give rise to new states on $\mathcal{A}$ by composition: $\omega_\rho := \rho \circ \pi_\omega$ is again a state on $\mathcal{A}$.
Elements $\Phi \in \mathcal{D} \subset \mathcal{H}_\pi$, for instance, define **vector states** on $\pi_\omega(\mathcal{A})$ by $\rho_\Phi(A) := \langle \Phi|A \cdot \Phi\rangle/\langle \Phi|\Phi\rangle$, $A \in \pi(\mathcal{A})$. Note that for $a \in \mathcal{A}$ it is $\omega_{\rho_\Phi}(a) = \langle \Phi|\pi(a) \cdot \Phi\rangle/\langle \Phi|\Phi\rangle$. Furthermore there exists $b \in \mathcal{A}$ such that $\Phi = \pi(b) \cdot \Omega$. This bears the interpretation that ω_{ρ_Φ} arises from $\omega = \omega_{\rho_\mathbf{1}}$ by local excitations created by physical operations b.

Another method of constructing new states out of old ones is taking convex combinations. If $\omega_1, \cdots, \omega_m$ are finitely many states on $\mathcal{A}$ and $q_1, \cdots, q_m \in [0, 1]$ satisfy $\sum_{k=1}^m q_k = 1$, then $\omega := \sum_{k=1}^m q_k\omega_k$ is again a state on $\mathcal{A}$. The set of states on $\mathcal{A}$ is convex.
Remark: The theory corresponding to ω may be realized on the direct sum of the GNS Hilbert spaces $\mathcal{H}_k$ of the ω_k via the representation $\bigoplus_{k=1}^m q_k\pi_k$. However, this is not in general the GNS representation induced by ω. E.g. in the case $\omega = \frac{1}{2}\omega + \frac{1}{2}\omega$ the representation $\frac{1}{2}\pi \oplus \frac{1}{2}\pi$ has the closed, invariant subspace $\{\, (\Phi, \Phi) \mid \Phi \in \mathcal{H} \,\} \subset \mathcal{H} \oplus \mathcal{H}$ and the corresponding subrepresentation is equivalent to the GNS representation π on $\mathcal{H}$ of ω.

States which can be constructed from a state ω on $\mathcal{A}$ by finite application of the two above constructions constitute the **folium** of ω.

D.3. Invariant States and Implementation of Symmetries

Definition D.4. Let $\mathscr{G} \subset \mathscr{L}_{n+1}$ be a subgroup of the Poincaré group. A state ω on the observable algebra $\mathcal{A}$ is **$\mathscr{G}$-invariant** $:\Leftrightarrow$

$$\forall\, g \in \mathscr{G}, a \in \mathcal{A} \,:\, \omega(\alpha_g(a)) = \omega(a)\,.$$

Invariant states are of interest, because they describe situations, where an active in setup of measuring devices has no impact on measuring results. In $\mathbb{R}^n$-invariant states, for instance, the shift of measuring devices has no impact on measurement results. In $\mathbb{R}$-invariant states, the time at which the measurement is performed does not matter, such states are static.

The following theorem exhibits another feature of the GNS representation.

Theorem D.4. *Let $\mathcal{A}$ be a $*$-algebr, $\mathscr{G} \subset \mathrm{aut}(\mathcal{A})$ a subgroup of the $*$-automorphisms of $\mathcal{A}$ and ω a $\mathscr{G}$-invariant state on $\mathcal{A}$. Then there exists a unitary representation U of $\mathscr{G}$ on the GNS Hilbert space $\mathcal{H}$ of the GNS representation induced by ω, leaving $\mathcal{D} \subset \mathcal{H}$ invariant, such that $\forall\, a \in \mathcal{A}$, $g \in \mathscr{G} \,:\, \pi(\alpha_g(a)) = U(g)\pi(a)U(g)^{-1}$.*

Proof. For $g \in \mathscr{G}$ and $a/\mathcal{I} \in \mathcal{D}$ define $U(g)\cdot a/\mathcal{I} := \alpha_g(a)/\mathcal{I}$ (here α stands for the defining representation of $\mathscr{G}$ as automorphisms on $\mathcal{A}$. This notation is used to be consistent with notation used to describe the action of $\mathscr{P}_{n+1}$ on the field algebra, where α is the representation of the Poincaré group as automorphisms of the field algebra). This is well-defined since α_g preserves left-ideals of $\mathcal{A}$. Each $U(g)$ leaves $\mathcal{D}$ invariant and for $g_1, g_2 \in \mathscr{G}$ one has

$$U(g_1)U(g_2)\cdot a/\mathcal{I} = \alpha_{g_1}(\alpha_{g_2}(a/\mathcal{I})) = \alpha_{g_1 g_2}(a/\mathcal{I}) = U(g_1 g_2)\cdot a/\mathcal{I}\,.$$

This shows that U is a group representation of $\mathscr{G}$ on $\mathcal{D}$. In particular, it is $U(g)\cdot\mathcal{D} = \mathcal{D}$. To show that each $U(g)$ is an isometry, compute

$$\begin{aligned}
||U(g)\cdot a/\mathcal{I}||^2 &= \langle U(g)\cdot a/\mathcal{I} | U(g)\cdot a/\mathcal{I}\rangle \\
&= \langle \alpha_g(a)/\mathcal{I} | \alpha_g(a)/\mathcal{I}\rangle \\
&= \omega(\alpha_g(a)^*\alpha_g(a)) \\
&= \omega(\alpha_g(a^* a)) \\
&= \omega(a^* a) \\
&= \langle a/\mathcal{I} | a/\mathcal{I}\rangle \\
&= ||a/\mathcal{I}||^2\,.
\end{aligned}$$

