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An Analysis of the 'Thermal-Time Concept' of Connes and Rovelli

Eine Analyse des 'Thermal-Time Konzepts' von Connes und Rovelli

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Notation

We work in the *geometrized unit system* with the speed of light c, the gravitational constant G, the Boltzmann constant k_B, and the reduced Planck constant $\hbar$ equal to unity. The signature of the metric $g_{\mu\nu}$ used throughout this thesis is $-+++$. The index notation uses Greek indices $\mu, \nu, \cdots = 0, 1, 2, 3$ to indicate 4-dim. spacetime indices, Latin indices $i, j, \ldots$ describe the spatial part and run from 1 to 3. We exploit Einstein's summation convention. If not explicitly mentioned, we assume to be in 3+1-dim. Minkowski space. The (complex) Hilbert spaces we are dealing with are supposed to be separable (only in appendix A this assumption is explicitly stated whenever it is necessary). For the benefit of the reader, we list below a number of symbols appearing more or less frequently in the course of this thesis.

$\mathcal{H}, \mathcal{K}, \ldots$	(separable) Hilbert spaces
$A, B, \ldots$	operators (observables)
$\mathcal{D}(A)$	domain of the operator A
$A \subset B$	$\mathcal{D}(A) \subset \mathcal{D}(B)$ and $A\|_{\mathcal{D}(A)} = B\|_{\mathcal{D}(A)}$
$A \leq B$	$A - B$ is a positive operator
$\ker(A)$	kernel of the operator A
$\mathrm{ran}(A)$	rank of the operator A
$E(\lambda)$	spectral decomposition
$\rho(A)$	resolvent of A
$R_\lambda(A)$	resolvent of A at λ, $R_\lambda(A) = (\lambda - A)^{-1}$
$\sigma(A)$	spectrum of A
P, Q	projection operators
P_Ω	projection operator on the subspace generated by the vector $\Omega \in \mathcal{H}$
$\mathcal{O}$	spacetime region
$\mathcal{O}'$	causal complement of $\mathcal{O}$
$\mathcal{O}_1 \subset\subset \mathcal{O}_2$	closure of $\mathcal{O}_1$ is contained in the interior of $\mathcal{O}_2$
M	Minkowski space
D	double-cone
W	wedge region
$V_\pm$	forward / backward lightcone
$\mathcal{A}$	algebra
$\mathcal{A}^+$	positive cone of $\mathcal{A}$
$\mathfrak{A}$	(quasi-local) C*-algebra
$\mathcal{R}$	von Neumann algebra
$\mathcal{M}$, $\mathcal{N}$	von Neumann algebras, mostly type I factors
$\mathcal{R}'$	commutant of $\mathcal{R}$
$\mathcal{R}^*$	dual space of $\mathcal{R}$
$\mathcal{R}_*$	predual space of $\mathcal{R}$
$\Gamma(\mathcal{R})$	Connes spectrum
$S(\mathcal{R})$	spectral invariant
$\vee_i \mathcal{R}_i$	$(\cup_i \mathcal{R}_i)''$
$\mathcal{B}(\mathcal{H})$	von Neumann algebra consisting of all bounded operators acting on $\mathcal{H}$
$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$	abstract local net
$\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$	weak closure of the representation of an abstract net
ω	state
ρ	density matrix associated with a normal state
$\omega \restriction \mathcal{R}(\mathcal{O})$	partial state obtained from the restriction of ω on $\mathcal{R}(\mathcal{O})$
$\mathfrak{C}^\omega$	centralizer of ω
ω^P	pertubed state, constructed from a perturbation $P = P^* \in \mathcal{R}$ of ω
π	representation
$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$	GNS representation induced by ω
$(\mathcal{H}_0, \pi_0, \Omega_0)$	vacuum representation
$\Omega_0 \in \mathcal{H}_0$	vacuum vector
$\omega_0 = \langle \Omega_0, \cdot\, \Omega_0 \rangle$	vacuum state

τ_t	one-parameter group of *-automorphisms
$\mathcal{R}^\tau$	fixed point algebra of τ
$\mathcal{R}_\tau$	set of entire analytic elements for τ
$\operatorname{spec}\tau$	Arveson spectrum
$\delta(A)$	infinitesimal generator of τ_t, $\delta(A) = \frac{\mathrm{d}}{\mathrm{d}t}\tau_t(A)\big\vert_{t=0}$
τ_t^P	perturbed *-automorphism group with perturbation $P = P^* \in \mathcal{R}$
α_t	physical time flow
H	(physical) Hamiltonian
$\mathrm{Ad}(e^{iHt})$	$\mathrm{Ad}(e^{iHt})(A) = e^{iHt}Ae^{-iHt}$
H_c	covariant Hamiltonian
$S = J\Delta^{1/2}$	Tomita operator
Δ	modular operator
$\tilde{\Delta}$	modular function
K	modular / thermal Hamiltonian
σ_s^ω	modular flow / thermal time flow induced by ω
$\Lambda_s^{\mathcal{O}}$	automorphism group describing a geometrical action of the modular group in $\mathcal{O}$
$\sigma_s(\mathcal{O})$	modular flow acting on $\mathcal{R}(\mathcal{O})$; we reserve this expression for the vacuum modular flow
$\tilde{\sigma}_s$	canonical flow associated with a von Neumann algebra
J	modular conjugation
j^ω	*-anti-isomorphism from $\mathcal{R}$ onto $\mathcal{R}'$ induced by ω
s	modular parameter / thermal time
$\mathcal{P}_+^\uparrow$	proper orthochronous Poincaré group
$\Lambda(\lambda)$	Lorentz boost with boost parameter λ
L	generator of the unitary representation of the Lorentz boosts
a	acceleration
τ	proper time
T	clock time
Γ	phase space
Γ_{ex}	extended phase space
Γ_c	covariant phase space
β	inverse temperature

Chapter 0

Introduction

"I strongly suspect that the key is time."
Lee Smolin, American physicist (*1955)

It is a long-standing dream of mankind to disclose the most fundamental secrets of Nature and to understand how it works. The ultimate goal is a "theory of everything" which gives a comprehensive picture how Nature is build up. In a first step one would like to unify general relativity with the principles of quantum theory into a theory of quantum gravity. One of the main issues appearing in attempts to do that is an incompatible treatment of the notion of time underlying both theories. A complete formulation of quantum gravity requires an understanding of the true nature of physical time in order to resolve this so-called *problem of time in quantum gravity*. One possible approach, advocated by several physicists, is to adopt the position that time is not a fundamental concept, which means that a theory of quantum gravity can be formulated and fully interpreted without the distinction of some flow of time. In fact, there is the hope that certain issues which have obstructed the unification of general relativity and quantum theory so far will disappear that way. However, perception of a flow of time is an everyday-experience. Time plays an utmost crucial role in the way we experience and describe the world, and it appears in many well-established physical theories, wherein it seems to be an indispensable ingredient. So if time is accepted as a "real" component of the physical world, how can this be combined with the expected universal timelessness of Nature? How does time emerge to become such a characteristic part of our macroscopic world?

A solution to this problem was suggested by Connes and Rovelli in 1994 in form of the *thermal time hypothesis* (TTH), which is the subject of this thesis. According to that hypothesis time is state-dependent and emerges on a thermodynamical scale: Given a physical system one generically does not know its precise microscopic state. By measurements of macroscopic observables one can assign a density matrix to the system instead, which contains all the knowledge the observer gained about the system and permits a statistical treatment as the best description available. Following the reasoning of Connes and Rovelli it is this ignorance about the microscopic details of the system, expressed via the mixed state, which distinguishes a flow on the algebra of observables. This *thermal time flow* is selected in such a way that the state becomes a thermal equilibrium state relative to this flow. An equilibrium flow has a variety of physical properties, which is the main reason why this flow is postulated to be a physical time flow. The TTH says that it is this flow which we perceive as the flow of time and which is best suitable to evolve the system. Thermal time represents a time concept as in surroundings of a thermalised system.

The TTH can be formulated under very general conditions in the algebraic scheme of quantum field theory where it is supported by strong and profound mathematical results such as the Tomita-Takesaki theorem and the cocycle Radon-Nikodým theorem. In this setting it is the modular flow induced by a faithful normal state on a local von Neumann algebra $\mathcal{R}(\mathcal{O})$ which is identified with the physical time flow of an observer who is confined to a spacetime region $\mathcal{O}$. This identification is due to the surprising discovery that the modular flow satisfies the KMS condition, which is usually consulted to characterise equilibrium states, and thus equips the state in view with KMS equilibrium dynamics. The modular flow has its origin in a symmetry structure revealed by the

Tomita-Takesaki theory and is up to now a little understood quantity. Under certain conditions it shows a geometrical realisation in the underlying spacetime manifold. In such a case the TTH predicts a local temperature which is supposed to be measurable by observers moving along the modular flow lines.

While the TTH has been successfully applied to a couple of examples, where it leads to reasonable physical time flows and recovers theoretically well-founded phenomena such as the Unruh effect or the Gibbons-Hawking effect, there are doubts if it can generally expected to give rise to a physical flow, in the sense that it is really exploited by an observer to describe the evolution of the system. In this thesis it is our intention to make a comprehensive analysis of the thermal time concept and the peculiarities of thermal time. We want to investigate to what extent the proceeding of Connes and Rovelli to justify the thermal time flow as a thermodynamical, physical time concept is plausible. In chapter 7 we study the relation between thermal time and ordinary notions of time such as Newtonian time or proper time (in the limit where the latter ones exist), and if thermal time defines a natural time concept in spite of the presence of those well-established times. We analyse further to what extent the thermal time flow shows physically meaningful properties, in particular we are interested whether the thermal time postulate suffices to distinguishes indeed an equilibrium flow with all its characteristics. Only a flow with typical dispersive properties can justify a *thermal* time concept in the proper sense. Moreover, we take a look at the physical content of the thermal time flow. Are the motivations leading to the TTH sufficient to permit an interpretation as physical time in the sense that this flow is realised in the system, that is observable? In our discussion we analyse the thermal time flow on local algebras appearing in algebraic quantum field theory, but we also address the mathematically much more tractable quantum mechanical limiting case of the thermal time postulate.

The thesis is organized as follows. In chapter 1 we describe the various notions and aspects of time appearing in physical theories but also in our conscious awareness. In particular we give a detailed account on the conceptually different appearance of time in general relativity and quantum theory which ultimately leads us to the problem of time in quantum gravity. Moreover, we motivate the fundamental timelessness approach being the starting point for the formulation of the TTH. Chapter 2 contains an introduction into algebraic quantum field theory. We present some main motivations and results, explain the emergence of local nets of von Neumann algebras, which are crucial for the applicability of the TTH, and we justify a useful property of local nets, the split property, which later admits some (heuristic) considerations concerning the behaviour of the thermal time flow on local algebras. In chapter 3 we introduce the KMS condition as a generalisation of the Gibbs condition to distinguish thermal equilibrium states. The main purpose of this chapter is to analyse properties of KMS states concerning above all their stability properties and to investigate to what extent the KMS condition alone is really a suitable selection criterion for physical equilibrium states. Modular theory is the subject of chapter 4. We state the Tomita-Takesaki theorem as well as the cocycle Radon-Nikodým theorem and explain the deep relation between modular theory and KMS theory, which altogether provide the mathematical input into the TTH. With regard to applications of the TTH and its discussion we treat physical implications and examples of modular theory afterwards, like e.g. a possible geometrical meaning of the modular objects. All these chapters provide the basis for studying the TTH and are intended to be self-contained introductions into the respective subject.

Eventually we come to the formulation of the TTH as it was postulated by Connes and Rovelli in chapter 5. We explicate in detail the motivations and ideas underlying the TTH and take a look at extensions and alternative formulations. All its various successful applications are discussed in chapter 6. An interpretation of thermal time turns out to be quite simply possible in these examples, which is why a main emphasis lies on the temperature aspect and a recovery of the above mentioned effects. Chapter 7 then contains the critical discussion of the TTH we already made some comments about. The thesis is closed by conclusions as well as an outlook.

We refer to appendix A for the most important mathematical tools appearing in the main text, like for instance the Borel functional calculus, the theory of almost periodic functions, and an overview over C*-algebras, von Neumann algebras, their classification theory, states and representations. In appendix B we present covariant approaches to classical and quantum mechanics, which among other things are supposed to underline that on a mechanical level the notion of time is not as much linked to the respective theories as it is suggested in their standard formulations, which may be rated as another hint for a fundamental timelessness of Nature.

Chapter 1

The Notion of Time

"If nobody asks me, I know what time is, but if I am asked then I am at a loss what to say."
St. Augustine of Hippo, Romanized Berber philosopher and theologian (345-430)

1.1 Introduction

What is time? Perception of a flow of time seems to be an elementary experience. Everybody agrees that time has something to do with change, growth and decay. But what is the precise nature of time, and why does time seem to play such an important role in our physical world as well as in our consciousness? What is the origin of time? All these fascinating questions belong to the most foundational ones aiming at the deepest secrets of our world. To quote Barbour, "*nothing is more mysterious and elusive than time*" ([8], p. 11).

Astonishingly, although the concept of time shows tantalising paradoxes and largely unresolved issues, physicists (and philosophers) have not worked much on this subject. Undoubtedly, one can find a huge variety of literature having the word "time" in their title, but very few elaborations really try to discuss the notion of time on a fundamental and scientific level. Most writers on this subject take time, or rather the most basic aspects of time, as being already well-understood. They primarily deal with descriptions of the various notions of time appearing in physics (and also biology, awareness, psychology, philosophy, etc.), they discuss measurement procedures, notions of simultaneity, the emergence of certain properties of time (like the arrow of time), how a metrization of a topological time may be realised, or they analyse the role of consciousness in the perception-of-time process as well as other epistemological problems. Nonetheless they do not give an idea what the very concept of (a flow of) time actually is, why it is fundamental, or if not, how it ultimately emerges to become such a "real" ingredient of our experience.

In most physical theories time is assumed to be an a priori given quantity, whose origin is not explained in the context of the respective theory. For instance, when Newton brought time into the domain of scientific inquiry, it proved a fruitful method of analysing physical processes, though little about time itself was taught. Anyway, there have been great contributions, apart from Newton by e.g. Galilei, Leibniz, Kant, Mach, Poincaré, Einstein, Reichenbach or Grünbaum, which altogether led to a better insight into some mysteries around the notion of time. Only in the last few decades this subject received increasing attention, when it turned out that an understanding of the nature of time is crucial for the development of more fundamental physical theories like a still hypothetical theory of quantum gravity. A topical treatment on the subject of time took place within the scope of the FQXi essay contest on *The Nature of Time* in 2008,[1] where many threads around the origin of time are discussed [9, 37, 48, 75, 111].

In this chapter it is our intention to describe various notions and properties of time as they appear in physics. In particular, we want to go in detail into the role of time in the two most important theories reflecting our knowledge about the world, general relativity and quantum theory, and explain the so-called *problem of time in quantum gravity*, which results from an incompatible treatment of time in these two theories. We shall also justify one possible ansatz to solve the problem of time, advocated by a couple of physicists, which is to abandon the notion of time

[1] www.fqxi.org

from fundamental physics. This way we provide the starting point for the introduction of the TTH. In the past, there has been for centuries a deep and fruitful connection between physics and philosophy to investigate fundamental questions in science. In a thesis which deals with such an issue, it seems reasonable to continue this tradition (at least for one section), which is why we want to begin with some rather philosophical considerations, which, to anticipate that, will immediately lead us to physics.

1.2 Time and Philosophy

How do we know about time [99]? The psychological perception of time to be flowing is of the very essence of time as experienced by us. It is rooted in the sense of an ongoing, allied with the conscious awareness of a transient state of affairs, the *present*, or *Now*, which appears to flow through time. According to WHITROW ([134], p. 77), "*awareness of the present is the most fundamental temporal experience*". Each person is confined to a Now and cannot experience other parts of time. Yet we speak of other parts called *future* and *past*, even if they are not real, and this brings us to another evidence of time. The Now causes a psychological distinction between past and future, which are completely different entities in our consciousness. We have memories of the past, and none of the future; we feel we cannot change the past, but can influence the future. The present has the character of a passing moment which separates the past from the future, it is the instant of transition between them. While it slips away, we enter into a new present, thus always remaining in the eternal Now. Preceding in time, a part of the undetermined, unreal future becomes real (this is also expressed by the term *Becoming*), and as soon as it is real, it slides into the past to become unreal again. The present is the only reality. All we have of time is this Now, gliding with us through the current of events that flows from the unchangeable past to the unknowable future. There is nothing whatever we can do to prevent time passing, to prevent events sliding into the past. This aspect expresses a third important property of experienced time, its *(psychological) irreversibility*, the sense of a unidirectional ongoing. It concerns the process of perception and cognition, once we have seen or known something we can never unsee or unknow it.[2] Besides these rather inner experiences, we believe to know about time from various (periodic) motions and changes in the external world, which already suggests that there is a link between the physical and the mental. A notion of time with all these very familiar peculiarities is called *common sense time*.

On the other hand, the introduction of a mathematical time with a different and more elementary structure turned out to be an extremely fruitful ingredient in physical theories. It drives the dynamics of a system, creates and destroys correlations between the various dynamical degrees of freedom, and defines an ordering relation on the experiments that have been done. The fact that notions of time which entered physics have mostly been regarded as very similar to space, but of a simpler character, because time is only one-dimensional, is the reason why problems concerning the nature of time have been much less investigated than those of space. According to REICHENBACH [100] this is wrong, time is much more associated with our consciousness than space. We feel a flow of time with all the peculiarities pointed out above, and we time-order not only events of the external world, but also all our subjective experiences occur in time, to which no spatial relation exist. Our observations of physical things, our feelings and emotions, and our thinking processes extend through time. MINKOWSKI's fusion of space and time seems to suggest that time is on an equal footing as space. REICHENBACH argues that this unification does by no means imply that the special role of time is lost (in the physical objective world). It is indicated by a different sign with which the time-part enters the spacetime-metric, and it is incorporated in the causal structure; the lightcones define a chronological future and a chronological past, while all spacelike directions are on an equal footing. Most importantly, spacelike separated objects are necessarily different, whereas timelike separated particles might be the same particle at different times. REICHENBACH comes to the conclusion that time is much more fundamental than space, and even in the physical world it has a structure much more special than that of space.

Time plays a decisive role to give a physical description of our world, but at the same time it seems to be closely related to our consciousness. This rises the substantial question to what extent time really belongs to the external world and to what extent it is rather a construction of our

[2] "to know" in the sense of "to experience"

mind. According to WHITROW [134] the idea of time is closely associated with the human mind. ST. AUGUSTINE considered purely temporal phenomena (auditory rather than visual) to conclude that the source of time is related to the mind. In his famous work *Kritik der reinen Vernunft* KANT believed (cf. [134]) that time is merely a part of our mental apparatus for imagining and visualising the world; it is essential to our experience of things in the world, but it has no claim of reality. He interpreted the flow of time as the form in which human beings experience reality.

However, following REICHENBACH [99, 100], emotive reaction to a time flow cannot reveal the true features of time. An understanding of time can only be achieved in a profound, scientific analysis of the nature of time in physics. Only in this frame a logical clarification becomes accessible. He argues that a reduction of the notion of time to a purely physical one, disregarding psychological aspects, is possible and in fact advantageous ([99], pp. 17-18):

"*There is no other way to solve the problem of time than the way through physics. [...] If time is objective the physicist must have discovered that fact, if there is Becoming the physicist must know it; but if time is merely subjective and Being is timeless, the physicist must have been able to ignore time in his construction of reality and describe the world without the help of time. [...] It is a hopeless enterprise to search for the nature of time without studying physics.*"

Most scientists believe [134] that our perception of time is based on an objective factor that provides an external control for the timing of our psychological processes. This objective part is what we call *physical time*. The path we follow in the course of this chapter and chapter 5 combines the alternatives of the quotation: On a fundamental level the physical world is assumed to be describable in a timeless language. Nevertheless, the thermal time hypothesis claims that time is not a subjective product of our mind, but emerges on a thermodynamical level; the statistical state of a physical system singles out a time flow which causes our perception of a flowing time.

As mentioned before, the most vital feature of our experience of time concerns the notion of a Now shifting through time. At some instant certain things exist and others do not. In most physical theories however this notion disappears completely. The time that enters bears only the vaguest resemblance to the common sense time of our everyday experience (no transient Now, no flowing, no arrow of time). Time is modelled as a dimension of reality similar to the spatial dimensions; all events, past, present and future, are treated as equally real. The theory describes them as simply being and sustaining relations of earlier and later, but not as "coming-into-being". The "fact" that we come across events, experiencing them to take place, has to be regarded as a mere peculiarity of consciousness. By abstracting time into a mathematical parameter, it has been robbed much of its original, human content.

It is extensively discussed in the literature (cf. [54] for more details) if the transiency of the Now passing through time should be regarded as a feature of physical time (as argued by GRÜNBAUM) or exclusively as a part of common sense time (as demanded by REICHENBACH), and in which way it could be implemented in physical theories. For EINSTEIN the notion of a Now is important, though it cannot be grasped by science, "*there is something essential about the Now which is just outside the realm of science*" ([8], p. 143). REICHENBACH puts forward that the transient Now of common sense time has an objective significance and hence that an essential content of time is omitted so far in physical theories.

This view is supported by some physicists, as well. EDDINGTON, PENROSE and DAVIES hold the view [44] that the impression of the flow of time is so central to our experience that it must correspond to something in the objective world, and that we are missing a vital aspect of great significance from the physics of time and our perception thereof. EDDINGTON, REICHENBACH and WHITROW argue [44] that the indeterminism in the quantum collapse process provides the key structure to model a physical flow of time. While there is no flow of time in deterministic theories, the passage of time transforms statistical expectations into real events. PENROSE [94] regards the flow of time as restricted to the human brain which gives the sense of the passage of time. He claims that the flowing is due to quantum processes and certain still mysterious cerebral processes in the brain, irreducibly quantum-mechanical in nature, that accompany acts of observation of the world. But PENROSE also takes into account that common sense time with all its very special features might have a conception quite different from that of a physical time determining physical reality. The temporal perception of consciousness may not coincide with the temporality of the external (relative to consciousness) time of physical reality. In that case the flow of time would be physical, though not a part of fundamental physics. Even more, GRÜNBAUM says that due to its

inherent dependence on consciousness, the transient Now is not at all a feature of physical time. It has no significance apart from the egocentric perspectives of a conscious organism [54], p. 324:

"*The coming-into-being or becoming of an event, as distinct from its merely being, is thus no more than the entry of its effect(s) into the immediate awareness of a sentient organism.*"

Becoming is an illusion, it is the way human beings experience time, but there is nothing in Nature which corresponds to it. The passing Now of our conscious awareness is a subjective phenomenon. The dependence of coming-into-being on consciousness is expressed by Weyl (cf. [54], p. 326):

"*The objective world simply is, it does not happen. Only to the gaze of my consciousness, crawling upward along the [. . .] [world-]line of my body, does a section of this world come to life as a fleeting image in space which continuously changes in time.*"

From this point of view a time concept appearing in physical theories which does not incorporate the notion of a transient Now, a flowing time, or the psychological distinction between past and future, should be still suitable to capture all physical, objective features of the "true" notion of time (nevertheless for stylistic convenience we shall frequently speak of a "time flow"). The richness of common sense time derives entirely from subjective factors and is unrelated to intrinsic qualities of the physical time. Dealing with a physical time disentangled from consciousness, one may disregard all these features, and in fact such "reduced" notions of time appears in many physical theories.

As opposed to that thermal time is regarded as something which does flow in the proper sense, and this way the thermal time hypothesis is intended to provide an explanation of the roots of one component of common sense time in the physical world. Anyway, what we conclude from this section is that the issue of the true nature of time has to be attacked in the objective physical world, where certain aspects of time might not be of primary significance.

1.3 Properties Ascribed to Time

The notion of time is associated with a number of peculiar features that make it a very special physical variable, some of which have already been mentioned above. The numerous properties come from our everyday experience. It is the aim of this section to make them more precise. We assort to what extent these properties are maintained in the various theoretical frameworks afterwards. Based on a classification of Rovelli [110], we assign the following possible structure to the set of all time instants, $\mathcal{T}$:

- *Set structure* – distinction between various instants of time
- *Topological structure* – existence of a notion of two time instants being close to each other
- *1-dimensionality* – possibility of arranging the instants of time in a one dimensional manifold
- *Metric structure* – possibility of stating that two distinct time intervals have equal duration
- *Ordering relation* – distinction between past and future (arrow of time)
- *Preferred instant of time* – selection of a preferred instant of time, representing the Now

Disregarding the countable time of the pre-Newtonian-era, all the concepts of time we are going to take a look at show the first four properties. The common sense time of our everyday experience includes all the features listed so far. If time is represented as a variable t in $\mathcal{T} = \mathbb{R}$ all the above properties are captured (the notion of a present corresponds to $t = 0$). A given notion of time may be characterised by some further properties we shall make extensively use of:

- *Absoluteness* – time is a per se existing reality
- *Uniqueness* – existence of one single distinguished time variable
- *External time* – independence of the notion of time from dynamical variables of the theory
  Internal time – the notion of time is affected by the dynamics, it is state-dependent
- *Spatial globality* – possibility of defining the same time variable in all space points
  Temporal globality – every motion passes every value of the time variable once and only once
  Globality – the notion of time is spatially and temporally global

1.4 Non-Generally Covariant Notion of Time

Newtonian Time and Classical Mechanics

Mechanics is usually understood to describe the laws of the change of physical systems in time. Thus, time is one of the fundamental ingredients in terms of which it is built. The notion of time has already been created by the ancients to cope with the fact that things are changing (cf. [134]). However, for a long time those aspects of time which were of primary significance for the human mind were not duration or irreversibility, but repetition, periodicity, rhythm and simultaneity. It was a countable time having its psychological origin in the fact that the time line consists of discrete moments of attention. Even for Greek philosophers time remained something vague and mysterious, a matter of mythology rather than mathematics, in particular is was not a mathematical parameter. The heavenly bodies were regarded as representatives of time.

In the 17th century GALILEI established time as a fundamental, measurable quantity. His invention of the pendulum clock provided mankind with an accurate timekeeper and strengthened belief in homogeneity of time. Another milestone in the Galileian era was to associate time with a continuum, represented as a geometrical straight line. This mathematical abstraction, the so-called *spatialization of time*, is one of the most basic concepts of modern science. It was NEWTON's great achievement to create a conceptual clarification of the notion of time in his *Philosophiae naturalis principia mathematica* from 1687, where he claimed time to be of an independent existence (cf. [54], p. 5):

"*Absolute, true, and mathematical time, of itself, and from its own nature, flows equably [$\hat{=}$ uniformly] without relation to anything external, and by another name is called duration; relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.*"

NEWTON postulated the existence of a unique, continuous, external, global quantity called time, which flows absolute and equal to itself. Newtonian time, t, is supposed to march on with perfect uniformity and without relation to anything external for ever and nothing in the world affects its flow. NEWTON introduced his absolute time as a correlative to absolute space and absolute motion to ensure the principle existence of an ideal rate-measurer (in contrast to days or years). We do not analyse here the consistency of this definition, in the sense of what is meant by a uniform flow if there is no relation to anything external, see [54, 134]. NEWTON's concept of time was necessary for his formulation of mechanics, and is still close to the way many people think about time. However, by definition one cannot directly access NEWTON's absolute time and it might well be that there is no equable motion whereby it may be accurately measured. It is unobservable.

What actually can be done is to construct suitable measuring devices, called *clocks*, which are supposed to approximate t within a certain accuracy (even NEWTON was aware of this point, though he claimed that his time can be deduced from relative motions). A clock is a physical object which refers to one specific observable quantity, T, which has a simple behaviour in t. A good clock shows for instance a linear behaviour, $T(t) = \alpha t$. Good clocks may be constructed from monotone or periodic processes. For this it is crucial, that there is no dynamical interaction between the clock and other degrees of freedom, an accomplishable assumption in classical mechanics. The evolution of observable quantities cannot be measured in terms of t, but only against the clock time T. Hence, in classical mechanics one really measures the correlation $A(T)$ of observables A with clock variables T; in other words, it is the relative evolution between the variables constituting the proper system and a clock which is observable. Since t does never appear in laboratory experiments, $A(t)$ or $T(t)$ do not. The relation between T and t can never be verified, the problem of constructing clocks is a delicate and non-trivial problem.

Absolute Newtonian time is an abstraction about which we have only an imprecise knowledge due to inadequate realisations by concrete physical processes. We never exactly know the "true" duration between two events. Any realistic physical clock emulates t merely down to a suitable scale. Below that scale higher-order physical effects, systematic or statistical errors jeopardise the performance of any clock. Nonetheless, Newtonian mechanics is grounded on the assumption that such an unobservable, absolute time with a unique topological and metrical structure exists, in terms of which the dynamics is defined.

It is a non-trivial empirical result of pre-relativistic physics that all the observations agree with what one computes using the evolution equations in t, which is why one came up to the conclusion that Newtonian time does exist and is uniquely defined (up to affine transformations). Indeed, this assumption makes classical mechanics mathematically elegant. A classical mechanical system can be cast into a *Hamiltonian system*, defined by a phase space and a Hamilton function. The latter one generates the evolution in Newtonian time and codes all dynamical information.

Note that despite the fact that Newtonian time is represented as a variable in $\mathbb{R}$, it does not capture all the properties ascribed to $\mathbb{R}$ via the above classification. Owing to the symmetries $t \mapsto -t$ and $t \mapsto t + a$, Newtonian time is just a 1-dimensional metric space, which does neither provide an arrow of time nor a preferred instant representing the present.

Criticism of Newton's Absolute Time

Since antiquity there has been physical and philosophical debates concerning the reality of time. Aristotle regarded time (in the sense of coming-into-being) as a fundamental and irreducible aspect of Nature; for him all motion was ultimately referred to the uniform circular motion of the celestial bodies. Later it was Archimedes who argued for the elimination of time. It is no surprise that also Newton's concept of absolute time (and space) was heavily criticised by his contemporaries and descendants, above all by Leibniz and by Mach [8, 54, 88, 134].

Newton believed that absolute time exists in its own right, independently of the existence of material clocks. The theory, that events are more fundamental than moments, which do not exist in their own right, but are classes of events defined by the concept of simultaneity, is known as the *relational theory of time*. It was formulated by Leibniz (~ 1716) who founded it on the principles of *sufficient reason*, *identity of indiscernibles*, and *pre-established harmony*. According to Leibniz neither absolute time nor space exist. The world is to be understood in terms of more fundamental entities that fuse space and matter into relative configurations of simultaneously existing bodies. Bodies and events define points and instants by conferring their identity upon them, and thus enabling them to serve as the loci of other bodies and events. The world does not contain things, it is things. Time is merely supposed to be the succession of such instantaneous configurations, and not something that flows quite independently of the bodies in the universe. The dynamics is exclusively based on observable elements. The disjoint viewpoints of Newton and Leibniz led to a historical debate whether time and space exist as real objects (absolute), or whether they are merely orderings upon actual objects (relational). It was held in a famous correspondence between Leibniz and Clark, a follower of Newton, known as the *Leibniz-Clark correspondence*.

It was particularly Mach who criticised that Newtonian time cannot be determined directly by experiment [88, 134], and for this reason cannot be of any practical and hence scientific use. It has to be regarded as a metaphysical concept. He was the first who demanded that one should only use those notions in physics which have a direct empirical meaning. All statements concerning the temporal behaviour of an object necessarily refer to a clock time, i.e. to a specific physical process. The impression that absolute Newtonian time exists, arises from many apparently periodic processes in Nature (like the motion of celestial bodies or the oscillations of a pendulum) which seem to march in step with Newtonian time: When they are used to measure time, Newton's laws are found to hold. According to Mach these processes only feign the existence of an abstract time, for instance we cannot decide if the period of a pendulum varies in Newtonian time.

The time variables defined by two equally well suitable clocks emulating Newtonian time can be related as $T_2 = f(T_1)$, where f is some monotonically increasing function. To get a unique measure of duration, one needs some universal criterion which fixes time up to some affine transformation. Although we know that one event is anterior to another, we cannot assign the same precise meaning to saying by how much it was anterior, unless there is a definition of duration. If the laws of Nature singled out some specific process, this could be used to define a distinguished metrization of time. Since this does not seem to be the case, the choice cannot be made physically and should be chosen by convention. Mach proposed to use a (subjective) *physiological time*.

Poincaré realised that periodic processes in Nature fix the metrization of time such that the equations of motion in classical mechanics take on their simple Newtonian form. As a general principle he proposed that time has to be defined in such a way, that the fundamental laws of physics become as simple as possible. This is the so-called *principle of simplicity*. There are powerful reasons of mathematical tractability and convenience for greatly preferring the time-metrization

in which all accelerations w.r.t. inertial frames are of dynamical origin, that is attributable to the action of specific masses (see [54, 88]). To make these considerations consistent one needs a weaker definition of an inertial frame, which does not fall back upon the notion of Newtonian time. In fact, it can be shown that in classical mechanics the principle of simplicity distinguishes Newtonian time (otherwise one has to deal with fictitious forces causing secular accelerations), which therefore represents an idealized clock time: If the motion of a body is uncoupled from external, distracting forces, one may regard the resulting motion as a measure of absolute Newtonian time. In principle, this gives a prescription to approximate Newtonian time. Every inertial motion may be regarded as a good clock to measure it.

By way of summary, Newtonian time, albeit of a rather metaphysical nature, is a powerful tool of much theoretical relevance which permits an elegant formulation of physical theories. The assumption that one can construct suitable idealized clocks to approximate it seems to be plausible. Historically, Newtonian time provided the starting for the construction of other modified notions of time.

Time in Quantum Mechanics

In classical mechanics a special status is assigned to the concept of time in the sense that it is treated as a background parameter, external to the system itself. This special role of time is reflected in standard quantum mechanics, as well (in the formulation of SCHRÖDINGER or HEISENBERG, see [39]). Time is not represented as an operator acting on the relevant Hilbert space, i.e. it is *not* a physical observable. Rather, it is regarded as part of an a priori given classical background with a well-defined value. It is the Newtonian notion of time which is used in quantum mechanics. In fact, the classical treatment of time is deeply intertwined with the Copenhagen interpretation of quantum mechanics, and, thus, with the conceptual foundations of quantum theory: All measurements of observables are made at certain instants of time and probabilities are only assigned to such measurements. The inner product on the Hilbert space is required to be conserved in time. Newtonian time is further needed to make sense of a complete set of observables commuting at a fixed value of time, or of the canonical equal-time commutation relations. Analogously to classical mechanics, the system can be described by a phase space and a Hamilton operator. The Hamiltonian generates the evolution relative to the classical external Newtonian time.

As in the classical case, one may ask if there are suitable observables, *quantum clocks*, which can be used as internal times to approximate the Newtonian background time. However, it could be shown by UNRUH and WALD [131] that quantum observables, which serve as good quantum clocks, do not exist. No quantum clock provides an adequate measure of Newtonian time. More precisely, the authors excluded the existence of an observable, T, which has a non-vanishing probability of running forward in Newtonian time, and a vanishing probability of running backward, provided that the energy of the physical system is bounded from below. There is no good quantum clock in the sense that its observed values increase monotonically with Newtonian time.

Time in Special Relativity and Quantum Field Theory

In the special theory of relativity, formulated by LORENTZ, POINCARÉ, EINSTEIN and MINKOWSKI, time is physically unified with space to a four-dimensional spacetime continuum, the Minkowski space, where spatial and temporal aspects are entangled. Minkowski space still constitutes an absolute, rigid background arena, but even more than the Newtonian notions of space and time, it is envisaged as a kind of hyperspace in which events do not happen, but are laid out all at once and one merely comes across them ("block spacetime picture"). The absolute Newtonian time is replaced by a class of related times, the Lorentz times. There is one external time for each Lorentz observer. Each observer has his own specific separation of spacetime into space and time.

Special relativity has modified the notion of time in several surprising ways, concerning particularly the relativity of simultaneity. The unique temporal order of events is lost, causing new phenomena such as *time dilatation* or the *twin paradox*. Clocks are affected by motion, and since physical time is measured by clocks, one has to conclude that time is stretched, as well (as opposed to the uniform unique Newtonian time). Anyway, from a fixed Lorentz observer's viewpoint time remains a distinguished, absolute, external, and global parameter. The Newtonian notion of time essentially carries over to special relativistic systems, where it is hidden in the spacetime structure.

Relativistic quantum field theory is the extension of quantum mechanics to systems with infinitely many degrees of freedom – in consideration of special relativity (cf. section 2.1). It is formulated on a background spacetime with fixed metric structure, namely Minkowski space. The peculiar time variable of a particular Lorentz frame is used to describe the evolution, which is generated by a Hamiltonian. The resulting QFT is consistent with the equivalence of Lorentz frames. This is expressed by the Poincaré covariance due to which QFTs constructed in different Lorentz frames are unitarily equivalent. Altogether, the external Lorentzian time is also the well-established concept of time prevalent in QFT. For convenience we shall from now on also include Lorentz times when we speak of the Newtonian notion of time.

1.5 General Covariance and the Notion of Time

Next, we want to describe the notion of time in the general theory of relativity (cf. [105, 110]), created by Einstein up to the end of 1915. Therein, gravity is treated as a manifestation of the spacetime geometry. There is a reactio of all matter and fields onto spacetime, and even an interaction of spacetime with itself (expressed by e.g. gravitational waves). Spacetime is no longer an absolute background object, but is dynamical. The dynamical nature of spacetime has far reaching consequences for the notion of time.

The Principle of General Covariance

Before we come to the treatment of time, let us pay attention to the peculiarities of *generally covariant theories*. These are theories which transform covariantly w.r.t. the action of the group of diffeomorphisms, Diff(M), acting on the underlying spacetime manifold (which does not necessarily need to be equipped with a metric). The *principle of general covariance* formalises the idea that the laws of Nature should be the same in all reference systems. General relativity is based on this ground-breaking idea, which according to Rovelli [110] "*reveals the true magic of general relativity*". The principle of general covariance is by no means restricted to general relativity, which undoubtedly constitutes the prime example of a generally covariant theory. General covariance refers to an invariance w.r.t. a relabelling of the underlying spacetime manifold, a structure which is usually expected to be preserved also in a theory of quantum gravity (cf. section 1.7).

Einstein's field equations describe the evolution of the gravitational field in terms of a coordinate time, usually denoted by x^0 (supplemented by the spatial coordinates x^i), but due to the diffeomorphism covariance the coordinate time has no physical meaning. A coordinate time is not gauge invariant. It was emphasized by Dirac [45] that an observable quantity has to be gauge invariant, which in this case means it has to be independent of the coordinates. This is closely related to *Einstein's hole argument* (see [106]): General covariance implies that there is no physical meaning in talking about an individual point in spacetime, but only in talking about locations determined by the dynamical elements of the theory (like the intersection point of two one-particle-wordlines). Motion is entirely relative. The spacetime manifold itself has no physical significance, it is a gauge artefact. According to Einstein, "*the requirement of general covariance takes away from space and time the last remnant of physical objectivity*" (cf. [110], p. 74).[3] Because of the existence of gauge degrees of freedom, the same physical situation can be described by different solutions of the field equations. A state does not correspond to a configuration of fields located in spacetime, but to a gauge equivalence class of field configurations.

Time in General Relativity

Because of general covariance, coordinate time is out of question as candidate for a physical time. It is a highly non-trivial question in which physical time (if at all) the evolution of the gravitational field is given. In non-generally covariant physics one has the notion of the idealized external Newtonian time, which is emulated by suitable classical clocks whose dynamics is assumed to be

[3] In particular scalar quantities at a certain spacetime point are not regarded as observable. In fact, there is no agreement in the literature concerning the question what precisely is observable in general relativity [106]. It is argued by some authors that the gauge reflects just a particular reference frame and, since any measurement can only be contemplated after a distinguished frame has been identified in the small region in which the measurement is to be made, different spacetime points *are* physically distinguishable, so that scalars would be observable quantities.

completely uncoupled from the proper system. Due to the dynamical nature of spacetime, such an assumption is not possible in general relativity; there are no non-dynamical objects that break diffeomorphism covariance. In general relativity, as in any other generally covariant theory, *all* variables including any clocks inevitably interact, they are affected by the gravitational field. Irrespective of the existence of some distinguished external time, there will be in general no observable which serves as a suitable clock to measure it (globally), whence such a time is neither directly nor indirectly accessible by observations. General relativity seems to be incompatible with the very idea of an external time [107]. If only gauge-invariant quantities are regarded as observable, any physical time to be in the run has to be identified with one of the degrees of freedom of the theory itself, i.e. with an internal time. In addition to the notion of internal times, the Lorentz metric equips the spacetime manifold with a causal structure, such that one can distinguish between timelike and spacelike separated regions, which is clearly much weaker than a notion of time.

The most important internal notion of time in general relativity is *proper time*, denoted by τ. It is defined along any timelike worldline $x^\mu(s)$ parametrised by an arbitrary parameter s,

$$\tau := \int \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu(s)\mathrm{d}x^\nu(s)} = \int \sqrt{-g_{\mu\nu}\dot{x}^\mu(s)\dot{x}^\nu(s)}\,\mathrm{d}s. \tag{1.1}$$

Given a solution of the field equations, proper time provides the definition of a time quantity along every timelike worldline. It is a complicated non-local function of the gravitational field. Proper time does not capture all the properties of Newtonian time, it is neither spatially global[4] nor external or unique, but dynamical and state-dependent. Since proper time is only defined along timelike curves, the full dynamics of the theory cannot be expressed as evolution in proper time. However, if one studies a system sufficiently localised in space, its evolution may be described in terms of τ. Due to the dynamical nature of proper time, it may happen that it ends after a finite interval (and is not continuable), i.e. its range is not necessarily the whole real line. In that case the corresponding trajectory runs into a *spacetime singularity* (see e.g. [63]).

Again, a priori there is no natural, physical time metric related to a distinguished concrete physical process. However, Poincaré's principle of simplicity leads directly to proper time as the preferred time in general relativistic systems [88]: In general relativity inertial motion is replaced by geodesic motion. It turns out that the geodesic equations of motion take on their simplest form when the system is parametrised w.r.t. an affine parameter, which precisely recovers proper time. Proper time is the natural generalisation of absolute Newtonian time. Moreover, it is supposed to flow along a given worldline independently of the presence of any clock. Hence, proper time resembles Newtonian time also in the sense that it is absolute [138]. As in the Newtonian case, one assumes that there exist *ideal clocks*, whose rate is unaffected by acceleration and which measure proper time along their worldlines. This is the *clock hypothesis of general relativity*. It can be motivated by an analogous *clock hypothesis of special relativity* together with the equivalence principle. By no means all clocks meet this criterion, but it was stressed by Sexl that the absoluteness of acceleration ensures that ideal clocks can be built, in principle, cf. [103].

Like in non-relativistic physics, one may select any observable quantity, T, as the independent one and describe the evolution of any other observable, A, as a function of it. Typically such an internal clock time will behave as a good clock at most in special states or for a limited amount of time, where it provides an approximate notion of time. Clock times will generally neither be global nor monotonically increasing along each orbit, and the theory does not single out one of them. A clock is a physical object and thus reacts on the metric and changes the geometry. In contrast to non-general relativistic physics, where one can choose a clock which is completely disentangled from the proper system and solve the dynamics for the clock independently, one has to consider the full coupled gravity-clock system in order to predict the evolution in clock time. To this end (cf. [42]) let us denote the solutions of the equations of motion in some reference system by $f_a(x^0, x^i)$, where the index a labels all the fields f_a of the theory. Starting from these solutions one has to construct coordinate-independent, gauge-invariant quantities. Let us assume that suitable physical reference objects are given by $f_0, \ldots, f_3$, in particular f_0 is supposed to be a clock (representing a laboratory clock, the pulses of a pulsar, etc.). At least locally the coordinates x^0, x^i can be expressed in terms of the fields $f_0, \ldots, f_3$,

$$f_0(x^0, x^i), \ldots, f_3(x^0, x^i) \quad \longrightarrow \quad x^0(f_0, \ldots, f_3),\ \ x^i(f_0, \ldots, f_3). \tag{1.2}$$

[4] In suitable reference frames proper time can be extended to a global time [78], cf. section 7.2.

Consider now any arbitrary field f_a, $a > 3$, and write $f_0 \equiv T$ for the clock variable. Then

$$A(T) \equiv f_a(T, f_1, f_2, f_3) = f_a(x^0(f_0, \ldots, f_3), x^i(f_0, \ldots, f_3)) \tag{1.3}$$

describes the evolution of an observable A in clock-time, $A(T)$, which is measurable – similar to non-relativistic physics.

We have seen that there are certain weaker notions of time available in general relativity. None of them incorporates all features of Newtonian time, and none of them can generally be used to evolve the system globally, i.e. leads to a fully deparametrised dynamical system. A general relativistic system cannot be formulated as a Hamiltonian system evolving in a physical time parameter (at least such a formulation is neither known nor expected). This led to the statement that general relativity is a *timeless* theory. Its physical interpretation does not rely on the concept of time. The notion of time plays a secondary role. This becomes particularly apparent in the canonical covariant formulation of general relativity, which is briefly sketched in appendix B.1. A Newtonian-like time is actually not needed, in different physical contexts one simply falls back on different clock times.

1.6 Some Special Notions of Time

Cosmic Time

General relativity seems to destroy the ancient concept of a universal time. In a universe full of chaotic motion and random concentrations of matter, there would be no global time. Mysteriously, on the largest scales the universe is not chaotic, the distribution of galaxies and their pattern motion, when averaged out, are surprisingly uniform (evidence for this uniformity is provided by the cosmic background radiation). Because of this special structure, in cosmological models a preferred global time variable can be restored in form of the *cosmic time* [134]. Weyl postulated that in every region of the universe which is sufficiently extensive there is a definite mean motion of matter. The proper times associated with these motions are expected to fit together into a cosmic time (although there are, at least mathematically, counter-examples like the homogeneous Gödel universe). Cosmic time is not fundamental, since its existence depends on peculiarities of the state.