$U(g)$ extends uniquely to an isometry on $\mathcal{H}$ with dense range. This extension is a unitary operator on $\mathcal{H}$. Finally for $a \in \mathcal{A}$, $g \in \mathcal{G}$ and $b/\mathcal{I} \in \mathcal{D}$ one has

$$\begin{aligned}
& U(g)\pi(a)U(g)^{-1}\cdot b/\mathcal{I} \\
=\; & U(g)\pi(a)\cdot\alpha_{g^{-1}}(b)/\mathcal{I} \\
=\; & U(g)\cdot(a\cdot\alpha_{g^{-1}}(b))/\mathcal{I} \\
=\; & \alpha_g(a\cdot\alpha_{g^{-1}}(b))/\mathcal{I} \\
=\; & (\alpha_g(a)b)/\mathcal{I} \\
=\; & \pi(\alpha_g(a))\cdot b/\mathcal{I}\,.
\end{aligned}$$

This means that $U(g)\pi(a)U(g)^{-1} = \pi(\alpha_g(a))$ on $\mathcal{D}$. $\square$

Theorem D.5. *Let $\mathscr{G} \subset \mathscr{P}_{n+1}$ be a Lie subgroup of the Poincaré group, $\mathcal{A}$ the field algebra and ω a regular $\mathscr{G}$-invariant state on $\mathcal{A}$. Then the unitary representation U of $\mathscr{G}$ on the GNS Hilbert space is strongly and weakly continuous.*

Proof. Since U is a unitary representation, weak and strong continuity are equivalent and it suffices to prove that U is weakly continuous. Note that for monomials $a = \phi(f_1)\cdots\phi(f_{m_1})$, $b = \phi(h_1)\cdots\phi(h_{m_2})$ and $\Psi := a/\mathcal{I}$, $\Phi := b/\mathcal{I}$ it is

$$\langle a/\mathcal{I}|U(g)\cdot b/\mathcal{I}\rangle = \omega(\phi(\overline{f}_{m_1})\cdots\phi(\overline{f}_1)\phi(h_{1,g})\cdots\phi(h_{m_2,g}))\,,$$

which is continuous in g, as $g \mapsto h_g \in \mathcal{S}(\mathbb{R}^{n+1})$ is continuous and $f_1,\cdots,f_m \mapsto \omega(\phi(f_1)\cdots\phi(f_m))$ is continuous by regularity of ω. This continuity extends to the case of generic $\Phi,\Psi \in \mathcal{D}$ and, since $\mathcal{D} \subset \mathcal{H}$ is dense and $||U(g)|| = 1$, to $\Phi,\Psi \in \mathcal{H}$. □

The strong continuity of the representation U of $\mathscr{G}$ allows the application of Stone's theorem:

Theorem D.6 (Stone)**.** *Let $t \mapsto U(t)$ be a strongly continuous unitary representation of $\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Then there exists a self-adjoint operator H on $\mathcal{H}$ such that $U(t) = e^{itH}$. Conversely if H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then $U(t) := e^{itH}$ defines a strongly continuous representation of $\mathbb{R}$ on $\mathcal{H}$.*

So if $\mathscr{G}$ is a one-parameter subgroup of $\mathscr{P}$ and ω is a $\mathscr{G}$-invariant state on the field algebra $\mathcal{A}$, then there exists a self-adjoint generator H which implements the unitary action of $\mathscr{G}$ in the GNS representation induced by ω. In the case of space-time translations, this allows to define the Hamiltonian P_0 and impulse operators P_k.

A proof of Stone's theorem can be found in [3].

D.4. Correlation Functions

For $m \in \mathbb{N}$, the correlation- or m-point functions $\mathcal{W}_m$, defined by

$$< \mathcal{W}_m, f_1 \otimes \cdots \otimes f_m >= \omega(\phi(f_1)\cdots\phi(f_m))\,,$$

are of special interest, as their knowledge allows to fully reconstruct a theory. Given the family $(\mathcal{W}_m)_{m\in\mathbb{N}}$, one can recover the state ω, which in turn defines a representation of $\mathcal{A}$ using the GNS construction.

The information of how measurements consequent excitations influence other measurements, is encoded in the correlation functions. A convenient way to measure the degree of influence various measurements have upon each other is given by the **truncated correlation functions** $\mathcal{W}_m^T$. They can be defined as the difference of the m-point functions and their respective uncorrelated parts. The uncorrelated part of a m-point function expresses the expectation value of the measurements in question, under the assumption of these measurements not perturbing each other. A precise definition of $\mathcal{W}_m^T$ requires some preparation.

For a set X, let $\mathfrak{P}(X)$ denote the power set of X.

Definition D.5. Let X be a set. A subset $\mathfrak{T} \subset \mathfrak{P}(X)$ is called a **partition of** X :⇔

$$\bigcup_{S \in \mathfrak{T}} S = X \quad \text{and} \quad \forall\, S_1, S_2 \in \mathfrak{T} \,:\, S_1 \cap S_2 \neq \varnothing \Rightarrow S_1 = S_2\,.$$

In other words a partition of X is a collection of disjoint subsets of X whose union is all of X. Denote the set of partitions of X by $\mathcal{P}(X)$.