In isotropic models, described by the Robertson-Walker metric, the smoothed-out universe (the cosmic background radiation), distinguishes an ultimate frame of reference on a cosmological scale. Cosmic time does give rise to a preferred well-defined global time. It is the proper time of the isotropic observers (cf. section 6.3). Imposing a suitable gauge condition, i.e. choosing suitable coordinates, one obtains a coincidence between T and the coordinate time t.

Arrow of Time and Thermodynamical Time

Let us come back to the Newtonian notion of time. The laws of classical and quantum mechanics are reversible and do not distinguish between past and future. There is no preferred direction of time, while it is an empirical observation that there exists a time asymmetry in Nature. One can find many processes which single out a preferred direction of time, an *arrow of time*, among them a psychological, a thermodynamical, and a cosmological arrow of time; moreover, the weak interaction provides a phenomenon on the microscale which is not invariant under time reversal. The utmost important thermodynamical arrow of time (expressed in form of the second law) can be understood in the context of statistical mechanics: It is overwhelmingly likely that the entropy of a closed system increases, defining thereby a preferred direction of time. In thermodynamical systems the Newtonian notion of time can therefore be extended to a time concept which does possess a distinguished direction of time, the *thermodynamical time*. The thermodynamical arrow might be related to the irreversible character of mental processes, and, thus, ultimately explain the psychological arrow of time, the sense of a unidirectional ongoing.

The anisotropy of time is the subject of many treatises, [95, 99, 138] just to mention a few. Prigogine [95] suggests to replace the traditional time-reversible laws by slight modifications leading to time-irreversible equations, which build in a temporal directionality at the most basic level. Penrose [94] sees the origin of the arrow of time in certain asymmetries in quantum

measurement processes. On the other hand, it is discussed in [54] if the anisotropy of time expressed by the existence of irreversible processes really justifies to speak of "the" direction of time.

In this thesis we are little concerned with the arrow of time. We just remark that a direction of time might be justified independently of the existence of a notion of time [74, 138]. The basic idea is to exploit a certain asymmetry in the *Wheeler-DeWitt equation* in quantum cosmological models w.r.t. a scale factor that describes the size of the universe.[5]

Time as an Operator

Time enters quantum mechanics not as an observable, i.e. as an operator, but as a parameter that has a well-defined value. We have mentioned that every realistic quantum clock has a non-vanishing probability of running backward. Even more fundamentally one may ask why the notion of time is not implemented as a Hamiltonian time observable in the sense that $[T, H] = i$. In that case T would be a function of the position and momentum operators derived from the corresponding classical expression (in general there is no unique way to construct T, which is already a drawback). This would also imply at once [69, 93] that the expectation value of T coincides with t, and that $e^{iHt_1}|t_2\rangle = |t_1 + t_2\rangle$, where $T|t\rangle = t|t\rangle$, which is precisely the type of behaviour required for a perfect clock; moreover, the energy-time uncertainty relation could be derived in a rigorous manner. T would represent an internal time, defined by the dynamical behaviour of the system itself (as opposed to the Newtonian time t measured by a macroscopic clock which has nothing to do with the system under consideration). However, by the *Stone-von Neumann uniqueness theorem* self-adjoint operators satisfying commutation relations of the above kind are known to have a continuous spectrum which coincides with the whole real line, and thus are incompatible with the requirement that a physical Hamiltonian has a spectrum bounded from below. The only way to circumvent this is to deal with a time operator with an abnormal domain of definition as well as a couple of other unphysical properties such that the uniqueness theorem does not apply. Using the example of the free particle Paul [93] comes to the following conclusions: In physically typical states the expectation value does not have a finite value, and small perturbations of the state lead to arbitrarily large changes in the expectation value of T. Hence, only special states permit a time measurement, that is an application of T. Since the expectation value of T is defined only for a very restricted class of states the energy-time uncertainty relation looses its significance.

An alternative, more promising way is to introduce time operators which are not observables and which act on a different space. This is realised in the covariant approach to quantum mechanics (appendix B.2), and it is also suggested by Prigogine [95] who introduces time as the conjugate variable to the Liouville operator $L = i[H, \cdot\,]$. His motivation is to capture the complexity at the microscopic level. The average value of Prigogine's operator time coincides with the ordinary time, however, it takes the age of the phase space distribution into account.

Summary

We have met several physical notions of time arising in various physical theories with different complexities. In the following we shall compare them by their characteristic properties:

- *Thermodynamical time*: Unique, global, absolute, external time with a preferred direction.
- *Newtonian time*: Unique, global, absolute, external time.
- *Lorentzian time*: Global, absolute, external time.
- *Cosmic time*: Unique, global, internal time.
- *Proper time*: Temporally global, absolute, internal time.
- *Clock time*: Internal time.

[5] The Wheeler-DeWitt equation is due to Wheeler and DeWitt, cf. [74]. Roughly speaking it is the quantum version of a Hamiltonian constraint, a generalisation of the time-independent Schrödinger equation.

1.7 The Problem of Time in Quantum Gravity

Quantum Gravity

Two seminal, path-breaking theories, general relativity and quantum theory, have drastically changed and widened our understanding of the world we live in. Both theories work in different regimes and explain different aspects of our world. It is a natural aim to combine them into a new one which incorporates all the knowledge about our world we have. In particular, one would like to develop a quantum version of general relativity, *quantum gravity*. Since it is strongly suspected that gravity, like all the other interactions, has a quantum nature, such a theory is expected to exist.

There are several motivations for developing a theory of quantum gravity (see e.g. [69]): First of all such a theory will be applicable in broader regimes, so there is the well-grounded hope to gain new insights into physics. This concerns for instance the nature of (quantum) spacetime, or the early history of our universe. It also may become possible to resolve inner problems of quantum theory (like the presence of certain infinities or the state reduction in measurement processes). Moreover, quantum gravity effects may clarify the existence of spacetime singularities, or the true origin of time. Unfortunately, the subject of quantum gravity turned out to be exceptionally difficult, both in regard to mathematical and conceptual issues. There have been many attempts to combine general relativity and quantum theory, just to mention the currently most popular ones:

- Direct quantisation of general relativity: loop quantum gravity. This scheme exemplifies the strategy of starting with a classical theory to which some quantisation algorithm is applied.
- Unified quantum theory of all interactions: superstring theory. This programme starts with a theory whose basic items are quantised one-dimensional strings propagating in a continuum spacetime; general relativity emerges in a low-energy limit from the theory.

More details and further literature on this subject may be found in [69, 76]. The source of problems in combining general relativity and quantum theory lies in incompatibilities between both theories, concerning above all the way they treat time (see below) and the absence of an absolute background structure in general relativity, such that a sweeping success has not yet been accomplished. A deeper understanding of both theories and what they really can tell us is necessary.

Another issue is to what extent one should assume a continuum manifold structure of the underlying spacetime in a theory of quantum gravity. For a general access towards this problem the following hierarchy structure was suggested by Butterfield and Isham [35]:

point set of events $\rightarrow$ *topological structure* $\rightarrow$ *smooth manifold* $\rightarrow$ *Lorentzian manifold*

In many approaches the manifold structure is fixed. For instance string theory assumes an underlying spacetime manifold, and it is also assumed in generally covariant AQFT (cf. section 2.8).

Origin of the Problem of Time

As discussed before, quantum theory relies on the notion of an external, non-dynamical Newtonian time, an absolute element of the theory, generated by a Hamiltonian. Such a time concept is manifestly incompatible with general covariance, but also with the idea of the quantum theory of a truly closed system (as it is required e.g. in quantum cosmology). General relativity, as a generally covariant theory, provides merely the notion of internal, dynamical times, coupled to the gravitational field. In particular, these internal times will generally fail to be global. That is no problem in the classical context, where one can simply choose a new clock. In quantum theory this is not possible, if one wants to define the operators on a dense subset of the entire Hilbert space, and if the time evolution is required to be unitary [42].

But even these weak internal notions of time are not transferable to a quantised theory because of their dynamical nature. Quantum fluctuations of the metric tensor and quantum superposition of different metric structures are a source of problems:[6] The concept of a trajectory disappears, there is no Lorentzian structure, and time becomes fuzzy. The notion of proper time and the

[6] According to Zeh [138] the metric itself is a clock, so that quantising the metric may be seen as quantising time.

whole causal structure are lost. A meaningful treatment of commutation relations is impossible (this concerns in particular the commutator of the quantised metric tensor). For most pairs of points on a manifold one can find a Lorentz metric w.r.t. which they are not spacelike separated, i.e. taking quantum fluctuations into account one could never demand a commutator to vanish [69]. Note that very similar problems arise when one studies general relativistic statistical systems (cf. [107]). If one describes a system in terms of a density matrix, one has to struggle with a superposition of geometries, and the whole causal structure is lost again.

Anyway, time is external, global and absolute in quantum theory, but internal and local in general relativity. The concepts of time contained in general relativity and quantum theory are drastically different, or to put it more dramatically they are incompatible. A true physical time, whatever it is, must be either dynamical or non-dynamical, it must be either absolute or non-absolute. For a theory of quantum gravity one has to bring together these disparate concepts of time. One has to understand the true origin of physical time. This conceptual issue is referred to as the *problem of time in quantum gravity*. It is one of the deepest issues that must be addressed in the search for a coherent theory of quantum gravity. At this stage we refer to the comprehensive reviews of Isham [69] and Kuchař [76]. Basically one distinguishes three solutions to the problem of time [76] when quantising general relativity:

(i) Determining the notion of time *before* quantisation,

(ii) determining the notion of time *after* quantisation, or

(iii) assuming a fundamental timelessness of Nature.

In both cases (ii) and (iii) the theory can be quantised without the notion of time. The difference is that only in (iii) it is possible to construct a technically coherent and conceptually complete quantum theory including a full interpretation without needing to make any reference to the concept of time. In this thesis we shall focus on the third option. We shall use the term *fundamental timelessness* when the notion of time is not defined on a fundamental level, or at the very last plays no fundamental role in the scope of the theory. A theory which admits various internal times, such that none of them can be preferred, neither requires the concept of time, nor admits a meaningful time concept which corresponds to our intuitive notion of time. It thus seems reasonable to regard those theories as timeless, as well.

1.8 Fundamental Timelessness

One did not succeed in constructing a global, absolute time for a general relativistic system. Evolution can only be defined w.r.t. one of the internal degrees of freedom of the theory itself, that is w.r.t. an internal time. Accepting general relativity to be indeed of a timeless nature, it seems plausible that also a theory of quantum gravity is timeless. To quote Rovelli [105], "*the only way out that we see is to completely abandon the idea of absolute time*". The notion of (a preferred) time is supposed to be not definable on a fundamental level. There is the hope that some important problems in physics arise from the assumption that time does exist fundamentally and can be resolved this way. In certain physical situations one may idealize internal clock times by an absolute, external time as a useful tool in order to formulate physical theories.

The idea that the notion of time is not fundamentally tied to the foundations of physics has been traced by many scientists, among others by Rovelli [105, 110, 111], Kiefer [74, 75], McTaggart [87], or Barbour [8, 9], see [105] for some more references. While the two last mentioned authors adopt the point of view that, since the concept of time is riddled with contradictions, it makes more sense to assume that time is unreal, an illusion of our mind, it is particularly argued by Rovelli that, albeit time is not an ingredient of fundamental physics, it does emerge physically on larger scales, namely in the context of thermodynamics.[7] However, we do not want to conceal

[7] This is the subject of this thesis. To make things complete, Kiefer proposes that time emerges as an approximate semi-classical concept through the process of decoherence from the separation of a system into two subsystems, one of such determines the time w.r.t. which the other evolves. Another opportunity to recover time is to treat it as an approximate concept which is due to very special initial conditions of our universe [44]. In [35] various ways are discussed how the notion of time might emerge, in particular the meaning of the expression "emergence" itself is investigated.

that there are critics [37, 48, 118] who argue that the notion of time is irreducible and crucial for a fundamental description of Nature. The starting point of this thesis is a fundamentally timeless world, which is why we adopt this hypothesis here rather uncritical. Nonetheless, we provide some arguments to justify the assumption of a fundamental timeless world in what follows.

As indicated by Unruh and Wald [131] there are no good quantum clocks. All physical clocks suffer from the same uncertainty, the same quantum fuzziness. The attempts of quantising general relativity suggest that in quantum gravity not just clocks but time itself will be subject to quantum fuzziness, which may be rated as a hint that there is indeed no meaningful time available.

In a fundamentally timeless world one has to make sense of the whole quantisation procedure, and one has to be able to interpret the resulting quantum theory. This is an open challenge for quantum gravity. However, it turns out, that even classical and (at least to a certain degree) quantum mechanics can be cast into a form in which the notion of time is *not* needed. This indicates that time is not as intrinsically linked to these theories as the standard formulations suggest, and it shows that the assumption of a timeless theory of quantum gravity could be meaningful. We present the timeless formulations of classical and quantum mechanics in appendix B.

Let us outline another heuristic argument supporting fundamental timelessness [110], advocated first by Fraser. Time seems to loose features when the theory becomes more fundamental. To describe Nature on an elementary level the physical laws do not need many properties of a concept of time and certain typical features of time seem to emerge only in specific regimes. So it is possible that on the bottom level even the remaining attributes assigned to the notion of time are lost, and that there does not exist any time concept at all. We give three examples to underline this.

- The uniqueness of time is only observable in ensembles of bodies moving slowly w.r.t. each other, in the Newtonian regime. In special relativity time becomes observer-dependent.
- In cosmological models there does exist a preferred global internal time variable, the cosmic time. As discussed above, in arbitrary general relativistic models one does not expect the existence of a distinguished global time variable.
- In statistical systems the time flow has a preferred direction, there is a thermodynamical arrow of time. Considering only a few particles, described via classical or quantum mechanics, both time directions are on an equal footing, cause and effect are interchangeable.

Another argument was put forward by Kiefer [75]. He draws conclusions about the nature of time in quantum gravity without having the final theory. To do so, he makes two assumptions about the hypothetical theory. Firstly, he assumes the universality of the superposition principle of quantum theory, and secondly he supposes that general relativity can be recovered in the classical limit. Then Kiefer argues that the Wheeler-DeWitt equation must hold, at least approximately on scales sufficiently large in comparison with the Planck scale. Since this equation does not make use of any time concept, he concludes that quantum gravity must be fundamentally timeless.

There remains one decisive question: If one accepts our world to be fundamentally timeless, how does time emerge? As explained at the beginning of this chapter, (the flow of) time is very familiar to our perception of the world and is a crucial ingredient of most physical theories. How can the timelessness on a fundamental level be related to our intuition of an (at least locally) distinguished unique physical time flow? What is it that singles out a particular flow as the physical time flow? What characterises *physical* time? One possible solution to this issue was suggested first by Rovelli [107] in the classical context and later by Connes and Rovelli [42] in the algebraic setting of quantum field theory in form of the *thermal time hypothesis*, which is the subject of this thesis and which regards time as a real macroscopic phenomenon tied to thermodynamics.

Chapter 2

Algebraic Quantum Field Theory

"There is no quantum world. There is only an abstract physical description. It is wrong to think that the task of physics is to find out how nature is, physics concerns what we can say about nature."
Niels Bohr, Danish physicist (1885-1962)

2.1 A Historical Look Back at Quantum Field Theory

After the formulation of quantum mechanics there were attempts to generalise quantum theory to systems with infinitely many degrees of freedom. At the same time the special theory of relativity was incorporated into the quantum framework. As the result *relativistic quantum field theory* (QFT) came out. The concepts of QFT were used for the formulation of the *standard model of elementary particle physics*, which is up to now the best description of our world available on small scales. It explains the electroweak as well as the strong interaction, and with it three of the four fundamental interactions known to penetrate our world. The standard model includes *quantum electrodynamics* (QED), which is one of the most successful theories at all.

Standard QFT is described by means of a Lagrangian, a well-established concept known from classical mechanics. The Lagrangian is constructed for instance from the free Dirac Lagrangian by exploiting the *local gauge principle*. Starting from the Lagrangian one derives the full system of field equations. The development of a manifestly covariant form of perturbation theory and the idea of renormalization to remove all infinities permitted computations of higher order corrections, which were in spectacular agreement with the experimental results and increased faith in QFT.

However, in spite of all successes, there remained open problems which caused dissatisfaction. Haag's theorem said that the interaction picture used throughout QFT does not exist mathematically, the expressions appearing in the Lagrangian merely had a formal meaning, and it was not understood how computations could be done without falling back upon perturbation expansions. A theory based entirely on perturbation theory was conceptually unsatisfactory, intensified by the fact that it was generally not applicable in cases in which the strong interaction was involved. In addition, perturbation theory as well as the renormalization procedure, which was regarded as very unnatural, turned out to be highly complicated. All these grievances led physicists to look for a deeper understanding of the underlying principles and to create a more concise mathematical foundation for QFT.

After the second world war many powerful mathematical tools have been developed, which on the one hand comprised the theory of distributions, and on the other hand the theory of C^*-algebras. Both tools found an application in QFT when it was put onto an axiomatic framework. The theory of distributions was used for the *Wightman approach* [120], while the theory of C^*-algebras was the crucial input for the algebraic formulation of QFT in the sense of Haag, Kastler and Araki [60, 5], a facet of QFT which is known as *algebraic quantum field theory* (AQFT), or, to emphasize that fields are not a primary concept anymore, *local quantum physics* (LQP). Both settings led to a mathematically much more rigorous access towards QFT and helped to gain more insights into the fundamentals of physics.

The thermal time hypothesis is formulated in the algebraic scheme of QFT. Hence, it seems to be useful to review the basic ideas and concepts which contributed to the development of AQFT

and to describe some of the main results. A comprehensive presentation of AQFT is given in the textbook of HAAG [60], a purely mathematical access towards this subject is contained in [10]. For a nice review article suppressing technical details we refer the reader to [34]. Since some applications of the time hypothesis are rested on those theories which are locally associated with Wightman fields, we shall also briefly recall the Wightman axioms. An outline over C*-algebras and von Neumann algebras as well as states and representations can be consulted in appendix A.

2.2 The Wightman Framework

In 1952 GÅRDING and WIGHTMAN started to isolate those features of quantum field theory which could be stated in mathematically precise terms and to extract physically trustworthy postulates. This ended up in the *Wightman axioms* [120], which have been proposed for charged fields of any spin. It turned out that not only observable fields, as counterpart of the quantum mechanical observables, but also unobservable fields, which are not gauge-invariant, are of interest in order to explain the existence of superselection rules.

We shall be concerned with a *finite component* Wightman field theory, described in terms of a finite number of fields each having a finite number of components, which can be associated with a local net of von Neumann algebras in the vacuum representation obeying the Haag-Kastler axioms (cf. section 2.3 and 2.4). Since Wightman fields are in general unbounded operators this entails technical difficulties. Using a strengthened form of the Wightman axioms, which we are going to present for convenience only for a neutral scalar field ϕ, the transition to von Neumann algebras is immediately possible [56]. For a general and comprehensive discussion when a quantum field is associated to a local net see [20]. Therein specific positivity properties of the n-point functions are isolated, which are required for the passage to algebras. A detailed physical discussion of the Wightman axioms may be found in [120].

Wightman Axioms

(W1) The states of the theory are described by unit rays in a separable Hilbert space $\mathcal{H}$, which carries a strongly continuous unitary representation $U(g)$ of the (universal covering of the)[1] proper orthochronous Poincaré group satisfying the *spectrum condition*.[2] Moreover, there is precisely one state Ω_0, the physical vacuum, which is invariant under $U(g)$ for all $g \in \mathcal{P}_+^\uparrow$.

(W2) The field ϕ is an operator valued tempered distribution over Minkowski space (the test function space is usually taken to be the Schwartz space). For all real-valued test functions f, the field operators $\phi[f]$ are essentially self-adjoint operators f acting on a common dense domain $\mathcal{D}$ in $\mathcal{H}$, which contains Ω_0 and is invariant under the action of $U(g)$ and $\phi[f]$.

(W3) Under the representation of the Poincaré group the field transforms as

$$U(g)\phi[f]U(g)^* = \phi[f_g], \quad f_g(x) = f(\Lambda^{-1}(x-a)) \quad \text{for all} \quad g = (\Lambda, a) \in \mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{R}^4.$$

(W4) If the supports of f_1 and f_2 are spacelike separated, $\phi[f_1]$ and $\phi[f_2]$ commute as self-adjoint operators.

2.3 Issues of Conventional Approaches and the Local Viewpoint

The quantisation procedure in quantum mechanics consists of replacing the Poisson brackets $\{\cdot,\cdot\}$ by commutators $-i[\cdot,\cdot]$. Doing that for the coordinates q_i of the classical configuration space and their conjugate momenta p_i, one obtains algebraic relations, the *canonical commutation relations*

$$[p_i, q_j] = -i\delta_{ij}, \quad [q_i, q_j] = 0 = [p_i, p_j], \quad i,j = 1,\ldots,n. \tag{2.1}$$

[1] The occurrence of a representation of the covering group stems from the fact that states correspond to rays rather than vectors in $\mathcal{H}$, so that one actually has to deal with a ray representation of the symmetry group.

[2] That means that the spectrum of the energy-momentum operators is confined to the closed forward cone.

It is established by the *Stone-von Neumann uniqueness theorem* that under mild assumptions these commutation relations fix the representation of the operators q_i and p_i up to unitary equivalence as long as $n < \infty$. Thus, from the physical point of view it does not matter which representation is chosen. If the system has infinitely many degrees of freedom, as it is prevalent in QFT or in the thermodynamical limit of statistical mechanics, the theorem does not hold anymore. The canonical commutation relations admit inequivalent irreducible representations, and considerable attention was attracted to the issue which representation has to be chosen. As we shall see, the algebraic approach provides the appropriate scheme to understand the existence of unitarily inequivalent representations.

In ordinary quantum mechanics a physical observable is represented in mathematical terms by a self-adjoint, in general unbounded operator, A say, acting on some separable Hilbert space [39, 91]. However, unbounded operators are somewhat nasty, since one has to struggle carefully with domain problems. It was VON NEUMANN [91] who emphasized that it is possible to replace the unbounded observables by bounded functions of them: Observables are measured by suitable measuring devices. The outcome of such a measuring procedure is a number which lies in the spectrum of the observable. Considering the observable $f(A)$ instead of A, f being a real-valued, Borel function, relabels the possible measuring results, $f(a)$ instead of a, but it does not change the apparatus with which the observable is measured, so that one does not give away something when working with bounded functions of A.

Guided by this result, SEGAL proposed in 1947 to use the algebra generated by all bounded observables equipped with the norm topology, i.e. a C*-algebra, as the primary object in the mathematical formalism of quantum theory, while the Hilbert space merely appeared as representation space of the algebra. Due to the Stone-von Neumann uniqueness theorem, the C*-algebra approach is equivalent to the ordinary Hilbert space formulation. This idea was adopted and enhanced by HAAG in 1959.[3] One can use the Wightman fields to associate to each open region $\mathcal{O}$ in spacetime an algebra $\mathcal{A}(\mathcal{O})$ of operators acting on a Hilbert space. To do so, let us consider an operator of the form

$$Q = \sum_n \int f^n(x_1, \ldots, x_n)\phi(x_1)\ldots\phi(x_n)\,\mathrm{d}^4x_1\ldots\mathrm{d}^4x_n. \tag{2.2}$$

Q is called *local*, with localisation region $\mathcal{O}$, if $\mathrm{supp}(f^n) \subset \mathcal{O}^{\times n}$. $\mathcal{A}(\mathcal{O})$ is the polynomial algebra generated by all operators localised in $\mathcal{O}$. According to HAAG, the elements of the local algebra $\mathcal{A}(\mathcal{O})$ are to be interpreted as physical operations performable in $\mathcal{O}$. Given the net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of local algebras one can for instance compute collision cross sections, which suggests that the intrinsic structure of the theory is already characterised by the algebraic relations in the net [60]; the primary physical interpretation of the theory is given in terms of local operations. Quantum fields are merely regarded as a kind of coordinatization of the net of local algebras. Choosing another field system which defines the same net simply corresponds to another coordinatization, the physics remains the same.[4] This break with quantum fields fits with an observation by BORCHERS that certain Wightman field systems are equivalent [120].

To avoid the use of polynomial algebras $\mathcal{A}(\mathcal{O})$ of unbounded operators, for which, in addition, no physically reasonable topology is available, one passes again to bounded operators. Indeed, algebras of bounded operators admit mathematically as well as physically natural topologies leading to concrete C*-algebras and von Neumann algebras. The local algebras are generated by the spectral projections of quantum fields $\phi[f]$, where ϕ and f run over all possibilities in a given region $\mathcal{O}$.[5] When we consider algebras generated that way in the remainder of this thesis, we speak of a *net locally associated with a Wightman quantum field theory*. Further developments and abstractions of these considerations finally resulted in the framework of AQFT in the sense of [57], which we are going to present next.

[3] As pointed out by RUDOLF HAAG on the conference on AQFT *Algebraic Quantum Field Theory – the first 50 years* in Göttingen 2009, this already happened in 1957.

[4] Note that this is quite a radical viewpoint, the field concept looses its physical significance.

[5] For unobservable fields one may have to take polynomials.

2.4 The Algebraic Approach

Haag-Kastler Axioms

The algebraic approach to QFT is concerned with *states* and *operations* (or their idealizations, *observables*, which also imply suitable limits of an infinite number of operations [57]). A state is a prescription for the preparation of a physical system, or equivalently, stands for a statistical ensemble. The term operation is used for a physical apparatus which may act on the systems of an ensemble producing a transformation from an initial state to a final state. If the operation is defined by means of laboratory experiments, an experimenter can gather pieces of information about the state by the determination of transmission probabilities. Now, each realistic measurement is performed in some specific limited spacetime domain, that is each physical operation is localised in some finite region $\mathcal{O}$. The key idea is to consider these local operations mathematically abstracted as self-adjoint elements of an abstract C*-algebra $\mathfrak{A}(\mathcal{O})$, whereby the Hilbert space concept is completely dropped. This yields a local net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ of observable algebras, labelled by spacetime regions. The algebraic relations between the observables move to the centre of the theory.

From now on we call all the elements of $\mathfrak{A}(\mathcal{O})$ *(local) observables*. The C*-algebra $\mathfrak{A}$ which contains all observables is called the *algebra of quasi-local observables*. Mathematically it is the unique C*-algebra obtained by the norm-completion of the set-theoretical union of all local algebras $\mathfrak{A}(\mathcal{O})$, i.e. $\mathfrak{A}$ is the C*-inductive limit of the local net. The quasi-local algebra $\mathfrak{A}$ is assumed to contain all observables of interest. States are identified in mathematical terms as normalized positive linear functionals over $\mathfrak{A}$, namely as elements in $\mathfrak{A}^{*+}$ with norm 1. The restriction of a state on a local algebra $\mathfrak{A}(\mathcal{O})$ is said to be a *partial state*. The net of algebras is supplemented by some natural axioms, the *Haag-Kastler axioms* [57, 60], which specify the scheme.

Haag-Kastler Axioms Let $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ be a net of abstract C*-algebras, where $\mathcal{O}$ denotes an open subset of Minkowski space with compact closure, a so-called *finite region*. For an arbitrary open set $\tilde{\mathcal{O}}$ the C*-algebra $\mathfrak{A}(\tilde{\mathcal{O}})$ is defined to be the C*-inductive limit of all $\mathfrak{A}(\mathcal{O})$ for which $\mathcal{O} \subset \tilde{\mathcal{O}}$. The net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is subject to the following conditions:

(HK1) *Isotony*: $\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$

(HK2) *Locality*: $\mathcal{O}_1 \subset \mathcal{O}_2' \;\Rightarrow\; \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$ [6]

(HK3) *Covariance*: The proper orthochronous Poincaré group is realised by a group of automorphisms of the net with the geometrical significance $\alpha_g \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(g\mathcal{O})$ for all $g \in \mathcal{P}_+^{\uparrow}$.[7]

(HK1)-(HK3) form a minimal set of requirements on a local quantum theory, and are completed by the demand for the existence of a vacuum representation (see below). The algebraic framework is convenient in dealing with conceptual issues and provides a general framework where suitable tools have to be developed for specific situations. The axioms show clear physical interpretations within the setting of quantum theory and special relativity. Thereto let us add some remarks.

(HK1) is a clear necessity by taking the interpretation of the self-adjoint elements in the local algebras $\mathfrak{A}(\mathcal{O})$ as localised physical operations performable in $\mathcal{O}$ into account. Postulate (HK2) is a manifestation of Einstein causality. If all the points in two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are mutually spacelike separated, measurements of observables localised in these regions cannot disturb each other, because signals and causal influences cannot propagate faster than the speed of light; the operators representing these observables must commute. (HK3) says that if an observable localised in $\mathcal{O}$ is transformed by α_g, the resulting observable is localised in the transformed region $g\mathcal{O}$.

The intrinsic structure of the theory is determined by the algebraic relations in the local net of abstract C*-algebras, which is regarded as the most fundamental object in the algebraic approach. Even more, one makes the assumption that the physical system is *fully characterised* by the local net, which contains *all* the relevant physical information. Once the net is given, in principle, an analysis of the physical properties and predictions can be carried out. For instance the net allows to construct the mathematical counterparts of coincidence arrangements of detectors. Exploiting

[6] The *causal complement* $\mathcal{O}'$ of an arbitrary subset $\mathcal{O}$ is the set of all points which lie spacelike to all points of $\mathcal{O}$. $\mathcal{O}''$ is called the *causal completion* of $\mathcal{O}$. $\mathcal{O}$ is *causally complete* if $\mathcal{O}'' = \mathcal{O}$. $\mathcal{O}'$ is always causally complete. The *commutant* of an algebra is introduced in appendix A.4.

[7] $\alpha_g(A)$ is usually assumed to be continuous in g in the norm-topology for all $A \in \mathfrak{A}(\mathcal{O})$.

the axioms one can extract information about the energy-momentum range of the detector as well as about its placement in spacetime. This suffices to disentangle particle structure and collision cross sections from the net structure.

In general one has to expect that a local net admits mathematical states which are physically meaningless; think e.g. of states which carry an infinite amount of energy in a finite spacetime region. Therefore, it could be meaningful to add a criterion which selects certain (folia of) states, or representations, as physically realisable (see [60] for a further discussion). Anyway, one considers particularly two classes of states: In particle physics one is interested in states differing from the vacuum only by (quasi-)local disturbances (cf. chapter IV in [60]); in quantum statistical mechanics one studies states which are close to thermal equilibrium, idealized by a medium with non-vanishing matter and energy distributions at infinity (cf. chapter V in [60] and the subsequent chapter).

Inequivalent Representations and Superselection Rules

Given a net of abstract C*-algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ one is interested in faithful representations of the quasi-local algebra $\mathfrak{A}$ by operator algebras acting on some Hilbert space $\mathcal{H}$.[8] A representation π of the quasi-local algebra $\mathfrak{A}$ induces a representation of the net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ by restriction, and, thus, gives rise to a net of operator algebras $\mathcal{O} \mapsto \pi(\mathfrak{A}(\mathcal{O}))$ over $\mathcal{H}$. This way one obtains back the conventional framework, though generally there will exist many unitarily inequivalent representations. One reason for their appearance can be heuristically explained as follows: Two states generate inequivalent representations via the GNS construction (cf. appendix A.3), if they cannot be transformed into each other by (quasi-)local operations. This is expected to occur if two states look different when restricted to arbitrary local algebras, which happens e.g. if the representations are induced by equilibrium states at different temperatures in an infinitely extended system; another important example stems from particle physics, when states carry different charge quantum numbers. However, it turns out that unitary equivalence is a too fine distinction to be of physical relevance.

One can never perform infinitely many experiments, nor can any experiment have an unlimited accuracy. Hence, it is not possible to determine the exact state of a physical system. Only a weak* neighbourhood in the set of all states, $\mathfrak{A}^{*+}$, is experimentally accessible, so that the weak* topology may be regarded as the physical topology. Recalling that every state on $\mathfrak{A}$ defines a folium of states over $\mathfrak{A}$ (cf. appendix A.4), we refer to an intriguing theorem proven by Fell in 1960:

Theorem 2.4.1 (Fell's Equivalence Theorem [60])**.** *The folium of a faithful representation of a C*-algebra is weakly* dense in the set of all states.*

Consequently, one cannot find out by experiment in which folium the state of the system lies; all faithful representations of $\mathfrak{A}$ are physically equivalent. Any state in another representation can be approximated by states in the representation under consideration. Inequivalent representations differ only in global aspects of their states which do not matter as long as one is interested in finite regions. The choice of a particular representation is largely a matter of convenience, the representation problem is irrelevant for physics. Nevertheless, the existence of inequivalent representation plays an important role, above all to give a natural explanation for the emergence of superselection rules.

It was recognised by Wigner that the superposition principle cannot hold universally.[9] This led to the concept of *superselection rules*. In conventional QFT superselection rules are linked to the existence of operators commuting with all observables. Typical examples of such superselection operators, or *(generalised) charges*, are the total electric charge, the total baryon number, or thermodynamical quantities of infinite systems. In that case the relevant Hilbert space can be decomposed into a direct sum of subspaces, the *superselection sectors*, such that the full superposition principle holds for each sector, whereas phase relations between different sectors are meaningless. One has to deal with observable and unobservable fields. The observable fields transform each sector into itself, while the unobservable, charge carrying fields lead from one sector into another.

[8] Actually it is physically meaningful to restrict oneself to faithful representations. If a non-faithful representation π_n contained all the physical information, one could argue that the C*-algebra $\mathfrak{A}/\ker(\pi_n)$ should have been taken as the relevant algebra instead of $\mathfrak{A}$ [57].

[9] For example the relative phase between two state vectors carrying different electric charges is unobservable. The phase is changed by an appropriate gauge transformation which has no effect on the observables.

It is a fundamental feature of the algebraic approach, that superselection rules can be understood in a natural way on the level of abstract observable algebras, without the appearance of unobservable fields. They arise whenever the abstract net possesses inequivalent Hilbert space representations. The equivalence classes of irreducible representations correspond to the superselection sectors. The charge structure with its composition laws as well as the exchange symmetry are encoded in the global algebra $\mathfrak{A}$ and its inequivalent representations. Different sectors cannot be distinguished by experiments confined to a finite region. They are distinguished by charges, which are global quantities.[10] The charge of a state can be changed by adding a charge arbitrarily far away ("on the moon") from the finite region $\mathcal{O}$ under consideration, which will change the physical situation in $\mathcal{O}$ arbitrarily little (Haag's "behind-the-moon argument"). In fact, this is an illustration of Fell's equivalence theorem: Each sector contains all accessible information (see [60] for more details).

The Vacuum Sector

To characterise the notion of *vacuum*, i.e. the absence of matter, we fall back upon the definition given in [22, 56]:

Vacuum Representation We assume that the local net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is equipped with a faithful representation π_0 acting on a separable Hilbert space $\mathcal{H}_0$ with the following properties:

(V1) *Covariance.* There exists a strongly continuous unitary representation U of the Poincaré group $\mathcal{P}_+^\uparrow$ such that

$$U(g)\pi_0(\mathfrak{A}(\mathcal{O}))U(g)^* = \pi_0(\mathfrak{A}(g\mathcal{O})) \quad \text{for all} \quad g \in \mathcal{P}_+^\uparrow. \tag{2.3}$$

(V2) *Vacuum.* There is an up to a multiplicative constant unique vector Ω_0 which is Poincaré invariant (sometimes the vacuum is only required to be invariant under the translation group),

$$U(g)\Omega_0 = \Omega_0 \quad \text{for all} \quad g \in \mathcal{P}_+^\uparrow. \tag{2.4}$$

(V3) *Spectrum Condition.* The joint spectrum of the self-adjoint generators of the translation subgroup (energy-momentum spectrum) is contained in the closed forward lightcone.

(V4) *Weak Additivity.* For every $\mathcal{O}$ the following relation holds:

$$\pi_0(\mathfrak{A})'' = \bigvee_{a\in\mathbb{R}^4} \pi_0(\mathfrak{A}(\mathcal{O}+a))''. \tag{2.5}$$

The representation $(\mathcal{H}_0, \Omega_0, \pi_0)$ is called *vacuum representation*, or *vacuum sector* of the net of observable algebras. $\omega_0 \equiv \langle\Omega_0, \pi_0(\cdot)\Omega_0\rangle$ is called the *vacuum state*. Vacuum representations have been explicitly constructed for many models (cf. [122] for the relevant literature).

Again, let us add some remarks concerning these additional axioms. (V1) and (V2) express the idea, that the vacuum should appear to be the same at every position and in every direction for all inertial observers. The physical vacuum is characterised by the property that one cannot extract energy from it, so that it should be the lowest possible energy state. The spectrum condition (V3) reflects a relativistically invariant way of requiring that the total energy is non-negative w.r.t. every inertial frame, and that the quantum system is stable in the sense that it cannot decay to energies below the vacuum state. Let us envisage a fixed inertial frame. Then the positive self-adjoint operator $H = P_0$ generating the time translation subgroup $U(t)$, i.e. the Hamiltonian, carries the interpretation of the total energy operator. According to (V2) it annihilates the vacuum, which therefore is the state of lowest energy. (V4) is a weak technical assumption, satisfied in most models. It holds e.g. in theories where the net is locally associated with Wightman fields [122].

Conformal Covariance We shall also deal with local nets of observable algebras which are supposed to be *conformally covariant*. In that case the Poincaré group has to be replaced in the axioms by the local conformal group (see [55] for more details).

[10] We remark that total energy, total charge, etc., are considered as unobservable, they are not contained in $\mathfrak{A}$.

2.5 AQFT in Terms of von Neumann Algebras and Reeh-Schlieder Theorem

The Appearance of von Neumann Algebras

Given any state ω over the quasi-local algebra $\mathfrak{A}$, one can construct from a net of abstract C*-algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ a net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$. To do that, denote by π_ω the corresponding GNS representation and put

$$\mathcal{R}(\mathcal{O}) \equiv \pi_\omega(\mathfrak{A}(\mathcal{O}))'', \quad \mathcal{R} \equiv \pi_\omega(\mathfrak{A})''. \tag{2.6}$$

The net obtained this way satisfies (HK1)-(HK3). Mathematically it is sometimes more convenient to work with a local net of von Neumann algebras instead of a net of concrete C*-algebras $\mathcal{O} \mapsto \pi_\omega(\mathfrak{A}(\mathcal{O}))$. Modular theory for example, which we shall extensively meet in chapter 4, works exclusively on von Neumann algebras. That is the reason why we focus our attention mainly on nets of von Neumann algebras. An approach which is completely based on them has been developed by Araki [2]. Some care is needed concerning the total von Neumann algebra $\mathcal{R}$, which turns out be too big to include only quasi-local elements. We shall come back to this issue in chapter 3.

Dealing with a net of von Neumann algebras induced by a state ω, physicists are usually contented to work only with normal states, constituting the folium of the state ω. A normal state can be associated to a density matrix and its GNS representation again leads to a von Neumann algebra (cf. appendix A.4). Consider a local von Neumann algebra $\mathcal{R}(\mathcal{O})$ together with the set of normal partial states on $\mathcal{R}(\mathcal{O})$. It is a typcial situation in AQFT that $\mathcal{R}(\mathcal{O})$ admits a separating vector in the underlying Hilbert space. For instance in the vacuum sector this is guaranteed by the Reeh-Schlieder theorem (coming next), and it is also known to be true in the thermal sector induced by a KMS state (see the next chapter, in particular theorem 3.3.9 and the remark thereinafter). In such a situation *every* normal state is represented as a vector, and it is possible to work exclusively with vector states (cf. theorem A.4.24 and proposition A.4.23).

Reeh-Schlieder Theorem

Consider the vacuum sector $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$, and let $\omega_0 = \langle \Omega_0, \cdot\, \Omega_0 \rangle$ be the vacuum state. One might expect that the vector $A\Omega_0$, $A \in \mathcal{R}(\mathcal{O})$, describes a state localised in $\mathcal{O}$, which looks like the vacuum w.r.t. measurements in $\mathcal{O}'$. Assuming certain clustering properties this is qualitatively true if A is picked up randomly and one carries out measurements at large spacelike distances from $\mathcal{O}$. However, a striking result shows that this interpretation has to be handled with care.

Theorem 2.5.1 (Reeh-Schlieder Theorem [5])**.** *Let $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ be a local net of von Neumann algebras in the vacuum sector. Then, for any non-void open region $\mathcal{O}$ the vacuum vector Ω_0 is cyclic for $\mathcal{R}(\mathcal{O})$. If the interior of $\mathcal{O}'$ is non-empty, Ω_0 is also separating for $\mathcal{R}(\mathcal{O})$.*[11]

The Reeh-Schlieder theorem was originally proven by Reeh and Schlieder [98] in the Wightman framework. The positivity of the energy leads to an analyticity property of the Wightman functions which in turn implies the stated cyclicity of the vacuum. In the algebraic setting the theorem is due to Borchers [17].

In fact, the Reeh-Schlieder theorem holds for any vector in the vacuum representation which is analytic for the energy (cf. [10, 17]). A vector $\psi \in \mathcal{H}_0$ is called *analytic for the energy operator* P_0 if ψ is in the domain of P_0^n for all $n \in \mathbb{N}$ and the series

$$\sum \frac{1}{n!} \|P_0^n \psi\| z^n \tag{2.7}$$

has a non-zero radius of convergence. The set of analytic vectors is dense in $\mathcal{H}_0$, in particular any vector of the form $E(I)\psi$, with $\psi \in \mathcal{H}_0$, $I \subset \mathbb{R}$ any bounded interval and E the spectral decomposition of P_0, is analytic. In particular, the Reeh-Schlieder theorem is valid for any vector state with finite energy content, and thus for all physically realisable vector states in the vacuum representation.

[11] For the proof of the theorem the weak additivity assumption is needed. Weak additivity and Reeh-Schlieder property are actually equivalent.

The cyclicity of the vacuum state Ω_0 implies that for any vector $\psi \in \mathcal{H}_0$ one finds an operator $A \in \mathcal{R}(\mathcal{O})$ which, applied to Ω_0, produces a state arbitrarily close to ψ, no matter how small $\mathcal{O}$ may be. The physical reason for this feature lies in small non-local correlations existing in the vacuum. Indeed, rigorous results, like the Reeh-Schlieder theorem, indicate that the vacuum is a highly entangled state (cf. [122] for an overview). As a consequence, for all non-empty spacelike separated regions $\mathcal{O}_1$, $\mathcal{O}_2$ with non-empty causal complements, there exist many projections $P_i \in \mathcal{R}(\mathcal{O}_i)$ which are positively correlated in the vacuum state ω in the sense that $\omega(P_1P_2) > \omega(P_1)\omega(P_2)$. Note that these non-local vacuum fluctuations do not influence the commensurability of observables localised in spacelike separated regions. However, they permit a preparation of any prescribed vector state in the vacuum representation up to any given accuracy by experiments designed to exploit them judiciously. Thus, an experimenter in a laboratory on Earth could, in principle, build a house on the moon. The class of states resulting from the action of arbitrary operations upon the vacuum is practically indistinguishable from that resulting from operations performed in an arbitrarily small spacetime region upon the vacuum.

The separability of the vacuum means that non-vanishing local observables cannot annihilate Ω_0. Let $A \in \mathcal{R}(\mathcal{O})$ be a non-vanishing, positive operator. Then there exists a self-adjoint $B \in \mathcal{R}(\mathcal{O})$ such that $A = B^2$. This entails $\omega_0(A) = \|B\Omega_0\|^2 > 0$. Any event represented by a positive operator $A \in \mathcal{R}(\mathcal{O})$ has a non-vanishing expectation value in the vacuum state, i.e. in the vacuum any local event can occur [122].

2.6 Nuclearity Condition and Split Property

Buchholz-Wichmann Nuclearity Condition

The basic postulates of AQFT provide just a general setting for studying QFT. They do not exclude models with manifestly unphysical properties. Therefore, one is looking for further conditions which restrict the possible theories to a certain class of physically interesting ones. Guided by the goal to characterise those theories which entail a reasonable particle interpretation,[12] Buchholz and Wichmann proposed the so-called *nuclearity condition* [29], which imposes restrictions on the number of local degrees of freedom. It is a physically well-motivated condition which reflects specific phase space properties of a particle theory. The underlying idea can be expressed as follows: If the theory describes particles, the number of states, which can be accommodated within some bounded region of configuration space, should be limited.

For its precise formulation (see [29, 60] for the details) one has to consider the analogue of classical phase space volumes, which describe those parts of state space corresponding to a simultaneous limitation of energy and space volume. As will be described in the next section, local algebras $\mathcal{R}(\mathcal{O})$ are not of type I, i.e. not unitarily equivalent to $\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1} \subset \mathcal{B}(\mathcal{H})$. That is the reason why it is not possible to define a subspace $\mathcal{H}_1 \subset \mathcal{H}$ which corresponds to the states localised in $\mathcal{O}$. While the localisation of observables is a fundamental concept of AQFT, the localisation of states is rather involved. To describe the set of states "well-localised" in a diamond D_r whose base is a ball with radius r at time $t = 0$, the vacuum Ω_0 is needed as a reference state, which is why we focus our attention to the vacuum sector. First of all, a state ψ, well-localised in D_r, should be generated from the vacuum by some local operation in D_r, that is $\psi \in \mathcal{R}(D_r)\Omega_0$, where $\mathcal{R}(D_r) = \pi_0(\mathfrak{A}(D_r))''$. Secondly ψ should be indistinguishable from the vacuum by measurements in the causal complement of D_r,

$$\langle \psi, A'\psi \rangle = \langle \Omega_0, A'\Omega_0 \rangle \quad \text{for all} \quad A' \in \mathcal{R}(D_r'). \tag{2.8}$$

These requirements eventually lead to the identification of the *set of well-localised states* in D_r as

$$\mathcal{L}_r = \{W\Omega_0 : W \in \mathcal{R}(D_r),\, W^*W = 1\}, \tag{2.9}$$

which characterises precisely those states. The "localisation" in momentum space is best realised by a smooth cut off function of the energy, so that the relevant sets whose size is to be analysed are of the form $e^{-\beta H}\mathcal{L}_r$, $\beta > 0$, H the Hamiltonian. A natural assumption concerning the size of these sets is motivated by an analogy to the Gibbs ensembles (cf. section 3.1): If the operator $e^{-\beta H}$

[12] Particles are regarded as stable quasi-local excitations of the vacuum.

is restricted to excitations of the vacuum in a bounded region, properties at spacelike infinity are not tested. Accordingly, if one confines the theory to a large (in comparison with D_r) but finite volume V, the size of $e^{-\beta H}\mathcal{L}_r$ should not vary substantially. Then $e^{-\beta H}$ is replaceable by the density matrix $e^{-\beta H_V}$ representing the Gibbs ensemble in V. The localised states corresponding to the finite volume theory are given by the unit ball $\mathcal{H}_{V,1}$ of the relevant Hilbert space $\mathcal{H}_V$. $e^{-\beta H_V}$ is a trace-class operator, for which the set $e^{-\beta H_V}\mathcal{H}_{V,1}$ is known to be small, namely *nuclear*.

Definition 2.6.1 ([29])**.** A subset $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ is called *nuclear* if there is a sequence of linear functionals l_n on the linear span of $\mathcal{N}$, and a sequence of unit vectors e_n, such that

(i) $\sum_n \lambda_n < \infty$, where $\lambda_n = \sup\{|l_n(\psi)| : \psi \in \mathcal{N}\}$,

(ii) $\sum_n l_n(\psi) \cdot e_n = \psi$ for all $\psi \in \mathcal{N}$.

The *nuclearity index* of $\mathcal{N}$ is defined to be $\nu(\mathcal{N}) := \inf \sum_n \lambda_n$, where the infimum is to be taken w.r.t. all functionals l_n and vectors e_n complying with the above conditions.

Assuming that the theory has a reasonable number of local degrees of freedom, the nuclearity property should persist for the sets $e^{-\beta H}\mathcal{L}_r$, as well, which provides a heuristic motivation for the sets $e^{-\beta H}\mathcal{L}_r$ to be nuclear. That is the core of the nuclearity condition.

Definition 2.6.2 ([29])**.** The *(Buchholz-Wichmann) nuclearity condition* is said to hold, if the sets $e^{-\beta H}\mathcal{L}_r$ are nuclear for all $\beta > 0$ and $r > 0$, and if there exist positive constants c, n, r_0 and β_0 such that

$$\nu(e^{-\beta H}\mathcal{L}_r) \leq e^{cr^3\beta^{-n}} \text{ for all } r \geq r_0 \text{ and } 0 < \beta \leq \beta_0. \tag{2.10}$$

In the high temperature limit the nuclearity index of $e^{-\beta H}\mathcal{L}_r$ for fixed r is regarded as a measure of the asymptotic energy-level density of the well-localised states, which governs the nature of correlations between observables in spacelike separated regions. The estimate of the nuclearity index originates from the finite case, too, and is linked to thermodynamical properties: $\nu(e^{-\beta H}\mathcal{L}_r)$ is dominated by $\operatorname{tr} e^{-\beta H_V}$, and there is the expectation that the pressure $p_{(V,\beta)} = (\beta V)^{-1}\log(\operatorname{tr} e^{-\beta H_V})$ stays bounded in the thermodynamical limit, which enforces $\nu(e^{-\beta H}\mathcal{L}_r) \leq e^{r^3\phi(\beta)}$. In the relevant cases $\phi(\beta)$ is supposed to be bounded by some $c\beta^{-n}$. Hence, requiring nuclearity for the sets $e^{-\beta H}\mathcal{L}_r$ together with a energy-level density which does not increase too fast with the energy as expressed via the bound on the nuclearity index, ensures sensible thermodynamical properties. Consequently, the nuclearity condition seems to be an adequate condition to distinguish local QFTs which admit a meaningful interpretation in terms of particles and exhibit a regular thermodynamical behaviour. It is rigorously proven to hold in free theories [29, 31].

There exist a couple of variants and generalisations of the nuclearity condition. One of them is the *energy nuclearity condition* proposed in [32], which can equivalently be expressed by using only modular operators (which will be introduced in chapter 4), without any knowledge of the global Hamiltonian. This *modular nuclearity condition* is a purely local one, in the sense that it only requires information about the restriction of a suitable state (not necessarily the vacuum) on local algebras. It thus permits and extension to QFTs on curved spacetimes or generally covariant QFTs. Our main intention to introduce the nuclearity condition is to physically motivate the split property, and it is the Buchholz-Wichmann nuclearity condition from which the split property can be deduced in a sufficiently strong form to meet our purposes.[13]

Split Property

Consider two spacelike separated spacetime regions $\mathcal{O}_1$ and $\mathcal{O}_2$, such that there is a region $\mathcal{O}$ which properly contains $\mathcal{O}_1$ and is disjoint from $\mathcal{O}_2$ [34, 60]. Let us assume that the von Neumann algebra generated by the local algebras $\mathcal{R}(\mathcal{O}_1)$ and $\mathcal{R}(\mathcal{O}_2)$ is isomorphic to their tensor product,

$$\mathcal{R}(\mathcal{O}_1) \vee \mathcal{R}(\mathcal{O}_2) \cong \mathcal{R}(\mathcal{O}_1) \otimes \mathcal{R}(\mathcal{O}_2). \tag{2.11}$$

This *statistical independence* between $\mathcal{R}(\mathcal{O}_1)$ and $\mathcal{R}(\mathcal{O}_2)$ may be regarded as a strengthened form of the locality principle, where algebras associated with spacelike separated regions merely have to

[13] The alternative version only implies a *distal split property*.

commute. It means that partial states on $\mathcal{R}(\mathcal{O}_1)$ and $\mathcal{R}(\mathcal{O}_2)$ are uncoupled and can be extended to a state on $\mathcal{R}(\mathcal{O}_1) \vee \mathcal{R}(\mathcal{O}_2)$ with no correlations between $\mathcal{O}_1$ and $\mathcal{O}_2$. The underlying Hilbert space of the corresponding local algebras can be written as a tensor product, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\mathcal{R}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}, \quad \mathcal{R}(\mathcal{O}_2) \subset \mathbb{1}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2). \tag{2.12}$$

In particular, there exists a total set of vectors $\psi \in \mathcal{H}$, the product states, such that

$$\langle \psi, AB\psi \rangle = \langle \psi, A\psi \rangle \cdot \langle \psi, B\psi \rangle \text{ for all } A \in \mathcal{R}(\mathcal{O}_1) \text{ and } B \in \mathcal{R}(\mathcal{O}_2). \tag{2.13}$$

Thus, for the vector states $\omega(\cdot) = \langle \psi, \cdot\, \psi \rangle$ measurements in $\mathcal{O}_1$ and $\mathcal{O}_2'$, respectively, are completely uncorrelated. The factorisation is in analogy to the notion of subsystems in quantum mechanics. It is a remarkable fact, that this form of statistical independence manifests itself in a specific algebraic structure of the local net, the so-called *split property* [29, 60]. One focuses on two bounded spacetime regions $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the closure of $\mathcal{O}_1$ is contained in the interior of $\mathcal{O}_2$, denoted by $\mathcal{O}_1 \subset\subset \mathcal{O}_2$. By isotony this implies $\mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)$. The split property asserts that there exists an intermediate type I factor $\mathcal{M}$ such that

$$\mathcal{R}(\mathcal{O}_1) \subset \mathcal{M} \subset \mathcal{R}(\mathcal{O}_2). \tag{2.14}$$

Note that one cannot assign a definite localisation region to $\mathcal{M}$. Statistical independence and split property are essentially equivalent (cf. [60], proposition V.5.2.1).

The question of whether local nets have the split property was originally raised by Borchers. In fact, the split property could be established in various models including theories of non-interacting particles, or interacting theories which are locally Fock (see [29] for the relevant literature). But there are also models in which the split property does not hold [29, 60]. These counter-examples are altogether characterised by a large number of local degrees of freedom. Indeed, if this number is limited, as it is prevalent in theories with a sensible particle interpretation and as it is mathematically grasped by the nuclearity condition, the split property holds. This surprising result, which establishes the split property as a physically very reasonable assumption on the local net, is due to Buchholz, D'Antoni and Fredenhagen [30].

Theorem 2.6.3 (Buchholz-D'Antoni-Fredenhagen)**.** *Let $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ be a local net in the vacuum sector which satisfies the Buchholz-Wichmann nuclearity condition. Then, the split property holds.*

A variation of the nuclearity condition which is supposed to work in charged superselection sectors and which also implies the split property is due to Borchers and Schumann [18].

2.7 Types of von Neumann Algebras Appearing in AQFT

An important question concerns the type of von Neumann algebras appearing in physics (the classification of von Neumann algebras is summarised in appendix A.5). For quantum systems with a finite number of degrees of freedom the simplest possibility, type I factors, is perfectly adequate, while in relativistic quantum field theory as well as in quantum statistical mechanics of infinite systems factors of type III occur naturally (see e.g. [60, 137]). Even more, the cumulative effort of many physicists in the course of several decades has revealed the following striking uniqueness result about the structure of local algebras appearing in AQFT:

Each local algebra appearing in AQFT is generically *-isomorphic to the unique hyperfinite type III_1 factor.

This universality of a local algebra, regardless of whether one considers particle states or thermal states, may be compared with the type I factors appearing in quantum mechanics, i.e. algebras which are *-isomorphic to $\mathcal{B}(\mathcal{H})$, irrespective of the system or subsystem of interest. It is the change from the materially defined systems in mechanics to open subsystems corresponding to sharply defined spacetime regions in relativistic local theories which forces the change in nature of the algebras from type I to type III_1.

The uniqueness result is a consequence of general physical requirements of AQFT, supplemented by further physically plausible assumptions (above all existence of a scaling limit and

nuclearity condition), which could be verified in various models. Accordingly, physical information distinguishing different theories or the algebras associated with different spacetime regions cannot be contained in the algebraic or topological structure of one single local algebra $\mathcal{R}(\mathcal{O})$. The specific features of a particular theory are encoded in the net structure, in the relation between the various local algebras, each of which *-isomorphic to the hyperfinite type III_1 factor. It is the way they are embedded into each other, which determines the physical properties of the theory, like particle structure, scattering cross sections, bound states, or superselection sectors (the split property for instance is a condition on the net, not on single algebras).

Here, we list some decisive works which led to the above uniqueness result.[14] The first result is due to Araki [3] who proved in 1963/64 that the local algebras for relativistic free fields are type III factors. If a *local normality assumption* is satisfied, the same holds for interacting fields. Hugenholtz [67] recognised in 1967 that factors generated by equilibrium states are of type III. In fact, the weak closure of the representation of the quasi-local algebra induced by an equilibrium state turned out to be a type III_1 factor [73].[15] In 1975/77 Driessler [46] established the type III property for wedge algebras by considering the dimension function of projectors, whenever the theory is *dilatation invariant*. This result could be sharpened by Longo [80] in 1979 to the statement that dilatation invariance forces the wedge algebras to be of type III_1. In 1985 Fredenhagen [50] could give a rigorous proof that local algebras associated with double-cones are of type III_1. He needed the additional assumption that the net is locally associated with a Wightman field with a *non-trivial scaling limit*,[16] as it is expected to be the case for instance in renormalizable field theories with an ultraviolet fixed point, and hence in particular in all theories which are asymptotically free. In 1987 Haagerup [61] succeeded to show that up to *-isomorphisms there is a unique hyperfinite type III_1 factor. In the same year Buchholz, D'Antoni and Fredenhagen [30] proved that the *nuclearity condition* implies the *split property*, which in turn revealed the hyperfiniteness of local algebras. Eventually Buchholz and Verch [33] carried out a comprehensive analysis of the concept of a scaling limit in the algebraic setting in 1996. They established that the existence of a *non-classical scaling limit* in a theory which satisfies *wedge duality* implies the type III_1 property for all local algebras corresponding to double-cones. Making the physically plausible assumption that superselection rules are not recognisable within bounded regions, the local algebras should have a trivial centre, i.e. local algebras should be factors [34].

Other criteria to derive the type III_1 property in various cases of physical interest are reviewed e.g. in [10]. To a certain degree these criteria are also applicable to curved spacetimes. Just to mention one example, the result of Fredenhagen concerning the type III_1 property for local algebras with a scaling limit was generalised by Wollenberg to local algebras over curved spacetimes.

2.8 Generalisations to Curved Spacetimes and Generally Covariant AQFT

The synthesis between the tenets of quantum physics and general relativity is still an unsolved problem. Up to now we have only considered quantum field theory in Minkowski space, and if we do not explicitly mention it, we will always understand under a local net a net which is associated with open regions in Minkowski space and which satisfies the Haag-Kastler axioms. However, it is a natural endeavour to look for generalisations. This concerns the semi-classical limit which treats gravitation as a classical background field, i.e. *QFT in curved spacetime* (see [12, 133]), but also a not yet discovered theory of *quantum gravity*. In the latter case the principle of general covariance has to be built into the quantum setting (assuming that it is preserved at the quantum level).

Let us begin with some comments about AQFT in curved spacetime (see [60] for some more details). At this level much structure of Minkowski space is retained. The metric field is given

[14] Some terms appearing in this section are not explained, since they are not needed anywhere else in this thesis. We refer to the cited papers for their definitions.

[15] More precisely, the statement holds for a representation induced by a stationary, hyperclustering, dynamically stable state over an asymptotically abelian C*-dynamical system.

[16] The theory has a *non-trivial scaling limit* if there exists a positive monotone function $N(\lambda)$, $\lambda > 0$, such that the scaled n-point functions, $W_n^\lambda = N(\lambda)^n \langle \Omega, \varphi(\lambda x_1) \dots \varphi(\lambda x_n) \Omega \rangle$, converge for $\lambda \to 0$ in the sense of tempered distributions with $\lim W_2^\lambda \equiv W_2^0$ different from zero. This condition makes sure that the local algebras $\mathcal{R}(\lambda\mathcal{O})$ are large enough for small λ, such that the properties of the theory at different scales can be compared.

classically in terms of a gravitational background field, so that spacetime, equipped with the fixed metric field, is a Lorentzian manifold which provides a rigid arena where physics takes place. The algebraic part of the theory carries over, so that the theory can still be defined in terms of a local net of observable algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ associated with open regions of the spacetime manifold, and satisfying isotony and locality. The local metric structure can be used to formulate the dynamical laws. The main problem which has to be resolved is caused by the absence of Poincaré (or translation) covariance, or even more of possibly any geometrical symmetry, which in turn is necessary to give a meaningful definition of energy-momentum, stability, or to distinguish the vacuum. Moreover, due to the lack of Poincaré symmetry there is no Hamiltonian defined; the dynamics cannot be formulated in terms of an evolution in a single external time parameter. However, the net structure, together with the possibility to pass to von Neumann algebras via the GNS construction, will prove to be rich enough to apply the thermal time hypothesis. In the course of this thesis we shall meet some examples of QFTs locally associated with Wightman fields in specific curved spacetimes, and we will go into some more details when it becomes necessary.

To formulate a generally covariant quantum field theory [51, 60] an elimination of the global features and a characterisation of the theory in terms of allowed germs of families of states is needed. Spacetime is still modelled as a 4-dimensional manifold. The theory is assumed to be given by a local net of involutive algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ associated to each open region of the spacetime manifold. A state ω is described mathematically in the usual manner by a positive linear functional on the net. Because of the ordering structure provided by the manifold the isotony axiom is still meaningful. In order to implement the principle of general covariance one cannot make use of an a priori given Lorentzian metric field, so that the locality axiom does not make sense. General covariance requires the field equations to be independent of the coordinates, and it is by no means clear that quantum theory is compatible with this principle, which imposes strong restrictions on the field. A first step towards an understanding of generally covariant quantum theories was undertaken by Fredenhagen and Haag [51], who could give an affirmative answer to this question.

Under the additional assumption that the states have a scaling limit when restricted to the neighbourhood of any point, the authors succeeded to show that each state defines a reduced theory in the tangent space of each point. While the tangent space theory contains kinematical information, it lacks to reflect the full dynamical laws. The reduced theory will in general be invariant under translations. Furthermore, there is the possibility that the physical state (the relevant folium) determines a macroscopic metric field.

It may happen that observables of an involutive algebra are represented in the GNS space as unbounded operators. Nevertheless, the transition from unbounded operator algebras to von Neumann algebras, crucial for our purposes, is still possible in a canonical way by taking account of the polar decomposition theorem [51]. Again, the opportunity to pass to von Neumann algebras will suffice for the emergence of thermal time. In fact, this will be the strength of the thermal time hypothesis, that it is applicable under very general conditions and, thus, opens the opportunity to define a notion of time in generally covariant theories, where common notions of time are not expected to exist. One just has to assume the minimal adaption of the algebraic approach that there is a net of involutive algebras, labelled by some (partially) ordered set, and that states are positive linear functionals over these algebras, such that QFT in terms of von Neumann algebras is available.

Chapter 3

Thermal Equilibrium and the KMS Condition

"Truth is much too complicated to allow anything but approximations."
John von Neumann, Austro-Hungarian mathematician (1903-1957)

Thermal equilibrium states arise as the simplest states of interest when one studies the properties of bulk matter. They are considered as an equally fundamental aspect of microphysics as vacuum states. For the formulation and motivation of the thermal time hypothesis, just as for its discussion, the notion of thermal equilibrium is going to play a decisive role. Therefore, we intend here to present the theory of equilibrium states of matter, that is equilibrium statistical mechanics. We open the chapter by a short account of the Gibbs prescription as a well-established equilibrium condition for finitely extended systems, which historically has been the starting point for equilibrium statistical mechanics. Then, we discuss how time evolution, being a key structure to define the notion of equilibrium, is modelled in AQFT. After giving some basic tools and definitions, we continue by introducing the KMS condition which is the natural generalisation of the Gibbs condition, and analyse several properties which led to the acceptance of this condition as a reasonable criterion for the selection of states representing thermal equilibrium. But we shall also underpin that to establish certain typical features of true equilibrium states, concerning above all a "return to equilibrium"-behaviour, apart from the KMS condition further conditions are necessary in order to obtain some form of dispersion. This analysis is of great importance with regard to properties of the thermal time. The main references used in this chapter are [24, 60, 73, 128], where many more details may be found.

3.1 The Traditional Approach and its Limitations

Gibbs' Ensemble Theory

The traditional form of statistical mechanics [128] is based on *standard Gibbs canonical ensembles* of finite systems, i.e. systems which are enclosed in a box of finite extension. This ansatz goes back to the path-breaking works of Boltzmann and Gibbs on classical statistical mechanics around the turn of the 19th century, and has been successfully transferred to quantum mechanics in the late 1920s. The powerful methods introduced at that time provide a highly satisfactory foundation for the development of modern statistical mechanics. We shall recall here the Gibbs prescription for quantum systems. Our motivation to do that is two-fold: We want to show off the need for a more general description of equilibrium as it is embodied by the KMS condition, but even more the Gibbs approach will be best suitable to explain the emergence of thermal time.

To make things more concrete, we envisage a system enclosed in a box containing a finite number N of particles in a finite volume V (typically the system consists of many particles and has a huge number of degrees of freedom).[1] In the usual manner one constructs the separable N-

[1] The considerations can be readily generalised to those cases where the particle number is not constant.

particle Hilbert space $\mathcal{H}_N$. The dynamical evolution $\alpha_t^{(V,N)}$ is generated by the box Hamiltonian $H_{(V,N)}$, which incorporates a description of the interactions and boundary conditions, via the *Heisenberg equation of motion*,

$$\alpha_t^{(V,N)}(A) = e^{iH_{(V,N)}t} A e^{-iH_{(V,N)}t}. \tag{3.1}$$

In order to obtain $H_{(V,N)}$ as a well-defined self-adjoint operator, one has to supplement carefully suitable boundary conditions on the walls of the box. The relevant von Neumann algebra is the algebra $\mathcal{B}(\mathcal{H}_N)$ of all bounded operators acting on $\mathcal{H}_N$.

Statistical methods have to be employed to predict the behaviour of a system about which merely an incomplete knowledge is available, not sufficient for a specification of its precise state. In fact, this is the generic case, both because observations on the state of the system are necessarily approximate in nature due to the finite resolution of realistic measuring devices, and because the exact treatment of complicated systems composed of many particles is impracticable due to computational limitations. Hence, approximate specifications have a better correspondence with the real situation to be studied than the highly idealized notion of a pure microscopic state.

It was Gibbs' great insight to treat not the behaviour of a single system, but to consider the properties of a *representative ensemble* of systems instead, each of similar constitution to the one of actual interest. The average behaviour and properties of the systems in this ensemble provide the best estimates as to those pertaining to the actual system under consideration. The condition of the ensemble at any time is specified by a *density matrix*, or *statistical operator*, denoted by $\rho_{(V,N)}$, which describes the distribution of the representative systems in phase space. It is assumed to be a positive trace-class operator acting on $\mathcal{H}_N$ with $\operatorname{tr}\rho_{(V,N)} = 1$. The evolution of the density matrix is determined by the *Liouville equation*

$$\partial_t \rho_{(V,N)} = -i[H_{(V,N)}, \rho_{(V,N)}]. \tag{3.2}$$

This way one is able to make predictions about the average future behaviour. The predictions are to be regarded as true on the average for the systems in the ensemble. The ensemble, which is set up to represent the individual system in its partially specified state, is selected in such a manner as to agree with the partial knowledge about the precise state of the system of interest, and is otherwise constructed in accordance with the reasonable postulates of *equal a priori probabilities* and *random a priori phases* (cf. [128]).

We are interested in the properties of those systems, which can be regarded from a gross point of view, which neglects the behaviour of individual particles, as being in a steady condition of macroscopic equilibrium. If $\rho_{(V,N)}$ does not explicitly depend on time, the ensemble is said to be in *statistical equilibrium*. This can be achieved either by taking an initial distribution with $\rho_{(V,N)}$ equal to a constant, or equal to some function of the integrals of motion, for which in systems with a specified number of particles (and assuming total momentum and angular momentum to vanish) the energy $H_{(V,N)}$ is the only remaining choice. Ensembles being in statistical equilibrium are important for the representation of systems which are themselves in a condition of macroscopic equilibrium, though they do not uniquely determine them. The most important ensemble which does it, is the *(canonical) Gibbs ensemble*, represented by the distribution

$$\rho_{(V,N,\beta)} = Z^{-1}\exp(-\beta H_{(V,N)}), \quad Z = \operatorname{tr} e^{-\beta H_{(V,N)}}. \tag{3.3}$$

Note that this formula makes sense, since $H_{(V,N)}$ is typically assumed to be a positive operator which has a discrete spectrum with level density increasing less than $\exp(\beta E)$ for energies $E \to \infty$ [60]. Hence, the distribution $\rho_{(V,N,\beta)}$ is a trace-class operator for positive β. The Gibbs ensemble is especially useful to study relations between statistical mechanics and thermodynamics, where it has been found to provide the appropriate ensemble for a system having a precisely specified temperature, which is β^{-1}. In the ordinary cases of interest with many degrees of freedom nearly all systems in the ensemble have energies close to the average energy. Therefore, it is possible to use Gibbs ensembles also as representatives of systems with an (almost) sharp energy. Most importantly, Gibbs ensembles are the best appropriate candidates to describe finite systems in a condition of macroscopic equilibrium at inverse temperature β.

To understand the mechanism by which the approach of a system to equilibrium takes place, *Boltzmann's H-theorem* proved to be very informing. The quantity H is related to the number

of states defined by equal volumes in phase space that correspond to the specified condition of the system. The H-theorem establishes that it is overwhelmingly likely that H decreases in time towards an equilibrium value as a result of particle collisions. An even more powerful method to study the approach to equilibrium is provided by *Gibbs' generalised $\bar{H}$-theorem*. While H applies directly to the system of interest, $\bar{H}$ characterises the condition on the representative ensemble. Hereto a distinction between *fine-grained* and *coarse-grained probabilities* for the possible states of the members of the ensemble is necessary: Since observations are of only limited accuracy it is not possible to distinguish between neighbouring states with similar properties. That motivates the introduction of coarse-grained probabilities for a state, which are defined as the mean of the fine-grained probabilities taken over neighbouring states of nearly identical properties. $\bar{H}$ is defined in terms of the coarse-grained probabilities and shows the same endeavour to decrease with time towards a steady value, where the ensemble would give a suitable representation for a system in equilibrium. To study the long time behaviour of an ensemble representing a system in contact with its surroundings, one applies the generalised $\bar{H}$-theorem to two interacting systems, modelling the proper system and its surroundings. Both, in cases of thermal equalisation and essential isolation suitable coarse-grained distributions of the proper system (resulting from approximate measurements of so-called *nearly steady states* [128]) show the tendency to approach an ensemble, whose fine-grained distribution is appropriately given by the Gibbsian one.

Another method of justifying the Gibbs ensemble as a suitable ensemble to represent thermal equilibrium is the *principle of maximal entropy*. In thermodynamics, entropy is a fundamental, extensive quantity, monotonically increasing for closed systems, whose existence is deduced from the empirical fact that irreversible processes exist in Nature. A statistical definition of entropy, which should be some functional of the density matrix, $S(t) = S[\rho_{(V,N)}(t)]$, was given by GIBBS for the classical case and extended to quantum systems by VON NEUMANN. In quantum statistical mechanics the *Gibbs entropy* is given as

$$S_{Gibbs}(t) = -\mathrm{tr}(\rho_{(V,N)}(t) \log \rho_{(V,N)}(t)). \tag{3.4}$$

The entropy is a measure for the lack of information about the exact microstate. The Gibbs ensemble is characterised by the property that it maximizes the Gibbs entropy of all distributions with the same average energy. In fact, the Gibbs entropy turns out to be a wholly satisfactory definition for equilibrium ensembles and shows all the properties of the thermodynamical entropy.

A third road to the Gibbs ensemble is obtained by decomposing the system into macroscopic subsystems. The system is supposed to be in equilibrium if each decomposition leads to subsystems which are themselves in equilibrium with the same temperature as the whole. This is an intrinsic characterisation of thermal equilibrium, which we shall meet again in the course of the thesis.

The methods of statistical mechanics provide an explanation for the principles of thermodynamics, which are devised for giving a phenomenological account on the gross behaviour of physical systems. The average properties of an appropriately chosen ensemble is in substantial agreement with the predictions of thermodynamics. The conditions of a thermodynamical system are specified by the values of a limited number of thermodynamical variables (such as volume, pressure, energy, or temperature). These are used to construct the representative ensemble which describes the macroscopic state of the system. Under usual laboratory conditions realistic systems are never in perfect isolation, but are either purposely placed in thermal contact with some other system such as a heat reservoir, or are necessarily in contact with their immediate surroundings such as the walls of a container. Hence, by way of summary, the condition of thermal equilibrium in finite systems can be best represented by the Gibbs ensemble.

Infinitely Extended Systems and the Breakdown of Gibbs' Approach

Although Gibbs' ensemble theory has been enormously successful in relating equilibrium bulk properties of matter to the laws of microphysics, it does suffer from certain basic deficiencies. Realistic thermodynamical systems have a couple of features (such a phase transitions, transport phenomena, etc.) which are absent in the finite model systems. Consequently, standard statistical mechanics does not possess sufficient structure to provide all phenomena. This is owing to the fact that a system enclosed in a box is by no means a suitable idealization of matter filling a finite volume in space. The box Hamiltonian has a discrete spectrum corresponding to excited states

which persist indefinitely in time, a behaviour quite different from a distinguished equilibrium state towards which any other excited state tends asymptotically in time ("return to equilibrium"). This discrepancy is due to the inadequacy of the "system enclosed in a box" as a physical model. Whatever the boundary conditions on the walls of the box are, they reflect the unphysical assumption that the system is isolated, neglecting the feeble but essential interaction with the box. It turns out that a capture of all wished features can be achieved by passing to infinite systems describing matter that fills all space with finite density. Thus, one is led to the study of states of infinitely extended systems as adequate approximation for finite systems. The first idea might be to directly extrapolate Gibbs' ensemble theory to infinite systems. However, this is not possible, since the energy operator, which implements the time evolution, becomes meaningless. Moreover, the state does not need to be described by a density matrix anymore. A generalisation is necessary.

One possibility to get a description for infinite systems is to supplement the Gibbs approach by the prescription of performing the *thermodynamical limit*. While the Gibbs ansatz itself gives at most an approximation of an equilibrium state, the exact thermal state at density d is obtained as the limit of the Gibbs states when $V, N \to \infty$ such that $N/V = d$. The attitude is to replace the walls by the same medium to surround and interact with finite portions of the substance under investigation. In that case one does not need to care about recurrences caused by reflections from the walls occurring in the finite case. Only in this limit fluctuations within the ensemble become zero for intensive quantities and the thermodynamical laws become true for the individual system. The existence of the thermodynamical limit is essentially a question of choosing a consistent set of boundary conditions. The sensitivity on these boundary conditions, i.e. on the precise definition of the box Hamiltonian, is related to the uniqueness of the limit. This ambiguity at certain critical values of the thermodynamical variables yields the appearance of the thermodynamical behaviour such as phase transitions in the infinite-volume limit states.

Instead of first considering finite systems and then performing the thermodynamical limit, it would be much more pleasant to consider the infinite system from the beginning and give a direct characterisation of equilibrium states. The formal substitute of the Gibbs prescription for infinite systems turned out to be the *KMS condition*, subject of the remainder of this chapter.

3.2 Time Evolution in AQFT and the Complementary Role of C*- and von Neumann Algebras

Usually one is looking for an approach which characterises an equilibrium state of an infinitely extended system as a state over the quasi-local algebra, which represents the observable algebra in the thermodynamical limit. As we shall explicate in the following we are interested in local dynamics of open subsystems of an infinitely extended system. Anyway, it is necessary to specify first the dynamical law governing the time evolution of the observables, because equilibrium states are determined by their properties with respect to the dynamics. This thesis is about time, which is why we use this occasion to say a few words concerning the time evolution in AQFT and explain our starting point for the description of equilibrium states.

Thereto let us pick out some inertial frame. The Poincaré group is realised on the quasi-local algebra $\mathfrak{A}$ as an automorphism group, acting geometrically on the local net. From this we can extract the time evolution of the observables, denoted by α_t, as used in the chosen inertial frame. We give the mathematically precise definition of such a *dynamical system* consisting of the algebra and a one-parameter group of *-automorphisms representing the time translations.

Definition 3.2.1 ([23])**.** A *C*-dynamical system* is a tuple $(\mathfrak{A}, \tau)$, where $\mathfrak{A}$ is a C*-algebra and τ a strongly continuous one-parameter group of *-automorphisms.

If we think of a one-parameter group of *-automorphisms as the group generating the dynamics on the observable algebra, i.e. the *physical* time evolution of the observables, we shall reserve throughout this thesis the symbol α_t; in particular to delimit it from the modular flow, which will be another important example of a *-automorphism group. For an unspecified *-automorphism group, as it is prevalent in this rather mathematical part, we prefer the somewhat more neutral expression τ_t; yet we will speak of time translations, as well.

Now, let us consider a representation by a concrete C*-algebra $\pi(\mathfrak{A})$, as it is defined e.g. by the GNS construction of the state under consideration. Moreover, we assume the time translation group

τ_t to be unitarily implemented (cf. corollary A.3.10). Then, by Stone's theorem A.1.22, the time translations are generated by a self-adjoint operator, the Hamiltonian H, $\alpha_t(A) = e^{iHt}Ae^{-iHt}$, $A \in \pi(\mathfrak{A})$. This is the usual way time evolution is treated in AQFT. H is a global, extensive quantity, which is not contained in the algebra $\pi(\mathfrak{A})$ of quasi-local observables, and it generally cannot be regarded as the energy operator, since the physical energy of an infinitely extended system can take on an infinite expectation value with infinite fluctuations, both being subtracted in the construction of H (cf. [58, 60]).

In a next step one may pass from $\pi(\mathfrak{A})$ to the von Neumann algebra $\mathcal{R} = \pi(\mathfrak{A})''$. The time translation *-automorphism group τ_t extends to $\pi(\mathfrak{A})''$ (see below). Whilst the time translations on $\mathfrak{A}$ are directly related to the physical system itself, one is only concerned with the analytical apparatus associated with a particular physical sector of the system after passing to the representation space [73]. The advantage is that the enriched structure of von Neumann algebras is sometimes mathematically more convenient. Moreover, the spectral projection of self-adjoint operators belong to the von Neumann algebra, useful for measurement theory (cf. proposition A.4.10).

Unfortunately, there is a physical deficiency: The von Neumann algebra associated with the quasi-local algebra is too big. Global quantities cannot be observed locally, and consequently they cannot be identified as physical observables. While all the elements in $\pi(\mathfrak{A})$ correspond to physical observables, there are representations which lead to von Neumann algebras containing non-observable global quantities (such as bounded functions of the Hamiltonian). One should therefore not regard the von Neumann algebra $\mathcal{R}$, together with the time translation group, as a physical system, but only as a mathematical extension of the physical system pertaining to certain particular physical situations (some more details on this are given in [73]). This argument is usually consulted to explain why equilibrium states on infinitely extended systems have to be studied on C*-dynamical systems, rather than on W*-dynamical systems to be defined below. The C*-algebra is the relevant physical object, while the von Neumann algebra is of particular mathematical importance.

However, it might also be meaningful to study local dynamics, i.e. time evolutions defined on local subalgebras which represent open subsystems of the infinitely extended medium. In that case the use of von Neumann algebras does not cause any problems. More specifically, one adopts the position that the time evolution is defined on the weak closure, $\mathcal{R}(\mathcal{O}) = \pi(\mathfrak{A}(\mathcal{O}))''$ of the relevant representation of the local algebra $\mathfrak{A}(\mathcal{O})$. One important example of a local flow will be the thermal time flow. The general nature of the dynamical law is inferred from the dynamics of finite systems. It is assumed to be a σ-weakly continuous one-parameter group of *-automorphisms acting on the local von Neumann algebra

Definition 3.2.2 ([23])**.** A one-parameter family $\mathbb{R} \ni t \mapsto \tau_t$ of bounded, linear maps on a von Neumann algebra $\mathcal{R}$ is called a *σ-weakly continuous one-parameter group of *-automorphisms* if

(i) $\tau_{t_1+t_2} = \tau_{t_1}\tau_{t_2}$ for all $t_1, t_2 \in \mathbb{R}$, and $\tau_0 = \mathbb{1}$.

(ii) $\|\tau_t\| = 1$ for all $t \in \mathbb{R}$.

(iii) $t \mapsto \eta(\tau_t(A))$ is continuous for all $A \in \mathcal{R}$ and $\eta \in \mathcal{R}_*$ (i.e. for each normal state).

Example 3.2.3 ([23])**.** We give an important example for such a group of *-automorphisms. Let $t \mapsto U_t$ be a strongly continuous group of unitary elements such that $U_t\mathcal{R}U_t^* \subset \mathcal{R}$ for all $t \in \mathbb{R}$, then one establishes that $t \mapsto \tau_t(A) \equiv U_tAU_t^*$ is a σ-weakly continuous group of *-automorphisms.

Requiring continuity of all numerical functions $t \mapsto \eta(\tau_t(A))$ in definition 3.2.2 means continuity for the time evolution of all expectation values of the observables. Eventually we make the notion of a W*-dynamical system precise, which is intended to model our (locally defined) physical system, including the dynamical law, on which the notion of equilibrium is to be defined.

Definition 3.2.4 ([23])**.** A *W*-dynamical system* is a tuple $(\mathcal{R}, \tau)$, where $\mathcal{R}$ is a von Neumann algebra and τ a σ-weakly continuous one-parameter group of *-automorphisms.

Example 3.2.5. Consider the quantum theory of a finite system with observable algebra $\mathcal{R} = \mathcal{B}(\mathcal{H}_N)$ on some separable Hilbert space $\mathcal{H}_N$. Let $H_{(N,V)}$ be the box Hamiltonian of the theory, which generates the dynamics of the system via

$$\alpha_t(A) = U_tAU_t^*, \quad U_t = e^{itH_{(N,V)}}. \tag{3.5}$$

By Stone's theorem U_t is a strongly continuous one-parameter unitary group. Thus, by example 3.2.3, $t \mapsto \alpha_t$ is a σ-weakly continuous group of *-automorphisms, and, by the last definition, $(\mathcal{B}(\mathcal{H}_N), \alpha_t)$ is a W*-dynamical system.

Indeed, the use of von Neumann algebras might be seen as advantageous, because it can be much harder to define a one-parameter group of *-automorphisms representing the time translations on a C*-algebra [73]. While an implementable strongly continuous *-automorphism group of a C*-dynamical system has a σ-weakly continuous extension to a *-automorphism group on the weak closure of the GNS representation of the C*-algebra (cf. [24], corollary 5.3.4), such that a KMS state on a C*-dynamical system is transformable into a KMS state on a W*-dynamical system, the reversed way is in general not available. Hence, this viewpoint offers the opportunity to treat time translations which are definable only on the weak closure of the representing physical algebra. Even if this view is not completely in accord with the common one in AQFT, it is an assumption of the time hypothesis that the physical time flow is solely defined on the representing von Neumann algebra.

This thesis is devoted to a discussion of the thermal time hypothesis, which is deeply intertwined with modular theory working only on von Neumann algebras, and, along with that, it seems preferable to focus exclusively on von Neumann algebras. Accordingly, we shall develop from the beginning the KMS theory merely for von Neumann algebras. The main purpose of the remainder of this chapter is to investigate the compatibility of the KMS condition with the notion of thermal equilibrium on von Neumann algebras, and to look for properties related to the KMS condition which later can be assigned to the thermal time flow. These investigations are mainly related to certain stability properties, irrespectively whether the corresponding von Neumann algebra is associated to some spacetime region. Possible confluences arising from such a relation will be treated later. For the time being we regard the W*-dynamical system above all as a mathematical object and forget about the precise setting in which it is embedded.

For the corresponding C*-algebra approach towards the KMS theory we refer the reader to the literature cited in the introduction. We just note that the mathematical results presented in the following (apart from certain differences concerning the dynamical stability behaviour) essentially apply to the C*-case, as well. Over and above that, dealing with C*-algebras offers some facets which are not covered by W*-dynamical systems. Some of them are mentioned in footnote 5.

3.3 Kubo-Martin-Schwinger Boundary Condition

States obtained from Gibbs ensembles in the thermodynamical limit are called *limit Gibbs states*. Limit Gibbs states have an important property which is left over from the ordinary Gibbs states. This so-called *Kubo-Martin-Schwinger boundary condition* (or simply *KMS condition*) was initially proposed in quantum statistical mechanics by Kubo in 1957, and by Martin and Schwinger in 1959 as a boundary condition determining the solution of an infinite set of differential equations satisfied by the thermodynamical Green's functions describing equilibrium states. Haag, Hugenholtz and Winnink [60] adapted the KMS condition to the algebraic framework, realised that it can serve as an adequate substitute to the Gibbs ansatz, and pointed out its importance especially in the context of quantum statistical mechanics of infinite systems, on which the KMS condition is directly applicable. The box, the boundary conditions, and the limiting procedures are not needed anymore. Before we are able to give a mathematically precise definition of the KMS condition, we need the technical definition of analytic elements.

Definition 3.3.1 ([23])**.** Let $t \mapsto \tau_t$ be a σ-weakly continuous group of *-automorphisms. An element A of a von Neumann algebra $\mathcal{R}$ is said to be *analytic for* τ_t, if there exists a strip

$$I_\lambda = \{z : |\mathrm{Im}\, z| < \lambda\} \subset \mathbb{C}, \tag{3.6}$$

and a function $f : I_\lambda \to \mathcal{R}$ which satisfies the conditions

(i) $f(t) = \tau_t(A)$ for all $t \in \mathbb{R}$;

(ii) $z \mapsto \eta(f(z))$ is analytic for all $\eta \in \mathcal{R}_*$.

The set of all analytic elements for τ is denoted by $\mathcal{R}_\tau$.

Lemma 3.3.2 ([24]). *$\mathcal{R}_\tau$ is a σ-weakly dense, τ-invariant *-subalgebra of $\mathcal{R}$.*

Example 3.3.3 ([23]). Consider a positive, invertible, self-adjoint operator Ξ, for which $\tau_t(A) \equiv \Xi^{it} A \Xi^{-it}$ defines a σ-weakly continuous group of *-automorphisms of a von Neumann algebra $\mathcal{R}$. For any analytic element $A \in \mathcal{R}$, τ_t can be extended to complex values in time, and $\tau_t(A)$ gives rise to a uniquely determined analytic extension into a strip around the real axis, $\tau_z(A) = \Xi^{iz} A \Xi^{-iz}$ for all z in the strip, when both sides of the equation are viewed as bilinear forms. Note that in general $\tau_z(A)$ will not be a bounded operator, which implies $\tau_z(A) \notin \mathcal{R}$.

Now, after all the preliminary work is done, we introduce the KMS condition.

Definition 3.3.4 ([24]). Let $(\mathcal{R}, \tau)$ be a W*-dynamical system. A state ω on $\mathcal{R}$ is called a (τ, β)-*KMS state*, $\beta \in \mathbb{R}$, if ω is normal and the relation

$$\omega(A\tau_{i\beta}(B)) = \omega(BA) \tag{3.7}$$

is valid for all A and B in a σ-weakly dense τ-invariant *-subalgebra of $\mathcal{R}_\tau$. One says that ω satisfies the *KMS condition* w.r.t. τ. If the precise value of β is irrelevant, we shall simply speak of a τ-*KMS state*.

ω is a (τ_t, β)-KMS state iff it is an $(\tau_{\beta t}, 1)$-KMS state for all $\beta \in \mathbb{R} \setminus \{0\}$. Thus, one can eliminate β by rescaling the group τ_t. KMS states do not need to exist for a given W*-dynamical system. To ensure their existence additional criterions like the nuclearity condition have to be imposed (cf. [60]). Consider e.g. a W*-dynamical system $(\mathcal{R}, \tau)$ with $\mathcal{R} = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, and $\tau_t = \mathrm{Ad}(e^{iHt})$. Then, a KMS state for positive β exists whenever H has a discrete spectrum, bounded below, with a density of eigenvalues increasing less that $e^{\beta E}$ for $E \to \infty$.

The KMS condition may be regarded as a generalisation of the defining property of a tracial state (for which $\omega(AB) = \omega(BA)$). For this note that the value $\beta = 0$ is distinct from the others, since ω is merely assumed to be such a trace-state in that case. Conversely, a tracial state is a $(\tau, 0)$-KMS state. If $\beta \neq 0$, τ can be viewed as providing a qualitative measure of the extent to which a given (τ, β)-KMS state fails to be a tracial state.

The definition of a KMS state given so far in terms of dense sets of analytic elements is preferred in many cases because it is easy to corroborate, but it is also possible to express the KMS condition equivalently via time correlation functions. Such a form is more natural and has the advantage of being statable for arbitrary elements, no analyticity being required.

Theorem 3.3.5 ([24]). *Let $(\mathcal{R}, \tau)$ be a W*-dynamical system, and let ω be a normal state over $\mathcal{R}$. Then, the following conditions are equivalent:*

(i) ω satisfies the (τ, β)-KMS condition.

(ii) For any given two elements $A, B \in \mathcal{R}$ there exists a complex-valued function f_{AB} which is bounded and continuous on the strip $\{z : 0 \leq \mathrm{Im}\, z \leq \beta\} \subset \mathbb{C}$, and analytic on the interior of that strip,[2,3] such that

$$f_{AB}(t) = \omega(A\tau_t(B)) \quad \textit{and} \quad f_{AB}(t + i\beta) = \omega(\tau_t(B)A) \quad \textit{for all} \quad t \in \mathbb{R}. \tag{3.8}$$

The theorem emphasizes the term "boundary condition": If the KMS condition holds, the time correlations functions $\omega(A\tau_t(B))$ and $\omega(\tau_t(B)A)$ are the boundary values of the analytic function f_{AB}. Later, the width of the analyticity strip associated with a KMS state will be interpreted as the inverse temperature. What eventually remains from the Gibbs prescription is an analyticity property of the time correlation functions, though equivalent to it on finitely extended systems as the subsequent section reveals.

In the next theorem we summarise some further alternative characterisations and formulations of the KMS condition. For this purpose we make the useful

Definition 3.3.6 ([24]). The infinitesimal generator of the automorphism group τ, denoted by δ, is defined via $\delta(A) = \frac{\mathrm{d}}{\mathrm{d}t}\tau_t(A)|_{t=0}$. The domain $\mathcal{D}(\delta)$ is the set of all $A \in \mathcal{R}$ for which $\delta(A)$ exists.

[2] For $\beta = 0$, f_{AB} is assumed to be bounded and continuous on $\mathbb{R}$.
[3] In fact, the boundedness-condition may be dropped.