Definition D.6. For $m \in \mathbb{N}$ let $M_m := \{1, \cdots m\}$. Define the **truncated correlation functions** by $\mathcal{W}_1^T = \mathcal{W}_1$ and

$$< \mathcal{W}_m, f_1 \otimes \cdots \otimes f_m >= \sum_{\mathfrak{T} \in \mathcal{P}(M_m)} \prod_{S \in \mathfrak{T}} < \mathcal{W}_m^T, \widehat{f_1}^S \otimes \cdots \otimes \widehat{f_m}^S >\,,$$

where $f_1, \cdots, f_m \in \mathcal{S}(\mathbb{R}^{n+1})$ and $\widehat{f_j}^S$ denotes omission if $j \notin S$.

This implicit definition of the $\mathcal{W}$ is to be understood as a recurrence relation in m and defines the $\mathcal{W}_m^T$ uniquely.

Proposition D.7. *Let ω be a regular $\mathbb{R}^{n+1}$-invariant state. There exists a tempered distribution $W \in \mathcal{S}'(\mathbb{R}^{n+1})$ such that for $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$ there holds*

$$< \mathcal{W}_2, f \otimes g >=< W, f * \check{g} > \;.$$

Proof. Since ω is invariant under space-time translations there holds for all $a \in \mathbb{R}^{n+1}$ and $f, g \in \mathcal{S}(\mathbb{R}^{n+1}) \,:< \mathcal{W}_2, f_a \otimes g_a >=< \mathcal{W}_2, f \otimes g >$. It remains but to apply C.1. □

Definition D.7. W is called the **reduced two-point function**.

Definition D.8. $C := W - \check{W}$ is called the **commutator function**.

Definition D.9. Let $T \in \mathcal{D}'(\mathbb{R}^n)$. T is of **positive type** :⇔

$$\forall\, f \in \mathcal{D}(\mathbb{R}^n) \,:< T, \overline{f} * \check{f} >\geqslant 0\,.$$

Theorem D.8 (Bochner-Schwartz Theorem)**.** *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. T is of positive type iff T is tempered and $\widehat{T}$ is a polynomially bounded positive measure.*

A proof can be found in [4].

Corollary D.9. *Let ω be a regular $\mathbb{R}^{n+1}$-invariant state. Then the reduced two-point function W is of positive type and $\widehat{W}$ is a polynomially bounded positive measure.*

Proof. Since ω is regular, W is tempered and for $f \in \mathcal{S}(\mathbb{R}^{n+1})$ it is

$$< W, \overline{f} * \check{f} >=< \mathcal{W}_2, \overline{f} \otimes f >= \omega(\phi(f)^* \phi(f)) \geqslant 0\,.$$

The rest follows from the Bochner-Schwartz theorem. □

D.5. Quasifree States

A fairly accessible class of states on the field algebra $\mathcal{A}$ is given by those states, which are fully determined by their two-point function.

Definition D.10. A **partition into pairs** of X is a partition $\mathfrak{T}$ of X, such that

$$\forall\, S \in \mathfrak{T} \,:\, |S| = 2\,.$$

Denote the set of partitions into pairs of X by $\mathcal{P}_2(X)$.

Definition D.11. For $m \in \mathbb{N}$ let $M_m := \{1, \cdots m\}$. A state ω on the field algebra $\mathcal{A}$ is called **quasifree** $:\Leftrightarrow\ \forall\, m \in \mathbb{N}\, \forall\, f_1, \cdots f_m \in \mathcal{S}(\mathbb{R}^{n+1})$:

$$\omega(\phi(f_1)\cdots\phi(f_m)) = \sum_{\mathfrak{T}\in\mathcal{P}_2(M_m)} \prod_{\substack{(s_1,s_2)\in\mathfrak{T}\\ s_1<s_2}} \omega(\phi(f_{s_1})\phi(f_{s_2}))\,.$$

Note that for odd m there are no partitions into pairs of M_m, so $\omega(\phi(f_1)\cdots\phi(f_m)) = 0$. It is apparent, that a quasifree state is entirely determined by its two-point function. Conversely, it is possible to define a quasifree state prescribing only the two-point function.

Theorem D.10. *Let*

$$\xi \,:\, \mathcal{S}(\mathbb{R}^{n+1}) \times \mathcal{S}(\mathbb{R}^{n+1}) \to \mathbb{C}$$

be bilinear such that

- $\forall\, f \in \mathcal{S}(\mathbb{R}^{n+1}) \,:\, \xi(\overline{f}, f) \geqslant 0,$
- $\forall\, f, g \in \mathcal{S}(\mathbb{R}^{n+1}) \,:\, \mathrm{supp}(f) \bowtie \mathrm{supp}(g) \Rightarrow \xi(f,g) = \xi(g,f).$

Then there exists a unique quasifree state ω on the field algebra $\mathcal{A}$, such that for all $f, g \in \mathcal{S}(\mathbb{R}^{n+1}) \,:\, \omega(\phi(f)\phi(g)) = \xi(f,g)$. Furthermore, if ξ is separately continuous, then ω is regular.