Theorem 3.3.7 ([24, 60]). *Let $(\mathcal{R},\tau)$ be a W^*-dynamical system, and let ω be a normal state over $\mathcal{R}$. Then, the following conditions are equivalent.*

(i) *ω satisfies the (τ,β)-KMS condition.*

(ii) *The Fourier transforms of the functions F_{AB} and G_{AB},*[4]

$$\begin{cases} F_{AB}(t) \equiv \omega(A\alpha_t(B)) - \omega(A)\omega(B) \\ G_{AB}(t) \equiv \omega(\alpha_t(B)A) - \omega(A)\omega(B) \end{cases} \qquad A,B \in \mathcal{R} \tag{3.9}$$

satisfy the relation

$$\hat{F}_{AB}(\varepsilon) = \exp(\beta\epsilon)\hat{G}_{AB}. \tag{3.10}$$

$\hat{F}_{AB}$, $\hat{G}_{AB}$ are considered as distributions over smooth test functions with compact support.

(iii) *For all $A,B \in \mathcal{R}$ and all real functions with smooth, compactly supported Fourier transform the following relation is valid:*

$$\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\omega(A\tau_t(B)) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(t+i\beta)\omega(\tau_t(B)A). \tag{3.11}$$

(iv) *$-i\beta\omega(A^*\delta(A)) \geq \omega(A^*A)\log(\omega(A^*A)/\omega(AA^*))$ for all $A \in \mathcal{D}(\delta)$.*

Theorem 3.3.8 ([24]). *Let $(\mathcal{R},\tau)$ be a W^*-dynamical system, and ω be a faithful τ-KMS state on $\mathcal{R}$. Moreover, let $\tilde{\omega}$ be any other normal state on $\mathcal{R}$. Then, the following statements are equivalent:*

(i) *$\tilde{\omega}$ is a τ-KMS state.*

(ii) *There exists a positive operator T affiliated with the centre $\mathcal{R}\cap\mathcal{R}'$ of $\mathcal{R}$ (cf. definition A.4.9), such that*

$$\tilde{\omega}(A) = \omega(T^{1/2}AT^{1/2}) \quad \text{for all} \quad A \in \mathcal{R}. \tag{3.12}$$

If these conditions are true, then T is unique. In particular, ω is the unique τ-KMS state on $\mathcal{R}$ iff $\mathcal{R}$ is a factor.

The theorem shows that on von Neumann factors (which is the generic case in AQFT) faithful KMS states are unique.[5] The next theorem concerns the GNS representation of KMS states.

Theorem 3.3.9 (cf. [24]). *Let $(\mathcal{R},\tau)$ be a W^*-dynamical system, ω a (τ,β)-KMS state over $\mathcal{R}$, and $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ the corresponding GNS representation. Then, Ω_ω is separating for $\pi_\omega(\mathcal{R})$.*

The theorem does also hold for C^*-dynamical system ($\pi_\omega(\mathcal{R})$ has to be replaced by $\pi_\omega(\mathcal{R})''$). It shows what we already mentioned in the previous chapter: The folium of a KMS state is fully described by the set of vector states on the algebras obtained from the weak closure of its GNS representation.

We close this section by taking a look at the classical situation. The KMS condition is not suitable in the classical limit when $\mathcal{R}$ is a commutative algebra, since a non-trivial time flow does not admit a faithful KMS state (cf. theorem 7.13.4). Nevertheless, it is possible to perform a classical limit from the quantum KMS condition, cf. [52]. For this purpose it is useful to rewrite the KMS condition and restore the Planck constant,

$$\omega(A\tau_{i\hbar\beta}(B)) = \omega(BA) \quad \Leftrightarrow \quad \omega\left(-\frac{i}{\hbar}[A,B]\right) = \beta\omega\left(-A\frac{\tau_{i\hbar\beta}B - B}{i\hbar\beta}\right). \tag{3.13}$$

Taking the classical limit prescription $\hbar \to 0$, replacing the objects by their classical counterparts, and using the Hamilton equations, one can convert this formula into the *classical KMS condition*,

$$\omega(\{A,B\}) = \beta\omega(A\{B,H\}). \tag{3.14}$$

Some care is needed in order to define Poisson brackets for infinitely extended systems.

[4] The subtraction of $\omega(A)\omega(B)$ takes away the uncorrelated part in the expectation value and could be omitted.

[5] This is one crucial difference in comparison with C*-dynamical systems, where the set of KMS states forms a convex, weak*-compact set admitting extremal states which are interpreted as pure thermodynamical phases and which the other mixed KMS states can be uniquely decomposed into. This provides an explanation for the various thermodynamical phases and their mixtures.

3.4 Some Basic Properties of KMS States

We start with the following *tentative* definition:

Definition 3.4.1. An *equilibrium state at inverse temperature* $\beta > 0$ is a state over $\mathcal{R}$ satisfying the (α, β)-KMS condition w.r.t. the physical time translations α_t.

Definition 3.4.1 is usually understood to be applicable to the quasi-local C*-algebra. As discussed before, we focus our considerations on (local) von Neumann algebras. Although KMS states are formal extensions of Gibbs states, by no means this implies that they intrinsically possess all the properties one would attribute to true equilibrium states, in particular under the premise that we have to deal with time flows whose physical content is not known a priori. Therefore, we want to investigate next to what extent the KMS condition alone is sufficient to characterise physical equilibrium situations, so that definition 3.4.1 is a reasonable one. As we shall see, it is necessary to demand additional conditions related to dispersive properties of the system to make sure that the KMS state shows the typical "return to equilibrium"-behaviour.

In this section we look into some very fundamental properties KMS states have to satisfy in order to be justifiable as sensitive physical generalisations of Gibbs states. First of all, the crudest property, which has most naturally to be expected by any equilibrium state, is stationarity, i.e. invariance under time translations. This is established by the next proposition.

Proposition 3.4.2 (Winnink, cf. [24])**.** *Let ω be a (τ, β)-KMS state over the W*-dynamical system $(\mathcal{R}, \tau)$, such that $\beta \neq 0$. Then, ω is τ-invariant,*

$$\omega(\tau_t(A)) = \omega(A) \text{ for all } A \in \mathcal{R} \text{ and } t \in \mathbb{R}. \tag{3.15}$$

Proof. By the KMS condition we have $\omega(A\tau_{i\beta}(B)) = \omega(BA)$ for all $A, B \in \mathcal{R}_\tau$. Putting $A = 1$ it follows that $\omega(\tau_{i\beta}(B)) = \omega(B)$, whence

$$\omega(\tau_{z+i\beta}(B)) = \omega(\tau_{i\beta}\tau_z(B)) = \omega(\tau_z(B)) \tag{3.16}$$

$$\Rightarrow \quad f(z + i\beta) = f(z) \ \ \forall z \quad \text{for} \quad f(z) \equiv \omega(\tau_z(B)) \tag{3.17}$$

f is analytic in the strip $0 \leq \operatorname{Im} z \leq \beta$, and because of the periodicity analytic for all $z \in \mathbb{C}$ for all analytic elements B. Taking advantage of the Cauchy-Schwarz inequality we obtain

$$|f(z)|^2 = |\omega(\tau_z(B))|^2 \leq \omega(\tau_z(B)^*\tau_z(B))^2 = \|\tau_z(B)\|^2 \tag{3.18}$$

$$\overset{z\equiv x+iy}{=} \|\tau_x\tau_{iy}(B)\|^2 \leq \|e^{iHx}\|^2\|\tau_{iy}(B)\|^2\|e^{-iHx}\|^2 = \|\tau_{iy}(B)\|^2. \tag{3.19}$$

$M \equiv \sup\{\|\tau_{iy}(B)\|^2 : 0 \leq y \leq \beta\} < \infty$, since the interval under consideration is compact. Thus, f(z) is bounded by M in the strip $0 \leq \operatorname{Im} z \leq \beta$ and due to the periodicity on the whole of $\mathbb{C}$. $f(z)$ is analytic and bounded on $\mathbb{C}$. We conclude $f \equiv$ const. for all analytic B by applying Liouville's theorem. That yields

$$\omega(\tau_t(B)) = \omega(B) \quad \text{for all} \quad B \in \mathcal{R}_\tau. \tag{3.20}$$

Since the analytic elements are dense by lemma 3.3.2, we are done. □

It is of equal importance to understand the relation between KMS condition and Gibbs prescription, which is a well-established description of equilibrium states in finite quantum systems. Such systems are described by the algebra of all bounded operators $\mathcal{R} = \mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. The Hamiltonian H generates the time translations group α_t acting on $\mathcal{B}(\mathcal{H})$. In fact, the equivalence between both approaches can be ensured:

Proposition 3.4.3 ([60])**.** *Consider a W*-dynamical system $(\mathcal{R}, \tau)$ with $\mathcal{R} = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, and $\tau_t = \operatorname{Ad}(e^{iHt})$. The Hamiltonian H is supposed to have discrete spectrum, bounded below, with a density of eigenvalues increasing less than $e^{\beta E}$ for $E \to \infty$, $\beta > 0$. Then, the (τ, β)-KMS condition and the Gibbs condition at inverse temperature β are equivalent.*

Proof. One can readily check that the Gibbs state $\rho = Z^{-1}\exp(-\beta H)$, $Z = \operatorname{tr} e^{-\beta H}$ satisfies the KMS condition. We sketch the other direction. Let ω be an arbitrary state on $\mathcal{R}$ which satisfies the KMS condition. In particular, ω is a normal state given by a density matrix ρ. Let B be an invariant element, i.e. $\tau_t(B) = B$. We exploit that ω is a KMS state (cf. example 3.3.3),

$$\omega(AB) = \omega(A\tau_{i\beta}(B)) = \omega(BA) \quad \Rightarrow \quad \operatorname{tr}(\rho AB) = \operatorname{tr}(\rho BA) \text{ for all } A \in \mathcal{R}_\tau \tag{3.21}$$

$$\Rightarrow \quad \operatorname{tr}([\rho, B]A) = 0 \text{ for all } A \in \mathcal{B}(\mathcal{H}) \quad \Rightarrow \quad [\rho, B] = 0. \tag{3.22}$$

Each fixed point of τ_t commutes with the density matrix. Thus, ρ has to be a bounded function of H, which implies that there exists an ONB $\{e_n\}$ consisting of simultaneous eigenvectors of ρ and H (note that H has a discrete spectrum). We use the following notation:

$$A_{jk} := |e_j\rangle\langle e_k|\,, \quad \rho\,|e_n\rangle = \rho_n\,|e_n\rangle\,, \quad H\,|e_n\rangle = H_n\,|e_n\rangle\,. \tag{3.23}$$

One more time we take advantage of the fact that ω is a KMS state:

$$0 = \operatorname{tr}(\rho(A_{lm}A_{jk} - A_{jk}\tau_{i\beta}(A_{lm})) = \rho_k\delta_{kl}\delta_{jm} - e^{-\beta H_k}e^{\beta H_m}\rho_j\delta_{jm}\delta_{kl} \tag{3.24}$$

Choosing $k = l$ and $j = m$ yields

$$e^{\beta H_k}\rho_k = e^{\beta H_j}\rho_j \text{ for all } j,k \quad \Rightarrow \quad e^{\beta H}\rho = \lambda\cdot\mathbb{1} \text{ for a suitable } \lambda. \tag{3.25}$$

This condition can only hold because of the assumption about the spectrum of H. ρ is normalized, hence it must be precisely a Gibbs state. □

3.5 Stability Properties and the KMS Condition

We have seen that all states singled out by the KMS condition are invariant under time translations, and that the KMS condition indeed generalises the Gibbs prescription. These properties justify the KMS condition at least as a necessary condition for an adequate characterisation of thermal equilibrium states. However, there are more features one expects from a state which claims to represent a physical equilibrium situation. Above all that concerns some sort of stability condition. An equilibrium state should not react too sensitive to small changes in the dynamical law (or to small perturbations of the state), and should approach the original state when the external intervention is switched off. In this section we want to analyse stability properties of KMS states, whereby we get closer to the question if the KMS condition can really be regarded as a physically meaningful description of thermal equilibrium.

We continue focussing on W*-dynamical systems, a corresponding treatment for C*-dynamical systems may be found in [24, 104]. First we describe a time-independent, kinematical stability property, an approach which was pursued by Araki [4] in 1973, and then pass to a time-dependent, dynamical stability concept, which was initiated by Robinson [104] in the same year.

Kinematical Stability

We consider persistent inner perturbations of the dynamics modelled by bounded symmetric derivations $\delta_P(\cdot) = i[P, \cdot\,]$ on $\mathcal{R}$, $P = P^* \in \mathcal{R}$, which are added to the infinitesimal generator δ of the σ-weakly continuous group of *-automorphisms of a W*-dynamical system. Physically one would require that small perturbations P of the dynamical law ("adding a few grains of dust") do not essentially modify the equilibrium state ω. Rather one would expect the perturbed system to admit a state ω^P invariant w.r.t. the new dynamics and such that $\omega^{\lambda P}$ approximates ω for sufficiently small λ, such that the system can react on the inner perturbation without being affected globally. This important property one would assign to equilibrium states is expressed by the term *kinematical stability*.

Proposition 3.5.1 ([24])**.** *Let $(\mathcal{R}, \tau)$ be a W*-dynamical system acting on a Hilbert space $\mathcal{H}$, such that τ is unitarily implemented,*

$$\tau_t(A) = U_t A U_t^*, \tag{3.26}$$

for a strongly continuous one-parameter group of unitaries $U_t = e^{itH}$. Then, for each $P = P^* \in \mathcal{R}$ there is a perturbed σ-weakly continuous one-parameter group of *-automorphisms τ^P whose infinitesimal generator is $\delta + \delta_P$, i.e. to each perturbation P one can associate a new W*-dynamical system $(\mathcal{R}, \tau^P)$. τ^P is unitarily implemented,

$$\tau_t^P(A) = U_t^P A U_t^{P*}, \tag{3.27}$$

where $U_t^P = e^{it(H+P)}$ is a strongly continuous one-parameter group of *-automorphisms of $\mathcal{R}$. Moreover, one has

$$\tau_t^P(A) = \Gamma_t^P \tau_t(A) \Gamma_t^{P*}, \tag{3.28}$$

where $\Gamma_t^P = U_t^P U_{-t}$ is a unitary one-cocycle.[6] If $\{P_n\}$ is a sequence of self-adjoint elements of $\mathcal{R}$ which converges strongly to zero, then

$$\lim_{n\to\infty} \|(\Gamma_t^{P_n} - \Gamma_t)\psi\| = 0, \quad \lim_{n\to\infty} \|(\tau_t^{P_n}(A) - \tau_t(A))\psi\| = 0, \tag{3.29}$$

for all $\psi \in \mathcal{H}$ and $A \in \mathcal{R}$, uniformly for t on compact intervals of $\mathbb{R}$.

Remark. The perturbation of the dynamical law is realised by perturbing the Hamiltonian with a bounded observable contained in the algebra, hence physically we consider local perturbations. The perturbed group τ^P can be computed in terms of τ and P via perturbation theory [24]. Note that there is a symmetry between τ and τ^P in the sense that each group is a perturbation of the other one with perturbation P and $-P$, respectively.

Definition 3.5.2. Let ω be a state on a W*-dynamical system $(\mathcal{R}, \tau)$. We define ω to be *kinematically stable*, whenever there is a map $P \mapsto \omega^P$, $0 \mapsto \omega$, from a neighbourhood $\mathcal{U}$ of 0 in the self-adjoint part of $\mathcal{R}$ into the state space over $\mathcal{R}$, fulfilling the following conditions for each $P \in \mathcal{U}$:

(i) ω^P is τ^P-invariant (in particular ω is τ-invariant);

(ii) $\lim_{\lambda\to 0} \|\omega^{\lambda P} - \omega\|$.

Theorem 3.5.3 ([24])**.** *Let $(\mathcal{R}, \tau)$ a be W*-dynamical system acting on a Hilbert space $\mathcal{H}$, such that τ is implemented by unitaries $U(t) = \exp(itH)$. Moreover, let Ω be a normalized U-invariant cyclic vector, such that the associated vector state $\omega = \langle \Omega, \cdot\, \Omega\rangle$ is a (τ, β)-KMS state. Let $P = P^* \in \mathcal{R}$, then $\Omega \in \mathcal{D}(e^{-\beta(H+P)/2})$ and the state*

$$\omega^P \equiv \langle \Omega^P, \cdot\, \Omega^P\rangle \tag{3.30}$$

is a (τ^P, β)-KMS state, where

$$\Omega^P \equiv \frac{e^{-\beta(H+P)/2}\Omega}{\|e^{-\beta(H+P)/2}\Omega\|}. \tag{3.31}$$

ω^P is the unique (τ^P, β)-KMS state iff $\mathcal{R}$ is a factor. Furthermore,

$$\lim_{n\to\infty} \|\omega^{P_n} - \omega\| = 0 \tag{3.32}$$

for each sequence $P_n = P_n^* \in \mathcal{R}$ such that $\|P_n\| \to 0$.

Remark. An analogous result is obtained if one replaces ω by an arbitrary KMS state, because the GNS construction represents ω as a normalized cyclic vector and admits a unitary implementation of the *-automorphism group leaving Ω_ω invariant. In that case ω^P is a vector state of ω, i.e. ω^P is represented as a vector, Ω^P, in the GNS space of ω.

Corollary 3.5.4. *Let $(\mathcal{R}, \tau)$ be a W*-dynamical system, and let ω be a (τ, β)-KMS state on $(\mathcal{R}, \tau)$. Then, ω is kinematically stable.*

[6] Since we do not need it here any further, we postpone the definition of a unitary one-cocycle to section 4.4. Let us just note that it is precisely the cocycle property which makes sure that $(\mathcal{R}, \tau^P)$ is a W*-dynamical system.

The corollary verifies kinematical stability of KMS states – without any further assumptions. In fact, theorem 3.5.3 says that KMS states show a stronger form of kinematical stability in the sense that for arbitrary perturbations $P = P^* \in \mathcal{R}$ one finds a state ω^P, which, in addition, is not only stationary but satisfies the τ^P-KMS condition.

However, kinematical stability implies not at all that the system really approaches ω^P when the perturbation is imposed on it, nor does it mean that the system returns to the original state when it is switched off. For this, a more stringent form of stability is needed.

Dynamical Stability

The time-dependent, dynamical approach starts with a kinematically stable state ω. Again, one imposes a small perturbation $P = P^*$ on the dynamics and assumes that the system somehow comes into the τ^P-KMS state ω^P, differing only slightly from ω. If the perturbation is switched off, one wants to identify ω as the evolute of the perturbed state ω^P under the original dynamics ("return to equilibrium"). This concept is formalised as the requirement of *dynamical stability*.

Definition 3.5.5 (cf. [73])**.** Let $(\mathcal{R}, \tau)$ be a W*-dynamical system. A state ω on $(\mathcal{R}, \tau)$ is called *dynamically stable*, if ω is kinematically stable and if

$$\omega(A) = \lim_{t \to \pm\infty} \omega^P(\tau_t(A)) \tag{3.33}$$

for all $P \in \mathcal{U}$, where $\mathcal{U}$ is the neighbourhood appearing in definition 3.5.2.

Alternatively, one may interpret P as a perturbation of the state, $\omega = \langle \Omega_\omega, \pi_\omega(\cdot)\Omega_\omega \rangle \rightsquigarrow \omega^P = \langle e^{-\beta(H+P)/2}\Omega_\omega, \pi_\omega(\cdot)e^{-\beta(H+P)/2}\Omega_\omega \rangle$ rather than of the dynamics. But note that the set of perturbed states one is led to consider is confined to only those states obtained by the action of $e^{-\beta(H+P)/2}$ on the representing vector Ω_ω. Pursuing the viewpoint that the system should return to equilibrium under arbitrary (sufficiently small, local) perturbations of the state, it might be reasonable to require this behaviour for a larger class of states such as the whole folium. Since such a view is quite natural, we shall bear in mind this view, as well.

Anyway, dynamical stability does not follow from the KMS condition as one can most easily see on type I factors (treated in chapter 7). It turns out to depend intimately on some form of dissipative behaviour of the dynamical system, instead. This is usually realised by an asymptotic abelianness assumption, a property which corresponds to dispersion of local disturbances in time. A sufficiently dispersive system is expected to ensure that the perturbed state returns to equilibrium under the original dynamics.

Definition 3.5.6 ([23])**.** A W*-dynamical system $(\mathcal{R}, \tau)$ is called *asymptotically abelian in the norm sense*, if

$$\lim_{t \to \pm\infty} \|[A, \tau_t(B)]\| = 0 \text{ for all } A, B \in \mathcal{R}. \tag{3.34}$$

Let ω be a state on $\mathcal{R}$. $(\mathcal{R}, \tau)$ is said to be *asymptotically abelian in the weak sense w.r.t.* ω if

$$\lim_{t \to \pm\infty} \langle \phi, \pi_\omega([A, \tau_t(B)])\, \psi \rangle = 0 \text{ for all } \phi, \psi \in \mathcal{H}_\omega \text{ and } A, B \in \mathcal{R}, \tag{3.35}$$

where $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denotes the GNS representation of ω.

The physical picture behind this algebraic property is as follows (cf. [60]): The mutual influence of local operations performed in distinct space regions should decrease in strength as the temporal distance increases, i.e. one demands a decrease of all correlations in time (which limits the type of interactions and excludes finite systems). Asymptotic abelianness can be justified in free theories, and there is the hope that the correlations induced by interactions do not dominate the spreading of wave packets over large times. Norm asymptotic abelianness is a property which characterises the behaviour of the system itself, independently of any special state, albeit weak asymptotic abelianness is implied for each state. For our purposes weak asymptotic abelianness turns out to be a sufficiently strong condition. Since the thermal time flow will be defined as a state-dependent flow and one thus has to deal with a special state anyway, it is a natural condition, too.

The next theorem asserts dynamical stability whenever the system is asymptotically abelian. Since it is usually formulated only for C*-dynamical systems, and since we present a slightly different version than the corresponding statement in [24], corollary 5.4.7 (we merely assume weak asymptotic abelianness), we attach the proof, which yet is closely related to [24].

Theorem 3.5.7. *Let $(\mathcal{R},\tau)$ be a W*-dynamical system, where $\mathcal{R}$ is assumed to be a von Neumann factor.*[7] *Furthermore, let ω be a faithful (τ,β)-KMS state, $\beta\neq 0$ (which then is unique by theorem 3.3.8), and assume $(\mathcal{R},\tau)$ to be asymptotically abelian in the weak sense w.r.t. ω. Let τ^P be the perturbed evolution corresponding to the perturbation $P=P^*\in\mathcal{R}$. We assume the system to be in the unique (cf. theorem 3.5.3) (τ^P,β)-KMS state ω^P. Then,*

$$\omega(A)=\lim_{t\to\pm\infty}\omega^P(\tau_t(A)) \quad \textit{for all} \quad A\in\mathcal{R}. \tag{3.36}$$

Proof. Consider the GNS representation π induced by ω, where ω is represented by the cyclic and separating vector Ω. Since ω is faithful and normal, $\pi(\mathcal{R})$ has to be a factor as well, which implies $\pi(\mathcal{R})\vee\pi(\mathcal{R})'=\mathcal{B}(\mathcal{H})$. We fix arbitrary $A,B\in\mathcal{R}$. $\psi_A\equiv\pi(A)\Omega-\omega(A)\Omega$ and Ω are orthogonal, which is why there exists a self-adjoint operator $T\in\mathcal{B}(\mathcal{H})$ such that

$$T\psi_A=0 \quad \text{and} \quad T\Omega=\Omega. \tag{3.37}$$

We define the operators

$$D_1\equiv T(\pi(A)-\omega(A)\mathbb{1}) \quad \text{and} \quad D_2\equiv(\mathbb{1}-T)(\pi(A)-\omega(A)\mathbb{1}), \tag{3.38}$$

which satisfy $D_1\Omega=0=D_2^*\Omega$ and $\pi(A)=\omega(A)\mathbb{1}+D_1+D_2$. That yields

$$\langle\Omega,[D_1,\pi(\tau_t(B))]\Omega\rangle=\omega(A\tau_t(B))-\omega(A)\omega(B). \tag{3.39}$$

Next, we define a unital *-subalgebra of $\mathcal{B}(\mathcal{H})$,

$$\mathcal{A}:=\{\sum_{j=1}^{k}A_jB_j' : A_j\in\pi(\mathcal{R}),B_j'\in\pi(\mathcal{R})',k<\infty\}. \tag{3.40}$$

Clearly $\pi(\mathcal{R})\cup\pi(\mathcal{R})'\subset\mathcal{A}\subset\mathcal{B}(\mathcal{H})$ and we have $\mathcal{A}''=\mathcal{B}(\mathcal{H})$. Hence, by the von Neumann density theorem A.4.7 $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$, and we can find for each $\varepsilon>0$ a finite family of $A_i\in\mathcal{R}$ and $B_i'\in\pi(\mathcal{R})'$, $i=1,\dots,n$, such that

$$\|(D_1-\sum_{i=1}^{n}\pi(A_i)B_i')\Omega\|\leq\varepsilon/(2\,\|B\|). \tag{3.41}$$

From this we deduce with the help of the Cauchy-Schwarz inequality

$$|\omega(A\tau_t(B))-\omega(A)\omega(B)|=|\,\langle\Omega,[D_1,\pi(\tau_t(B))]\Omega\rangle\,|= \tag{3.42}$$

$$|\,\langle\Omega,\left[\left(D_1-\sum_{i=1}^{n}\pi(A_i)B_i'\right)\pi(\tau_t(B))-\pi(\tau_t(B))\left(D_1-\sum_{i=1}^{n}\pi(A_i)B_i'\right)+[\sum_{i=1}^{n}\pi(A_i)B_i',\pi(\tau_t(B))]\right]\Omega\rangle\,|$$

$$\leq\varepsilon+\sum_{i=1}^{n}|\,\langle\Omega,B_i'\pi([A_i,\tau_t(B)])\Omega\rangle\,| \tag{3.43}$$

Because of weak asymptotic abelianness that can only happen if

$$\lim_{t\to\pm\infty}(\omega(A\tau_t(B))-\omega(A)\omega(B))=0. \tag{3.44}$$

The last expression is the so-called *strong clustering condition*, which expresses statistical independence of any two observables whenever their temporal distance is sufficiently large. Repeating the whole procedure we obtain

$$\lim_{t\to\pm\infty}(\omega(\tau_t(B)AC)-\omega(B)\omega(AC)=0, \tag{3.45}$$

[7] Since we are interested in applications in AQFT we do not hesitate to restrict ourselves to factors when it becomes advantageous.

which gives

$$|\omega(A\tau_t(B)C) - \omega(AC)\omega(B)| \leq | \underbrace{\omega([A,\tau_t(B)]C)}_{=\langle\Omega,\pi([A,\tau_t(B)])\cdot\pi(C)\Omega\rangle} | + |\omega(\tau_t(B)AC) - \omega(AC)\omega(B)| \overset{t\to\pm\infty}{\longrightarrow} 0.$$

$\pi(\tau_t(B))$ thus converges to $\omega(B)\mathbb{1}$ in the weak sense for all $B \in \mathcal{R}$. According to theorem 3.5.3 and the remark thereinafter ω^P is a vector state of ω, so that the desired result comes out,

$$\omega(A) = \lim_{t\to\pm\infty} \omega^P(\tau_t(A)) \quad \text{for all} \quad A \in \mathcal{R}. \qquad \square$$

Corollary 3.5.8. *Let $(\mathcal{R},\tau)$ be a W^*-dynamical system, where $\mathcal{R}$ is assumed to be a factor. Then, a faithful (τ,β)-KMS state ω is dynamically stable, supposed that $(\mathcal{R},\tau)$ is asymptotically abelian in the weak sense w.r.t. this state.*

In fact, weak asymptotic abelianness implies even more, and this leads us eventually to the study of reactions of the system to perturbations of the state.

Theorem 3.5.9. *Let $(\mathcal{R},\tau)$ be a W^*-dynamical system, where $\mathcal{R}$ is assumed to be a factor. Moreover, let ω be a τ-invariant faithful normal state and suppose that the system is asymptotically abelian in the weak sense w.r.t. ω. Denote by $\tilde{\omega}$ any state in the folium of ω. Then,*

$$\omega(A) = \lim_{t\to\pm\infty} \tilde{\omega}(\tau_t(A)) \quad \textit{for all} \quad A \in \mathcal{R}. \tag{3.46}$$

Proof. We exploit corollary A.4.25 according to which any state in the folium of a faithful normal state ω is a vector state of ω. The assertion is then established by a calculation identical with that in the proof of theorem 3.5.7. $\square$

Note that the theorem itself does not make any reference to KMS states. One might ask why a stationary state ω w.r.t. which the system is weakly asymptotically abelian is not sufficient to characterise equilibrium situations. If one considers a perturbation P of the dynamics, $\tau \rightsquigarrow \tau^P$, it might happen that a state ω', stationary w.r.t. the perturbed dynamics, is not contained in the folium of ω (cf. [104]). Since the perturbation is bounded the states differ by a finite amount of energy, which can be either transferred by a finite number of particles, then ω' would generically be normal to ω, or by an infinite number of particles with infinitesimally small energy, which is when normality is expected to be destroyed. Taking ω as a KMS state this problem is avoided, for kinematical stability is known to hold.

However, if the algebra admits a faithful KMS state, $\hat{\omega}$ say, it necessarily has to coincide with ω. More general, let ω and $\hat{\omega}$ be two faithful normal states on a von Neumann algebra $\mathcal{R}$, and let $(\pi,\mathcal{H},\Omega)$ and $(\hat{\pi},\hat{\mathcal{H}},\hat{\Omega})$ be the corresponding GNS representations. By the unitary representation theorem A.4.18 there exists a unitary operator $U : \hat{\mathcal{H}} \to \mathcal{H}$ such that $\pi(A) = U\hat{\pi}(A)U^*$ for all $A \in \mathcal{R}$. That yields

$$\hat{\omega}(A) = \langle\hat{\Omega},\hat{\pi}(A)\hat{\Omega}\rangle = \langle U\hat{\Omega}, U\hat{\pi}(A)U^*U\hat{\Omega}\rangle = \langle U\hat{\Omega},\pi(A)U\hat{\Omega}\rangle\,. \tag{3.47}$$

Thus, $\hat{\omega}$ is a vector state of ω (and vice versa). If $\mathcal{R}$ is a factor and the system is asymptotically abelian in the weak sense w.r.t. ω, theorem 3.5.9 applies. If there is apart from ω any other stationary faithful normal state $\hat{\omega}$, such as a faithful KMS state, the theorem enforces $\omega = \hat{\omega}$.

Taking into account that von Neumann algebras appearing in AQFT are supposed to be factors, the foregoing investigations show that weak asymptotic abelianness is a sufficient condition to ensure the characteristic "return to equilibrium"-behaviour of a system in a faithful KMS state ω under quite general circumstances, namely from all states in the folium of ω, that is from all states obtained by local excitations of Ω. Thus, the stability behaviour stemming from the weak asymptotic abelianness criterion is very strong. Demanding this stronger version of dynamical stability, which we simply want to call *"return to equilibrium"-property*,[8] weak asymptotic abelianness is in fact also a necessary condition. More specifically we have the following result:

[8] The "return to equilibrium"-property together with kinematical stability is a strengthened form of dynamical stability.

Theorem 3.5.10. *Let $(\mathcal{R},\tau)$ be a W^*-dynamical system, $\mathcal{R}$ a factor, and ω a τ-invariant, faithful normal state. Then, the following conditions are equivalent:*

(i) $(\mathcal{R},\tau,\omega)$ *has the "return to equilibrium"-property;*

(ii) $\lim_{t\to\pm\infty}\omega(A^*\tau_t(B)A)=\omega(A^*A)\omega(B)$ *for all* $A,B\in\mathcal{R}$*;*

(iii) $(\mathcal{R},\tau)$ *is asymptotically abelian in the weak sense w.r.t.* ω*;*

(iv) ω *is strongly clustering, i.e.* $\lim_{t\to\pm\infty}(\omega(A\tau_t(B))-\omega(A)\omega(B))=0$ *for all* $A,B\in\mathcal{R}$*.*

Proof. (i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii): Denote by π the GNS representation of ω, and let $\mathcal{H}$ be the underlying Hilbert space containing the representative vector Ω. Let $\psi\in\mathcal{H}$, $B\in\mathcal{R}$ and $\varepsilon>0$ be arbitrary. Due to the cyclicity of Ω for $\pi(\mathcal{R})$ there exists an $A\in\mathcal{R}$ such that $\|\psi-\pi(A)\Omega\|<\varepsilon/(4\|B\|)$. W.l.o.g. let us assume that A is chosen such that $\|\psi\|=\|\pi(A)\Omega\|$. Using the Cauchy-Schwarz inequality yields

$$\begin{aligned}|\langle\psi,\pi(\tau_t(B))\psi\rangle-\langle\psi,\psi\rangle\omega(B)|\leq{}&|\langle\psi-\pi(A)\Omega,\pi(\tau_t(B))\psi\rangle|+|\langle\pi(A)\Omega,\pi(\tau_t(B))(\psi-\pi(A))\Omega)\rangle|\\&+|\langle\pi(A)\Omega,\pi(\tau_t(B))\pi(A)\Omega\rangle-\langle\psi,\psi\rangle\omega(B)| &(3.48)\\\leq{}&2\|\psi-\pi(A)\Omega\|\|B\|+|\omega(A^*\tau_t(B)A)-\|\pi(A)\Omega\|^2\omega(B)| &(3.49)\\\leq{}&\varepsilon/2+|\omega(A^*\tau_t(B)A)-\omega(A^*A)\omega(B)|. &(3.50)\end{aligned}$$

Because of (ii) we can find a t_0, such that $|\omega(A^*\tau_t(B)A)-\omega(A^*A)\omega(B)|<\varepsilon/2$ for all $|t|>t_0$. This justifies

$$\lim_{t\to\pm\infty}\langle\psi,\tau_t(B)\psi\rangle=\langle\psi,\psi\rangle\omega(B)\text{ for all }B\in\mathcal{R}\text{ and }\psi\in\mathcal{H}. \tag{3.51}$$

The polarization identity implies that for all $B\in\mathcal{R}$ and for all $\phi,\psi\in\mathcal{H}$

$$\lim_{t\to\pm\infty}\langle\phi,\pi(\tau_t(B))\psi\rangle=\langle\phi,\psi\rangle\omega(B)\quad\Rightarrow\quad\lim_{t\to\pm\infty}\langle\phi,(\pi(\tau_t(B))-\omega(B)\mathbb{1})\psi\rangle=0. \tag{3.52}$$

From this expression we conclude

$$\begin{aligned}0&=\lim_{t\to\infty}\langle\pi(A)^*\phi,(\pi(\tau_t(B))-\omega(B)\mathbb{1})\psi\rangle-\lim_{t\to\infty}\langle\phi,(\pi(\tau_t(B))-\omega(B)\mathbb{1})\cdot\pi(A)\psi\rangle &(3.53)\\&=\lim_{t\to\infty}\langle\phi,\pi([A,\tau_t(B)])\psi\rangle\,. &(3.54)\end{aligned}$$

(iii) $\Rightarrow$ (i) is the statement of theorem 3.5.9. (iii) $\Leftrightarrow$ (iv) can be found in [23], p. 403. □

Remark. If Ω is cyclic for a subset $\pi(\mathcal{R}_1)$ of $\pi(\mathcal{R})$, the elements A in condition (ii) may equally well be taken only from $\mathcal{R}_1$.

Weak asymptotic abelianness is not only a sufficient condition to ensure that the system returns to equilibrium, but characterises this behaviour. This suggests to impose weak asymptotic abelianness as an additional condition to get a reasonable stability behaviour and to justify KMS states on von Neumann factors to represent thermal equilibrium. However, it could also be meaningful to demand the "return to equilibrium"-property only for excitations induced by some suitable chosen subalgebra (such as all observables which are localised in a proper subregion of $\mathcal{R}$'s localisation region, to make sure that the horizon does not play any role), or merely for small or infinitesimal perturbations. In these cases weaker notions of dispersion might be sufficient to cause all the typical features of equilibrium situations.

Stability Conditions

There is an endeavour to gain a better understanding of the relation between the kinematical and the dynamical approach. Thereto, we consider a physical system which is in a (τ^P,β)-KMS state $\omega^P=\langle\Omega_P,\cdot\,\Omega_P\rangle$ on a von Neumann factor $\mathcal{R}$. Let $\omega=\langle\Omega,\cdot\,\Omega\rangle$ be a β-KMS state w.r.t. the unperturbed dynamics τ, Ω a cyclic and separating vector. We assume that ω^P evolves under τ into a state $\tilde{\omega}$,

$$\tilde{\omega}(A)=\lim_{t\to\pm\infty}\omega^P(\tau_t(A))\quad\text{for all}\quad A\in\mathcal{R}. \tag{3.55}$$

If ω^P is contained in a family of states $\omega^{\lambda P}$, $0 < \lambda \leq 1$, we can do perturbation theory in λ. It is possible to rewrite τ as a norm convergent series expansion of $\tau^{\lambda P}$, cf. [24]. This can be used to expand $\tilde{\omega}$,

$$\tilde{\omega}(A) = \omega^{\lambda P}(A) - \lambda \cdot \lim_{T\to\pm\infty} i \int_{-T}^{0} \mathrm{d}t\, \omega^{\lambda P}([A, \tau_t^{\lambda P}(P)]) + \mathcal{O}(\lambda^2). \tag{3.56}$$

On the other hand, for sufficiently small perturbations λP it is possible to relate ω and $\omega^{\lambda P}$ by a uniformly convergent series [24],

$$\omega(A) = \omega^{\lambda P}(A) - \lambda \cdot \int_0^{\beta} \mathrm{d}s\, \omega_T^{\lambda P}(A, \tau_{is}^{\lambda P}(P)) + \mathcal{O}(\lambda^2), \tag{3.57}$$

where ω_T is the *truncated function*

$$\omega_T^{\lambda P}(A, \tau_{is}^{\lambda P}(P)) = \omega^{\lambda P}(A\tau_{is}^{\lambda P}(P)) - \omega^{\lambda P}(A)\omega^{\lambda P}(P). \tag{3.58}$$

To obtain dynamical stability, we have to demand the identity of these series, term by term. That yields the so-called *stability conditions*. If one restricts oneself to marginal perturbations, it might be satisfactory when the first-order stability condition holds,

$$\lim_{T\to\pm\infty} i \int_{-T}^{0} \mathrm{d}t\, \omega^{\lambda P}([A, \tau_t^{\lambda P}(P)]) = \int_0^{\beta} \mathrm{d}s\, \omega_T^{\lambda P}(A, \tau_{is}^{\lambda P}(P)). \tag{3.59}$$

In particular this implies

$$\lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}t\, \omega^{\lambda P}([A, \tau_t^{\lambda P}(P)]) = 0. \tag{3.60}$$

As $\lambda \to 0$, we have seen that $\omega^{\lambda P} \to \omega$ and $\tau^{\lambda P} \to \tau$ in a suitable sense. This way one formally finds an expression which was originally introduced by Haag, Kastler and Trych-Pohlmeyer [59],

$$\lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}t\, \omega([A, \tau_t(B)]) = 0 \quad \text{for all} \quad A, B \in \mathcal{R} \tag{3.61}$$

(or at least for a sufficiently large subset). The authors called it *consistency condition*. Physically it results from the requirement that small local perturbations should produce only small local changes in the stationary state. We want to regard it as a kind of minimal stability condition on W*-dynamical systems, a basic prerequisite which is necessary to expect (at least some weakened form of) dynamical stability. If ω is a τ-KMS state and if the system is asymptotically abelian in the weak sense, or equivalently if the strong clustering condition holds, the consistency condition follows (as well as all the stability conditions), see theorem 5.4.12 and proposition 5.4.13 in [24].

There is a second way to motivate the consistency condition as a weak form of dynamical stability [24]. Let us consider a state ω on a W*-dynamical system $(\mathcal{R}, \tau)$ which is dynamically stable. Then for every perturbation $P = P^* \in \mathcal{R}$ and every sufficiently small λ, $\omega^{\lambda P}$ evolves back into the state ω under the perturbed dynamics τ,

$$\omega(A) = \lim_{t\to\pm\infty} \omega^{\lambda P}(\tau_t(A)). \tag{3.62}$$

Using the relation (cf. [60])

$$\frac{\mathrm{d}}{\mathrm{d}t} \tau_{-t}^{\lambda P} \tau_t(A) = -i\lambda \tau_{-t}^{\lambda P}[P, \tau_t(A)], \tag{3.63}$$

we obtain

$$\lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}t\, \omega^{\lambda P}([P, \tau_t(A)]) = \frac{i}{\lambda} \lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \omega^{\lambda P}(\tau_t(A)) \tag{3.64}$$

$$= \frac{i}{\lambda} \left(\lim_{T\to\infty} \omega^{\lambda P}(\tau_T(A)) - \lim_{T\to-\infty} \omega^{\lambda P}(\tau_T(A)) \right) \overset{(3.62)}{=} 0. \tag{3.65}$$

Again, by taking the limit $\lambda \to 0$ formally the consistency condition falls out. This derivation one more time underlines it as a necessary condition a kinematically stable state has to satisfy in order to be dynamically stable under infinitesimal perturbations.

3.6 Passivity

A natural approach to the KMS condition as a direct characterisation of thermal equilibrium based on the second law of thermodynamics was given by Pusz and Woronowicz in 1978 [96], see [24] for the treatment of W*-dynamical systems. They showed that a KMS state is essentially distinguished by the condition that the system is unable to perform mechanical work in cyclic processes. We sum up their results.

A state ω on $\mathcal{R}$ is said to be *passive* if

$$-i\omega(U^*\delta(U)) \geq 0 \tag{3.66}$$

for all $U \in \mathcal{D}(\delta)$ which are contained in the connected identity component of the group of all unitary elements of $\mathcal{R}$. δ is supposed to be the infinitesimal generator of the *-automorphism group. Passivity reflects another form of stability which is basically kinematical. It is related to the maximum entropy principle satisfied by Gibbs equilibrium states and is in fact a consequence of it in finite systems. $i\omega(U^*\delta(U))$ can be interpreted as the energy difference between ω and the perturbed state $\omega_U(\cdot) \equiv \omega(U^* \cdot U)$, that is excitations by the unitaries U lead necessarily to states with more energy.

To get a better intuition of what passivity means, a W*-dynamical system is supposed to undergo a series of external changes, modelled by a norm-continuous one-parameter family of self-adjoint elements $\mathbb{R} \ni t \mapsto P_t \in \mathcal{R}$ which describes local changes in the external conditions. Similarly to proposition 3.5.1 the family $\{P_t\}$ defines a one-parameter family of *-automorphisms τ^P of $\mathcal{R}$, uniquely determined by the differential equation

$$\frac{\mathrm{d}\tau_t^P}{\mathrm{d}t}(A) = \tau_t^P(\delta(A) + i[P_t, A]) \quad \text{for all} \quad A \in \mathcal{D}(\delta). \tag{3.67}$$

Furthermore, there exists a unique unitary element Γ_t^P such that

$$\tau_t^P(A) = \Gamma_t^P \tau_t(A) \Gamma_t^{P*}. \tag{3.68}$$

Here we are interested in *cyclic processes* with a smooth and temporary alteration of the external conditions, i.e. we assume $P_t = 0$ for $t \leq 0$ and $t \geq T$. This way we end up with a closed system at $t = T$, whose external conditions coincide with the original ones at $t = 0$. Changing the external conditions transmits energy to the system since mechanical work is performed by external forces. The transmitted energy of the cyclic process is given as

$$E^P(\omega) = \int_0^T \omega(\tau_t^P\left(\frac{\mathrm{d}P_t}{\mathrm{d}t}\right))\mathrm{d}t. \tag{3.69}$$

See [60] for the motivation why this is interpreted as an energy difference. If $E^P(\omega) \geq 0$ for all cyclic processes, one is unable to extract energy from the state by cyclic processes. This condition is suggested by the second law of thermodynamics. The inability to perform work in cyclic processes turns out to characterise the passivity property (for W*-dynamical systems implicitly contained in the proof of theorem 5.4.28 in [24]):

Theorem 3.6.1. *Let $(\mathcal{R}, \tau)$ be a W*-dynamical system, δ the infinitesimal generator of the group τ, and ω a state on $\mathcal{R}$. Then, ω is passive iff $E^P(\omega) \geq 0$ for all cyclic processes $\{P_t\}$.*

The theorem explains the physical meaning of passivity. Indeed, the term "passive" originates from this equivalence. We are completely content to give an indication of this equivalence. For all those cyclic processes $\{P_t\}$ which are contained in $\mathcal{D}(\delta)$, it can be shown that

$$E^P(\omega) = \int_0^T \omega(\tau_t^P\left(\frac{\mathrm{d}P_t}{\mathrm{d}t}\right))\mathrm{d}t = -i\omega(\Gamma_t^P \delta(\Gamma_t^{P*})), \tag{3.70}$$

which fills the gap to the definition of passivity (3.66). Passivity is another important property one would attribute to equilibrium states. In fact, it is coded in the KMS structure, and, even more, characterises KMS states to a large extent:

Theorem 3.6.2 (Pusz-Woronowicz [24, 96]). *Let $(\mathcal{R}, \tau)$ be a W^*-dynamical system, and ω a state on $\mathcal{R}$. The following statements are true:*

(i) Let ω be a (τ, β)-KMS state for some $\beta \in (0, \infty)$. Then, ω is passive.

(ii) Let ω be weakly clustering[9] and passive. Then, ω is a (τ, β)-KMS state for some $\beta \in [0, \infty]$.