Sketch of proof. For $m \in \mathbb{N}$, $f_1, \cdots f_m \in \mathcal{S}(\mathbb{R}^{n+1})$ define

$$\omega(\phi(f_1)\cdots\phi(f_m)) = \sum_{\mathfrak{T}\in\mathcal{P}_2(M_m)} \prod_{\substack{(s_1,s_2)\in\mathfrak{T}\\ s_1<s_2}} \xi(f_{s_1}, f_{s_2})\,,$$

which extends linearly to the field algebra $\mathcal{A}$. The second condition on ξ in the statement of the theorem is used to show that this is well-defined (causality). The first (positivity) condition is used to show that ω, extended in this way, is in fact positive and thus defines a state on $\mathcal{A}$. ω is by definition quasifree with two-point function $\omega(\phi(f)\phi(g)) = \xi(f,g)$. $\square$

D.6. Weyl Operators and Relations

Quasifree states have the property, that for $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$ the operator $\pi(\phi(f))$ on the GNS Hilbert space is essentially self-adjoint and has a unique closure $\pi(\phi(f))_c$. By Stone's theorem the unitary operator $W(f) := e^{i\pi(\phi(f))_c}$ allows to recover $\pi(\phi(f))_c$ by differentiation. The **Weyl operators** $W(f)$ obey **Weyl relations**, which fully contain the structure of the algebra $\pi(\mathcal{A})$. The algebra generated by $W(f)$, subject to the Weyl relations, is a C^*-algebra, called the **Weyl algebra**. Proceeding to the weak closure of the Weyl algebra, one can resort to W^*-algebraic methods to study quantum field theories. Here, the complete theory is neither fully developed nor applied, but a brief introduction is given and some results will be used.

Lemma D.11. *Let $m \in \mathbb{N}$, $f, g, h_1, \cdots h_m \in \mathcal{S}(\mathbb{R}^{n+1})$ and ω be a quasifree state on the field algebra $\mathcal{A}$. The following holds for $1 \leqslant k \leqslant m$:*

$$\begin{aligned} & \omega(\phi(h_1)\cdots\phi(h_k)[\phi(f),\phi(g)]\phi(h_{k+1})\cdots\phi(h_m)) \\ = \; & \omega([\phi(f),\phi(g)])\omega(\phi(h_1)\cdots\phi(h_m))\,. \end{aligned}$$

Sketch of proof. It is

$$\begin{aligned} & \omega(\phi(h_1)\cdots\phi(h_k)[\phi(f),\phi(g)]\phi(h_{k+1})\cdots\phi(h_m)) \\ = \; & \omega(\phi(h_1)\cdots\phi(h_k)\phi(f)\phi(g)\phi(h_{k+1})\cdots\phi(h_m)) \\ & -\omega(\phi(h_1)\cdots\phi(h_k)\phi(g)\phi(f)\phi(h_{k+1})\cdots\phi(h_m))\,. \end{aligned}$$

Partitions into pairs of the latter two terms either pair $\phi(f)$ and $\phi(g)$ together, or pair both with one of the $\phi(h_j)$. In the latter case, there exists for each partition of the first term $\phi(h_1)\cdots\phi(h_k)\phi(f)\phi(g)\phi(h_{k+1})\cdots\phi(h_m)$ a corresponding partition of the second term $\phi(h_1)\cdots\phi(h_k)\phi(g)\phi(f)\phi(h_{k+1})\cdots\phi(h_m)$ appearing with the opposite sign, so the only non-cancelling contributions arise for partition into pairs in which $\phi(f)$ and $\phi(g)$ are paired together. These sum to the r.h.s of the equation in the claim. □

This has the consequence that commutators of generators of $\mathcal{A}$ are multiples of the identity in GNS representations induced by quasifree states:

Proposition D.12. *Let $\mathcal{A}$ be the field algebra, ω a quasifree state on $\mathcal{A}$ and π the GNS representation of $\mathcal{A}$ on $\mathcal{H}$. For $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$ it is $\pi([\phi(f),\phi(g)]) \in \mathbb{C}\mathbf{1}$.*

Proof. Let $a/\mathcal{I}, b/\mathcal{I} \in \mathcal{D} \subset \mathcal{H}$. Write

$$\begin{aligned} a^*/\mathcal{I} &= \sum_{\substack{i \\ \text{finite}}} \prod_{\substack{k_i \\ \text{finite}}} \phi(h_{a,k_i}) \\ b/\mathcal{I} &= \sum_{\substack{j \\ \text{finite}}} \prod_{\substack{l_j \\ \text{finite}}} \phi(h_{b,l_j})\,. \end{aligned}$$

Using the preceding lemma it is

$$
\begin{aligned}
& \langle a/\mathcal{I}|\pi([\phi(f),\phi(g)])\cdot b/\mathcal{I}\rangle \\
= & \sum_{\substack{i,j \\ \text{finite}}} \omega\Big(\prod_{\substack{k_i \\ \text{finite}}} \phi(h_{a,k_i})[\phi(f),\phi(g)]\prod_{\substack{l_j \\ \text{finite}}} \phi(h_{b,l_j})\Big) \\
= & \ \omega([\phi(f),\phi(g)]) \sum_{\substack{i,j \\ \text{finite}}} \omega\Big(\prod_{\substack{k_i \\ \text{finite}}} \phi(h_{a,k_i}) \prod_{\substack{l_j \\ \text{finite}}} \phi(h_{b,l_j})\Big) \\
= & \ \omega([\phi(f),\phi(g)])\langle a/\mathcal{I}|b/\mathcal{I}\rangle \\
= & \ \langle a/\mathcal{I}|\omega([\phi(f),\phi(g)])\mathbf{1}\cdot b/\mathcal{I}\rangle\,.
\end{aligned}
$$

$\pi([\phi(f),\phi(g)])$ and $\omega([\phi(f),\phi(g)])\mathbf{1}$ agree on the dense subspace $\mathcal{D}\subset\mathcal{H}$, so they are the same in $\pi(\mathcal{A})$. □

It is clear (cf. proof of the GNS construction) that for $f\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$ the operator $\pi(\phi(f))$ is hermitian. The proof that it is also essentially self-adjoint relies on Nelson's analytic vector theorem (cf. [4]) and the observation that every element in $\mathcal{D}$ is analytic for $\pi(\phi(f))$. Since the operators $\pi(\phi(f))$ are essentially self-adjoint, they possess a unique self-adjoint closure, denoted $\pi(\phi(f))_c$. By Stone's theorem, these self-adjoint operators give rise to a strongly continuous representation of $\mathbb{R}$ as unitary operators given by $t\mapsto e^{it\pi(\phi(f))_c}$.

Definition D.12. Let ω be a quasifree state on the field algebra $\mathcal{A}$ and π the GNS representation induced by ω. For $f\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$ define the unitary **Weyl operator**:

$$W_\pi(f) := e^{i\pi(\phi(f))_c}$$

Theorem D.13. *The Weyl operators satisfy the* ***Weyl relations****:*

$$
\begin{aligned}
W_\pi(f)W_\pi(g) &= e^{-\frac{1}{2}\sigma(f,g)}W_\pi(f+g)\,, \\
W_\pi(f)^* &= W_\pi(-f)\,,
\end{aligned}
$$

where $f,g\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$ *and* $\sigma(f,g):=\omega([\phi(f),\phi(g)])$.

Sketch of proof. Noting that $t\mapsto W_\pi(tf)$ is the unitary representation of $\mathbb{R}$ associated to the self-adjoint operator $\pi(\phi(f))_c$, the second relation is a consequence of the representation being a group homomorphism and $U^{-1}=U^*$ for unitary U.
The first relation can be proven by considering the Lie group generated by the $W_\pi(f)$ and establishing that the universal enveloping algebra of its Lie algebra is generated by $\pi(\phi(f))$ for $f\in\mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$. The relation is then a consequence of the BHC formula using that $[\pi(\phi(f)),\pi(\phi(g))]=\omega([\phi(f),\phi(g)])\mathbf{1}$ is central. □

Definition D.13. The $*$-algebra generated by $W_\pi(f)$ is called the **Weyl algebra**.

Remark: The Weyl algebra is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$. Hence, its closure in the operator norm of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

The Weyl algebra contains all information necessary to recover the algebra $\pi(\mathcal{A})$. For $f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$, the operator $\pi(\phi(f))_c$ can be recovered by differentiation. For general $f \in \mathcal{S}(\mathbb{R}^{n+1})$, the operator $\pi(\phi(f))$ is given by $\pi(\phi(\operatorname{Re} f)) + i\pi(\phi(\operatorname{Im} f))$, where $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^{n+1})$.

D.7. Vacuum States

Theorem D.14. *For $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$ and $m > 0$ define*

$$\xi(f,g) := (2\pi)^{-n} \int \mathrm{d}p\, \theta(\underline{p})\delta(p^2 - m^2)\, \widehat{\check{f}} \cdot \widehat{g}(p)\,.$$

This ξ fulfills all conditions of theorem D.10 and thus defines a regular quasifree state on $\mathcal{A}$.

Proof. Clearly ξ is bilinear. $\mathrm{d}p\, \theta(\underline{p})\delta(p^2 - m^2)$ defines a positive measure, so it is $\xi(\overline{f}, f) = (2\pi)^{-n} \int \mathrm{d}p\, \theta(\underline{p})\delta(p^2 - m^2)\, |\widehat{f}|^2(p) \geqslant 0$. Let $\operatorname{supp}(f) \bowtie \operatorname{supp}(g)$. To see that $\xi(f,g) = \xi(g,f)$, first note that $\check{f} * g$ vanishes on $V^+ \cup V^-$:
Let $x \in V^+ \cup V^-$. For $y \in \mathbb{R}^2$ it is $f(y) = 0$ or $g(x+y) = 0$ (if both $f(y) \neq 0$ and $g(x+y) \neq 0$ then $y \in \operatorname{supp}(f)$ and $x + y \in \operatorname{supp}(g)$ and consequently $x = (x+y) - y \notin V^+ \cup V^-$ since $\operatorname{supp}(f) \bowtie \operatorname{supp}(g)$), which implies that $\check{f} * g(x) = \int \mathrm{d}y\, f(y) g(x+y) = 0$. By continuity, $\check{f} * g(x)$ vanishes on $\overline{V^+} \cup \overline{V^-}$. It is

$$\begin{aligned}
\xi(f,g) - \xi(g,f) &= (2\pi)^{-n} < \varepsilon(\underline{p})\delta(p^2 - m^2), \widehat{\check{f}} \cdot \widehat{g}(p) > \\
&= (2\pi)^{-n} < \varepsilon(\underline{p})\delta(p^2 - m^2), \mathcal{F}(g * \check{f})(p) > \\
&= (2\pi)^{-n} < \mathcal{F}(p \mapsto \varepsilon(\underline{p})\delta(p^2 - m^2))(x), g * \check{f}(x) >
\end{aligned}$$

in a slightly sloppy notation. The distribution $\varepsilon(\underline{p})\delta(p^2 - m^2)$ is proportional to the difference of two distributions $\widehat{G}_\pm$ who are given as boundary values of holomorphic functions:

$$\widehat{G}_\pm(p) = \lim_{\substack{\epsilon \searrow 0 \\ \epsilon \in V^+}} \frac{1}{(p \pm i\epsilon)^2 - m^2}\,.$$