3.7 Concluding Remarks

KMS states are introduced as formal generalisations of canonical Gibbs states. In this chapter we have seen that general KMS states have a marvellous variety of properties one would assign to equilibrium states, including stationarity, kinematical stability, and passivity. However, other typical equilibrium features like dynamical stability and the characteristic "return to equilibrium"-behaviour are not included in the KMS condition and have to be ensured by an additional requirement reflecting dispersion. We have found that dispersive properties are best modelled by weak asymptotic abelianness or in a weaker form by the consistency condition. In chapter 7 we shall come back to this lack of a sufficiently strong stability behaviour of general KMS states which is to be expected from true thermal equilibrium states.

The last section showed that passivity together with a weak clustering assumption leads to the KMS condition. In other words, the KMS condition is derivable from assumptions which may be not exclusively reserved for equilibrium situations. We want to rate this, together with the lack of a sufficiently strong dispersive behaviour, as an indication that the KMS condition is above all a necessary condition which provides us with the most promising candidates for the description of thermal equilibrium. Only if the relevant flow is a priori known to be a physical time flow, the KMS condition might be sufficient, as well, to single out equilibrium states; we shall take up this point in chapter 7.

A basic concept like that of an equilibrium state should ultimately be defined in simple physical terms. The historically first derivation of the KMS condition from first principles was given in 1974 by Haag et al. [59], see also [24, 73]. According to these authors, the natural assumptions one has to impose on a state to represent an equilibrium situation include stationarity and some form of stability, which in its weakest form is the consistency condition motivated above. Additionally, they assume a clustering as well as an ergodicity condition to ensure that correlations die out with increasing temporal difference. There are a couple of results which strengthen either the asymptotic abelianness assumption or the clustering condition in order to derive the KMS condition. The most cogent result has been obtained for factor states on L^1-asymptotically abelian systems,[10] for which the consistency condition is essentially equivalent to the KMS condition.

[9] Let η be an invariant mean on $\mathbb{R}$. A state ω is weakly clustering if $\eta_t(\omega(A\tau_t(B))) = \omega(A)\omega(B)$ for all $A, B \in \mathcal{R}$. In particular strongly clustering states are weakly clustering.

[10] In that case $t \mapsto \|[A, \tau_t(B)]\|$ is assumed to be an L^1-function.

Chapter 4

Tomita-Takesaki Modular Theory

"No human investigation can be called real science if it cannot be demonstrated mathematically."
LEONARDO DA VINCI, Italian polymath (1452-1519)

The foundations of the Tomita-Takesaki modular theory were developed by TOMITA in two unpublished papers between 1957 and 1967. Motivated by the harmonic analysis of non-unimodular locally compact groups, he studied the algebraic relation between a von Neumann algebra $\mathcal{R}$ and its commutant $\mathcal{R}'$ in the case where both algebras have a common cyclic vector. His results were eventually published by TAKESAKI in a refined and extended form not until 1970 [124]. Subsequently, the theory turned out to be an extremely fruitful, powerful and indispensable tool in the theory of operator algebras and has found numerous applications in mathematical physics. Being the most essential ingredient for the formulation of the thermal time hypothesis the Tomita-Takesaki theory is going to play a key role in the sequel of this thesis. Therefore, this chapter is devoted to a thorough introduction into modular theory. We present the theory, explore some of its striking consequences and consider applications in physics.

4.1 Tomita-Takesaki Theorem

In this section we focus attention on the mathematical ingredients while we treat physical implications afterwards. The proofs are omitted, they can be consulted for instance in [23, 72, 124], which besides have been served as main references. A few foundations concerning von Neumann algebras, states, representations, and, even more fundamental, the functional analytical background as well as the theory of conjugate linear operators can be found in appendix A.

We start off by considering a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$, which admits a cyclic and separating vector Ω. By lemma A.4.12, Ω is cyclic and separating for $\mathcal{R}'$, as well. This implies both involutions S_0 and F_0, given by

$$S_0 : \mathcal{H} \supset \mathcal{D}(S_0) = \mathcal{R}\Omega \longrightarrow \mathcal{H}, \quad A\Omega \longmapsto S_0 A\Omega := A^*\Omega, \tag{4.1}$$

$$F_0 : \mathcal{H} \supset \mathcal{D}(F_0) = \mathcal{R}\Omega \longrightarrow \mathcal{H}, \quad A'\Omega \longmapsto F_0 A'\Omega := A'^*\Omega, \tag{4.2}$$

to be well-defined and densely defined operators on $\mathcal{H}$. Clearly, S_0 and F_0 are conjugate-linear operators. The first step is achieved by the following stimulating observation:

Proposition 4.1.1 ([23]). *The involutions S_0 and F_0 are closable as conjugate-linear operators on the Hilbert space $\mathcal{H}$, and the relations $\overline{S}_0 = F_0^*$ and $\overline{F}_0 = S_0^*$ hold.*

From now on, let us denote the closure of S_0 and F_0 by S and F, respectively. S is usually called *Tomita operator*. We exploit the fact that closed operators allow for a unique polar decomposition (cf. theorem A.1.20 as well as the remark thereinafter).

Definition 4.1.2 ([23]). Let $S = J\Delta^{1/2}$ be the polar decomposition of the Tomita operator S. The positive, invertible, self-adjoint, and in general unbounded operator Δ is called the *modular operator of the pair* $(\mathcal{R}, \Omega)$; the conjugate-linear partial isometry J is called *modular conjugation* or *modular involution of the pair* $(\mathcal{R}, \Omega)$. Since $\mathrm{ran}(S) = \mathcal{D}(S)$ is dense in $\mathcal{H}$ and $\ker(S) = \{0\}$, one concludes $S = S^{-1}$, which implies that J is in fact an anti-unitary operator.

Due to the self-adjointness of Δ the Borel functional calculus may be applied, so that $f(\Delta) := \int_0^\infty f(\lambda)\,\mathrm{d}E(\lambda)$ is well-defined for all complex Borel functions f defined on $(0,\infty)$. The normal operator Δ^z is defined for all $z = s + it \in \mathbb{C}$ and satisfies essentially all the rules suggested by this notation (see appendix A.1 for some more details). Δ^s is a positive self-adjoint operator, Δ^{it} is unitary. Since log is a real function, the self-adjointness of Δ ensures that of $\log\Delta$, and $t \mapsto U(t) = \Delta^{it} = \exp(it\log\Delta)$ is a strongly continuous one-parameter unitary group.

Proposition 4.1.3 ([23, 124])**.** *The following relations can be established:*

$$\Delta = FS, \quad \Delta^{-1} = SF, \tag{4.3}$$

$$Jf(\Delta)J = \bar{f}(\Delta^{-1}), \ \textit{in particular}\ J\Delta^{is}J = \Delta^{is} \ \textit{and}\ J\Delta^{s}J = \Delta^{-s}\ \forall s \in \mathbb{R}, \tag{4.4}$$

$$S = J\Delta^{1/2} = \Delta^{-1/2}J, \quad F = J\Delta^{-1/2} = \Delta^{1/2}J, \tag{4.5}$$

$$J = J^* = J^{-1}, \tag{4.6}$$

$$\Delta\Omega = \Omega, \quad J\Omega = \Omega. \tag{4.7}$$

Eventually, we are in a position to state the central theorem of this section which opens the door towards modular theory:

Theorem 4.1.4 (Tomita-Takesaki Theorem [23])**.** *Let $\mathcal{R}$ be a von Neumann algebra with a cyclic and separating vector Ω. Furthermore, let Δ be the associated modular operator and J the modular conjugation. Then*

(i) $J\mathcal{R}J = \mathcal{R}'$, $J\mathcal{R}'J = \mathcal{R}$,

(ii) $\Delta^{-is}\mathcal{R}\Delta^{is} = \mathcal{R}$ *for all* $s \in \mathbb{R}$,[1]

(iii) $\Delta^{-is}\mathcal{R}'\Delta^{is} = \mathcal{R}'$ *for all* $s \in \mathbb{R}$.

Since $s \mapsto \Delta^{is}$ is strongly continuous, the same must be true for the function $s \mapsto \Delta^{-is}A\Delta^{is}$ for each $A \in \mathcal{R}$. For a given cyclic and separating vector, the Tomita-Takesaki theorem equips a von Neumann algebra as well as its commutant with a strongly continuous one-parameter group of *-automorphisms. The unitary group Δ^{is} is called *modular group*. Moreover, the theorem establishes the existence of a *-anti-isomorphism between $\mathcal{R}$ and its commutant $\mathcal{R}'$, obtained by conjugating $\mathcal{R}$ with J, which expresses an interplay between algebraic and analytic structures.

One would like to apply the theorem to states instead of vectors, because the former ones are more natural in the algebraic formulation of QFT and a more general description is permitted this way. To do so, one has to transfer the notions of cyclicity and separability to states. This is provided by faithfulness and normality. The GNS construction induced by a faithful normal state ω leads to a von Neumann algebra $\pi_\omega(\mathcal{R})$ over a Hilbert space $\mathcal{H}_\omega$ which contains the state as a cyclic and separating vector Ω_ω, such that the Tomita-Takesaki theory is applicable (cf. proposition A.4.23). Moreover, the GNS representation π_ω is faithful, in particular invertible, so that the resulting *-automorphism group on $\pi_\omega(\mathcal{R})$ can be translated back to the original algebra $\mathcal{R}$. Conversely, a cyclic and separating vector Ω gives rise to a faithful normal state via $\omega = \langle\Omega, \cdot\,\Omega\rangle$ (in fact only separability is required). Thus, faithful normal states are precisely those we have been looking for.[2] There remains the question under which conditions a von Neumann algebra admits a faithful normal state. The answer turns out to be contained in the following definition.

Definition 4.1.5 ([23])**.** A von Neumann algebra $\mathcal{R}$ is said to be *σ-finite* if every family of non-zero mutually orthogonal projections of $\mathcal{R}$ has at most a countable cardinality.

Lemma 4.1.6 ([23])**.** *A von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ is σ-finite iff there exists a countable subset of $\mathcal{H}$ which is separating for $\mathcal{R}$.*

The subsequent theorem combines the notion of σ-finiteness with that of faithful normal states, and cyclic and separating vectors, respectively. It shows that the property of σ-finiteness of von Neumann algebras is the characteristic feature for the applicability of the Tomita-Takesaki theorem.

[1] Note that the usual definition prevalent in mathematics uses another sign convention.

[2] One may extend modular theory to normal non-faithful states by restricting the state to a suitable subalgebra.

Theorem 4.1.7 ([23]). *Let $\mathcal{R}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then, the following three conditions are equivalent:*

(i) $\mathcal{R}$ *is σ-finite.*

(ii) There exists a faithful normal state on $\mathcal{R}$.

(iii) $\mathcal{R}$ *is *-isomorphic to a von Neumann algebra which admits a cyclic and separating vector.*

In particular, σ-finite von Neumann algebras are *-anti-isomorphic to their commutants. Von Neumann algebras encountered in quantum field theory fulfil this property. Note for this that on a separable Hilbert space every von Neumann algebra is σ-finite.

We have seen that any faithful normal state defines a one-parameter group of *-automorphisms on the corresponding von Neumann algebra. This important result is summarised in the next

Definition 4.1.8 ([23]). Let $\mathcal{R}$ be a von Neumann algebra and ω a faithful normal state on $\mathcal{R}$. Besides, let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the corresponding GNS representation with Ω_ω the representative vector of ω, cyclic and separating for $\pi_\omega(\mathcal{R})$, and Δ_ω the modular operator of $(\pi_\omega(\mathcal{R}), \Omega_\omega)$. Then, by the Tomita-Takesaki theorem there is a one-parameter group of *-automorphisms $s \mapsto \sigma_s^\omega$ on $\mathcal{R}$,[3]

$$\sigma_s^\omega(A) \equiv \pi_\omega^{-1}(\Delta_\omega^{-is}\pi_\omega(A)\Delta_\omega^{is}), \quad A \in \mathcal{R}. \tag{4.8}$$

This group is called *modular automorphism group of the pair* $(\mathcal{R}, \omega)$. One also uses the term *modular flow* or *Tomita flow*. The parameter s is usually referred to as *modular parameter*.

The modular automorphism group shows the following continuity properties:

Proposition 4.1.9 ([23, 72]). *Let σ_s^ω be the modular automorphism group of the pair $(\mathcal{R}, \omega)$. Then, for each $A \in \mathcal{R}$, $s \mapsto \sigma_s^\omega(A)$ is strongly continuous. Moreover, $s \mapsto \sigma_s^\omega$ is a σ-weakly continuous one-parameter group of *-automorphisms on $\mathcal{R}$.*

Remark. $(\mathcal{R}, \sigma_s^\omega)$ provides another example of a W^*-dynamical system.

One can establish that the modular automorphism group is trivial ($\sigma_s^\omega(A) = A$ for all $A \in \mathcal{R}$ and $s \in \mathbb{R}$) iff ω is a tracial state. In particular, for an abelian algebra the modular group becomes completely invisible.

By proposition 4.1.3, the modular automorphism group of the pair $(\mathcal{R}', \omega)$, which we denote by $\hat{\sigma}_s^\omega$, reads

$$\hat{\sigma}_s^\omega(A) = \sigma_{-s}^\omega(A), \quad A \in \mathcal{R}'. \tag{4.9}$$

The *-anti-isomorphism which maps $\mathcal{R}$ onto its commutant is given by

$$j^\omega(A) \equiv \pi_\omega^{-1}(J_\omega \pi_\omega(A) J_\omega). \tag{4.10}$$

It has to be stressed that the existence of the modular objects for von Neumann algebras is a highly non-trivial result. The modular automorphism group contains information about both $\mathcal{R}$ and ω. Later on we shall see to what extent it really depends on the state.

Writing $K \equiv -\log \Delta$ the unitary operators implementing the modular automorphisms adopt the form $U(s) = \Delta^{-is} = e^{isK}$. This formally corresponds to unitary operators implementing a time evolution on the von Neumann algebra $\mathcal{R}$ which is generated by a Hamiltonian. For this reason the self-adjoint operator K is called *modular Hamiltonian* or *Tomita Hamiltonian*.

Ω_ω is an eigenvector of K to eigenvalue 0. The spectrum of K may extend from $-\infty$ to $+\infty$ (this necessarily happens in the type III_1 case, see appendix A.5). However, as the following proposition shows, the negative part of the modular Hamiltonian is suppressed w.r.t. $\mathcal{R}$, while the positive part is suppressed w.r.t. $\mathcal{R}'$:

[3] The Tomita-Takesaki theorem can be stated for generalised states, namely so-called weights, as well [23, 72, 125]. Also in that case, in order to apply the theorem, the weights have to show certain properties. More specifically, they have to be faithful, normal and semi-finite. However, it can be shown that *any* von Neumann algebra admits such a weight, such that modular theory becomes an even more powerful tool since it is applicable to *every* von Neumann algebra. Nonetheless, for our purposes it suffices to restrict ourselves to states, which is the usual form modular theory is presented in mathematical physics.

Another variant of the Tomita-Takesaki theorem concerns its extension to unbounded operators, which naturally appear e.g. in the Wightman framework. This has been investigated by Inoue [68] in the framework of *-algebras of closable operators, so-called *O^*-algebras*.

Proposition 4.1.10 ([60]). *Let E_κ^- be the spectral projection of K onto the interval $[-\infty, -\kappa]$, and E_κ^+ the spectral projection onto $[\kappa, \infty]$, where $\kappa \in \mathbb{R}_+$. Then*

$$\|E_\kappa^- A\Omega\| \leq e^{-\kappa/2}\|A\| \quad \textit{for all } A \in \mathcal{R}, \tag{4.11}$$

$$\|E_\kappa^+ A'\Omega\| \leq e^{-\kappa/2}\|A'\| \quad \textit{for all } A' \in \mathcal{R}'. \tag{4.12}$$

Note that the physical Hamiltonian implementing the time translations does not need to be positive definite in the representation space, as well (cf. [58, 60]). Think e.g. of a representation induced by a thermal state, where the spectrum coincides with the whole real axis.

4.2 Modular Theory by Example

In general, it is quite tricky and involved to determine the modular objects explicitly. To gain some intuition what they actually do, we use this section to discuss some elementary, more or less concrete examples where it can be done.

Quantum Mechanical Limit

We begin with the undoubtedly most vital example for our purposes. If the algebra is chosen to be $\mathcal{R} = \mathcal{B}(\mathcal{H})$, with a separable Hilbert space $\mathcal{H}$, the action of the modular group can be derived explicitly, which is what we intend to do next (cf. [42]). From the physical point of view such a type I factor is used to describe quantum mechanical systems with a finite number of degrees of freedom. Hence, we shall obtain the quantum mechanical limit of the modular flow. In that case the underlying Hilbert space $\mathcal{H}$ is identified by the N-particle space.[4]

Let ω be an arbitrary normal state on $\mathcal{B}(\mathcal{H})$. According to theorem A.4.14, ω is described in terms of a density matrix $\rho \in \mathcal{B}(\mathcal{H})$ which represents the mixed quantum state of the system.

Lemma 4.2.1 ([23]). *A normal state ω on the algebra of all bounded operators on a separable Hilbert space is faithful iff the corresponding density matrix ρ is invertible.*

The separability is in fact necessary, since on a non-separable Hilbert space ρ cannot be invertible due to the existence of at most countably many non-zero eigenvalues:

Lemma 4.2.2 ([23]). *The von Neumann algebra $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is σ-finite iff $\mathcal{H}$ is separable.*

We assume ω from now on to be faithful and normal, i.e. given by an invertible density matrix ρ. To carry out the Tomita-Takesaki construction, one has to construct the GNS representation space, in which ω is given by a cyclic and separating vector.[5] An abbreviation for this was suggested by Haag, Hugenholtz and Winnink [58], see also [60], which we shall recall in the following. The set of all Hilbert-Schmidt operators on $\mathcal{H}$, $\{\kappa \in \mathcal{B}(\mathcal{H}) : \operatorname{tr}\kappa^*\kappa < \infty\}$ together with the inner product $\langle\kappa|\kappa'\rangle \equiv \operatorname{tr}\kappa^*\kappa'$ form a Hilbert space, which we denote by $\mathcal{K}$. $\mathcal{K}$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$,

$$\kappa \in \mathcal{K}, \;\; A \in \mathcal{B}(\mathcal{H}) \quad \Rightarrow \quad A\kappa \in \mathcal{K}, \;\; \kappa A \in \mathcal{K}. \tag{4.13}$$

We construct a (reducible) representation of $\mathcal{B}(\mathcal{H})$ by operators acting on $\mathcal{K}$, in which ω is given by a pure vector. Since ρ is positive, $\kappa_0 = \rho^{1/2}$ gives rise to a well-defined operator (cf. theorem A.1.15 and A.1.16), which, owing to the finiteness of the trace of ρ, is of Hilbert-Schmidt class. We exploit the fact that $\mathcal{K}$ is an ideal to define

$$\pi_\omega(A)\,|\kappa\rangle := |A\kappa\rangle \in \mathcal{K}, \quad \kappa \in \mathcal{K}, \;\; A \in \mathcal{B}(\mathcal{H}). \tag{4.14}$$

That yields

$$\omega(A) = \operatorname{tr}(\rho A) = \operatorname{tr}(\kappa_0 A \kappa_0) = \langle\kappa_0|A\kappa_0\rangle = \langle\kappa_0|\pi_\omega(A)|\kappa_0\rangle\,, \tag{4.15}$$

i.e. κ_0 is a state vector corresponding to ω. It is a cyclic and separating for the representation π_ω:

[4] Or, if one prefers the field algebra, by the Fock space.

[5] As we shall see soon, the easiest way to obtain the modular flow is to cleverly guess it and show that it satisfies the KMS condition; at this point we prefer a more elementary way. Nevertheless, in the course of this thesis we shall also take advantage of the easier variant.

- Separability: Let $\pi_\omega(A)\,|\kappa_0\rangle = 0$. Thence $\omega(A^*A) = 0 \;\Rightarrow\; A = 0 \;\Rightarrow\; \pi_\omega(A) = 0$.
- Cyclicity: Let $\kappa \in \mathcal{K}$, such that $\langle\kappa|\pi_\omega(A)|\kappa_0\rangle = 0$ for all $A \in \mathcal{B}(\mathcal{H})$. Then choose $A = \kappa\kappa_0$,

$$0 = \langle\kappa|\pi_\omega(A)|\kappa_0\rangle = \langle\kappa|\kappa\rho\rangle = \mathrm{tr}(\rho\kappa^*\kappa) = \omega(\kappa^*\kappa) \quad \Rightarrow \quad \kappa = 0.$$

π_ω can be identified with the GNS representation of ω. Let $J\,|\kappa\rangle := |\kappa^*\rangle$ and $\Delta\,|\kappa\rangle := |\rho\kappa\rho^{-1}\rangle$ for all $\kappa \in \mathcal{K}$. Then one checks that $S = J\Delta^{1/2}$ is the polar decomposition of the Tomita operator S, defined by $S\pi_\omega(A)\,|\kappa_0\rangle = \pi_\omega(A^*)\,|\kappa_0\rangle$. From this we can immediately derive the action of the modular flow induced by a faithful normal state corresponding to an invertible density matrix on the algebra $\mathcal{R} = \mathcal{B}(\mathcal{H})$:

$$\pi_\omega(\sigma_s^\omega A)\,|\kappa\rangle = \Delta^{-is}\pi_\omega(A)\Delta^{is}\,|\kappa\rangle = |\rho^{-is}A\rho^{is}\kappa\rangle = \pi_\omega(\rho^{-is}A\rho^{is})\,|\kappa\rangle \quad \text{for all} \quad \kappa \in \mathcal{K} \tag{4.16}$$

$$\Rightarrow \quad \sigma_s^\omega(A) = \rho^{-is}A\rho^{is} = e^{-is\log\rho}Ae^{is\log\rho}. \tag{4.17}$$

The modular flow is generated by the Hamiltonian $K := -\log\rho$. Note that this is not the modular Hamiltonian, which is defined on the GNS space and which maps κ_0 to zero. Anyway, K generates the modular evolution on $\mathcal{B}(\mathcal{H})$.

Classical Limit

We can also perform a classical limit of the modular action [42]. For this purpose let us start again with a quantum mechanical system with finitely many degrees of freedom. We fall back upon the result just obtained, a faithful normal state ω is described by an invertible density matrix, and its modular flow reads

$$\sigma_s^\omega(A) = e^{-is\log\rho}Ae^{is\log\rho} \tag{4.18}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s}\sigma_s^\omega(A) = -i\log\rho\cdot\sigma_s^\omega(A) + i\sigma_s^\omega(A)\cdot\log\rho = -i[\sigma_s^\omega(A), -\log\rho]. \tag{4.19}$$

The classical limit is obtained by the usual substitutions: observables $A \mapsto$ smooth real functions A on phase space, density matrix $\rho \mapsto$ smooth positive normalized function ρ on phase space, commutators $-i[\cdot,\cdot] \mapsto$ Poisson brackets $\{\cdot,\cdot\}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}A(s) = \{A(s), -\log\rho\} = \{A(s), K\}, \quad K := -\log\rho. \tag{4.20}$$

Modular Flow Induced by Perturbed States

Let us assume that the modular operator as well as the modular conjugation are known for some state ω. One may ask for the modular objects associated with the state ω^P, induced by a perturbation $P = P^* \in \mathcal{R}$ (cf. section 3.5). In fact, the modular operator of the perturbed state can be related to the modular objects of the unperturbed state.

Proposition 4.2.3 ([24]). *Let $\mathcal{R}$ be a von Neumann algebra and $\omega = \langle\Omega, \cdot\,\Omega\rangle$ a state represented by a cyclic and separating vector Ω. Furthermore, let $\Delta = e^{-K}$ be the modular operator, and J the modular conjugation associated with the pair $(\mathcal{R}, \Omega)$. For $P = P^* \in \mathcal{R}$ consider the state*

$$\omega^P(\cdot) = \langle\Omega^P, \cdot\,\Omega^P\rangle, \quad \Omega^P = \frac{e^{-(K+P)/2}\Omega}{\|e^{-(K+P)/2}\Omega\|}. \tag{4.21}$$

Then, Ω^P is cyclic and separating, and

$$\Delta_P = e^{-K_P}, \quad K_P = K + P - JPJ \tag{4.22}$$

is the modular operator associated with $(\mathcal{R}, \omega^P)$. The modular flow is given by

$$\sigma_s^{\omega^P}(A) = \Delta_P^{-is}A\Delta_P^{is} = e^{is(K+P-JPJ)}Ae^{-is(K+P-JPJ)}, \tag{4.23}$$

$$\Delta_P^{is}\psi = \lim_{n\to\infty}(\Delta^{is/n}e^{-isP/n}Je^{isP/n}J)^n\psi. \tag{4.24}$$

Remark. A similar statement holds via the GNS construction for arbitrary faithful normal states.

Some Concrete Examples

There do exists a couple of concrete examples, where the modular objects are explicitly determined for certain states on specific von Neumann algebras [10, 41, 56, 79]. Not all these examples are of physical relevance. Let us just mention two important ones.

Example 1 Let $\mathcal{G}$ be a metrizable, separable, locally compact topological group. Then, there exists a left invariant measure μ satisfying $\mu(gA) = \mu(A)$ for each $g \in \mathcal{G}$ and every μ-measurable set $A \subset \mathcal{G}$, the so-called *left Haar measure* $\mathrm{d}\mu$. One may establish the existence of a continuous group homomorphism

$$\tilde{\Delta} : \mathcal{G} \to (0, \infty), \text{ such that } \mu(Ag) = \tilde{\Delta}(g)\mu(A). \tag{4.25}$$

$\tilde{\Delta}(g)$ is called the *modular function*. It constitutes the difference between the left and the right Haar measure, ρ, the latter one being analogously defined,

$$\mathrm{d}\rho(g) = \tilde{\Delta}(g^{-1})\mathrm{d}\mu(g). \tag{4.26}$$

Consider the left regular representation $L(\cdot)$ on the Hilbert space $L^2(\mathcal{G}, \mathrm{d}\mu(g))$, which is induced by multiplication with the inverse on the left. The vector Ω, defined by $\Omega(g) \equiv 1$, is cyclic and separating for $\mathcal{R} := \{L(g) : g \in \mathcal{G}\}'' \subset \mathcal{B}(L^2(\mathcal{G}, \mathrm{d}\mu(g)))$. The modular objects corresponding to the pair $(\mathcal{R}, \Omega)$ turn out to be given by [10, 79]

$$(J\varphi)(g) = \tilde{\Delta}(g)^{-1/2}\overline{\varphi(g^{-1})}, \quad \varphi \in L^2(\mathcal{G}, \mathrm{d}\mu(g)), \tag{4.27}$$

$$(\Delta\varphi)(g) = \tilde{\Delta}(g)\varphi(g), \quad \varphi \in L^2(\mathcal{G}, \mathrm{d}\mu(g)). \tag{4.28}$$

The modular operator is just multiplication by the modular function of the group. This example explains the origin of the name *modular theory*, the modular operator is a generalisation of the modular function.

Example 2 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional Hilbert spaces with $\dim\mathcal{H}_1 = \dim\mathcal{H}_2 = n$ and ONBs $\{e_i\}$ and $\{f_i\}$, respectively. Consider the von Neumann algebra $\mathcal{R} = \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}$ acting on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$$\Omega := \sum_i e^{\lambda_i}\, e_i \otimes f_i \in \mathcal{H}_1 \otimes \mathcal{H}_2, \text{ with } \lambda_i \in \mathbb{R} \text{ and } \sum_i e^{2\lambda_i} = 1, \tag{4.29}$$

is a cyclic and separating normalized vector for $\mathcal{R}$. The modular objects for $(\mathcal{R}, \Omega)$ read [79]

$$J(e_i \otimes f_j) = e_j \otimes f_i, \tag{4.30}$$

$$\Delta(e_i \otimes f_j) = e^{2(\lambda_i - \lambda_j)}(e_i \otimes f_j), \tag{4.31}$$

$$\sigma_s(A \otimes \mathbb{1}_{\mathcal{H}_2})(e_i \otimes f_j) = \sum_k A_{ki} e^{2is(\lambda_i - \lambda_k)}(e_k \otimes f_j), \text{ where } A_{ki} := \langle e_k, Ae_i \rangle\,. \tag{4.32}$$

Later on we shall extend this example to infinite-dimensional separable Hilbert spaces. An application of the Tomita-Takesaki theorem shows

$$(\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2})' = J\,(\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2})\,J = \mathbb{1}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1)' \otimes (\mathbb{1}_{\mathcal{H}_2})'. \tag{4.33}$$

This relation in fact holds in the general case. For two arbitrary von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ one has $(\mathcal{R}_1 \otimes \mathcal{R}_2)' = \mathcal{R}_1' \otimes \mathcal{R}_2'$, which is a highly non-trivial result, inferred with the decisive help of modular theory (see e.g. [72]).

Further physical examples, where the modular objects are explicitly determined, concern the CAR algebra [79] as well as the CCR algebra [56].

4.3 Link between Modular Theory and KMS Condition

There is a completely unexpected and outermost remarkable relation between modular theory and KMS theory, and with it between mathematics and physics, which in additional provides the decisive input to formulate the time hypothesis. This relation was realised and worked out independently by Takesaki and Winnink in 1970. To quote Haag ([60], p. 216), it is "*a beautiful example of "prestabilized harmony" between physics and mathematics*". Since the next two theorems form the basis for the formulation of the time hypothesis we add the proofs.

Theorem 4.3.1 ([72, 124])**.** *An arbitrary faithful normal state ω on a von Neumann algebra $\mathcal{R}$ satisfies the KMS condition for $\beta = 1$ w.r.t. the modular flow σ_s^ω the state generates itself.*

Proof. We sketch the proof. Considering the GNS representation of $\mathcal{R}$ induced by ω we may assume that $\mathcal{R}$ acts on a Hilbert space $\mathcal{H}$ which contains a cyclic and separating vector Ω_ω. For convenience we omit the π_ω's. Using $\mathcal{R}\Omega_\omega \subset \mathcal{D}(\Delta^{1/2})$ one concludes [72] that each of the functions

$$g_1(z) \equiv \langle A^*\Omega_\omega, \Delta^{-iz} B\Omega_\omega\rangle\,, \quad 0 \leq \operatorname{Im} z \leq 1/2, \tag{4.34}$$

$$g_2(z) \equiv \langle B^*\Omega_\omega, \Delta^{1+iz} A\Omega_\omega\rangle\,, \quad 1/2 \leq \operatorname{Im} z \leq 1, \tag{4.35}$$

is defined, bounded and continuous on the respective strip, and analytic on the interior of that strip. Falling back upon the relations from proposition 4.1.3 we deduce that for all $s \in \mathbb{R}$

$$g_1(s + i/2) = \langle A^*\Omega_\omega, \Delta^{-is}\Delta^{1/2} B\Omega_\omega\rangle = \langle SA\Omega_\omega, \Delta^{-is} J\cdot J\Delta^{1/2} B\Omega_\omega\rangle \tag{4.36}$$

$$= \langle J\Delta^{1/2} A\Omega_\omega, J\Delta^{-is}\cdot SB\Omega_\omega\rangle = \langle \Delta^{-is} B^*\Omega_\omega, \Delta^{1/2} A\Omega_\omega\rangle \tag{4.37}$$

$$= \langle B^*\Omega_\omega, \Delta^{is}\Delta^{1/2} A\Omega_\omega\rangle = g_2(s + i/2). \tag{4.38}$$

Thus, g_1 and g_2 can be combined into one single function f, bounded and continuous on $\{0 \leq \operatorname{Im} z \leq 1\}$. Exploiting that $\Delta\Omega_\omega = \Omega_\omega$ we obtain for all $A, B \in \mathcal{R}$:

$$\omega(A\sigma_s(B)) = \langle \Omega_\omega, A\Delta^{-is} B\Delta^{is}\Omega_\omega\rangle = \langle A^*\Omega_\omega, \Delta^{-is} B\Omega_\omega\rangle = g_1(s) = f(s), \tag{4.39}$$

$$\omega(\sigma_s(B)A) = \langle \Omega_\omega, \Delta^{-is} B\Delta^{is} A\Omega_\omega\rangle = \langle B^*\Omega_\omega, \Delta^{is} A\Omega_\omega\rangle = g_2(s + i) = f(s + i). \tag{4.40}$$

In the final step one shows [72] that f is analytic on the interior of that strip. Then f satisfies all the conditions demanded in theorem 3.3.5 which constitute the KMS condition. □

This way modular theory enters naturally into equilibrium thermodynamics. The KMS condition at value $\beta = 1$ is therefore often referred to as *modular condition*. The modular group is endowed with the analyticity associated with the KMS condition. There is a certain converse to this theorem, the modular group is uniquely determined by the KMS condition.

Theorem 4.3.2 (Takesaki [10, 124])**.** *Let $(\mathcal{R}, \tau)$ be a W*-dynamical system. Moreover, let ω be a faithful (τ, β)-KMS state on $\mathcal{R}$. Then, $\tau_t = \sigma^\omega_{t/\beta}$, i.e. τ_t coincides up to rescaling with the modular group of $\mathcal{R}$ associated with ω.*

Proof. We outline the proof for $\beta = 1$. Again, we work in the GNS space and skip all the π_ω's. It has to be shown that each automorphism group τ_t which satisfies the KMS condition at $\beta = 1$ coincides with σ_s^ω. ω is τ_t-invariant, hence by corollary A.3.10 τ_t is implemented by a unitary group $U(t)$ with $U(t)\Omega_\omega = \Omega_\omega$. One can establish the existence of σ-weakly dense subalgebras $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$, such that $\mathcal{R}_1\Omega_\omega$ and $\mathcal{R}_2\Omega_\omega$ are analytic vectors for the generators of Δ^{is} and $U(t)$, respectively (cf. section 2.5.1). We consider the analytic function

$$g_{AB}(z) \equiv \langle \Delta^{-i(\bar z + i)} B^*\Omega_\omega, U(z) A\Omega_\omega\rangle = \omega(\sigma_{z-i}(B)\tau_z(A)) \text{ for } A \in \mathcal{R}_1 \text{ and } B \in \mathcal{R}_2 \tag{4.41}$$

$$\Rightarrow \quad g_{AB}(t + i) = \omega(\sigma_t(B)\tau_{t+i}(A)) \overset{\text{KMS cond. for } \tau}{=} \omega(\tau_t(A)\sigma_t(B)) \tag{4.42}$$

$$\Rightarrow \quad g_{AB}(t) = \omega(\sigma_{t-i}(B)\tau_t(A)) \overset{\text{KMS cond. for } \sigma}{=} \omega(\tau_t(A)\sigma_t(B)). \tag{4.43}$$

It follows that $g_{AB}(z + i) = g_{AB}(z)$ for all $z \in \mathbb{R}$, and thereby also for all $z \in \mathbb{C}$. By the same argument used in proposition 3.4.2 one obtains $g_{AB} \equiv \text{const}$. In particular, $g_{AB}(t) = g_{AB}(0)$, i.e.

$$\omega(\tau_t(A)\sigma_t(B)) = \omega(AB) \tag{4.44}$$

$$\Rightarrow \quad \langle A^*\Omega_\omega, U(t)^*\Delta^{-it} B\Omega_\omega\rangle = \langle A^*\Omega_\omega, B\Omega_\omega\rangle \text{ for } A \in \mathcal{R}_1 \text{ and } B \in \mathcal{R}_2. \tag{4.45}$$

Sine $\mathcal{R}_1$ and $\mathcal{R}_2$ are dense in $\mathcal{R}$ and Ω_ω is cyclic for $\mathcal{R}$, we conclude $U(t) = \Delta^{-it}$. □

This theorem is very useful to check if a given automorphism group coincides with the modular flow. Summing up, the last two theorems can be combined into the crucial

Corollary 4.3.3. *A faithful normal state ω on a von Neumann algebra $\mathcal{R}$ is a β-KMS state w.r.t. the *-automorphism group τ_t iff the modular flow σ_s^ω relative to ω is related to this group via the formula $\tau_t = \sigma_{t/\beta}^\omega$.*

This results in the striking conclusion that a (faithful) KMS state at inverse temperature β may be *characterised* as a faithful normal state over the observable algebra $\mathcal{R}$ whose modular group σ_s^ω coincides with the time translation group τ_t up to rescaling and whose modular parameter s is related to the time t by $s = t/\beta$. In particular, any faithful normal state ω on a von Neumann algebra $\mathcal{R}$ is a KMS state for a unique automorphism group of $\mathcal{R}$. Moreover, as theorem 3.3.8 shows, the modular flow for a given state can satisfy the KMS condition only in states differing from the original one by the action of an element affiliated with the centre of $\mathcal{R}$. Consequently, two different faithful normal states cannot share the same modular automorphism group, granted that $\mathcal{R}$ is a factor.

4.4 Cocycle Radon-Nikodým Theorem

Any faithful normal state ω acting on a von Neumann algebra $\mathcal{R}$ determines a σ-weakly continuous one-parameter group of *-automorphisms σ_s^ω on $\mathcal{R}$. In this section we investigate to what extent the modular flow really depends on the reference state and, on the other hand, to what extent it is fixed by the algebra. The results presented here motivated the formulation of the thermal time hypothesis, and, in addition, permit an alternative formulation. The details concerning the proofs may be found in the pertinent literature [40, 4, 60]. In particular the role of this topic in the setting of Connes' classification of factors is described therein, which is also touched in appendix A.5.

Definition 4.4.1 ([60])**.** An automorphism of an algebra $\mathcal{A}$ is called *inner automorphism*, τ_{inner}, if there exists a unitary element $U \in \mathcal{A}$, such that

$$\tau_{inner} A = U^* A U \quad \text{for all} \quad A \in \mathcal{A}. \tag{4.46}$$

Two automorphisms τ_1 and τ_2 of the algebra $\mathcal{A}$ are said to be *inner equivalent*, if there exists an inner automorphism τ_{inner}, such that

$$\tau_2 = \tau_{inner} \circ \tau_1. \tag{4.47}$$

Hereby an equivalence relation is defined on $Aut(\mathcal{A})$. The group $Out(\mathcal{A}) = Aut(\mathcal{A})/\sim$ is called the *group of outer automorphisms of* $\mathcal{A}$.

Definition 4.4.2 ([10])**.** A *unitary one-cocycle* U of a W*-dynamical system $(\mathcal{R}, \tau)$ is a strongly continuous map $t \mapsto U_t$ from $\mathbb{R}$ into the unitaries of $\mathcal{R}$, such that

$$U_{t_1+t_2} = U_{t_1} \tau_{t_1}(U_{t_2}) \quad \text{for all} \quad t_1, t_2 \in \mathbb{R}. \tag{4.48}$$

Therefore, a unitary one-cocycle U generates a new W*-dynamical system $(\mathcal{R}, \tau^U)$, where

$$\tau_t^U(A) = U_t \tau_t(A) U_t^*, \quad t \in \mathbb{R}, \quad A \in \mathcal{R}. \tag{4.49}$$

Let ω and ω' be two faithful normal states on a von Neumann algebra $\mathcal{R}$, and let $(\pi, \mathcal{H}, \Omega)$ and $(\hat{\pi}, \hat{\mathcal{H}}, \hat{\Omega})$ be the corresponding GNS representations. The same argument used on page 42 shows that ω' is represented by a state vector $\Omega' \equiv U\hat{\Omega}$ on $\mathcal{H}$ which is cyclic and separating for $\pi(\mathcal{R})$. One proceeds by defining a conjugate-linear operator $S_{\Omega',\Omega}$ via

$$S_{\Omega',\Omega} \pi(A) |\Omega\rangle := \pi(A)^* |\Omega'\rangle \quad \text{for all} \quad A \in \mathcal{R}. \tag{4.50}$$

$S_{\Omega',\Omega}$ turns out to be closable, its closure, denoted by the same symbol, has a polar decomposition

$$S_{\Omega',\Omega} = J_{\Omega',\Omega} (\Delta_{\Omega',\Omega})^{1/2}. \tag{4.51}$$

The positive, invertible, self-adjoint operator $\Delta_{\Omega',\Omega}$ is called the *relative modular operator of the pair* $(\Omega,\ \Omega')$. Let us consider a triple of such vectors Ω, Ω' and Ω'' and put

$$u_{\Omega',\Omega}(s) \equiv \Delta_{\Omega',\Omega''}^{-is} \Delta_{\Omega,\Omega''}^{is}. \tag{4.52}$$

Lemma 4.4.3 ([60]). *$u_{\Omega',\Omega}(s)$ is a unitary one-cocycle of $(\pi(\mathcal{R}), \mathrm{Ad}(\Delta_\Omega^{-is}))$ which does not depend on Ω''.*

The unitary one-cocycle $u_{\Omega',\Omega}(t)$ of $(\pi(\mathcal{R}), \mathrm{Ad}(\Delta_\Omega^{-is}))$ gives rise to a unitary one-cocycle of $(\mathcal{R}, \sigma_s^\omega)$, $(D\omega' : D\omega)_s \equiv \pi^{-1}(u_{\Omega',\Omega})(s)$, which is called *Radon-Nikodým cocycle of ω' w.r.t. ω*.

Theorem 4.4.4 ([60]). *For two arbitrary faithful normal states ω_1 and ω_2 on a von Neumann algebra $\mathcal{R}$ the Radon-Nikodým cocycle $s \mapsto (D\omega_2 : D\omega_1)_s$ has the following properties:*

(i) Cocycle identity: $(D\omega_1 : D\omega_2)_{s_1+s_2} = (D\omega_1 : D\omega_2)_{s_1} \sigma_{s_1}^{\omega_2}((D\omega_1 : D\omega_2)_{s_2})$

(ii) Intertwining property: $(D\omega_1 : D\omega_2)_s(\sigma_s^{\omega_2} A) = (\sigma_s^{\omega_1} A)(D\omega_1 : D\omega_2)_s$

(iii) Chain rule: $(D\omega_1 : D\omega_2)_s(D\omega_2 : D\omega_3)_s = (D\omega_1 : D\omega_3)_s$

(iv) $(D\omega_2 : D\omega_1)_s^* = (D\omega_1 : D\omega_2)_s$

Combining (ii) and (iv) one immediately gets the intriguing *cocycle Radon-Nikodým theorem*, which was proven by Connes [40], and which answers the question about the state dependence of the modular flow.

Theorem 4.4.5 (Cocycle Radon-Nikodým Theorem). *Let ω_1 and ω_2 be two faithful normal states on a von Neumann algebra $\mathcal{R}$. Then,*

$$\sigma_s^{\omega_2} A = (D\omega_2 : D\omega_1)_s (\sigma_s^{\omega_1} A)(D\omega_2 : D\omega_1)_s^*, \tag{4.53}$$

the modular flows defined by two faithful normal states on $\mathcal{R}$ are inner equivalent.[6]

The theorem originates from the Radon-Nikodým theorem, which is a well-known theorem in measure theory. Since the theory of operator algebras replaces ordinary measure theory on non-commutative spaces, e.g. the classical notion of probability measure is substituted by the notion of a state, it is regarded as a non-commutative extension of this theorem. Eventually, by dividing out the inner automorphisms, we end up with the crucial

Corollary 4.4.6. *All faithful normal states ω on a von Neumann algebra $\mathcal{R}$ determine the same one-parameter subgroup $\tilde{\sigma}_s$ in $Out(\mathcal{R})$, that is every (σ-finite) von Neumann algebra is endowed with a canonical one-parameter group of outer automorphisms.*[7]

Up to now it is possible that the canonical flow $\tilde{\sigma}_s$ is altogether trivial. However, Takesaki [124] has shown that this does not happen if the composition of the algebra contains a type III factor.

Theorem 4.4.7 ([72]). *Let ω be a faithful normal state on a von Neumann algebra $\mathcal{R}$. Then, $\mathcal{R}$ is semi-finite iff there is a positive, invertible, self-adjoint operator T, affiliated with $\mathcal{R}$, such that*

$$\sigma_s^\omega(A) = T^{is} A T^{-is} \text{ for all } A \in \mathcal{R} \text{ and } s \in \mathbb{R}. \tag{4.54}$$

In particular, the modular automorphism group σ_s^ω is inner for all s. Supposed that, in addition, $\mathcal{R}$ has a separable predual, $\mathcal{R}$ is semi-finite iff σ_s^ω is inner for all s.

Corollary 4.4.8 ([124]). *Let $\mathcal{R}$ be a type III factor with separable predual. Then, one finds a faithful normal state on $\mathcal{R}$. Since $\mathcal{R}$ is not semi-finite, there is an $s \in \mathbb{R}$, such that σ_s^ω is an outer automorphism.*

If $\mathcal{R}$ is not semi-finite, the modular automorphism group σ_s^ω is not contained in the group of inner automorphisms. Its projection into the group of outer automorphisms, $\tilde{\sigma}_s$, is a non-trivial one-parameter subgroup. If $\mathcal{R}$ is semi-finite, the canonical flow is trivial. Studying the canonical flow historically led to the subclassification of type III algebras, about which a survey is given in appendix A.5.

By way of summary, the combination of the Tomita-Takesaki theorem and the cocycle Radon-Nikodým theorem shows that the non-commutativity of a von Neumann algebra manifests itself in a canonical, unique, intrinsic flow which converts the algebra into a "dynamical" object. Since von Neumann algebras emerging in QFT are of type III_1, this is undoubtedly a result of interest.

[6] As in the case of the Tomita-Takesaki theorem, the cocycle Radon-Nikodým theorem can be formulated in the more general case of weights, see for instance [40, 41, 125]. Moreover, there exists a certain converse to it. To any unitary one-cocycle U_t of $(\mathcal{R}, \sigma^{\omega_1})$ one finds a unique faithful semi-finite normal weight ω_2 on $\mathcal{R}$ such that $U_t = (D\omega_1 : D\omega_2)_t$ [41]. Once again, for our purposes the stated version is sufficient.