An argument involving the residue theorem and Lorentz invariance of $\widehat{G}_\pm$ shows that $\widehat{\widehat{G}}_\pm$ has support in $\overline{V^\mp}$, i.e. $G_\pm$ has support in $\overline{V^\pm}$. It immediately follows that the support of $\mathcal{F}(p \mapsto \varepsilon(\underline{p})\delta(p^2 - m^2))$ lies in $\overline{V^+} \cup \overline{V^-}$. Since $\check{f} * g(x)$ vanishes on $\overline{V^+} \cup \overline{V^-}$ it is $\xi(f,g) - \xi(g,f) = 0$.
As $\theta(\underline{p})\delta(p^2 - m^2)$ is a tempered distributions and the Fourier transform and convolution are continuous on $\mathcal{S}(\mathbb{R}^2)$, ξ is separately continuous, so the quasifree state given rise to by ξ is regular.

□

Definition D.14. The state on $\mathcal{A}$ defined by the above two-point function is called the **free mass m vacuum** state and is denoted by $\omega_{\infty,m}$. If $\pi_{\infty,m}$ is the GNS representation induced by $\omega_{\infty,m}$, then $\mathcal{A}_m := \pi_{\infty,m}(\mathcal{A})$ is called the **free mass m field algebra**. The reduced two-point function is denoted by $W_{\infty,m}$ and the commutator function by C_m. A theory is called **free**, iff it is given by a state on $\mathcal{A}_m$.

The free vacuum state on $\mathcal{A}$ has a few remarkable properties:

- $\omega_{\infty,m}$ is a $\mathscr{P}^{+,\uparrow}_{n+1}$-invariant state on $\mathcal{A}$ (Poincaré invariance). This means that the vacuum is stationary, homogeneous and looks the same for every Lorentz observer.

- The joint spectrum of the generators P_ν is contained in $\overline{V^+}$ (relativistic spectrum condition). Consequently, in every Lorentz frame the spectrum of the Hamiltonian (self-adjoint generator of time translations) is a positive operator.

- If $\alpha_{\mathbf{y}}$ denotes spatial translation by $\mathbf{y} \in \mathbb{R}^n$ and a, b are in $\mathcal{A}$, then

$$\lim_{|\mathbf{y}|\to\infty} |\omega_{\infty,m}(a\,\alpha_{\mathbf{y}}(b)) - \omega(a)_{0,m}\omega(b)_{0,m}| = 0$$

  (spatial clustering). The spatial clustering property ensures the uniqueness of the vacuum state in its folium. More generally, the folium of a Poincaré invariant state satisfying the relativistic spectrum condition and the spatial clustering property contains no other such state. In a stronger W^*-theoretic setting this is a feature of purity of $\omega_{\infty,m}$.

The Poincaré invariance can be directly computed. The clustering property is shown by noting that $\xi(f, g_{\mathbf{y}})$ vanishes in the limit of large $|\mathbf{y}|$ and exploiting the quasifreeness of $\omega_{\infty,m}$. To prove the relativistic spectrum condition note that, by the spectral theorem, it is $\omega_{\infty,m}(a\,\alpha_x(b)) = \int_{\sigma(P)} \mathrm{d}\mu_{a,b}(p)\, e^{ipx}$, where $\sigma(P)$ is the joint spectrum of the P_ν. The right hand side is the Fourier transform of the density of the measure $\mu_{a,b}$, so $\sigma(P)$ lies is the union of supports of the the inverse Fourier transforms of $x \mapsto \omega_{\infty,m}(a\,\alpha_x(b))$ as a, b range though $\mathcal{A}$. These supports can be shown to all lie in in $\overline{V^+}$, so $\sigma(P) \subset \overline{V^+}$.

Poincaré invariance, the relativistic spectrum condition and the spatial clustering property are expected to be features of any physical vacuum and motivate the definition of a vacuum state on $\mathcal{A}$.

Definition D.15. A state ω on the field algebra $\mathcal{A}$ is a **vacuum state** $:\Leftrightarrow$
ω is Poincaré invariant, satisfies the relativistic spectrum condition and has the spatial clustering property.

Vacuum states are of special importance, as one expects states describing local excitations of the vacuum to cover a wide range of physically interesting situations and to be suitable (under additional assumptions) to describe scattering processes.

The relativistic spectrum condition implies analyticity properties of the correlations functions which provide a strong tool for the study of vacuum states.

Theorem D.15. *Let ω be a state on $\mathcal{A}$ invariant under time- and spatial translations satisfying the relativistic spectrum condition. Then for all $a, b \in \mathcal{A}$ the map*

$$\begin{aligned} F_{a,b} : \mathbb{R}^{n+1} &\to \mathbb{C} \\ x &\mapsto \omega(a\,\alpha_x(b)) \end{aligned}$$

admits an analytic continuation (also denoted F) to the domain $\mathcal{R} := \{z \in \mathbb{C}^{n+1} \mid \operatorname{Im} z \in V^+\}$.

Sketch of proof. It follows from the spectral theorem and the relativistic spectrum condition that for $a, b \in \mathcal{A}$ it is

$$F_{a,b}(x) = \int_{\overline{V^+}} \mathrm{d}\mu_{a,b}(p)\, e^{ixp} \,. \tag{D.3}$$

For $z = x + iy \in \mathbb{C}^{n+1}, p \in \mathbb{R}^{n+1}$ define $\eta(z,p) = \eta(p,z) := \eta(x,p) + i\eta(y,p)$ ($zp = xp + iyp$ in short). The map $z \mapsto e^{izy} = e^{ixp}e^{-yp}$ is holomorphic and uniformly bounded in $\mathcal{R}$ for p varying in $\overline{V^+}$, as $\eta(y,p) \geqslant 0$ if $p \in \overline{V^+}$, $y \in V^+$. Partial derivatives ∂_{z_j} of the integrand exist and differentiation and integration can be interchanged. Hence $z \mapsto F_{a,b}(z)$ is holomorphic in $\mathcal{R}$. □

Another consequence of the relativistic spectrum condition is that the two-point function can be recovered from the commutator function: Assuming ω is a vacuum state, it is

$$F_{f,g}(x) := \omega(\phi(f)\,\alpha_x(\phi(g))) =< \widehat{W}, p \mapsto \check{\hat{f}} \cdot \hat{g}(p) e^{ipx} > \;.$$

A comparison of this equation to $D.3$ shows that $\widehat{W}$ has support in $\overline{V^+}$. The spatial clustering property implies that $\widehat{W}$ has no discrete part at the origin. Hence $\hat{C} = \widehat{W} - \check{\widehat{W}}$ can be split without ambiguity into two parts with support in $\overline{V^+}$ and $\overline{V^-}$ respectively. It is clear that the former is given by $\widehat{W}$, i.e.

$$\widehat{W} = \theta(\underline{p})\hat{C} \,.$$

D.8. KMS States

Another interesting class of states is that of thermal equilibrium states, which are characterized by the KMS condition (named after *R. Kubo, P. Martin, J. Schwinger*, see [2]). It is well established and motivated that the KMS condition is the good generalization of the Gibbs- von Neumann condition $\omega(a) = \operatorname{tr}(e^{-\beta H}a)/\operatorname{tr}(e^{-\beta H})$ surviving, in contrast to the latter, the thermodynamic limit.

Definition D.16 (KMS condition)**.** Let $\beta > 0$. A state ω on the field algebra $\mathcal{A}$ is called a **β-KMS state** or said to have the **KMS property** at inverse temperature β :⇔

For all $a, b \in \mathcal{A}$ there exists a continuous function

$$F_{a,b} : \overline{\mathcal{S}_\beta} \to \mathbb{C},$$

which is holomorphic on $\mathcal{S}_\beta := \{ z \in \mathbb{C} \,|, 0 < \operatorname{Im} z < \beta \}$ and for $t \in \mathbb{R}$ satisfies

$$F_{a,b}(t) = \omega(a\, \alpha_t(b)), \quad F_{a,b}(t + i\beta) = \omega(\alpha_t(b)\, a),$$

where α_t denotes time translation.

Definition D.17. A β-KMS state is called **proper** $:\Leftrightarrow$
$\forall\, a, b \in \mathcal{A} \,:\, F_{a,b}$ is bounded in the boundary of $\mathcal{S}_\beta$ and $\exists\, c, \gamma \in \mathbb{R}\, \forall\, z = x + iy \in \mathcal{S}_\beta \,:$ $|F_{a,b}(z)| \leqslant c \cdot |x|^\gamma$.

Properness is a technical assumption made to ensure that KMS states are stationary: by renaming the field $\phi(f) - \omega(\phi(f)) \to \phi(f)$ it can w.l.o.g. be assumed that one-point functions vanish. For $a \in \mathcal{A}$, the map $t \mapsto F_{\mathbf{1},a}(t) = \omega(\alpha_t(a))$ can be extended to $\overline{\mathcal{S}_\beta}$ by the KMS condition and it is $F_{\mathbf{1},a}(t + i\beta) = F_{\mathbf{1},a}(t)$. It follows that $F_{\mathbf{1},a}$ can be further extended to a continuous function on $\mathbb{C}$, which is $i\beta$ periodic and holomorphic on $\mathbb{C}\backslash \bigcup_{k\in\mathbb{Z}} \mathbb{R} + ik\beta$ and thus entire. Properness implies that one can apply Lindelöf's theorem (cf. [16]) to show that $F_{\mathbf{1},a}$ is bounded on $\overline{\mathcal{S}_\beta}$, so by periodicity $F_{\mathbf{1},a}$ is bounded on $\mathbb{C}$. Liouville's theorem states that $F_{\mathbf{1},a}$ is constant, which means that ω is stationary. In a stronger C^*-algebraic setting the properness assumption is not needed to show this. In this work, all KMS states are assumed to be proper.

There is a variety of arguments corroborating that KMS states describe thermal equilibrium. One is that in settings where $e^{-\beta H}a$ is defined and has finite trace, Gibbs-von Neumann states are KMS states and that the KMS property is a distinctive feature of Gibbs- von Neumann states. A number of arguments exist intrinsically motivating the KMS condition as characteristic of thermal equilibrium by linking it to passivity or stationarity and stability under small perturbations ([2], [6]).