[7] Since each von Neumann algebra admits a faithful normal semi-finite weight, it can be shown in the more general setting of weights that *every* von Neumann algebra possesses such a canonical (possibly trivial) flow [40, 41, 125].

4.5 Geometrical Meaning of the Modular Flow and Bisognano-Wichmann Theorem

Let us pass to physics and consider a local net of von Neumann algebras together with a faithful normal (partial) state $\omega \restriction \mathcal{R}(\mathcal{O})$. Up to now, it is not known whether the modular flow generally permits a physical interpretation.[8] While the modular flow $\sigma_s^{\omega\restriction\mathcal{R}(\mathcal{O})}$ always maps $\mathcal{R}(\mathcal{O})$ onto itself, it is a fruitful question if $\sigma_s^{\omega\restriction\mathcal{R}(\mathcal{O})}$ applied to a local subalgebra $\mathcal{R}(\mathcal{O}_1)$, $\mathcal{O}_1 \subset \mathcal{O}$, leads to another local subalgebra. In that case the modular flow would permit a simple interpretation in terms of a geometrical transformation in the underlying space. Indeed, there do exist cases in which the modular flow has a purely geometrical action, whereby it obtains a physical meaning. In the current section we intend to present the most prominent examples of this kind. In this context we refer to the comprehensive overviews by Borchers [22] and Guido [56].

Definition of Geometrical Action

For the time being we focus our attention on the vacuum sector of an AQFT in (3+1-dim.) Minkowski space M. The vacuum representation is given by a local net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O}) = \pi_0(\mathfrak{A}(\mathcal{O}))''$ and the total algebra $\mathcal{R} = \pi_0(\mathfrak{A})''$ on M. The vacuum state ω_0 is represented by the state vector Ω_0 in the relevant Hilbert space, which is left invariant under the action of the Poincaré group. At the end of this section we shall say a few words concerning a geometrical action of the modular group on more general classes of spacetimes. The essential features can be found in the examples we are going to discuss in Minkowski space, anyway.

We are interested in the modular flow induced by the restriction of the vacuum state to certain subregions $\mathcal{O}$ of Minkowski space. The applicability of the Tomita-Takesaki theorem, and with it the existence of the modular objects, is established by the Reeh-Schlieder property as soon as $\mathcal{O}$ is non-empty and has non-empty causal complement.[9] First of all we make the term "geometrical action" precise. For this, we fall back upon the definition given by Trebels [129], the modular flow acts geometrically if it acts geometrically on double-cones (also called diamonds):

Definition 4.5.1 ([129])**.** Let $\mathcal{O}$ be a causally complete, connected, open set in M. A modular automorphism group $\sigma_s^{\omega\restriction\mathcal{R}(\mathcal{O})}$ induced on $\mathcal{R}(\mathcal{O})$ acts *geometrically* if there exists a one-parameter group of automorphisms $\Lambda_s^{\mathcal{O}}$ acting on $\mathcal{O}$ such that

$$\sigma_s^{\omega\restriction\mathcal{R}(\mathcal{O})}(\mathcal{R}(D)) = \mathcal{R}(\Lambda_s^{\mathcal{O}}(D)) \text{ for all double-cones } D \in \mathcal{O}. \tag{4.55}$$

The modular conjugation acts *geometrically* if $j^{\omega\restriction\mathcal{R}(\mathcal{O})}(\mathcal{R}(\mathcal{O})) = \mathcal{R}(\hat{\mathcal{O}})$ and if there exists an isomorphism $g : \mathcal{O} \to \hat{\mathcal{O}}$, such that for every double-cone $D \subset \mathcal{O}$ one has

$$j^{\omega\restriction\mathcal{R}(\mathcal{O})}(\mathcal{R}(D)) = \mathcal{R}(g(D)). \tag{4.56}$$

Bisognano-Wichmann Theorem and Related Results

Virtually all known results where the modular group acts geometrically find there origin in the pathbreaking work of Bisognano and Wichmann [13, 14] from 1975/76, where $\mathcal{O}$ corresponds to the Rindler wedge W. Historically, the Bisognano-Wichmann theorem was the first explicit determination of a modular group in physics.

Theorem 4.5.2 (Bisognano-Wichmann [13, 14, 60])**.** *Let $W = \{x^1 > |x^0|\} \subset M$ be the (right) Rindler wedge. Moreover, consider the vacuum state ω_0 restricted to the wedge algebra $\mathcal{R}(W)$, and assume that the net of observable algebras $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ is locally associated with a finite component quantum field theory satisfying the Wightman axioms. Then, the modular conjugation is given by*

$$J_{\omega_0\restriction\mathcal{R}(W)} = U(R_1(\pi))\Theta. \tag{4.57}$$

[8] In fact, it is the topic of this thesis to take a look at that issue.

[9] Note that the vacuum state ω_0 does not define a modular flow on the algebra $\mathcal{R}$ associated to the entire Minkowski space, since Ω_0 is not separating in that case.

Θ is supposed to be the anti-unitary PCT-operator.[10] *$U(R_1(\vartheta))$ denotes the unitary operator implementing the spatial rotation through an angle of ϑ around the x^1-axis,*

$$R_1(\pi)(x^0, x^1, x^2, x^3) = (-x^0, -x^1, x^2, x^3). \tag{4.58}$$

The modular operator satisfies the equation

$$\Delta_{\omega_0 \restriction \mathcal{R}(W)} = e^{-2\pi L}, \tag{4.59}$$

where L is the generator of the Lorentz boosts in x^1-direction,

$$\Lambda(\lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.60}$$

in the unitary representation of the Poincaré group $\mathcal{P}_+^\uparrow$, defined on the Hilbert space of the theory, that is $U(\Lambda(\lambda)) = e^{i\lambda L}$, where λ is the boost parameter.

According to the Bisognano-Wichmann theorem the modular objects assigned to the Rindler wedge possess a geometrical interpretation under the condition that the net is generated by a finite component Wightman field theory. The geometrical meaning Λ_s^W of the modular flow corresponds to the Lorentz boosts $\Lambda(2\pi s)$ in the wedge region (see figure 6.1 on page 77), induced by the modular Hamiltonian $K = 2\pi L$. For a scalar field this means

$$\sigma_s^{\omega_0 \restriction \mathcal{R}(W)}(\varphi[f]) = \varphi[f_s], \quad f_s(x) = f(\Lambda^{-1}(2\pi s)(x)), \tag{4.61}$$

for suitable test functions f supported in W, or expressed in a purely algebraic manner

$$\sigma_s^{\omega_0 \restriction \mathcal{R}(W)}(\mathcal{R}(\mathcal{O})) = \mathcal{R}(\Lambda_s^W \mathcal{O}) = \mathcal{R}(\Lambda(2\pi s)\mathcal{O}) \quad \text{for all } s \in \mathbb{R} \text{ and } \mathcal{O} \subset W. \tag{4.62}$$

The modular conjugation implements a combination of charge conjugation, time and spatial reflection. It has a geometrical meaning which reads

$$j^{\omega_0 \restriction \mathcal{R}(W)}(\mathcal{R}(\mathcal{O})) = \mathcal{R}(\theta_R \mathcal{O}) \quad \text{for } \mathcal{O} \subset W, \tag{4.63}$$

where $\theta_R \in \mathcal{P}_+$ is the reflection through the edge $\{(0, 0, x^2, x^3) : x^2, x^3 \in \mathbb{R}\}$ of wedge:

$$\theta_R(x^\mu) = \begin{cases} -x^\mu & \text{for } \mu = 0, 1 \\ x^\mu & \text{for } \mu = 2, 3 \end{cases} \tag{4.64}$$

In particular *wedge duality* holds, $\mathcal{R}(W)' = j^{\omega_0 \restriction \mathcal{R}(W)}(\mathcal{R}(W)) = \mathcal{R}(W')$. Similar results can be verfied for a general wedge region gW, $g \in \mathcal{P}_+^\uparrow$. Since the vacuum is Poincaré invariant, the Tomita operator $S(gW)$ for the pair $(\mathcal{R}(gW), \Omega_0)$ coincides with $U(g)S(W)U(g)^*$, when $S(W)$ is the Tomita operator associated with $(\mathcal{R}(W), \Omega_0)$. Accordingly, the modular objects corresponding to the wedge gW have a geometrical action obtained by adjoining that of the Rindler wedge W with g. For the modular flow that means $\Lambda_s^{gW} = g\Lambda_s^W g^{-1}$.

Definition 4.5.3. One says that *modular covariance* holds, whenever the modular operators induced by the vacuum state on wedge algebras have the same action as in the Bisognano-Wichmann case, i.e. if the modular group coincides with the unitary group representing the wedge-preserving boosts. For e.g. the right Rindler wedge the relation $\Delta_{\omega_0 \restriction \mathcal{R}(W)}^{-is} = U(\Lambda(2\pi s))$ has to be satisfied.

[10] The PCT-operator is an anti-unitary involution which physically represents a composition of space reflection, charge conjugation and time reversal. In the Wightman framework the existence of Θ was proven by Jost ("PCT-theorem", cf. [60, 120]). Θ is uniquely determined by the relations

$$\Theta\varphi(x)\Theta^{-1} = (-1)^m(-i)^F\varphi^*(-x) \quad \text{and} \quad \Theta\,|\Omega\rangle = |\Omega\rangle\,,$$

where m is the number of dotted spinor indices, and F is 0 or 1 if φ is a Bose or Fermi field, respectively. Θ satisfies the relation $\Theta U(\Lambda, a) = U(\Lambda, -a)\Theta$, U being the representation of the Poincaré group and $(\Lambda, a) \in \mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{R}^4$.

For the quite technical proof of the Bisognano-Wichmann theorem one particularly takes advantage of the geometrical simplicity of the wedge regions. For that reason, these special regions seem to be the only ones in Minkowski space permitting a geometrical interpretation under such general conditions. On the other hand, putting additional assumptions on the theory, wider classes of regions can be found for which the modular action is characterised by a simple geometrical meaning.

In a massless free theory every influence travels along the boundary of a lightcone. Thus, besides spacelike commutativity, timelike commutativity holds, as well. This implies that the vacuum vector Ω_0 is cyclic and separating for the algebra $\mathcal{R}(V_+)$ of the future-cone. Proceeding similarly to the Bisognano-Wichmann case and taking account of the dilatation invariance of massless free fields, the modular operators were determined by Buchholz [28].

Theorem 4.5.4 (Buchholz [28]). *Consider the vacuum sector of a massless free scalar field theory. Let $\mathcal{R}(V_+)$ be the von Neumann algebra of the forward lightcone. Then, the pair $(\mathcal{R}(V_+), \Omega_0)$ induces on V_+ the modular objects*

$$J_{\omega_0 \restriction \mathcal{R}(V_+)} = W\Theta, \tag{4.65}$$

$$\Delta^{-is}_{\omega_0 \restriction \mathcal{R}(V_+)} = U(D(2\pi s)). \tag{4.66}$$

Herein Θ is again the anti-unitary PCT-operator, W is supposed to be the unitary operator reducing to $+1$ or -1 on subspaces with an even or odd number of particles, respectively, and U denotes the unitary implementation of the global dilatation group $D(\lambda)$ on the Hilbert space of the theory,

$$D(\lambda)\varphi(x)D(\lambda)^{-1} = e^{\lambda}\varphi(e^{\lambda}x). \tag{4.67}$$

We conclude that the geometrical action is given by the dilatations:

$$\sigma_s^{\omega_0 \restriction \mathcal{R}(V_+)}\mathcal{R}(\mathcal{O}) = \mathcal{R}(D(2\pi s)\mathcal{O}) \quad \text{for all } s \in \mathbb{R} \text{ and } \mathcal{O} \subset V_+, \tag{4.68}$$

$$\Lambda_s^{V_+}x^\mu = D(2\pi s)x^\mu = e^{2\pi s}x^\mu. \tag{4.69}$$

The modular conjugation corresponds to the geometrical significance

$$j^{\omega_0 \restriction \mathcal{R}(V_+)}(\mathcal{R}(\mathcal{O})) = \mathcal{R}(-\mathcal{O}) \quad \text{for } \mathcal{O} \subset V_+. \tag{4.70}$$

This proves *timelike duality*, $\mathcal{R}(V_+)' = j^{\omega_0 \restriction \mathcal{R}(V_+)}(\mathcal{R}(V_+)) = \mathcal{R}(V_-)$. Of course, analogous results hold for the backward lightcone V_- and Poincaré transformed future-cones.

As a result of Swieca and Völkel [123] massless free theories are conformally covariant. In their paper the existence of a unitary representation of the conformal group, or rather of the universal covering group of the conformal group, is proven. Starting from these investigations Hislop and Longo [65] could deduce another example in which the modular group is explicitly known by establishing conformal relations between double-cone, forward lightcone, and wedge.[11]

Theorem 4.5.5 (Hislop-Longo [65]). *Consider again the vacuum sector of a massless free scalar field theory. Let $\mathcal{R}(D_1)$ be the von Neumann algebra associated with a double-cone of radius equal to 1 centred around the origin, $D_1 = \{x : |x^0| + \|\vec{x}\| < 1\}$. Future-cone V_+ and double-cone D_1 are related by a conformal transformation,*

$$V_+ = \rho(D_1 - e_0) - \frac{1}{2}e_0, \quad D_1 = \rho(V_+ + \frac{1}{2}e_0) + e_0, \tag{4.71}$$

where ρ is the relativistic ray inversion map

$$\rho(x^\mu) \equiv (x^\nu x_\nu)^{-1}x^\mu. \tag{4.72}$$

The unitary operator $T(1)U(\rho)T(\frac{1}{2})$ implements the equivalence of $\mathcal{R}(D_1)$ and $\mathcal{R}(V_+)$, and the modular objects are given by

$$J_{\omega_0 \restriction \mathcal{R}(D_1)} = T(1)U(\rho)T(\frac{1}{2})J_{\omega_0 \restriction \mathcal{R}(V_+)}T(-\frac{1}{2})U(\rho)T(-1) \tag{4.73}$$

$$= T(1)U(\rho)T(\frac{1}{2})W\Theta T(-\frac{1}{2})U(\rho)T(-1) = I_t U(\rho), \tag{4.74}$$

[11] The von Neumann algebras $\mathcal{R}(W)$, $\mathcal{R}(V_\pm)$ and $\mathcal{R}(D)$ of wedge, forward/backward lightcone, and double-cone are spatially isomorphic. The unitary equivalences are realised by unitary representations of suitable conformal transformations (see [65]).

$$\Delta_{\omega_0\restriction\mathcal{R}(D_1)} = T(1)U(\rho)T(\frac{1}{2})\Delta_{\omega_0\restriction\mathcal{R}(V_+)}T(-\frac{1}{2})U(\rho)T(-1), \tag{4.75}$$

$$\Delta^{-is}_{\omega_0\restriction\mathcal{R}(D_1)} = T(1)U(\rho)T(\frac{1}{2})U(D(2\pi s))T(-\frac{1}{2})U(\rho)T(-1). \tag{4.76}$$

U denotes the unitary implementation of the respective transformation on the Hilbert space of the theory. $T(\lambda)$ is supposed to be the one-parameter unitary group representing the time-translations, I_t is the time reversal operator, and, as usual, Θ is the anti-unitary PCT-operator.

After carrying out a few straightforward manipulations one eventually ends up with a geometrical significance of the following manner:

$$\Lambda_s^{D_1}x^\mu = \frac{2x^\mu + \delta_0^\mu(2x^0\cosh(2\pi s) + (1 - x^\mu x_\mu)\sinh(2\pi s) - 2x^0)}{2x^0\sinh(2\pi s) + (1 - x^\mu x_\mu)\cosh(2\pi s) + (1 + x^\mu x_\mu)} \tag{4.77}$$

for the modular group (see figure 6.2 on page 79), as well as

$$x^\mu \mapsto \begin{cases} -(x^\nu x_\nu)^{-1}x^\mu & \text{for } \mu = 0, \\ (x^\nu x_\nu)^{-1}x^\mu & \text{for } \mu = 1, 2, 3. \end{cases} \tag{4.78}$$

for the modular conjugation. By applying suitable Poincaré transformations, the Hislop-Longo theorem is easily extended to arbitrary shaped double-cones. Moreover, the result of Hislop and Longo implies *spacelike duality* for the local algebras associated to double-cones, $\mathcal{R}(D)' = \mathcal{R}(D')$.

The Bisognano-Wichmann theorem, as well as its two offshoots, theorem 4.5.4 and 4.5.5, have a natural formulation in the algebraic framework. The modular objects, at least for wedge and double-cone, do exist because of the Reeh-Schlieder theorem. Paradoxically, these theorems have been proven originally only in the Wightman setting. However, meantime there do exist some results in the algebraic formalism, as well.[12] The first of such was obtained by Brunetti, Guido and Longo [27] for the conformally covariant case. To treat the presence of singularities of the conformal transformations on certain submanifolds, it is necessary to extend the theory to a universal covering of a suitable compactification of M. The work of these authors is based on a result of Borchers [19], who could give conditions when modular objects satisfy those commutation relation with the translation operators which are expected from modular covariance.

Theorem 4.5.6 (Borchers [19])**.** *Let $\mathcal{R}$ be a von Neumann algebra with a cyclic and separating vector Ω, and let $U(t)$ be a strongly continuous unitary group which satisfies $U(t)\mathcal{R}U(t)^* \subset \mathcal{R}$ for all $t \geq 0$. Then any two of the following conditions imply the third:*

(i) $U(t) = e^{itH}$ *with* $H \geq 0$.

(ii) $U(t)\Omega = \Omega$ *for all* $t \in \mathbb{R}$.

(iii) $\Delta^{-is}U(t)\Delta^{is} = U(e^{2\pi s}t)$ *and* $JU(t)J = U(-t)$ *for all* $s, t \in \mathbb{R}$.

The theorem shows that in the vacuum sector of a theory, satisfying translation covariance and the spectrum condition, the modular objects commute with the translation operators in the same way as Lorentz boosts and the PCT operator would commute with. While in 1+1 dimensions this is sufficient to conclude modular covariance, in higher dimensions additional assumptions are needed.

Theorem 4.5.7 (Brunetti-Guido-Longo [27])**.** *Consider the vacuum sector of a conformally covariant theory, given by a local net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$. Then, the modular groups as well as the modular conjugations generated by the vacuum state Ω_0 restricted to wedge, future-cone and double-cone have a geometrical meaning which coincides with the corresponding modular actions considered above.*

Another case where the Bisognano-Wichmann theorem holds is due to Mund [89]. The author establishes modular covariance for massive models. More precisely, the theory is supposed to contain only massive particles, it has to be *asymptotically complete*, within each charge sector all particle masses must be isolated eigenvalues of the mass operator, and for each mass, charge and

[12] We do not explain all the terms appearing in this paragraph, because our intention is above all to emphasize the universality of the Bisognano-Wichmann theorem.

spin there has to be one particle multiplet under the gauge group. We want to mention two other cases. KUCKERT [77] has shown that whenever the modular flow acts as a symmetry, which means that it maps for all s double-cone algebras onto algebras associated with some open region, then modular covariance necessarily holds. An analogous result is true for the modular conjugation. It is due to BORCHERS (cf. [22]) that wedge duality together with a *reality-condition*, implies modular covariance.

Although the Bisognano-Wichmann theorem does not follow from the most basic principles of AQFT as the counter-example of YNGVASON illustrates [136], all these results underline that it remains valid under very general conditions. In fact, the Bisognano-Wichmann theorem is regarded to be generic in the algebraic framework, and in the following we shall assume that modular covariance holds.

Approximate Geometrical Meaning of the Modular Group

The geometrical meaning of the modular group for double-cone algebras holds merely for the conformally covariant case. The interacting or massive case is more involved (see below). Nevertheless, it is possible to derive an approximate geometrical meaning in specific subregions. To this end, let us consider again the von Neumann algebra $\mathcal{R}(W)$ associated with the Rindler wedge. Let $\mathcal{R}(D) \subset \mathcal{R}(W)$ be the algebra of a diamond D which contains the origin in its closure. Due to the observation that in the massless free case the geometrical actions of wedge and double-cone approximately coincide sufficiently close to the origin as well as the expectation that a general theory exhibits conformal covariance in the short distance limit, the asymptotic coincidence should persist in the general case. Even more, if one knows the action of the modular group of a certain spacetime region, one can show that the modular group of a suitable subregion admits approximately the same action in a specific sufficiently small part of this subregion. This result is due to FREDENHAGEN [50]. A similar result for the modular conjugation goes back to WOLLENBERG [10].

Theorem 4.5.8 (Fredenhagen-Wollenberg [50, 10])**.** *Let $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ be a local net. Let $\mathcal{O} \subset \mathcal{U}$ be two open regions, and assume that the state ω corresponds to a vector Ω which is cyclic and separating for both $\mathcal{R}(\mathcal{O})$ and $\mathcal{R}(\mathcal{U})$. Moreover, suppose that there exists a sequence of sets $\mathcal{O}_n \subset \mathcal{O}$ with $\mathcal{O}_{n+1} \subset \mathcal{O}_n$, such that Ω is cyclic for all $\mathcal{R}(\mathcal{O}_n)$, and a sequence of positive numbers τ_n with $\tau_n \to \infty$ for $n \to \infty$ such that the relation $\Delta^{-is}_{\omega\restriction\mathcal{R}(\mathcal{U})}\mathcal{R}(\mathcal{O}_n)\Delta^{is}_{\omega\restriction\mathcal{R}(\mathcal{U})} \subset \mathcal{R}(\mathcal{O})$ holds for all $|s| \leq \tau_n$.*

(i) For each $f \in L^1(\mathbb{R})$ and $n \in \mathbb{N}$ there exists a constant $a_f(n) > 0$ with $\lim_{n\to\infty} a_f(n) = 0$ such that

$$\| \int \mathrm{d}s\, f(s) \left(\Delta^{-is}_{\omega\restriction\mathcal{R}(\mathcal{O})} - \Delta^{-is}_{\omega\restriction\mathcal{R}(\mathcal{U})} \right) A\Omega_\omega \|^2 \leq a_f(n) \left(\|A\Omega_\omega\|^2 + \|A^*\Omega_\omega\|^2 \right) \quad \text{for all } A \in \mathcal{R}(\mathcal{O}_n).$$

(ii) For each continuous function g on $\mathbb{R}_+$ vanishing at 0 and ∞, and for each $n \in \mathbb{N}$, there exists a constant $b_g(n)$ with $\lim_{n\to\infty} b_g(n) = 0$ such that

$$\| \left(J_{\omega\restriction\mathcal{R}(\mathcal{O})} - J_{\omega\restriction\mathcal{R}(\mathcal{U})} \right) g(\Delta_{\omega\restriction\mathcal{R}(\mathcal{U})}) A\Omega_\omega \|^2 \leq b_g(n) \left(\|A\Omega_\omega\|^2 + \|A^*\Omega_\omega\|^2 \right) \quad \text{for all } A \in \mathcal{R}(\mathcal{O}_n).$$

Note that $a_f(n)$ and $b_g(n)$ neither depend on the details of the theory nor on the size of $\mathcal{O}$. The assumptions made in the theorem are satisfied if one chooses for $\mathcal{U}$ the Rindler wedge W, for $\mathcal{O}$ a diamond D in the wedge whose closure contains the origin and $\omega = \omega_0$; for $\mathcal{O}_n$ one may take the diamonds $\frac{1}{n}D$. Then, the theorem shows that the action of $\sigma_s^{\omega_0\restriction\mathcal{R}(D)}$ on the elements of $\mathcal{R}(\frac{1}{n}D)$ becomes almost geometrical for large n and approaches the geometrical action of $\sigma_s^{\omega_0\restriction\mathcal{R}(W)}$.

Possible Geometrical Actions for a Given Spacetime Region

We consider the issue, which was studied by TREBELS [129], to what extent the possible geometrical actions induced by the vacuum state for various AQFTs are fixed. For this purpose a somewhat more restrictive notion of geometrical action is useful, namely a *causally* geometrical action.

Definition 4.5.9 ([22, 129])**.** The *causal group*, causal($\mathcal{O}$), of a region $\mathcal{O} \subset M$ is defined to be the group of all automorphisms g of $\mathcal{O}$ satisfying the following condition:

(i) For all spacelike separated points $x, y \in \mathcal{O}$ both $g(x)-g(y)$ and $g^{-1}(x)-g^{-1}(y)$ are spacelike;

(ii) $x - y \in V_+$ implies $g(x) - g(y) \in V_+$ and $g^{-1}(x) - g^{-1}(y) \in V_+$.

An element $g \in \text{causal}(\mathcal{O})$ maps double-cones onto double-cones with the same light orientation. Since double-cones form a topological base of Minkowski space, $g : \mathcal{O} \to \mathcal{O}$ has to be continuous.

Theorem 4.5.10 ([129]). *The causal group of Minkowski space M, is the group generated by the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ together with the dilatations. The same is true for the causal group of the forward lightcone V_+.*

Corollary 4.5.11 ([129]). *Let $\mathcal{O} \subset M$ be a connected region which is related to the forward lightcone by a conformal orthochronous transformation k, then*

$$\text{causal}(\mathcal{O}) = k \circ \text{causal}(V_+) \circ k^{-1}. \tag{4.79}$$

Definition 4.5.12 ([129]). The modular automorphism group acts *causally geometrically* on a causally complete, connected, open set $\mathcal{O} \subset M$ if the automorphism in definition 4.5.1 is causal.

Theorem 4.5.13 ([129]). *Let k be a conformal orthochronous transformation which maps V_+ onto a connected set $\mathcal{O}$. Moreover, let $\Lambda_s^{\mathcal{O}}$ be a non-trivial one-parameter subgroup of* causal($\mathcal{O}$). *Suppose that $\frac{\mathrm{d}}{\mathrm{d}x}\Lambda_s^{\mathcal{O}}(x) \in \overline{V}_\pm$, for all $x \in \mathcal{O}$. Then, there exists a $\lambda > 0$ such that $\Lambda_s^{\mathcal{O}} = k \circ e^{\pm\lambda s} \circ k^{-1}$.*

Theorem 4.5.14 ([129]). *Let $\mathcal{O} \subset M$ be a region on which the action of the modular group is causally geometrical. Then, for all $x \in \mathcal{O}$ there exists an interval $[0, s_0)$, such that*

$$\Lambda_s^{\mathcal{O}} x \in x + \overline{V}_+ \quad \textit{for all} \quad s \in [0, s_0). \tag{4.80}$$

$\Lambda_s^{\mathcal{O}}$ acts in the direction of the forward lightcone. As consequence of theorem 4.5.13 and 4.5.14 one obtains that there is essentially only one possible geometrical meaning for the modular group.

Corollary 4.5.15 ([129]). *Let $\mathcal{O}$ be a connected set which is the image of the future-cone V_+ under a conformal orthochronous transformation k. Assume further that the modular group acts causally geometrically on $\mathcal{O}$. Then, there exists a $\lambda > 0$ such that $\Lambda_s^{\mathcal{O}} = k \circ e^{\lambda s} \circ k^{-1}$.*

If the modular group has a causally geometrical meaning, the orbits of each point are fixed by analyticity properties of the modular operators. Up to some positive rescaling of the parameter s it has to be the one known from the conformally covariant theory. For wedge, future-cone, and cuts of lightcones with wedge regions one can show [129] that the scaling parameter necessarily has to coincide with the one in conformal theories. So there is precisely one possible geometrical action. For other regions, like double-cones, no proof is available.

The Bisognano-Wichmann Theorem in Curved Spacetimes

At the end of this section we briefly want to mention some generalisations of the Bisognano-Wichmann theorem to certain classes of curved spacetimes. Quantum field theories in curved background spacetimes are usually formulated by adapting the Wightman axioms in a suitable manner, where the main problem is to distinguish a vacuum state and to find a substitute for the spectrum condition, which is not available in this general context. All generalisations of the Bisognano-Wichmann theorem are based on the peculiarity that the underlying spacetimes admit wedge-like regions which have similar properties to the related regions in flat spacetime. We shall need these generalisations to discuss applications of the thermal time hypothesis in curved spacetime.

Schwarzschild Space and Related Examples

The first generalisation was given in 1982 by Sewell [117]. The author proposed a framework for a rigorous general theory of quantum fields on a class of manifolds with topology $X = \mathbb{R}^2 \times Y$ (choose coordinates $(t, w; y)$), and with metric

$$\mathrm{d}s^2 = f(w^2 - t^2, y)(-\mathrm{d}t^2 + \mathrm{d}w^2) + g(w^2 - t^2, y)\mathrm{d}\sigma^2(y), \tag{4.81}$$

where f, g are positive, smooth functions on $\mathbb{R} \times Y$ and $\mathrm{d}\sigma^2(y)$ is a Riemannian metric on Y. Note that this class of spacetimes includes Minkowski space as well as Kruskal space. SEWELL continues by defining wedge-like regions on X, namely $X^+ \equiv \{w > |t|\}$. For this class of spacetimes he formulates a system of axioms by adopting the Wightman axioms, supplemented by certain regularity assumptions. All these spacetimes have a global time coordinate t, although the time-translations are, in general, no isometries of the manifold. Let G be a Lie group of isometries of X. It is assumed that there exists a strongly continuous unitary representation U of G in the Hilbert space of the theory. But whenever G does not contain the time translations it is not possible to define a Hamiltonian (the spectrum condition) in the usual way. To compensate for this deficiency, the peculiar structure of these spacetimes is exploited, namely that the restriction of the time translations on the surface $\{t + w = 0\}$ are isometries. The quantum fields induce essentially a Wightman theory on this surface, and a spectrum condition can be imposed. By adding some dynamical postulates the QFT can be extended to X^+ and X. One defines Lorentz-like boosts via

$$L(\tau)(t, w; y) = (t \cosh \tau + w \sinh \tau, t \sinh \tau + w \cosh \tau; y). \tag{4.82}$$

The boosts turn out to be a one-parameter group of isometries of X. Instead of the ordinary time translations in t the vacuum state is supposed to be invariant under the unitary representation of the boosts. The restriction of L on X^+, denoted by L^+, gives the usual time translations in a wedge.

In this framework SEWELL derives an analogue of the Bisognano-Wichmann theorem for the wedge-like regions. Let K^+ be the generator of the unitary representation of the time translations $U(L^+(\tau))$ in X^+. Under the additional assumption that either the dynamics of the system determines the quantum field in terms of that on $\{t + w = 0\}$, or the quantum field on X^+ enjoys certain stability properties, SEWELL shows that the restriction of the vacuum state ω_0 to the field on X^+ induces the modular operator

$$\Delta_{\omega_0 \restriction \mathcal{R}(X^+)} = e^{-2\pi K^+}. \tag{4.83}$$

The proof is very close to the original one by BISOGNANO and WICHMANN in Minkowski space.

De Sitter Space

De Sitter space [12, 63] is a geodesically complete spacetime of constant positive curvature $R > 0$ with topology $\mathbb{R} \times S^3$ obeying the vacuum Einstein equations. It can be visualised as a hyperboloid,

$$-(z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = \alpha^2, \quad \alpha^2 = 3/\Lambda = 12/R \tag{4.84}$$

in 4+1-dim. Minkowski space with metric

$$\mathrm{d}s^2 = -(\mathrm{d}z^0)^2 + (\mathrm{d}z^1)^2 + (\mathrm{d}z^2)^2 + (\mathrm{d}z^3)^2 + (\mathrm{d}z^4)^2. \tag{4.85}$$

Λ is the cosmological constant, which is assumed to be positive. Moreover, de Sitter space is the unique maximally symmetric curved spacetime with the same degree of symmetry as Minkowski space. Its symmetry group (in the embedding space $\mathbb{R}^5$) is the de Sitter group $SO(1, 4)$ admitting ten Killing fields. When we come to an application of the time hypothesis in de Sitter space, we shall go a bit more into the physical relevance of this spacetime.

BROS and MOSCHELLA [25], and BROS, EPSTEIN and MOSCHELLA [26] suggested an axiomatic scheme for QFT in de Sitter space. Taking advantage of the global symmetry group of de Sitter space, which is closely related to the Poincaré group of Minkowski space, they formulate an approach towards a QFT closely related, one more time, to Wightman's approach. The authors replace the spectrum condition, by a so-called *weak spectral condition* requiring appropriate analytic continuation properties of the Wightman n-point functions. The distinguished vacuum state ω_0, the so-called *Euclidean vacuum*, is selected to be the one which is invariant under the full de Sitter group and has best analyticity properties. Infinitesimally the state looks like the Minkowski vacuum (see also [90]).

By identifying wedge-like regions, such as static de Sitter space $\mathcal{S}_{stat}$, $\{z^4 > |z^0|\}$, the authors derive the Bisognano-Wichmann theorem for the de Sitter case. The modular action induced by $\omega_0 \restriction \mathcal{R}(\mathcal{S}_{stat})$ corresponds to Lorentz boosts in z^4-direction of the 5-dim. embedding space.

4.6 Non-Geometrical and Non-Local Modular Flows

Some General Considerations

We have met several examples so far where the modular flow admits a purely geometrical interpretation. Nevertheless, one does not expect this sort of examples to be of a generic nature, because there are not enough spacetime symmetries that preserve $\mathcal{O}$. To this purpose let us pick up an argument put forward by Fredenhagen [50]. Since the modular group induces automorphisms on the algebras of observables $\mathcal{R}(\mathcal{O})$ over the spacetime region $\mathcal{O}$, and for this reason preserves locality, the associated spacetime transformations should respect the causal structure of spacetime. But then only conformal transformations should be admissible at all, which is quite restrictive.

Knowing the action of the modular group is of paramount interest and gives new insights into physics. However, under general conditions, like in massive or interacting theories, the wedge is the only region, for which the modular action is known. One particularly would like to determine the modular flow induced by the vacuum state restricted to a double-cone algebra in the massive case, since double-cones are the most important regions in Minkowski space besides the wedges. This way one would gain a deeper understanding of the physical meaning of the modular structure, which undoubtedly would also give new insights into the thermal time hypothesis (cf. the subsequent chapters). Massive theories are not invariant under conformal transformations. Together with the result of Trebels that the conformal transformation predicted by the Hislop-Longo theorem 4.5.5 is essentially the only possible geometrical action, this suggests that one presumably cannot assign a geometrical meaning to the modular flow in that case.

It is also argued by Saffary that one has to suspect the modular flow to act non-geometrically on the massive double-cone algebra [115]. In his PhD thesis the author investigates a conjecture, put forward by Schroer and Wiesbrock, that the infinitesimal generator δ_m of the modular group, generated by the vacuum state confined to a double-cone, is a pseudo-differential operator in the case of massive Wightman fields.[13] More precisely, it is expected that δ_m is a perturbation of the massless infinitesimal generator by a pseudo-differential operator. This additional term is supposed to lead to the breakdown of conformal invariance and to cause a non-geometrical action. Saffary confirms the pseudo-differential character of the infinitesimal generator in the examples proposed by Yngvason [136] and by Borchers and Yngvason [21] (cf. the next subsection), where the modular flow is known to be non-geometrical. A strategy to compute δ_m on the massive double-cone algebra is to determine in a first step the modular group generated by the massless vacuum state and relate it to the modular group generated by the "correct" massive vacuum state with the help of the cocycle Radon-Nikodým theorem. For this one makes use of the unitary equivalence of local algebras $\mathcal{R}_m(\mathcal{O})$ for all masses m. The massive free scalar fields can be seen as different fields acting on the same Fock space, and the massless vacuum state is also cyclic and separating for the massive algebras. This result enables Saffary to determine the modular flow up to a cocycle. Anyway, the considerations made in [115] underline again, that the modular group associated with a double-cone cannot generally expected to show a geometrical meaning.

Modular Flow Induced by Thermal States

In all the examples considered so far the modular group acts locally in the sense that local algebras are mapped onto local ones. A first example which establishes the existence of modular groups with a non-local action (and non-geometrical action) was put forward by Yngvason [136]. Here we want to focus on another one, given by Borchers and Yngvason in [21], where all the details can be found. In their paper the authors study the modular flow induced by a KMS state in a thermal sector of an AQFT. The main results read as follows.

Theorem 4.6.1 ([21])**.** *Let $(\mathfrak{A}, \tau_t)$ be a C^*-dynamical system and $\mathfrak{B}$ a subalgebra such that $\tau_t\mathfrak{B} \subset \mathfrak{B}$ for $t \geq 0$ and assume $\cup_t \tau_t\mathfrak{B}$ to be norm dense in $\mathfrak{A}$. Let ω be a (τ, β)-KMS state on $\mathfrak{A}$, denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the corresponding GNS representation of $\mathfrak{A}$, and put $\mathcal{R} := \pi_\omega(\mathfrak{A})''$ and $\mathcal{R}_1 := \pi_\omega(\mathfrak{B})''$. Ω_ω is cyclic and separating for $\mathcal{R}$ as well as for $\mathcal{R}_1$ (cf. theorem 3.3.9). Moreover, let $U(t)$ be the*

[13] A pseudo-differential operator is a generalisation of a differential operator, for the precise mathematical definition we refer to [115]. In all the known cases in which a geometrical meaning exists the infinitesimal generator is an ordinary differential operator of first order.

unitary group implementing the time translations α_t, and define $\mathcal{R}_1(t) := U(t)\mathcal{R}_1U^*(t)$ Then, the modular flow induced by $\omega \restriction \mathcal{R}_1$ satisfies

$$\sigma_s^{\omega\restriction\mathcal{R}_1}(\mathcal{R}_1(t)) \equiv \Delta_{\mathcal{R}_1}^{-is}\mathcal{R}_1(t)\Delta_{\mathcal{R}_1}^{is} = \mathcal{R}_1(\varphi_+(s,t)), \quad \textit{where} \tag{4.86}$$

$$\varphi_+(s,t) = \frac{\beta}{2\pi}\log\left(1+e^{2\pi s}(e^{2\pi t/\beta}-1)\right) \quad \textit{for all } s,t \textit{ for which } 1+e^{2\pi s}(e^{2\pi t/\beta}-1)>0. \tag{4.87}$$

In a next step Borchers and Yngvason investigate first 1-dim. models and then pass to 2-dim. models that factorise in lightcone variables. Exploiting theorem 4.6.1 they show that the modular flow induced by a thermal state restricted to the algebra of a forward cone V_+, or a wedge region W, maps the algebras of translated forward cones or wedges, respectively, into another such algebra.

Theorem 4.6.2 ([21]). *Let φ_+ be as in theorem 4.6.1, and let $\varphi_-(s,t) \equiv -\varphi_+(-s,-t)$, then*

$$\sigma_s^{\omega\restriction\mathcal{R}(V_+)}(\mathcal{R}(V_+ + x^\mu)) = \mathcal{R}(V_+ + \varphi^\mu_{V_+}(s,x^\mu)), \quad \varphi^\mu_{V_+}(s,x^\mu) = (\varphi_+(s,x^0-x^1),\varphi_+(s,x^0+x^1)),$$
$$\sigma_s^{\omega\restriction\mathcal{R}(W)}(\mathcal{R}(W + x^\mu)) = \mathcal{R}(W + \varphi^\mu_{W}(s,x^\mu)), \quad \varphi^\mu_{W}(s,x^\mu) = (\varphi_-(s,x^0-x^1),\varphi_+(s,x^0+x^1)).$$

Far away from the boundary of the respective domains the flow lines of $\varphi^\mu_{V_+}$ and φ^μ_W essentially coincide with the ordinary time translations. Conversely, close to the apex of the forward lightcone, and near the edge of the wedge the modular flow approaches the vacuum modular flow.

For more concrete calculations concerning e.g. the behaviour of double-cone algebras under the modular flow the authors focus on the Weyl algebra of a generalised free Bose field in two dimensions. They determine explicitly the modular action induced by a quasi-free KMS state ω restricted to the algebras $\mathcal{R}(V_+)$ and $\mathcal{R}(W)$, respectively. While it can be shown that for a field of minimal scaling dimension the modular action is geometrical, it turns out that for fields of higher scaling dimensions $n > 0$ the modular flow is a non-local transformation, in the sense that double-cone algebras are no longer localised in double-cones after the transformation $\sigma_s^{\omega\restriction\mathcal{R}(V_+)}$ or $\sigma_s^{\omega\restriction\mathcal{R}(W)}$, respectively, is applied. The non-local modular flow causes a delocalisation of the observables.

Further Examples

Other concrete examples where the modular flow does not have a purely geometrical meaning were found by Casini and Huerta [38], and by Longo, Martinetti and Rehren [81]. In [38] (cf. also [81]), it is shown that the modular flow induced by the vacuum state of a free Fermi field in two dimensions restricted to spacelike separated double-cones is composed of a geometrical part and a non-local mixing term, although the mixing between the separated regions turns out to be weak. The appearance of such a "fuzzy" geometrical action is also proven in [81] in the case of a 1+1-dim. conformal QFT with boundary when the vacuum is restricted to a double-cone.

4.7 Physical Applications of Modular Theory

There are many fields where modular theory has entered mathematics as well as physics and led to great breakthroughs. From the mathematical point of view it particularly played an indispensable role in Connes' analysis and classification of von Neumann algebras. We close this chapter by compiling the most prominent consequences of modular theory for AQFT, for more details and the relevant literature we refer to [22, 60].

For our purposes the relation to equilibrium statistical mechanics is of utmost importance. Furthermore, modular theory disclosed new insights into the type of local algebras and played a significant role in the type specification of local algebras presented in section 2.7. Other applications concern algebraic proofs of the *PCT theorem* by Borchers and by Guido and Longo, and of the *spin-statistics theorem* by Guido and Longo. Moreover, Brunetti, Guido and Longo succeeded in showing that all spacetime symmetries are encoded intrinsically in the local net. Modular covariance thus implies that the representation of the Poincaré group is intrinsically contained in the net. The modular conjugations associated with the vacuum were used by Buchholz, Summers et al. to derive the spacetime itself, in which the quantum system is viewed as evolving, as well as its isometry group. In chapter 5 and 7 we shall discuss in detail a possible physical meaning of the modular flow as a flow describing some sort of local dynamics.

Chapter 5

Thermal Time Hypothesis

"Time is nature's way to keep everything from happening at once."
John A. Wheeler, American physicist (1911-2008)

5.1 The Starting Point

What we have seen so far is that for a formulation of a theory of quantum gravity (as well as of a generally covariant statistical theory) one has to face conceptual problems concerning the true origin of time, known as the issue of time in quantum gravity. We have further argued that it could be reasonable to adopt the position that Nature is penetrated by a fundamental timelessness. This corresponds to the assumption that in a hypothetical generally covariant quantistic or statistical theory one does not have a notion of a (distinguished) physical time, which, however, is not needed for the full formulation and interpretation of the theory. The dynamical laws are determined by correlations which are sufficient to make predictions. Nevertheless, the universal timelessness of such theories has to be related to the preferred physical time flow which apparently (or seemingly) characterises our non-generally covariant world. A mechanism is needed which explains how the familiar notion of time eventually emerges from the timeless structure to become such an important ingredient of the macroscopic world we live in as well as of our conscious experience.

An entirely new solution to this issue of time within the framework of algebraic quantum field theory was suggested in 1994 by Connes and Rovelli in form of the thermal time hypothesis (TTH) [42], see also [83, 110]. A similar ansatz had already been put forward for classical systems in 1993 by Rovelli [107], which also supplies some physical foundations for the formulation of the TTH. In this chapter we review in detail motivation and implementation of the TTH, that is to say in the spirit of Connes and Rovelli. It thus represents the core of this thesis. In the subsequent chapter we shall discuss those cases on which the time hypothesis has been successfully tested. Only in chapter 7 we will pass to a critical analysis of the thermal time concept and present our own point of view.

5.2 Formulation of the Thermal Time Hypothesis

State-Dependent Notion of Time

We start with three assumptions creating the appropriate setting to postulate the TTH, which partly reflect requirements already made before.

(i) The physical time flow is not an a priori given fundamental property of the theory. It has no mechanical meaning, but is a macroscopic feature of thermodynamical origin.

(ii) The notion of time is state-dependent. A (statistical) state singles out a preferred time flow, and it is this flow that we naturally associate to the system being in that state.

(iii) The time flow is realised as a σ-weakly continuous one-parameter group of *-automorphisms on the observable algebra (associated to the spacetime region the observer is confined to).

Accepting these conditions, Connes and Rovelli proposed an extension of the notion of time to generally covariant theories, and, thus, gave a new perspective of the origin of time in non-generally covariant theories, as well. Before continuing let us add a few remarks.

- (i) can be motivated by considering mechanical systems (cf. [107, 110]): There exist approaches towards classical mechanics and also quantum mechanics without any reference to a notion of time (these covariant formulations may be found in appendix B). However, this is only possible unless thermodynamics comes into play, for e.g. the definition of a Gibbs equilibrium state crucially relies on the notion of a preferred physical Hamiltonian which generates the dynamics of the system. This indicates that in classical and quantum mechanical systems most properties of time become really necessary only for the formulation of statistical mechanics, whereas one cannot single out a time variable by purely mechanical measurements. On the mechanical level all variables are on the same footing. The notion of a (preferred) physical time flow is only meaningful on ensembles or on single systems with many degrees of freedom, that is, time emerges statistically.

- (i) realises the old idea that thermodynamics and the notion of time are deeply intertwined. A statement of Eddington says e.g. that clocks are necessarily thermodynamical; a pendulum is not a clock unless it is equipped with a device that registers the number of oscillations [107]. It also fits in the picture that many peculiarities characterising the special time flow of our experience become visible particularly in a thermodynamical context (such as irreversibility, arrow of time, return to equilibrium, memory). The physical time flow has a thermodynamical origin and retains its validity even if a classical background metric is not available.

- To explain (ii), let us consider a concrete physical system with a large, possibly infinite, number of degrees of freedom. Usually one does not have access to the full microscopic state, but only to a limited number of macroscopic coarse-grained variables. As described in section 3.1 what actually can be done is to make a small number of observations from which one constructs a distribution, which represents the generically impure state, compatible with all the experimental knowledge (in the algebraic setting the observations are carried out in some finite spacetime region $\mathcal{O}$ and combined into a partial state over $\mathcal{R}(\mathcal{O})$). Following Connes and Rovelli it is the impurity of the state, representing the lack of knowledge of the observer about the microscopic details of the system, which causes the emergence of a preferred physical time flow; time is a quantity tied to our ignorance.

- As explicated in section 3.1, one cannot make infinitely many measurements with an infinitely accurate precision, and hence does never have access to the full microscopic configuration. The description of a state is intrinsically incomplete, and one is inevitably in a thermodynamical setting, which assigns to the system a statistical distribution function and thereby a preferred time flow.

- In a timeless world a priori *all* flows are potential candidates to represent a time flow, none being preferred. Only a concrete physical configuration, determined by the coarse-grained description of the physical fields, singles out a flow, which is physically most suitable.

- A state-dependent notion of time had already been studied by Rovelli in a preceding paper [107] in the classical context, which we shall briefly discuss at the end of this chapter.

- One motivation for (ii) explicitly mentioned by Connes and Rovelli is the observation that in classical generally covariant theories, such as general relativity, the notion of time tends to be state-dependent, as well (like e.g. proper time).

- (iii) reflects our usual assumption about the mathematical representation of a time flow. It has to be a *-automorphism group, i.e. a one-parameter group which preserves the structure of the laws of Nature, because otherwise time cannot be what is measured by clocks [90].

To summarise, a solution to the issue of understanding the origin of the physical time flow is proposed by the idea to ascribe the time flow (the selection of a preferred flow) to statistical properties of the state. The relevant statistical distribution determines a flow which coincides with what we perceive as the physical flow of time. We want to emphasize that Connes and Rovelli

are concerned to assign a physical time to a system in the sense of the specification of a physical time flow, not to define a *direction* of time. The crucial question to be answered next concerns the mechanism by which the state singles out a time flow from the set of all σ-weakly continuous one-parameter groups of *-automorphisms on the observable algebra. As hinted above, the solution is of course thermodynamics.

To this end, let us pay attention for the time being to a finite classical non-generally covariant statistical system, defined by the Hamiltonian H. Furthermore, let ρ be the (nowhere vanishing) distribution function corresponding to an equilibrium state in the sense of Gibbs, i.e. ρ is a smooth positive normalized function on phase space which satisfies $\rho = Z^{-1}e^{-\beta H}$, $Z = \operatorname{tr} e^{-\beta H}$. A decisive observation is that ρ contains all the information about the time flow as the Hamiltonian does, apart from an unimportant constant overall rescaling. The essential dynamical information is completely coded in the state. Indeed, due to the symplectic structure of the phase space the distribution ρ defines a flow generated by $-\log\rho$ which coincides up to a factor β with the physical time flow. β just sets the unit in which time is measured. The Hamiltonian can be inferred from experiment by repeated measurements on copies of the system, without any need of observing a time evolution. Hence, there is an operational procedure for determining which one is the time variable (in contrast to a purely mechanical context where such a procedure is not available). Clearly the same holds for a quantum mechanical system.

In a non-generally covariant quantum field theory in the algebraic framework with time evolution α_t, a thermal equilibrium state is defined via the KMS condition.[1] One more time, as the results of section 4.3 establish, if one knows such a state (which is assumed to be faithful) in combination with the full kinematics, it contains all relevant information concerning the dynamics of the system, and one can forget about α_t without loss of information. Since the dynamics pushes the system into an equilibrium state ("return to equilibrium"), it is a preferred state of particular interest. This importance is supported by the huge variety of stability properties being inherent to an equilibrium state. Hence, given a physical system (e.g. in terms of a net of observable algebras) with a physical time flow α_t, one searches for states satisfying the KMS condition w.r.t. α_t.

In a generally covariant quantum theory, one has a net of algebras containing the physical observables and the set of states, together determining the kinematics, but there is no dynamical information available whatsoever. The idea proposed by Connes and Rovelli is now to reverse the proceeding just described: Instead of searching for an equilibrium state for a given physical time flow, which is not available anymore, one gets a preferred flow for a given state ω by demanding ω to become a thermal equilibrium state w.r.t. this flow. Furthermore, they postulate this flow to be the physical time flow perceived by an observer. Of course, a priori it is not evident that such a flow exists. However, at this point the powerful results of modular theory apply. By theorem 4.3.1 and 4.3.2 for a given faithful normal state ω this flow does exist uniquely, and is precisely the modular flow σ_s^ω generated by ω – supposed that equilibrium states are identified with KMS states. From now on, we call this flow *thermal time flow*, the modular parameter s is said to be the *thermal time*, a Hamiltonian generating the thermal time evolution on the relevant algebra is called *thermal Hamiltonian* (this does not necessarily need to be the modular Hamiltonian).

Thermal Time Hypothesis ([42]). *The physical time flow governing the macroscopic behaviour of a quantum system being in a faithful normal state ω is given by the thermal time flow σ_s^ω.*

Let us emphasize the main input: According to Rovelli ([110], p. 143) the key observation is that given a (faithful normal) state "*there exists always a [flow] [...], w.r.t. which the system is in equilibrium and whose physics is the same as in the conventional [...] statistical case*". The idea is to select a preferred time flow in such a way that the resulting dynamical system is in thermal equilibrium. In that case the time evolution (expressed by the thermal time correlation functions), the equations of motion, etc. look the same as in well-known equilibrium situations described by ordinary time flows in non-generally covariant quantum statistical mechanics. To put it differently, such a distinguished *thermal* time flow simply represents equilibrium dynamics. Consequently, the thermal time flow has all the properties that characterise an ordinary equilibrium time flow, it describes the same physics, and should thus be regarded as a physical time flow. Even more, because of its thermodynamical origin, thermal time is regarded as properly "flowing" [113], and, thus, is supposed explain one of our most fundamental experiences with time (cf. section 1.2).

[1] In this chapter we disregard that additional restrictions are necessary in order to ensure a typically thermodynamical, dispersive behaviour of the system.

The thermal time postulate proposes a unifying perspective between the timelessness of a generally covariant quantum theory with the evidence of the flow of time. It acts on an intersection point which addresses various areas of physics. With the help of modular theory it is possible to define an intrinsic concept of time, which according to the cocycle Radon-Nikodým theorem is unique up to inner automorphisms. The canonical evolution associated with the algebra is the root of physical time. The deep mathematical results about the structure of von Neumann algebras were actually one main motivation for Connes and Rovelli to postulate the TTH, apart from the wish to develop a solution to the problem of time in quantum gravity.

A state gives rise to completely different time flows when restricted to different members of the net of local algebras. The characteristics of the time evolution therefore may change substantially when probing the hypothesis on different regimes (cf. the examples discussed in the next chapter). Since the hidden symmetry exploited by the modular flow can generally be expected to open the door for highly unconventional flows (in comparison with an ordinary Hamiltonian flow in AQFT which is simply interpretable by its geometrical meaning), the thermal time gives rise to an entirely new concept of time (see section 7.5). Nevertheless, one might hope, and this hope is nurtured particularly by an application of the TTH to cosmological models we are going to discuss in the next chapter, that it is at least a promising candidate to explain the true nature of our time, which recovers familiar notions of time in important cases of physical interest.

The fact that the time flow is chosen such that the system is always in equilibrium does, according to Connes and Rovelli, not imply that evolution is frozen and one cannot detect any dynamical changes. What actually can be measured is the effect of fluctuations around the thermal state. Wightman showed [120] that in conventional QFTs satisfying the Wightman axioms one can extract all physical information in terms of vacuum expectation values of products of field operators (n-point functions), i.e. by means of the single vacuum state Ω_0. Returning to a (possibly generally covariant) AQFT in a state ω on the relevant von Neumann algebra $\mathcal{R}$ a non-trivial evolution in thermal time can be expressed in terms of correlation functions of the form

$$F_{A,B}(s) = \omega(\sigma_s^\omega(B)A) \quad \text{for} \quad A, B \in \mathcal{R}. \tag{5.1}$$

Physically this is related to the amplitude of detecting $\sigma_s^\omega(B)\,|\Omega_\omega\rangle$ if one prepares $A\,|\Omega_\omega\rangle$ (in the GNS space). An admittedly speculative ansatz how a time asymmetry may be implemented in spite of the equilibrium dynamics generated by the thermal time can be found in [107].

Relation to Conventional Notions of Time and the Emergence of Temperature

In a generally covariant quantum theory thermal time seems to be the only time available. However, in a theory which admits, apart from the thermal time flow, another meaningful notion of time, such as a *geometrical time flow* with geometrical realisation on the spacetime manifold (as it is e.g. determined by a representation of a suitable subgroup of the Poincaré group), geometrical time and thermal time have to be related. The issue of comparing these two different flows was already addressed by Connes and Rovelli. We adopt here the proposal by Martinetti and Rovelli [83] from 2003, who postulated an extension of their ansatz. When a physical time t is measured in a non-generally covariant system being in an equilibrium state at inverse temperature β by a clock which measures just a rescaled physical time $\tilde{t} \propto t$, the state formally looks like an equilibrium state at the rescaled temperature. This observation was taken up by Martinetti and Rovelli and supplemented in the sense that also time-dependent rescalings should be permissible.

Thermal Time Hypothesis – Addendum ([83]). *Consider a system in a state ω which admits a thermal time flow σ_s^ω as well as a geometrical time flow α_t. If the modular flow has a geometrical meaning and either flows are proportional to each other, where the proportionality constant may vary along the flow, a local notion of temperature is interpreted to be the ratio of the two flows:*

$$\beta(s) := \frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{\|\partial_s\|}{\|\partial_t\|}. \tag{5.2}$$

∂_s and ∂_t are the corresponding tangent vector fields.

Provided that both times are just two different parametrisations of the same curve, when time is measured in geometrical time, the temperature is linked to its deviation from thermal time.

Temperature is the speed of thermal time w.r.t. geometrical time [112]. In accordance with this hypothesis there is an intricate relation between the notions of time and thermodynamics. While in conventional theories for a given time evolution α_t and an inverse temperature β one is looking for (α_t, β)-KMS states, in generally covariant theories it is the state in collaboration with the thermal time flow it generates itself, that determines the temperature by comparison with a geometrical time flow. Later on we shall meet the example which originally motivated this hypothesis (section 6.5). As discussed in section 4.5, the modular flow does generally not have a geometrical interpretation whereby leading to the notion of temperature.

Albeit this thesis focuses on the physical interpretation of the thermal time, let us touch the (open) issue of giving this local temperature a physical interpretation (cf. [84, 85] and the references cited therein). For instance a geometrical thermal time flow identified as the proper time of a non-eternal observer (which then has bounded range) necessarily leads to a non-trivial rescaling and thus to the notion of a local temperature. In principle there is no obstacle not to talk of a local, time-dependent temperature. Of course, one would require this sort of temperature to be somehow measurable. Note, however, that one deals here with temperatures defined at a point which may vary in a neighbourhood of this point, while any realistic measuring device ("thermometer") has a spatial extension. A consistent thermal interpretation may be a delicate point. On the other hand, the notion of temperature should not rely on the presence of any detector. The KMS condition e.g. asserts a temperature independently of the very concept of a detector. The problem in which way detectors respond to thermal states, supplemented by the issue that the response may depend on the specific details of the detector as well as the measuring process itself, and that a detector may need an infinite amount of time to get in equilibrium, would go beyond the scope of this thesis. We remark that the local notion of temperature defined here does *not* coincide with other such notions [81]. In section 6.5 we shall briefly come back to the temperature aspect.

In addition to the proposed physical relation between time and temperature we mention their etymological relation emphasized by Rovelli [107]. The word 'temporal' originates from the Latin word 'tempus' whose meaning is 'time', but also 'section', 'part of', or 'division of'. 'Temperature' is connected to the Latin verb 'temperare', which means 'to mix in due proportions'. The relation between 'temperare' and 'tempus' is then expressed in the idea of 'partition in due proportion'.

Classical and Quantum Mechanical Limit

In section 4.2 we have explicitly calculated the action of the modular flow in the quantum mechanical limit. The result can be used to set up the TTH for quantum mechanical systems. A faithful normal state ω is described via an invertible density matrix ρ, and the thermal time flow, acting on the observable algebra, is generated by the thermal Hamiltonian $K = -\log\rho$,

$$A(s) = \sigma_s^\omega(A) = e^{-is\log\rho} A e^{is\log\rho}. \tag{5.3}$$

The TTH changes ρ into a $(\beta = 1)$-Gibbs state w.r.t. the thermal time flow. As mentioned before, in contrast to the usual viewpoint in non-generally covariant systems, the Hamiltonian does not determine the thermal state, but it is the state which determines the Hamiltonian.

Next, we pass to the classical limit and derive the statement of the TTH in that case. The state ω is represented as a smooth positive normalized function on phase space. We have seen that the "modular flow" corresponds to a Hamiltonian flow, generated by the thermal Hamilton function $K = -\log\rho$. Accordingly, ρ induces the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}s} A(s) = \{A(s), -\log\rho\} = X_s(A), \tag{5.4}$$

where A is any observable, which is an element taken from the abelian algebra of smooth real functions on phase space, $C^\infty(\Gamma)$, and $X_s(\cdot) \equiv \{\,\cdot\,, -\log\rho\}$ is the flow vector field. To X_s one may add a null vector field of the presymplectic form corresponding to the classical system. Again, ρ resembles a Gibbs ensemble relative to the thermal Hamiltonian.

The TTH can be applied to the covariant formulation of classical and quantum mechanics (appendix B and section 7.9). Solely the state singles out a time variable in a system fully characterised by correlations between observables, and wherein a priori all variables are on an equal footing. While the weakly vanishing covariant Hamiltonian reflects the absence of a preferred time, the thermal Hamiltonian, which is (formally) a proper Hamiltonian, generates a time flow.

5.3 Alternative Versions of the Thermal Time Hypothesis

Intrinsic Equilibrium States as the Source of Thermal Time

An interesting variant of the TTH for classical systems represented in the covariant formalism (cf. appendix B.1) had already been developed by Rovelli [107] in 1993. It differs from the classical limit obtained here in the sense that it presupposes the system to be in equilibrium in order to distinguish a thermal time. Let us briefly recall the idea for completeness, but also because we think that it much more emphasizes the main idea underlying the TTH. Suppose $(\Gamma_{ex}, \sigma_c, H_c)$ to be a classical system represented in the covariant language. The Hamiltonian $H_c \approx 0$, weakly vanishing in the sense of Dirac [45], defines a constraint surface $\mathcal{C}$ in the extended phase space Γ_{ex}, on which the 2-form σ_c induces a presymplectic form ω. A state on this system is given as a distribution function ρ on $\mathcal{C}$ satisfying $\{\rho, H_c\} \approx 0$. This condition ensures the invariance along the orbits generated by the Hamiltonian constraint, and thus the consistency with the equations of motion. Again, the state selects a preferred time flow, generated by the vector field X_ρ on $\mathcal{C}$. X_ρ is fixed (up to a null vector field of ω) by the equation

$$\omega(X_\rho) = -\mathrm{d}\log\rho. \tag{5.5}$$

However, this flow is only regarded as a physically meaningful time flow (or *thermodynamical time flow* in the original terminology) if the state ρ is *a priori* an equilibrium state. For the definition of an equilibrium state an intrinsic, covariant notion of equilibrium is needed, not depending on the knowledge of a time flow. A first ansatz to implement such a characterisation is to require all macroscopic subsystems ρ' to be in equilibrium with the rest of the system ρ'', in the sense that the distribution function (approximately) factorises for these subsystems, $\rho = \rho' \cdot \rho''$.[2]

Consequently, the crucial difference to the TTH of Connes and Rovelli is that not an arbitrary (faithful normal) state but only an intrinsic equilibrium state generates a physical thermal time flow. This sets much greater store by the observation that in the non-generally covariant case precisely the equilibrium states contain the full information about the physical dynamics. We shall pick up this point in our discussion.

State-Independent Notion of Time

Different states define different thermal time flows, but recall that taking the cocycle Radon-Nikodým theorem into account, it is possible to define a canonical state-independent flow $\tilde{\sigma}_s$ as soon as inner equivalent flows are identified. Thereby an alternative formulation of the TTH, also suggested by Connes and Rovelli, becomes possible, which fully exploits the striking mathematical results of the Tomita-Takesaki theorem and the cocycle Radon-Nikodým theorem, which together ensure existence and uniqueness of the intrinsic flow.

Thermal Time Hypothesis – Alternative Version ([42])**.** *The physical time flow of a quantum system described by a von Neumann algebra $\mathcal{R}$ is given by the canonical flow $\tilde{\sigma}_s$*

This hypothesis changes a von Neumann algebra into a true intrinsically dynamical object. Von Neumann algebras inherit a natural physical time evolution from their non-commutativity. To get a non-trivial canonical flow it is necessary to deal with algebras of type III, as it happens to be the case in AQFT. But it also means, in contrast to the original version, that the hypothesis has no counterpart in classical or quantum mechanics (where all time flows are in the same equivalence class as the identity transformation). Nevertheless, since the modular automorphisms are inner equivalent, the alternative version may seem to be quite close to the original one, but note that there is a decisive change in perspectives: It is not the state anymore which determines, in combination with the algebra, the physical time, but it is exclusively the algebraic structure of the observables contained in one single local algebra. In the remainder of this thesis we shall mainly deal with the first version, and assume that the thermal time corresponds to the modular flow.

[2] Rovelli actually demanded the corresponding components of X_ρ to have (weakly) vanishing Lie brackets, which is a consequence of the factorisation property of the distribution function.

Chapter 6

Tests of the Thermal Time Hypothesis

"We've learned from experience that the truth will come out."
Richard P. Feynman, American physicist (1918-1988)

According to Connes and Rovelli the strongest support for the thermal time hypothesis stems from its applications to concrete physical situations, which we are going to study next. We consider systems which are already equipped with a physical time flow and analyse the nature of the thermal time flow as predicted by the TTH. As we shall see it works perfectly well in a couple of physically relevant examples, where it leads to the identification of thermal time with some well-established physical time, and, moreover, provides a slightly new perspective on well-known phenomena such as the Unruh effect or the Gibbons-Hawking effect. Thus, this section serves so to speak as an exploration and affirmation of the TTH by example.

The presented examples on which the TTH is probed are taken from Rovelli and Smerlak [112], Rovelli [108], Connes and Rovelli [42], Martinetti and Rovelli [83], Narnhofer, Peter and Thirring [90] and Tian [126], and include all cases known so far to the best knowledge of the author.

6.1 Equilibrium States and Tolman-Ehrenfest Effect

Systems being left to their own approach thermal equilibrium. That is the reason why in many cases physics can be described perfectly well by small perturbations around such a thermal state ω. Following Connes and Rovelli, in the non-generally covariant limit, where effects from the gravitational field can be disregarded, the TTH consequently has to be applied to equilibrium states w.r.t. the conventional Hamiltonian time evolution, that is to KMS states.

Clearly, in these cases the TTH works perfectly well by construction. Nonetheless, let us exemplary take a short look on a faithful normal (α_t, β_0)-KMS state ω over a quantum field theoretical system. ω is a $(\beta = 1)$-KMS state w.r.t. the thermal time flow and we have

$$\sigma_s^\omega = \alpha_{\beta_0 s} = \alpha_{t(s)} \quad \Rightarrow \quad t(s) = \beta_0 s. \tag{6.1}$$

The TTH applied to a KMS state predicts a thermal time flow which coincides up to a constant factor β_0 with the Hamiltonian standard time, which clearly is a physical time flow. We pay attention to the second part of the TTH: Thermal time and Hamiltonian time are proportional. The constant of proportionality, β_0, is precisely the inverse temperature of the KMS state. This is in accord with the second prediction of the TTH.

A slightly more general result is obtained by passing to general static spacetimes. They were studied by Rovelli and Smerlak [112] in the classical context, we focus on the quantum case. Static spacetimes are characterised by the existence of a globally defined, timelike Killing field ξ^μ,[1] which additionaly satisfies the equation $\xi_{[\mu}\nabla_\nu\xi_{\sigma]} = 0$ expressing *hypersurface orthogonality*

[1] A *Killing field* ξ_μ is defined by Killing's equation $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$.

(see [132] and section 7.2). This Killing field can be used to introduce a particularly convenient coordinate system:

$$\mathrm{d}s^2 = -g_{00}(\vec{x})\mathrm{d}t^2 + g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j. \tag{6.2}$$

We assume the existence of a QFT where the one-parameter group of isometries defined by the Killing field ξ^μ is unitarily implemented on the observable algebra whereby defining the time translations α_t of the theory. [2] Let us further assume the system to be in a faithful (α_t, β_0)-KMS equilibrium state ω. Then the modular flow is given by $\sigma_s^\omega = \alpha_{\beta_0 s}$, i.e. $(\partial_s)^\mu = \beta_0 \xi^\mu$. Let us focus on the temperature prediction of the TTH. The geometrical time is the proper time of observers moving on the flow lines of the Killing field. The corresponding tangent field reads $(\partial_\tau)^\mu = (-g_{00})^{-1/2}\xi^\mu$. Hence, we obtain for the inverse temperature measured by a static observer

$$\beta = \beta(\vec{x}) = \|\partial_s\|/\|\partial_\tau\| = \sqrt{-g_{00}(\vec{x})}\,\beta_0. \tag{6.3}$$

Accordingly, the TTH predicts a temperature which depends on the gravitational potential at that point where the measurement is made; in general the temperature is not constant in space. The dependence on position at thermal equilibrium is such as to make the quantity $T\sqrt{-g_{00}(\vec{x})}$ a constant throughout the system. This effect was first discovered by Tolman and Ehrenfest [127] in 1930 who used the pressure of electromagnetic radiation to construct a thermometer which measures the local temperature via the Stefan-Boltzmann law. It is known as the *Tolman-Ehrenfest effect* (see [112] for references to alternative derivations).

6.2 Special Relativistic System

Consider a single relativistic particle with rest mass m interacting with a heat reservoir of inverse temperature β in the sense that it may exchange energy with the reservoir (cf. [108]). In the presymplectic formalism the system is defined by the extended phase space $\Gamma_{ex} = \langle x^\mu, p_\mu\rangle$ together with the Hamiltonian constraint $H_c = p_\mu p^\mu + m^2 \approx 0$, and the presymplectic 2-form

$$\omega = \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu = \mathrm{d}p_0(\vec{p}) \wedge \mathrm{d}x^0 + \mathrm{d}p_i \wedge \mathrm{d}x^i = \frac{p^i}{\sqrt{\vec{p}^2 + m^2}}\mathrm{d}p_i \wedge \mathrm{d}x^0 + \mathrm{d}p_i \wedge \mathrm{d}x^i. \tag{6.4}$$

The mechanics of the particle is manifestly Lorentz invariant. Up to now all Lorentz frames are on an equal footing; each of them defines a Hamiltonian time, none of which being preferred.

Let us assume the distribution ρ describing the state of the particle to be fixed as a Gibbs distribution of inverse temperature β. The heat reservoir defines a preferred Lorentz frame O', the unique frame in which the average momentum of the gas vanishes (Lorentz invariance is broken this way). In that Lorentz frame the distribution is given by

$$\rho(x'^\mu, p'_\mu) = \exp\left(-\beta\sqrt{\vec{p}'^2 + m^2}\right). \tag{6.5}$$

First of all we observe that $\{\rho, H_c\} \approx 0$, which is important for the consistency of the state. After the distribution has been fixed, we apply the TTH. The state determines the thermal time flow generated by the thermal Hamiltonian $K = -\log\rho = \beta\sqrt{\vec{p}'^2 + m^2}$,

$$X_s = \{\,\cdot\,, K\}_{x',p'} = \frac{\beta p'_i}{\sqrt{\vec{p}'^2 + m^2}}\frac{\partial}{\partial x'^i}. \tag{6.6}$$

We are allowed to add to this vector field a null vector field Y of ω, which has the general form

$$Y \propto \frac{p'_i}{\sqrt{\vec{p}'^2 + m^2}}\frac{\partial}{\partial x'^i} - \frac{\partial}{\partial x'^0}. \tag{6.7}$$

This leads to a more convenient form of the flow vector field:

$$X_s = \beta\frac{\partial}{\partial x'^0} = \beta\frac{\partial}{\partial t'} \quad \Rightarrow \quad t'(s) = \beta s. \tag{6.8}$$

[2] See e.g. [133] for the construction of the QFT of a massive Klein-Gordon field in globally hyperbolic, stationary spacetimes.

Thus, in agreement with the TTH, thermal time s is a $1/\beta$ rescaling of the Hamiltonian time t' corresponding to the Lorentz frame in which the particle is in equilibrium and in which the heat reservoir is at rest. The ratio between the flows gives precisely the temperature of the heat reservoir. While a priori all Lorentz times were on the same level, the state breaks Lorentz invariance and distinguishes one special Lorentz time as the physically most reasonable one.

6.3 Robertson-Walker Space Filled with a Dynamical Maxwell Field

Next, let us apply the TTH to a Robertson-Walker space filled with an arbitrary non-homogeneous electromagnetic radiation. Such a spacetime could be regarded as a phenomenological model of our universe, where the Maxwell field models the cosmic microwave background. The description is based on [108] with the difference that, like already in the previous example, we apply the TTH in the form as it was suggested in [42] and as it is used throughout this thesis. Moreover, we add a cosmological constant Λ. The relevant basics concerning general relativity and the ADM formalism can be found in many textbooks [63, 132, 74], they are not repeated here. We use standard notation of general relativity (see p. v and e.g. [74]). Let us start with the *Robertson-Walker metric*, which has the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\underbrace{[\mathrm{d}\chi^2 + f^2(\chi)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\,\phi^2)]}_{=:\tilde{g}_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j}, \text{ where } f(\chi) = \begin{cases} \sin\chi & \text{for } k=1 \\ \chi & \text{for } k=0 \\ \sinh\chi & \text{for } k=-1 \end{cases} \tag{6.9}$$

$$t\in\mathbb{R}, \quad \vartheta\in[0,\pi], \quad \phi\in\ [0,2\pi].$$

The coordinate χ runs from 0 to ∞ if $k=0,-1$, and from 0 to 2π if $k=1$. The Robertson-Walker metric comes out if one wishes to describe a spatially homogenous and isotropic spacetime.[3] It represents a class of spacetimes with constant curvature, and is in fact up to local isometry the most general metric which does that. The constant k is the normalized spacetime curvature. Depending on whether $k=-1,0,1$ the spatial section of the metric describes a geometry which is hyperbolic, flat, or closed, respectively. For a given k, $R(t)$ is the only remaining quantity which needs to be specified via the Einstein equations. To obtain an exact solution, the energy-momentum tensor must be required to be of the form of a perfect fluid whose density and pressure are functions of t only, and whose flow lines are the curves $(\chi,\vartheta,\phi)=\text{const}$. A Robertson-Walker space is expected to be a good approximation to the large scale geometry of our universe, where the fluid is thought of as a smoothed out approximation to the matter of the universe. R then corresponds to the radius of the universe. In the standard gauge (6.9) the coordinate time t can be interpreted as the proper time of an isotropic observer. Comparing the above expression with the ADM-metric

$$\mathrm{d}s^2 = -(N^2 - N_iN^i)\mathrm{d}t^2 + 2N_i\mathrm{d}t\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{6.10}$$

one reads off that the Robertson-Walker metric in standard form corresponds to a gauge with vanishing shift vector $N^i=0$ and lapse function $N=1$. The spacetime is supposed to contain an arbitrary Maxwell field, which in particular does neither need to be homogeneous nor spherically symmetric. Of course, such a system cannot be described by a solution of the Einstein equations, where the metric satisfies the Robertson-Walker ansatz. Let us assume instead that the Maxwell field interacts only with the $R(t)$-degree of freedom, while the interaction with the remaining degrees of freedom is neglected. This is justified if the electromagnetic field influences the expansion of the universe without really disturbing the symmetry of the Robertson-Walker space. Due to the decrease of the number of variables the Robertson-Walker ansatz has to be plugged in directly

[3] *Spatially homogeneity* is characterised by the existence of a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that for all t and $p,q\in\Sigma_t$ there is an isometry of the metric $g_{\mu\nu}$ mapping p on q, i.e. any point on Σ_t looks the same. A spacetime is *spatially isotropic* if there is a congruence of timelike curves ($\rightarrow$"isotropic observers") with tangent u^μ, filling the spacetime and satisfying the property that given any point p and unit tangent vectors s_1^μ, s_2^μ at p which are orthogonal to u^μ, then there exists an isometry of $g_{\mu\nu}$ which leaves p and u^μ invariant and rotates s_1^μ into s_2^μ. This is the mathematical formalisation of the physical picture that it is impossible to construct a geometrically preferred tangent vector orthogonal to u^μ. There is an exact spherical symmetry about *every* point. The surfaces Σ_t turn out to be orthogonal to u^μ. See [63, 132] for more details.

into the action rather than the equations of motion to get the right number of equations in the end. In other words, the model is implemented as a coupled Einstein-Maxwell system by freezing all the gravitational degrees of freedom except R.

The action of the system consists of the Einstein-Hilbert action of the gravitational field supplemented by the action of the electromagnetic field, $S = S_g + S_{em}$. After plugging in the Robertson-Walker ansatz, the system is described in the usual way by the full set of equations obtained by varying the action w.r.t. R as well as the electromagnetic variables. We want to determine the full Hamiltonian of the system. First, we extract the gravitational Hamiltonian from the action. For this it is most convenient to use the ADM action of the gravitational field,

$$S_g = \frac{1}{16\pi}\int \mathrm{d}^4x\,\mathcal{L}_g = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}(R-2\Lambda) = \frac{1}{16\pi}\int \mathrm{d}t\mathrm{d}^3x\,N\sqrt{h}(K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda).$$

Herein $h_{\mu\nu}$ denotes the Riemannian metric induced by g on the spacelike hypersurface Σ_t by restricting the action of $g_{\mu\nu}$ at each $p \in \Sigma_t$ to vectors tangent to Σ_t. One verifies $h_{ij} = R^2\tilde{g}_{ij}$. ${}^{(3)}R$ is the scalar curvature stemming from the Riemann tensor ${}^{(3)}R_{\mu\nu\sigma}{}^{\rho}$ constructed from $h_{\mu\nu}$ on Σ_t. Furthermore, K_{ij} is supposed to be the extrinsic curvature on Σ_t,

$$K_{ij} = \frac{1}{2N}[\dot{h}_{ij} - D_iN_j - D_jN_i] = \frac{1}{2}\dot{h}_{ij} = \frac{1}{2}(R^2\tilde{g}_{ij})^{\cdot} = R\dot{R}\,\tilde{g}_{ij} = \frac{\dot{R}}{R}h_{ij} \tag{6.11}$$

$$\Rightarrow \quad K_{ij}K^{ij} = \frac{3\dot{R}^2}{R^2}, \quad K = h^{ij}K_{ij} = \frac{3\dot{R}}{R}. \tag{6.12}$$

D_i is the spatial covariant derivative on Σ_t associated with $h_{\mu\nu}$. The dot denotes the Lie derivative w.r.t. the coordinate time t. Exploiting ${}^{(3)}R = -\frac{6k}{R^2}$ and $\sqrt{h} = R^3\sqrt{\tilde{g}}$ the Einstein-Hilbert action takes the form

$$S_g = \frac{1}{16\pi}\int \mathrm{d}t\mathrm{d}^3x R^3\sqrt{\tilde{g}}\left(\frac{3\dot{R}^2}{R^2} - \frac{9\dot{R}^2}{R^2} - \frac{6k}{R^2} - 2\Lambda\right) = -\int \mathrm{d}t\frac{3RV}{8\pi}\left(\dot{R}^2 + k + \frac{R^2\Lambda}{3}\right), \tag{6.13}$$

where $V := \int \mathrm{d}^3x\sqrt{\tilde{g}}$ describes the space volume for $R = 1$. From this we can read off the Lagrangian of the gravitational field

$$L_g = -\frac{3RV}{8\pi}(\dot{R}^2 + k + \frac{R^2\Lambda}{3}). \tag{6.14}$$

The conjugate momentum P of R is given as

$$P = \frac{\partial L_g}{\partial\dot{R}} = -\frac{3R\dot{R}V}{4\pi} \quad \Rightarrow \quad H_g = \dot{R}P - L_g = -\frac{2\pi P^2}{3RV} + \frac{3RVk}{8\pi} + \frac{R^3V\Lambda}{8\pi} \tag{6.15}$$

for the Hamiltonian corresponding to the gravitational field. Next, we shall determine the Hamiltonian of the source-free electromagnetic field for the Robertson-Walker space. The Lagrange density of the electromagnetic field reads

$$\mathcal{L}_{em} = -\frac{1}{4}\sqrt{-g}F_{\mu\nu}F^{\mu\nu} = -\sqrt{-g}\,\nabla_{[\mu}A_{\nu]}\nabla^{[\mu}A^{\nu]} = -\sqrt{-g}\,\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]}, \tag{6.16}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. Since the Maxwell field A_μ has a gauge arbitrariness, only the Maxwell potential A_i forms the configuration field, while A_0 has to be regarded as a non-dynamical variable. The conjugate momenta E^i of A_i are given as

$$E^i = \frac{\partial\mathcal{L}_{em}}{\partial(\dot{A}_i)} \doteq \frac{\partial\mathcal{L}_{em}}{\partial(\partial_0A_i)} = -\sqrt{-g}\,(\partial^0A^i - \partial^iA^0). \tag{6.17}$$

Again, the dot denotes the Lie derivative. Now we can construct the electromagnetic Hamiltonian,

$$\begin{aligned} H_{em}[R,A,E] &= \int \mathrm{d}^3x(E^i\dot{A}_i - \mathcal{L}_{em}) = \int \mathrm{d}^3x\left(\frac{1}{2\sqrt{-g}}E_iE^i + \sqrt{-g}\,\partial_{[i}A_{j]}\partial^{[i}A^{j]}\right) \\ &= \int \mathrm{d}^3x\left(\frac{1}{R}\frac{\tilde{g}_{ij}}{2\sqrt{\tilde{g}}}E^iE^j + \frac{1}{R}\tilde{g}_{ij}\tilde{g}^{ik}\tilde{g}^{jl}\partial_{[i}A_{j]}\partial_{[k}A_{l]}\right) =: \frac{V}{R}\tilde{H}[A,E]. \end{aligned} \tag{6.18}$$

It is supplemented by the Gauss constraint which is enforced by the non-dynamical scalar potential,

$$\nabla_i E^i \approx 0. \tag{6.19}$$

Altogether, the Hamiltonian constraint of the full system reads

$$H_c = H_g[R,P] + H_{em}[R,A,E] = -\frac{2\pi P^2}{3RV} + \frac{3RVk}{8\pi} + \frac{R^3V\Lambda}{3} + \frac{V}{R}\tilde{H}[A,E] \approx 0. \tag{6.20}$$

From (6.20) we deduce the presymplectic 2-form on the constraint surface

$$\omega = \int \mathrm{d}^3x\, \mathrm{d}E^i(x) \wedge \mathrm{d}A_i(x) + \mathrm{d}P[R,A,E] \wedge \mathrm{d}R \tag{6.21}$$

$$= \int \mathrm{d}^3x\, \mathrm{d}E^i(x) \wedge \mathrm{d}A_i(x) - \frac{3V^2}{4\pi P}\mathrm{d}\tilde{H}[A,E] \wedge \mathrm{d}R. \tag{6.22}$$

$\Gamma_{ex} = \langle R, P, A_i(x), E^i(x)\rangle$ is the extended phase space of the system which contains all the dynamical variables of the model. The definition of the dynamical system is completed by the Hamiltonian constraint, the Gauss constraint and the presymplectic form ω. The average energy density e of the electromagnetic field (i.e. of the cosmic background radiation) is given by

$$e(t) = \frac{H_{em}(t)}{\text{volume}} = \frac{H_{em}(t)}{R^3(t)V}. \tag{6.23}$$

Inserting (6.23) in the Hamiltonian constraint yields

$$3\dot{R}^2 - 3k - R^2\Lambda - 8\pi e R^2 \approx 0. \tag{6.24}$$

This is a generalisation of the Friedman equation to the average value of a spatially non-constant energy density. Since $\tilde{H}$ neither depends on R nor on P, it commutes with the Hamiltonian constraint, $\{H_c, \tilde{H}\}_{R,P,A,E} = 0$, $\tilde{H}$ is a constant of motion. Plugging (6.18) into (6.23) gives

$$e = \frac{\tilde{H}}{R^4} \propto R^{-4}. \tag{6.25}$$

Combining this expression with (6.24), we conclude

$$3\dot{R}^2 - 3k - R^2\Lambda - 8\pi\frac{\tilde{H}}{R^2} \approx 0, \tag{6.26}$$

which is nothing else but the radiation-dominated Friedman universe. We emphasize that the evolution of the radius R of the universe is independent of the dynamics of the electromagnetic field and only affected by the constant $\tilde{H}$, just as a change of the radius has no influence on the dynamics of the electromagnetic field.

Eventually the state of the Maxwell field has to be fixed. Its statistical distribution is intended to describe the cosmic background radiation of the universe. For that one makes the realistic assumption that the universe is filled with black body radiation. Then, the energy density is spatially constant and equal to its average value e. From thermodynamics it is known that

$$e = 4\sigma T^4, \quad \sigma = \text{const.} \quad \overset{(6.25)}{\Rightarrow} \quad T = T(R) = \frac{1}{R}\left(\frac{\tilde{H}}{4\sigma}\right)^{1/4}. \tag{6.27}$$

The result is one more time a slight generalisation of the standard result in the Friedman-Robertson-Walker model. The distribution that defines the state is supposed to be a Gibbs distribution at temperature T. The Hamiltonian appearing in the Gibbs state has to be chosen in such a way that it generates the time flow w.r.t. which the temperature of the background radiation is defined. Since the temperature was introduced as a function of the energy density e of the electromagnetic field, which in turn was defined in terms of the standard Robertson-Walker time t, the only Hamiltonian coming into question is H_{em}, which generates the evolution of A and E in t:

$$\rho(R,A,E) = \exp\left(-\frac{1}{T}H_{em}\right) = \exp\left(-(4\sigma)^{1/4}V\tilde{H}[A,E]^{3/4}\right) = \rho(A,E). \tag{6.28}$$

We observe that the dependence of the Hamiltonian H_{em} on the intrinsic time R is cancelled by that of the temperature, such that ρ is actually independent of R. Since $\tilde{H}$ is constant we obtain the important intermediate result that ρ commutes with the constraint H_c, $\{\rho, H_c\} = 0$, such that we have a well-defined state.

At this stage we apply the TTH. While a priori, at least locally, any functional of the form $f[R, P, A, E]$ is a suitable candidate as internal time, the TTH provides us with a preferred variable on the phase space as internal time: The thermal Hamiltonian reads $K = -\log\rho = (4\sigma)^{1/4} V \tilde{H}^{3/4}$ and generates the time flow

$$X_s = \{\,\cdot\,, K\} = \frac{3}{4} V \left(\frac{4\sigma}{\tilde{H}}\right)^{1/4} \{\,\cdot\,, \tilde{H}\} = \frac{3V}{4TR} \int \mathrm{d}^3x \left(\frac{\partial \tilde{H}}{\partial E^i(x)} \frac{\partial}{\partial A_i(x)} - \frac{\partial \tilde{H}}{\partial A_i(x)} \frac{\partial}{\partial E^i(x)} \right). \quad (6.29)$$

Again, we may add a null vector field Y of the presymplectic form ω

$$Y \propto \frac{\partial}{\partial R} + \frac{3V^2}{4\pi P} \int \mathrm{d}^3x \left(\frac{\partial \tilde{H}}{\partial E^i(x)} \frac{\partial}{\partial A_i(x)} - \frac{\partial \tilde{H}}{\partial A_i(x)} \frac{\partial}{\partial E^i(x)} \right). \quad (6.30)$$

An appropriately chosen Y yields

$$X_s = -\frac{\pi P}{TVR} \frac{\partial}{\partial R} = \frac{3}{4} \frac{\dot{R}}{T} \frac{\partial}{\partial R} \doteq \frac{3}{4} \frac{1}{T} \frac{\partial}{\partial t} \quad (6.31)$$

$$\Rightarrow \quad t(s) = \frac{3}{4} T^{-1} s. \quad (6.32)$$

t is the standard Robertson-Walker time, the proper time of isotropic observers. According to the TTH the standard Robertson-Walker time is the physical time. The constant of proportionality between thermal time flow and Robertson-Walker time flow is (almost) the temperature as measured by an isotropic observer.

The thermal time defined by the cosmic microwave background is precisely the conventional Robertson-Walker time (up to an affine transformation). In accordance with the TTH the statistical state of the universe we live in distinguishes a flow which corresponds to the physical time flow of our perception. According to Connes and Rovelli this is the strongest physical evidence for the TTH.

6.4 Rindler Wedge and Unruh Effect

In this subsection we return to AQFT. We consider the vacuum sector of an AQFT defined by a net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ in Minkowski space. The total algebra $\mathcal{R}$ associated with the full Minkowski space is assumed to be in the vacuum state $\omega_0 = \langle \Omega_0, \cdot\, \Omega_0 \rangle$. We apply the TTH to the vacuum state restricted to the von Neumann algebra $\mathcal{R}(W)$ corresponding to the Rindler wedge W [42].

Let O be a rectilinear, uniformly accelerated observer in Minkowski space with acceleration $a \in \mathbb{R}_{>0}$. W.l.o.g. we assume the worldline of this observer to be given as

$$x^0(\tau) = a^{-1} \sinh(a\tau), \;\; x^1(\tau) = a^{-1} \cosh(a\tau), \;\; x^2 = x^3 = 0. \quad (6.33)$$

In this parametrisation τ is O's proper time. The trajectory of O is an orbit of the group of all Lorentz boosts Λ in x^1-direction, the boost-parameter is $a\tau$. We have

$$\Lambda(a\tau) x^\mu(\tau_0) = x^\mu(\tau + \tau_0). \quad (6.34)$$

O has no access to the whole Minkowski space, but causally interacts only with events inside the Rindler wedge $W \equiv \{x^1 > |x^0|\}$, while the edge of W acts as a *causal horizon*. That means O can exchange information precisely with the events in W in the sense that he sends a signal to this event and an immediate reply can still reach him. For this reason he will describe the system only in terms of the subalgebra $\mathcal{R}(W) \subset \mathcal{R}$ consisting of those observables which are localised in W. The state he will ascribe to the system, which for him is a net ending with $\mathcal{R}(W)$, is the partial

state $\omega_o \restriction \mathcal{R}(W)$, obtained from the restriction of ω_0 to the wedge algebra. The wedge has to be regarded as a spacetime in its own right.

We determine the thermal time flow associated to this state. Recall that by the Reeh-Schlieder theorem $\omega_o \restriction \mathcal{R}(W)$ generates a modular flow $\sigma_s^{\omega_o \restriction \mathcal{R}(W)}$ in $\mathcal{R}(W)$. The Bisognano-Wichmann theorem assures that the modular operator is given by

$$\Delta = e^{-2\pi L}, \tag{6.35}$$

when L denotes the generator of the Lorentz boosts in x^1-direction in the representation of the Poincaré group. Consequently, the thermal time flow satisfies

$$\sigma_s^{\omega_o \restriction \mathcal{R}(W)}(A) = e^{2\pi s i L} A e^{-2\pi s i L}. \tag{6.36}$$

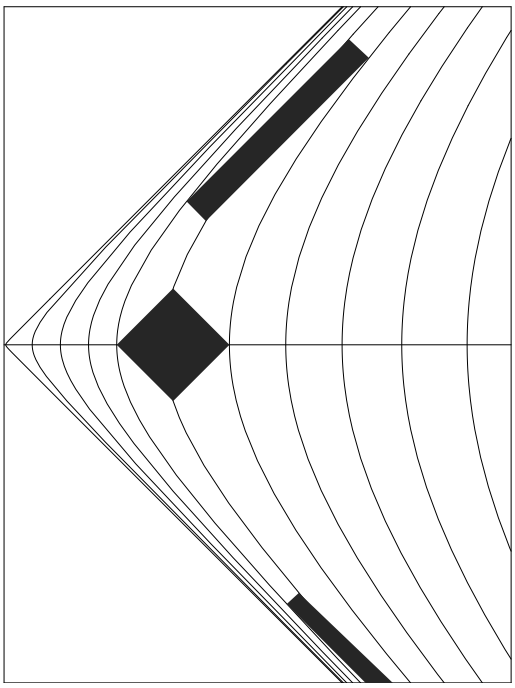

Figure 6.1: Flow lines of the modular flow induced by the vacuum state in the Rindler wedge. In addition, the evolution of the localisation region of a double-cone subalgebra under the modular flow is illustrated.

Now, let us think about the physical time flow perceived by O. We have seen that the Lorentz transformation $\Lambda(a\tau)$ carries O along his trajectory, it describes the proper time evolution of our observer. Since we are in the vacuum sector, there exists a strongly continuous one-parameter unitary representation $U(\Lambda(a\tau))$ which implements this evolution on the algebra. The generator of the unitary group is precisely aL. This suggests to interpret aL as the relevant Hamiltonian for O, which generates the evolution in physical (geometrical) time perceived by O.

$$\alpha_\tau(A) = e^{i\tau a L} A e^{-i\tau a L}. \tag{6.37}$$

We compare the two flows obtained so far,

$$\sigma_s^{\omega_o \restriction \mathcal{R}(W)}(A) = e^{2\pi s i L} A e^{-2\pi s i L} = \alpha_{2\pi a^{-1} s}(A), \quad \text{which implies} \quad \sigma_s^{\omega_o \restriction \mathcal{R}(W)} = \alpha_{2\pi a^{-1} s}. \tag{6.38}$$

Thermal time coincides up to an irrelevant rescaling with the geometrical time, which is used by the rectilinear, uniformly accelerated observer to describe the evolution of the system,

$$\tau(s) = 2\pi a^{-1} s. \tag{6.39}$$

The thermal Hamiltonian $K = 2\pi L$ is proportional to the physical Hamiltonian $H = aL$ generating the physical time of that observer. This result can be rated as another successful test of the TTH.