Definition D.18. A state ω on $\mathcal{A}$ is **time-clustering** $:\Leftrightarrow\ \forall\, a, b \in \mathcal{A}$:

$$\lim_{t\to\pm\infty} |\omega(a\, \alpha_t(b)) - \omega(a)\omega(b)| = 0\,.$$

In a stronger W^*-theoretic setting, the set of KMS states is a simplex. The time-clustering property can there be shown to be satisfied by states which are extremal in set of KMS states. These are interpreted to describe pure phase ([2]). A simple interpretation of the time-clustering property is given by the heuristic idea that the nature of a thermal background is such that two measurements performed in a thermal state become uncorrelated in the limit of the time elapsed between the two measurements tending to infinity.

For some existing relativistic theories, the map $t \mapsto \omega(a\, \alpha_t(b))$ can even be further extended and exhibits stronger analyticity properties beyond those conveyed by the KMS property. It is expected that states with this extended analyticity properties cover a wide range of thermal equilibrium situations (cf. [10]).

Definition D.19 (Relativistic KMS condition)**.** Let $\beta > 0$. A state ω on the field algebra $\mathcal{A}$ is called a **relativistic β-KMS state** or said to have the **relativistic KMS property** at inverse temperature β :⇔
There exists $e \in V^+, e^2 = 1$, such that for all $a, b \in \mathcal{A}$ there is a continuous function

$$F_{a,b} : \overline{\mathcal{R}_{\beta e}} \to \mathbb{C},$$

which is holomorphic on $\mathcal{R}_{\beta e} := \{ z \in \mathbb{C}^{n+1} \mid \operatorname{Im} z \in V^+ \cap (\beta e + V^-) \}$ and for $x \in \mathbb{R}^{n+1}$ satisfies

$$F_{a,b}(x) = \omega(a\, \alpha_x(b)), \quad F_{a,b}(t + i\beta e) = \omega(\alpha_x(b)\, a),$$

where α_x denotes space-time translation.

The relativistic KMS condition implies the KMS condition (choose a Lorentz frame in which e lies on the time axis). A relativistic β-KMS state is defined to be proper, if it is proper as a β-KMS state. Properness is assumed for all relativistic KMS states.

The relativistic KMS condition can be seen as the thermal counterpart to the relativistic spectrum condition of vacuum states. If the temperature tends to 0, i.e. $\beta \to \infty$, then the domain of analyticity $\mathcal{R}_{\beta e}$ of the function $x \mapsto \omega(a\, \alpha_x(b))$ tends to $\mathcal{R}$. The analogy can also be seen from the regular KMS condition in the Fourier transform of the commutator function. Assuming the underlying thermal state ω is invariant under space-time translations, it is

$$\omega(\phi(f))\, \alpha_t(\phi(g))) = < \widehat{W}, p \mapsto \check{\hat{f}} \cdot \hat{g}(p) e^{it\underline{p}} > \tag{D.4}$$

and by the KMS condition

$$\omega(\alpha_t(\phi(g))\, \phi(f))) = < \widehat{W}, p \mapsto \check{\hat{f}} \cdot \hat{g}(p) e^{it\underline{p}} e^{-\beta \underline{p}} > \ .$$

Setting $t = 0$ one obtains

$$\check{\widehat{W}} = \widehat{W} e^{-\beta \underline{p}}$$

and consequently

$$\hat{C} = \widehat{W} - \check{\widehat{W}} = (1 - e^{-\beta \underline{p}}) \widehat{W}\, . \tag{D.5}$$

Further assuming vanishing one-point functions and time-clustering, the positive measure $\widehat{W}$ does not have a discrete part at 0 (If it did, the right hand side of equation D.4 could not vanish for all f, g), so equation D.5 can be divided unambiguously by $(1 - e^{-\beta \underline{p}})$:

$$\widehat{W} = \frac{1}{(1 - e^{-\beta \underline{p}})} \hat{C}\, . \tag{D.6}$$

In the limit of zero temperature, i.e. $\beta \to \infty$ equation D.6 becomes

$$\widehat{W} = \theta(\underline{p}) \hat{C}\, ,$$

which is characteristic for states satisfying the relativistic spectrum condition.

Relation $D.6$ allows to recover quasifree KMS states from the commutator function in that state. Since $[\phi(f), \phi(g)]$ is a multiple of the identity in the free field algebra $\mathcal{A}_m$ (i.e. $\pi_{\infty,m}([\phi(f), \phi(g)]) \in \mathbb{C}\mathbf{1} \subset \mathcal{A}_m$), the free mass m commutator function C_m is the same for all free theories (theories which aries from states on $\mathcal{A}_m$). It follows that if there are translationally invariant β-KMS states on $\mathcal{A}_m$, their reduced two-point functions are fixed by

$$\widehat{W}_{\beta,m} := \frac{1}{1 - e^{-\beta \underline{p}}} \widehat{C}_m = 2\pi \frac{\varepsilon(\underline{p})\delta(p^2 - m^2)}{1 - e^{-\beta \underline{p}}}\,, \tag{D.7}$$

which is to be understood in the sense of distributions. $W_{\beta,m}$ satisfies the conditions of theorem D.10 and gives in fact rise to a quasifree KMS state on $\mathcal{A}$.

Definition D.20. The quasifree β-KMS state given by the reduced two-point function $W_{\beta,m}$ is called **free mass** m **β-KMS state**.

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