Let us pass to the addendum of the TTH. Since thermal time and geometrical time are proportional, their ratio, which is $\beta^{-1} = \frac{a}{2\pi}$, should be interpretable as the temperature the observer would assign to the system. But this is exactly the *Unruh temperature* predicted by the *Unruh effect* [130]: A uniformly accelerated observer in the Minkowski vacuum feels himself immersed in a thermal bath of quanta at temperature $\beta^{-1} = \frac{a}{2\pi}$.

The original derivation proposed by Unruh [130] in 1976 in the scheme of conventional free QFT is based on a comparison of two different, unitarily non-equivalent quantisation schemes in Minkowski space and in the Rindler wedge, see also [133]. For this one exploits the fact that apart from the globally timelike Killing vector fields generating the ordinary time translation isometries in the whole Minkowski space, there are additionally boost Killing fields, which are timelike in certain portions of spacetime, such as a globally hyperbolic wedge region, in which the orbits correspond to the trajectories of Lorentz boost isometries. This provides the basis for the uniformly accelerated observer to construct in the usual fashion a QFT living merely in the wedge region accessible to him. Tracing out over the degrees of freedom outside the wedge, one finds that the Minkowski vacuum formally becomes a thermal density matrix over the wedge Fock space. The thermalisation is due to vacuum correlations across the edge of the wedge. The two subsystems are highly entangled. For the accelerated observer ω_0 is not the pure vacuum state anymore, but corresponds to an (accelerated) thermal distribution of Rindler particles.

In an alternative approach (cf. [130, 135]) one integrates the interaction term of a simple discrete-level model detector coupled to the quantum field along the observer's world line. It turns out that the detector gets excited as if it was at rest in a thermal bath of Rindler particles at temperature $\beta = \frac{2\pi}{a}$. The remarkable relation between the Bisognano-Wichmann theorem and the Unruh effect, crucial for the derivation presented here, was firstly pointed out in 1980 by Sewell [116], and permits a much more general and rigorous foundation of the vacuum thermalisation: The vacuum state restricted to the wedge algebra satisfies the KMS condition at the Unruh temperature w.r.t. the boost time translations.

Summing up, from the theoretical point of view there is much evidence that the Unruh effect is in fact "real", such that we can regard the second part of the TTH as confirmed, too. In addition, the TTH sheds some new light on the nature of the Unruh effect. The amplitude of the temperature is due to a dissenting parametrisation induced by proper time and thermal time, i.e. to the speed of thermal time measured w.r.t. proper time.

Let us dwell on for a few more lines on the Unruh effect. First of all note that the Unruh temperature depends on the acceleration, i.e. two uniformly but differently accelerated observers in a given wedge region will ascribe different temperatures to the thermal bath. Furthermore, as indicated above, the Unruh effect has its origin in the fact that a uniformly accelerated observer does not have access to the full algebra but solely to those degrees of freedom which are inside the Rindler wedge W. A careful analysis (see e.g. [133]) reveals that the key geometrical structure in Minkowski space which causes the vacuum thermalisation is the Killing horizon generated by the Lorentz boost isometries.[4] That is the reason why the Unruh effect is generalisable to curved spacetimes possessing a one-parameter group of isometries which have an associated Killing horizon. In fact, we shall meet examples of those generalisations such as the Gibbons-Hawking effect.

6.5 Minkowski Diamond

Application of the Thermal Time Hypothesis

Another successful application of the TTH was studied by Martinetti and Rovelli [83]. Similar to the preceding subsection, it is to be probed on the vacuum state ω_0, restricted to the von Neumann algebra $\mathcal{R}(D)$ associated with a double-cone D. Beside the wedge, double-cones are of particular physical interest as very natural objects appearing in AQFT. Since the action of the modular flow is only known for conformally covariant theories we have to restrict ourselves to just those theories such as the massless free case.

Consider an observer O with *finite* lifetime. Let x_i be the event of his birth, and $x_f \in I^+(x_i)$ the event of his death.[5] The maximal region the observer can exchange signals with during his

[4] A *Killing horizon* is a null surface to which the Killing field generating the isometries is normal.
[5] I^+ denotes the chronological future – so this is a wise assumption.

lifetime is the diamond $D := V_+(x_i) \cap V_-(x_f)$. Thus, he has access to the local algebra $\mathcal{R}(D)$. According to the TTH, the physical time flow is singled out by the state $\omega_0 \restriction \mathcal{R}(D)$. By application of an appropriate Poincaré transformation we may assume w.l.o.g.

$$x_i^\mu = (-L, 0, 0, 0), \quad x_f^\mu = (L, 0, 0, 0). \tag{6.40}$$

The observer is supposed to be rectilinear, uniformly accelerated with acceleration $a \in \mathbb{R}_{>0}$. For convenience let us assume that the motion of the observer takes place completely in the $\{x_2 = x_3 = 0\}$-plane. The worldline can be parametrised by O's proper time τ,

$$x^0(\tau) = a^{-1}\sinh(a\tau), \;\; x^1(\tau) = a^{-1}\left(\cosh(a\tau) - \sqrt{1 + a^2L^2}\right), \;\; x^2(\tau) = x^3(\tau) = 0. \tag{6.41}$$

O's trajectory is an orbit of the group of Lorentz boosts Λ in x^1-direction with boost-parameter $a\tau$. In this case, τ is merely running from $-\tau_0$ to τ_0, where

$$\tau_0 = \frac{\mathrm{Arsinh}(aL)}{a}. \tag{6.42}$$

The geometrical modular flow is obtained from the Hislop-Longo theorem 4.5.5. Let $\mathcal{L}_L$ be the dilatation by the factor L, then the geometrical action of the modular group for a diamond D_L with radius L centred around the origin is given by

$$\begin{aligned} x^\mu(s) &= \Lambda_s^{D_L} x^\mu = \mathcal{L}_L \Lambda_s^{D_1} \mathcal{L}_L^{-1} x^\mu \\ &= L\frac{2Lx^\mu + \delta_0^\mu(2Lx^0\cosh(2\pi s) + (L^2 - x^\mu x_\mu)\sinh(2\pi s) - 2Lx^0)}{2Lx^0\sinh(2\pi s) + (L^2 - x^\mu x_\mu)\cosh(2\pi s) + (L^2 + x^\mu x_\mu)}. \end{aligned} \tag{6.43}$$

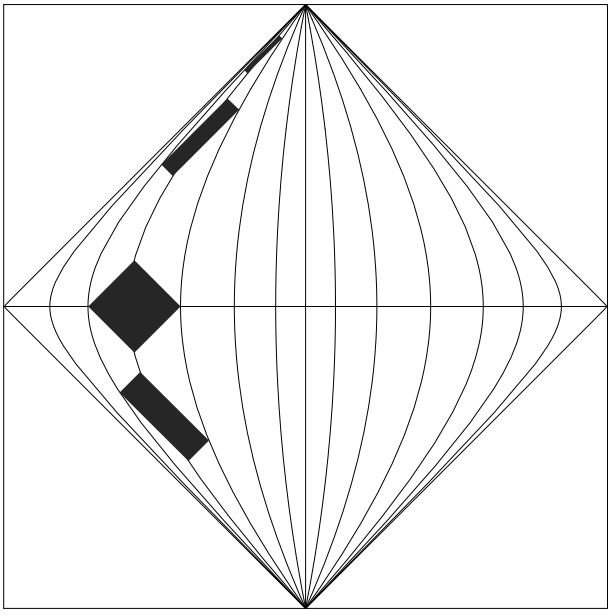

Figure 6.2: Flow lines of the modular flow induced by the vacuum state in a double-cone in the conformally covariant case. Moreover, the evolution of the localisation region of a double-cone subalgebra is illustrated.

For a given observer we choose $x^\mu \equiv x^\mu(s=0)$ such that

$$x^\mu(s=0) = x^\mu(\tau=0) \overset{(6.41)}{=} (0, a^{-1}(1-\sqrt{1+a^2L^2}), 0, 0) \tag{6.44}$$

$$\Rightarrow \quad x^0(s) = \frac{L\sinh(2\pi s)}{\cosh(2\pi s)+\sqrt{1+a^2L^2}}, \quad x^1(s) = -\frac{aL^2}{\cosh(2\pi s)+\sqrt{1+a^2L^2}}, \tag{6.45}$$

$$x^2 = x^3 = 0.$$

It can be shown that the curves (6.45), parametrised by thermal time, and the trajectories (6.41) of the uniformly accelerated observers, parametrised by proper time, coincide. Each orbit induced by thermal time is identical with the worldline of such an observer. Clearly, the parametrisations are different. However, we diagnose that thermal time corresponds up to rescaling to the unitary implementation of the proper time of these observes and this way gets a meaning as physical time. By equating (6.41) and (6.45) we obtain a relation between thermal time s and proper time τ:

$$\tau(s) = \frac{1}{a}\text{Arsinh}\left(\frac{aL\sinh(2\pi s)}{\cosh(2\pi s)+\sqrt{1+a^2L^2}}\right) \tag{6.46}$$

$$\Rightarrow \quad \frac{\mathrm{d}\tau(s)}{\mathrm{d}s} = \frac{2\pi L}{\cosh(2\pi s)+\sqrt{1+a^2L^2}}. \tag{6.47}$$

According to the second part of the TTH the vacuum looks for O like a thermal state, $\mathrm{d}\tau(s)/\mathrm{d}s$ has to be interpreted as the inverse temperature β of the thermal bath as observed by O.[6] It is useful to rewrite β as a function of O's proper time,

$$\beta(\tau) = \frac{2\pi L}{\cosh(2\pi s(\tau))+\sqrt{1+a^2L^2}} \overset{(6.45)}{=} -\frac{2\pi}{aL}x^1(s(\tau)) \overset{(6.41)}{=} \frac{2\pi}{a^2L}\left(\sqrt{1+a^2L^2}-\cosh(a\tau)\right). \tag{6.48}$$

Note that (6.42) makes sure that the inverse temperature is always positive and vanishes precisely at $-\tau_0$ and τ_0. Instead of the parameter L, it is sensible to express β as a function of the lifetime $\mathcal{T} = 2\tau_0 = \frac{2}{a}\text{Arsinh}(aL)$ of the observer:

$$\beta(\tau) = \frac{2\pi}{a\sinh\left(\frac{a\mathcal{T}}{2}\right)}\left[\cosh\left(\frac{a\mathcal{T}}{2}\right)-\cosh(a\tau)\right]. \tag{6.49}$$

Local Temperature and Causality

Eventually we have come upon an example where a local notion of temperature appears (actually this example motivated the formulation of the addendum to the TTH in the way presented here). However, it is not clear to what extent the local ratio between both time flows can be understood as a physical local temperature, which besides is related to the response of a model thermometer. The observer does not really see a true equilibrium state but rather a kind of local equilibrium, though a closer look on (6.49) shows that the temperature is to a large extent approximately constant and undergoes significant changes only near the events x_i and x_f of birth and death [83].

The dependence of the temperature on the lifetime of the observer seems to cause a second problem. An intelligent observer should be able to predict his own lifetime by measuring the temperature and using formula (6.49), which would be a clear violation of causality.[7] Similarly, two observers at the same position with the same velocity and constant acceleration but different lifetime may come to different conclusions about the local temperature at this position. The reason for all these problems lies in the assumption, that the observer has access to the full diamond during the whole of his life (which in turn is associated with the block-spacetime picture). In fact, this happens only at the end of his life. Anyway, falling back upon the energy-time uncertainty relation Martinetti could show [84] that there is actually no violation of causality, *the non-eternal observer does not live long enough to realise that he is non-eternal.* As consequence the deviations from the inverse Unruh temperature $2\pi/a$ calculated in this subsection would be not measurable.

[6] Another mathematically more elegant way of justifying this local ratio to be a temperature was suggested by Martinetti [85]. Starting from the conformal transformation relating wedge and double-cone it is shown there, that the diamond's temperature can simply be interpreted in terms of the Unruh temperature of the wedge rescaled by the conformal factor of this map.

[7] Even if one prefers to interpret x_i and x_f more humane as beginning and end of certain measuring processes, the problem would remain.

Unruh Effect for an Observer with Finite Lifetime

As for the Rindler wedge, there exists a kind of Unruh effect for the diamond, too. But note that the boundary of the double-cone is just a conformal Killing horizon,[8] whereas for the ordinary Unruh effect it is a Killing horizon. Moreover, in this case there are no alternative derivations known (using e.g. an inequivalent quantisation scheme for fields confined to D), which could be interpreted as a foundation of this result, so that the thermalisation of the double-cone is a new prediction made by the TTH.[9] Let us briefly mention some limiting cases of the Unruh effect for observers with finite lifetime.

- If the observer is sufficiently close to the edge ($\tau = 0$, $a \gg 0$), the temperature $\beta^{-1} \approx a/(2\pi)$ corresponds to the Unruh temperature. Taking theorem 4.5.8 of Fredenhagen into account this limit case does actually not require the conformal covariance of the theory.

- Consider an observer for which $L \gg 0$ is satisfied (e.g. by a very large lifetime). Then the inverse temperature can be well-approximated by the Unruh temperature and a first-order correction term,

$$\beta(\tau) \approx \frac{2\pi}{a} - \frac{2\pi \cosh(a\tau)}{a^2 L}, \tag{6.50}$$

  which disappears for $L \to \infty$.

- Finally let us consider the case in which the acceleration is small, or rather the limit of vanishing acceleration:

$$\beta(\tau) \xrightarrow{a\to 0} \frac{\pi}{L}(L^2 - \tau^2), \quad \tau \in [-L, L]. \tag{6.51}$$

  The TTH predicts a non-vanishing temperature even for an inertial observer, provided that his lifetime $\mathcal{T} = 2L$ is finite. It stands to reason to lead this back to the fact that the observer, again, does not have access to all degrees of freedom of the quantum field, which underlines that it is the presence of a causal horizon (the edge of the diamond), rather than the acceleration, which causes the thermalisation of the vacuum.

6.6 Minkowski Forward-Lightcone

Proceeding in the same way as for the double-cone, we apply the TTH to the vacuum state ω_0 restricted to the von Neumann algebra $\mathcal{R}(V_+)$ of the forward lightcone V_+ in the case of a conformally covariant theory, where we know the geometrical meaning of the modular group [83].

Consider an inertial observer which is born in x_i and lives forever. Suppose he performs a rectilinear, uniform motion with constant 4-velocity u^μ. The region he causally interacts with is then given by the future lightcone $V_+(x_i)$. The observer has access to the subalgebra $\mathcal{R}(V_+(x_i))$. We put w.l.o.g. $x_i = 0$, $V_+(0) := V_+$. The worldline of the observer reads

$$x^\mu(\tau) = \tau\gamma(\vec{v})u^\mu, \quad (u^\mu) = (1, \vec{v}), \quad \gamma(\vec{v}) = (1 - \vec{v}^2)^{-1/2}, \quad \tau \in [0, \infty). \tag{6.52}$$

The action of the modular flow is known from theorem 4.5.4 of Buchholz:

$$x^\mu(s) = e^{2\pi s} x^\mu, \quad x^\mu \in V_+, \quad s \in (-\infty, \infty). \tag{6.53}$$

Each orbit of the thermal time flow coincides with a wordline of one of the observers described above, so that one more time we assign a physical meaning to the thermal time s as a rescaling of

[8] A *conformal Killing horizon* stems from a *conformal Killing field* which is defined by the conformal Killing equation $\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu = \frac{1}{2}g_{\mu\nu}\nabla^\kappa\zeta_\kappa$ [132].

[9] There are several adaptions of the Unruh effect for observers with finite lifetime (see [84] for the relevant literature). Therein one mostly considers a detector, like e.g. a two-level system, coupled to the vacuum for a finite period of time. The result depends on the coupling as well as the switching on/off process. These investigations seem to indicate that the vacuum thermalises but the detector has not enough time to reach equilibrium with it. For that we remark that in the ordinary Unruh effect the detector needs an infinite amount of time to thermalise. One has to distinguish carefully between what the detector is displaying and the peculiarities of the state itself. Altogether, the thermalisation of the vacuum in the diamond region is at least a plausible result.

the proper time of an inertial observer in V_+. To see this and to determine the relation between thermal time and proper time, select any curve generated by the thermal time flow, i.e. fix an x^μ. Then choose $\vec{v}$ such that $x^\mu = c\gamma(\vec{v})u^\mu$ for some constant c. The corresponding curves coincide, but have different parametrisations,

$$\tau(s)\gamma(\vec{v})u^\mu = e^{2\pi s}x^\mu \quad \Rightarrow \quad \tau(s) = c\, e^{2\pi s} \tag{6.54}$$

$$\Rightarrow \quad \beta(s) = \frac{\mathrm{d}\tau}{\mathrm{d}s} = 2\pi c\, e^{2\pi s} \tag{6.55}$$

$$\Rightarrow \quad \beta(\tau) = 2\pi\tau, \quad T(\tau) = \frac{1}{2\pi\tau}. \tag{6.56}$$

The temperature diverges at the birth of the observer and decreases to zero during his life, so that this case becomes coherent with the eternal one.

6.7 Static De Sitter Space and Gibbons-Hawking Effect

We have discussed some generalisations of the Bisognano-Wichmann theorem to certain classes of curved spacetimes in section 4.5. In all these cases the TTH singles out a thermal time which corresponds up to a rescaling with the proper times of an appropriately chosen class of observers and thereby gets a physical meaning. Since these considerations are very close to the examples just discussed, they do not lead to new insights and we omit a thorough discussion. An exception in this respect are the de Sitter space as well as the extended Schwarzschild space having in store the Gibbons-Hawking effect and the Hawking effect, respectively, which we are going to pick up next.

Let us start with the former one. The highly symmetric de Sitter space was introduced in section 4.5 as a hyperboloid in 4+1-dim. Minkowski space

$$-(z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = \alpha^2, \quad \alpha^2 = 3/\Lambda = 12/R, \tag{6.57}$$

with metric

$$\mathrm{d}s^2 = -(\mathrm{d}z^0)^2 + (\mathrm{d}z^1)^2 + (\mathrm{d}z^2)^2 + (\mathrm{d}z^3)^2 + (\mathrm{d}z^4)^2. \tag{6.58}$$

Physically it is considered to play a central role during the early inflationary expansion of our universe. Introducing suitable coordinates $(\eta, \chi, \vartheta, \phi)$ the metric can be cast into the form [12]

$$\begin{aligned} &\mathrm{d}s^2 = \alpha^2 \sin^{-2}\eta \left(-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2)\right), \\ &\eta \in (0,\pi), \;\; \chi \in [0,\pi], \;\; \vartheta \in [0,\pi], \;\; \phi \in [0, 2\pi]. \end{aligned} \tag{6.59}$$

One can choose another coordinate system on de Sitter space, the *static coordinates*, with metric

$$\begin{aligned} &\mathrm{d}s^2 = -\left(1 - \frac{r^2}{\alpha^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2), \\ &t \in \mathbb{R}, \;\; r \in [0,\infty), \;\; \vartheta \in [0,\pi], \;\; \phi \in [0, 2\pi]. \end{aligned} \tag{6.60}$$

These coordinates do not cover the whole space but only a proper subspace $\mathcal{S}_{stat} \subset \mathcal{S}$, the so-called *static de Sitter space*. The coordinate transformation

$$\begin{aligned} &z^0 = \sqrt{\alpha^2 - r^2}\sinh(t/\alpha), \quad z^1 = r\cos\vartheta, \quad z^2 = r\sin\vartheta\sin\phi, \\ &z^3 = r\sin\vartheta\cos\phi, \quad z^4 = \sqrt{\alpha^2 - r^2}\cosh(t/\alpha) \end{aligned} \tag{6.61}$$

imbeds static de Sitter space in the above hyperboloid of 4+1-dim. Minkowski space, where it corresponds to the $z^4 > |z^0|$-part of that hyperboloid.

According to the Bisognano-Wichmann theorem for de Sitter space (section 4.5), the geometrical action $\Lambda_s^{\mathcal{S}_{stat}}$ of the modular group $\sigma_s^{\omega_0 \restriction \mathcal{R}(\mathcal{S}_{stat})}$ associated with the Euclidean vacuum state

ω_0 of de Sitter space restricted to the algebra $\mathcal{R}(\mathcal{S}_{stat})$ of static de Sitter space corresponds to Lorentz boosts in z^4-direction, and thus satisfies in the embedding coordinates z^A, $A = 1, \ldots, 4$:

$$\Lambda_s^{\mathcal{S}_{stat}} = \begin{pmatrix} \cosh(2\pi s) & 0 & 0 & 0 & \sinh(2\pi s) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sinh(2\pi s) & 0 & 0 & 0 & \cosh(2\pi s) \end{pmatrix}. \tag{6.62}$$

Using the transformations (6.61) we can immediately express the geometrical action w.r.t. the static coordinates

$$\begin{aligned}(t(s), r(s), \vartheta(s), \phi(s)) &= \Lambda_s^{\mathcal{S}_{stat}}(t, r, \vartheta, \phi) \\ &= (\alpha \cdot \mathrm{Arsinh}\left(\frac{z^0 \cosh(2\pi s) + z^4 \sinh(2\pi s)}{\sqrt{\alpha^2 - r^2}}\right), r, \vartheta, \phi).\end{aligned} \tag{6.63}$$

Because of the spherical symmetry we assume w.l.o.g. $\vartheta = 0$. This includes $z^2 = z^3 = 0$ as well as $z^1 = r$. All Lorentz-boosted curves in the wedge $z^4 > |z^0|$ of 4+1-dim. Minkowski space intersect $\{z^0 = 0\}$, thus we still capture all curves by putting $z^0 = 0$. Exploiting (6.57) and $z^4 > 0$ gives $z^4 = \sqrt{\alpha^2 - r^2}$. Hence, it remains to consider

$$\Lambda_s^{\mathcal{S}_{stat}}(t, r, \vartheta, \phi) = (2\pi\alpha s, r, 0, \phi). \tag{6.64}$$

The thermal time s is related to the proper time of observers moving along these timelike curves. Next, we have to understand the physical meaning of these curves. Let us start to calculate the proper time τ. Fixing the origin of proper time such that it vanishes when thermal time vanishes we get

$$\tau(s) = \sqrt{1 - \frac{r^2}{\alpha^2}}\, t(s) = 2\pi s \sqrt{\alpha^2 - r^2}. \tag{6.65}$$

From this we determine the absolute value $a = \sqrt{a^\mu a_\mu}$ of the 4-acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$, $u^\mu = \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau}$. After a straightforward calculation we find

$$a = \frac{r}{\alpha\sqrt{\alpha^2 - r^2}}. \tag{6.66}$$

Thus, the observers at rest w.r.t. the static coordinates, who are geodesic observers, are in fact uniformly accelerated. This is linked to the expansion of de Sitter space. Thermal time and proper time are proportional and coincide up to a constant factor. According to the TTH the observer should be immersed in a thermal bath of temperature

$$\beta = \frac{\mathrm{d}\tau}{\mathrm{d}s} = 2\pi\sqrt{\alpha^2 - r^2} \overset{(6.66)}{=} \frac{2\pi}{\sqrt{\alpha^{-2} + a^2}} = \frac{2\pi}{\sqrt{\Lambda/3 + a^2}}. \tag{6.67}$$

This temperature was first computed by Narnhofer, Peter and Thirring [90] who studied thermal properties of the Euclidean vacuum state in de Sitter space. Since the de Sitter universe is homogeneous and isotropic the temperature depends only on the acceleration of the observer, but not on his position or velocity. In particular, the result shows that, in contrast to Minkowski space, a freely moving eternal observer in de Sitter space will see thermal radiation of temperature $\sqrt{\Lambda/12}/\pi$. This result has already been found by Gibbons and Hawking [53] ("Gibbons-Hawking effect") who exploited the path-integral method developed by Hartle and Hawking. It is due to a peculiarity of de Sitter space, namely its exponential expansion, which implies that there exist no observer having access to the whole spacetime. The existence of causal horizons is a fundamental feature in de Sitter space. In analogy with the Unruh effect, the horizons prevent the observer from measuring the complete state and it is this loss of information which in the end is responsible for the thermalisation. Altogether, the predictions of the TTH seem to be realised in de Sitter space, too.

6.8 De Sitter Diamond

A related application of the TTH was given by Tian [126]. We remain in de Sitter space, and we restrict ourselves to conformally covariant quantum theories. In contrast to Tian we shall transform everything back to de Sitter space before applying the TTH, because it makes calculations a bit easier. Utilising the conformal covariance, a local net on Minkowski space gives rise to a local net on de Sitter space (and vice versa) [55], and it enables one to transform the notion of vacuum from Minkowski space to de Sitter space.

To explain the second point, we start with a general idea [12, 36]. Consider an arbitrary globally hyperbolic spacetime M and a globally hyperbolic flat spacetime M'. Suppose there exist subspaces $V \subset M$ and $V' \subset M'$ containing Cauchy surfaces Σ and Σ' of M and M', respectively. Assume further that there exists a conformal transformation $\alpha : V \to V'$ such that α is a diffeomorphism and $\alpha(\Sigma) = \Sigma'$. Then one can establish that for every globally timelike Killing field ξ^μ in M' there is a globally timelike conformal Killing field ζ^μ in M. A vacuum state in M' defined w.r.t. ξ^μ defines a conformal vacuum state in M via the conformal transformation w.r.t. ζ^μ. Thus, in conformally covariant theories defined over suitable conformally flat spacetimes one has a natural notion of vacuum. If M' is the Minkowski space its unique globally timelike Killing field ∂_t gives rise to a unique conformal vacuum in M.

We have seen that the de Sitter metric can be put into the form

$$\begin{aligned} &\mathrm{d}s^2 = \alpha^2 \sin^{-2}\eta \left(-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2)\right), \\ &\eta \in (0,\pi), \quad \chi \in [0,\pi], \quad \vartheta \in [0,\pi], \quad \phi \in [0,2\pi]. \end{aligned} \tag{6.68}$$

Comparison with the *Einstein static universe* $\mathcal{E}$ with metric

$$\begin{aligned} &\mathrm{d}s^2 = -\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2), \\ &\eta \in (-\infty,\infty), \quad \chi \in [0,\pi], \quad \vartheta \in [0,\pi], \quad \phi \in [0,2\pi] \end{aligned} \tag{6.69}$$

shows that de Sitter space can be conformally embedded into this spacetime. Minkowski space is conformal to a region of the Einstein universe, as well. This region is given by $\{(t,\chi,\vartheta,\phi) : -\pi < t+\chi < \pi, -\pi < t-\chi < \pi\} \subset \mathcal{E}$. The Einstein static universe may be imbedded in a cylinder in 4+1-dim. Minkowski space. The conformal images of Minkowski space and de Sitter space on the unwrapped cylinder shell constitute the *Penrose diagrams* of these spacetimes which represent the respective spacetime on a compact manifold [63].

The two conformal images have the common Cauchy surface $\{t = 0\} \subset \mathcal{E}$. Falling back on the considerations above there exists a unique conformal vacuum state ω_c on de Sitter space defined by the unique vacuum state ω_0 on Minkowski space.[10] Static de Sitter space,

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2}{\alpha^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2), \tag{6.70}$$

turns out to be conformal to a Minkowski diamond $D_{2\alpha}$. Denote by x^μ the coordinates of Minkowski space. The conformal transformation is realised by the pseudo-stereographic projection P,

$$x^\mu = P(z^\mu) = \frac{2\alpha}{\alpha + z^4} z^\mu, \quad \mu = 0,\ldots,3. \tag{6.71}$$

The inverse P^{-1} of the projection reads

$$z^\mu = P^{-1}(x^\mu) = \Omega(x) x^\mu, \quad \mu = 0,\ldots,3, \tag{6.72}$$

where $\Omega(x) = \left(1 + \frac{x^\mu x_\mu}{4\alpha^2}\right)^{-1}$ is the conformal factor of the transformation. By the Hislop-Longo theorem 4.5.5 the geometrical action of the modular flow $\sigma_s^{\omega_0 \restriction \mathcal{R}(D_{2\alpha})}$ associated with the vacuum state ω_0 confined to the local algebra $\mathcal{R}(D_{2\alpha})$ of the diamond is known. From that one derives the geometrical action $\Lambda_s^{\mathcal{S}_{stat}}$ of the modular flow $\sigma_s^{\omega_c \restriction \mathcal{R}(\mathcal{S}_{stat})}$ associated with the conformal vacuum ω_c of de Sitter space restricted to the algebra $\mathcal{R}(\mathcal{S}_{stat})$ of static de Sitter space, $\Lambda_s^{\mathcal{S}_{stat}} = P^{-1}\Lambda_s^{D_{2\alpha}} P$.

[10] Note that a globally timelike Killing field does not exist in de Sitter space.

This action coincides with the one calculated above, equation (6.62), so that the rest of the analysis is in exact agreement with the more general and fundamental justification obtained before.

However, the result generalises in a manner similar to the diamond case in Minkowski space to non-eternal observers. For this purpose let us consider a uniformly accelerated observer moving on a part of the worldline (6.63) in static de Sitter space $\mathcal{S}_{stat}$. Again, let us assume w.l.o.g. $z^0 = z^2 = z^3 = 0$, $z^1 = r$, and $z^4 = \sqrt{\alpha^2 - r^2}$. The proper time τ of the observer runs only over a finite interval, $[-\tau_0, +\tau_0]$ say. One has to determine the diamond region $\mathcal{S}_D$ in static de Sitter space which represents the causal completion of the non-eternal observer's wordline. His worldline can be identified with an orbit of the modular flow associated with the partial conformal vacuum state $\omega_c \restriction \mathcal{R}(\mathcal{S}_D)$, and the ratio between thermal time and proper time gives the temperature measurable by the observer.

One can transform the problem to Minkowski space. The conformal image of the uniformly accelerated eternal observer's wordline in Minkowski space is that of a uniformly accelerated non-eternal observer. The region he causally interacts with is $D_{2\alpha}$, the conformal image of $\mathcal{S}_{stat}$. This suggests that $\mathcal{S}_D$ corresponds to a reduced diamond D_{2M}, $M < \alpha$, which is shifted in x^1-direction in such a way that the modular action of the vacuum state restricted to $\mathcal{R}(D_{2M} + c\,\hat{e}_1)$ describes a part of the wordline in $D_{2\alpha}$, i.e. the wordline has to pass the edges of $D_{2M} + c\,\hat{e}_1$. For a given acceleration a, the shift c has to take on the value

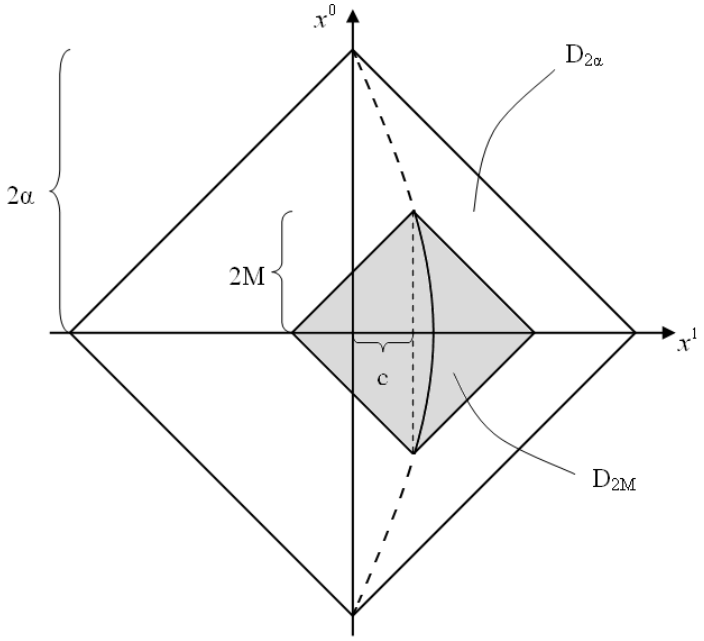

Figure 6.3: Motion of a non-eternal de Sitter observer, which perceives thermal time, represented in a Minkowski diamond.

$$c = \frac{2}{a}(\sqrt{1 + a^2\alpha^2} - \sqrt{1 + a^2 M^2}). \tag{6.73}$$

The conformal preimage of $D_{2M} + c\,\hat{e}_1$ is $\mathcal{S}_D \subset \mathcal{S}_{stat}$. The values of z^A under the projection P are

$$x^0 = 0, \quad x^1 = \frac{2\alpha r}{\alpha + \sqrt{\alpha^2 - r^2}}, \quad x^2 = x^3 = 0. \tag{6.74}$$

Let $\mathcal{T}_c$ be the translation by c in x^1-direction, then the modular action of the Minkowski vacuum

state confined to $\mathcal{R}(D_{2M}+c)$ according to the Hislop-Longo theorem 4.5.5 is

$$\Lambda^{D_{2M}+c}(s) = \mathcal{T}_c \Lambda^{D_{2M}}(s) \mathcal{T}_{-c} \tag{6.75}$$

$$\Rightarrow \quad \Lambda^{D_{2M}+c}(s)x^0 = \frac{2M\sinh(2\pi s)}{\cosh(2\pi s)+\sqrt{1+a^2M^2}}, \tag{6.76}$$

$$\Lambda^{D_{2M}+c}(s)x^1 = \frac{2}{a}\frac{(\sqrt{1+a^2\alpha^2}-\sqrt{1+a^2M^2})\cosh(2\pi s)+\sqrt{1+a^2M^2}\sqrt{1+a^2\alpha^2}-1}{\cosh(2\pi s)+\sqrt{1+a^2M^2}}.$$

The modular action transformed to de Sitter space reads in the coordinates z^A:

$$\Lambda^{\mathcal{S}_D}(s) = P^{-1}\Lambda^{D_{2M}+c}(s)P, \tag{6.77}$$

$$\Lambda^{\mathcal{S}_D}(s)z^0 = z^4\frac{a^2\alpha M\sinh(2\pi s)}{(\sqrt{1+a^2\alpha^2}-\sqrt{1+a^2M^2})\cosh(2\pi s)+\sqrt{1+a^2M^2}\sqrt{1+a^2\alpha^2}-1}, \tag{6.78}$$

$$\Lambda^{\mathcal{S}_D}(s)z^1 = z^1 = r, \quad \Lambda^{\mathcal{S}_D}(s)z^2 = \Lambda^{\mathcal{S}_D}(s)z^3 = 0,$$

$$\Lambda^{\mathcal{S}_D}(s)z^4 = z^4\sqrt{1+\frac{a^4\alpha^2M^2\sinh^2(2\pi s)}{(\sqrt{1+a^2\alpha^2}-\sqrt{1+a^2M^2})\cosh(2\pi s)+\sqrt{1+a^2M^2}\sqrt{1+a^2\alpha^2}-1}}.$$

Transforming back to the static coordinates yields

$$(t(s),r(s),\vartheta(s),\phi(s)) = \Lambda^{\mathcal{S}_{stat}}(s)(t,r,0,\phi) \tag{6.79}$$

$$= (\alpha\mathrm{Arsinh}\left(\frac{a^2\alpha M\sinh(2\pi s)}{(\sqrt{1+a^2\alpha^2}-\sqrt{1+a^2M^2})\cosh(2\pi s)+\sqrt{1+a^2M^2}\sqrt{1+a^2\alpha^2}-1}\right), r, 0, \phi).$$

Indeed, the trajectory corresponds to an observer with finite lifetime $2\tau_0$ at rest w.r.t. the static coordinates. From (6.65) and (6.66) we conclude

$$t(s) = \sqrt{1+a^2\alpha^2}\,\tau(s), \quad t_0 \equiv \lim_{s\to\infty} t(s) = \lim_{s\to\infty}\sqrt{1+a^2\alpha^2}\,\tau(s) \equiv \sqrt{1+a^2\alpha^2}\,\tau_0 \tag{6.80}$$

Using formula (6.79) we derive a relation between τ_0 and M from this expression,

$$M = \frac{\alpha\sinh(\tau_0\sqrt{\alpha^{-2}+a^2})}{\cosh(\tau_0\sqrt{\alpha^{-2}+a^2})+\sqrt{1+a^2\alpha^2}}. \tag{6.81}$$

Plugging this into (6.79) simplifies the expression significantly to

$$t(s) = \alpha\mathrm{Arsinh}\left(\frac{\sinh(2\pi s)\sinh(t_0/\alpha)}{\cosh(2\pi s)+\cosh(t_0/\alpha)}\right). \tag{6.82}$$

An application of the TTH gives the temperature of the thermal bath perceived by the non-eternal observer with lifetime $\mathcal{T} = 2\tau_0$:

$$\beta = \frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{1}{\sqrt{1+a^2\alpha^2}}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{2\pi\alpha}{\sqrt{1+a^2\alpha^2}}\frac{\sinh(\tau_0\sqrt{\alpha^{-2}+a^2})}{\cosh(2\pi s)+\cosh(\tau_0\sqrt{\alpha^{-2}+a^2})} \tag{6.83}$$

$$= \frac{2\pi}{\sqrt{\Lambda/3+a^2}}\frac{\cosh(\frac{\mathcal{T}}{2}\sqrt{\Lambda/3+a^2})-\cosh(\tau\sqrt{\Lambda/3+a^2})}{\sinh(\frac{\mathcal{T}}{2}\sqrt{\Lambda/3+a^2})}. \tag{6.84}$$

We observe that substituting $a \rightsquigarrow \sqrt{\Lambda/3+a^2}$ this is the same formula as the corresponding one for the Minkwoski diamond (6.49). The interpretation may be carried out in the same way as it was done there.

6.9 Schwarzschild Space and Hawking Effect

For the last application of the TTH we study *Schwarzschild space*. The Kruskal extension of the Schwarzschild metric is usually written in the form [132]

$$\mathrm{d}s^2 = \frac{32M^3e^{-\frac{r}{2M}}}{r}(-\mathrm{d}u^2+\mathrm{d}w^2)+r^2(\mathrm{d}\vartheta^2+\sin^2\vartheta\,\mathrm{d}\phi^2), \tag{6.85}$$

$$u,w\in\mathbb{R}, \quad \vartheta\in[0,\pi], \quad \phi\in[0,2\pi],$$

where the positive function $r = r(w^2 - u^2)$ is implicitly given by the expression

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = w^2 - u^2. \tag{6.86}$$

It is an exact solution of the vacuum Einstein equations. A deeper analysis (see e.g. [63, 132]) shows that it describes an empty spacetime containing a stationary, non-rotating, spherically symmetric, uncharged, eternal black hole of mass M with event horizon at $r = 2M$, which hides a physical singularity at $r = 0$. The Kruskal metric is manifestly of type (4.81), such that the Bisognano-Wichmann theorem is available. As a matter of course we assume a QFT satisfying all the axioms needed for this. The wedge $X^+ = \{w > |u|\}$ is simply the exterior Schwarzschild solution,

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\phi^2), \\ t &\in \mathbb{R}, \quad r > 2M, \quad \vartheta \in [0, \pi], \quad \phi \in [0, 2\pi]. \end{aligned} \tag{6.87}$$

Physically it is interpreted as representing the exterior gravitational field of the black hole. The Schwarzschild metric is transformed into the Kruskal metric by the coordinate transformation (6.86) together with

$$\frac{t}{2M} = 2\mathrm{Artanh}\frac{u}{w}. \tag{6.88}$$

The geometrical action $\Lambda_s^{X^+}$ of the modular flow induced by the vacuum state ω_0 on the extended Schwarzschild space restricted to the exterior Schwarzschild solution X^+ corresponds to the orbits in X^+ generated by the boosts $L^+(2\pi s)$,

$$\begin{aligned} \Lambda_s^{X^+}(u, w, \vartheta, \phi) &= L^+(2\pi s)(u, w, \vartheta, \phi) \\ &= (u\cosh 2\pi s + w \sinh 2\pi s, u \sinh 2\pi s + w \cosh 2\pi s, \vartheta, \phi). \end{aligned} \tag{6.89}$$

To understand which class of observers is captured by these curves, it is more convenient to express them in terms of the Schwarzschild coordinates (t, r, ϑ, ϕ). For this note that $w(s)^2 - u(s)^2 = w^2 - u^2 = \text{const.}$, and hence $r(s) = r = \text{const.}$ by equation (6.86).

$$\begin{aligned} \Lambda_s^{X^+}(t, r, \vartheta, \phi) &= L^+(2\pi s)(t, r, \vartheta, \phi) \\ &= (4M\mathrm{Artanh}\left(\frac{u\cosh(2\pi s) + w\sinh(2\pi s)}{u\sinh(2\pi s) + w\cosh(2\pi s)}\right), r, \vartheta, \phi). \end{aligned} \tag{6.90}$$

As for de Sitter space all Lorentz-boosted curves in X^+ intersect $\{u = 0\}$ which is why we are allowed to assume w.l.o.g. $u = 0$,

$$\Lambda_s^{X^+}(t, r, \vartheta, \phi) = (8\pi M s, r, \vartheta, \phi). \tag{6.91}$$

These curves are the worldlines of stationary observers in Schwarzschild coordinates (t, r, ϑ, ϕ). They perceive (rescaled) thermal time. The proper time of these observers is easily determined:

$$\tau(s) = \sqrt{1 - \frac{2M}{r}}\, t(s) = 8\pi M s \sqrt{1 - \frac{2M}{r}}. \tag{6.92}$$

We proceed by calculating the absolute value of the 4-acceleration which turns out to be

$$a = \sqrt{a^\mu a_\mu} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2}. \tag{6.93}$$

Now, we can specify the motion of the observers under consideration. They are at rest w.r.t. the gravitational centre of the spacetime which is hidden in the Schwarzschild black hole. Because of the gravitational field the observers must accelerate uniformly in order to remain stationary. The acceleration vanishes at $r = \infty$ when Schwarzschild space goes over into Minkowski space. This is due to the decrease of the gravitational field. At the event horizon $r = 2M$ of the black hole it

becomes impossible to remain stationary, or rather it would require an infinite acceleration (that is an infinite amount of energy).

Eventually, we apply the TTH to determine the local temperature which is related to the thermal bath as perceived by the observer:

$$\beta = \frac{\mathrm{d}\tau}{\mathrm{d}s} = 8\pi M\sqrt{1-\frac{2M}{r}}. \tag{6.94}$$

This temperature is in coincidence with the one predicted by the generalised Unruh effect for extended Schwarzschild spacetime [133]. It was further calculated by Unruh [130] that a detector stationed near the event horizon $r = 2M$ of a Schwarzschild black hole has a transition rate which corresponds to an accelerated detector immersed in Minkowski vacuum, a limiting case which can be recovered from (6.94),

$$a \approx \frac{1}{4M}\left(1-\frac{2M}{r}\right)^{-1/2} \quad \Rightarrow \quad \beta \approx \frac{2\pi}{a}. \tag{6.95}$$

From the general relativistic point of view the Schwarzschild Unruh effect and the (Minkowski) Unruh effect together are a manifestation of Einstein's principle of equivalence. For an accelerated observer the vacuum thermalises in exactly the same way as for an observer at rest in a gravitational field. (6.94) also implies that a stationary observer at infinity will see thermal radiation of inverse temperature

$$\beta_\infty = \lim_{r\to\infty} 8\pi M\sqrt{1-\frac{2M}{r}} = 8\pi M, \tag{6.96}$$

which is precisely the *Hawking temperature* of a Schwarzschild black hole as predicted by the *Hawking effect*, derived by Hawking [64] in 1975. The approach to the Hawking effect presented here is most originally due to Sewell [116].

Because all these results are theoretically (quite) well-founded, the thermalisation of Schwarzschild spacetime for static observers can be rated as another confirmation of the TTH.[11] Indeed, the Unruh effect was discovered in an attempt to gain more insights into the nature of the Hawking effect by replacing the difficulties associated with a collapsing star by natural boundary conditions on the past horizon of full empty Kruskal space. The close mathematical relationship between Unruh effect and (this special case of the) Hawking effect stems from the fact that the event horizon of a stationary black hole is a Killing horizon.

But note that the analogy between Unruh effect and Hawking effect, the latter one making a statement about the particle creation by a black hole at asymptotically late times, is limited (cf. [133] for more details). This is in essence due to the different notions of vacuum underlying both effects ("Hartle-Hawking vacuum" and "Unruh vacuum"). Considering a "realistic" spacetime which describes a gravitational collapse, or paying attention to a general Kerr-Newman black hole,[12] there is no analogue of the Unruh effect available, the Hawking effect on the other hand can be derived for the general case in which a gravitational collapse settles down to a Kerr-Newman black hole. Similarly, a sensible generalisation of the Bisognano-Wichmann theorem is not known for these class of spacetimes (disregarding the grievance that one first of all has to define a meaningful QFT with a distinguished vacuum state). Altogether, we conclude that the TTH may be suitable to provide an explanation for the (geneneralised) Unruh effect, while for the generic Hawking effect this is out of sight.

[11] In principle, it is also possible to apply the TTH to a "Schwarzschild diamond", conformal covariance assumed. Since it cannot be expected to give new insights concerning an understanding of the thermal time, we renounce it.

[12] The Kerr-Newman metric is known to be the most general solution of Einstein's field equations describing a stationary black hole in the electrovacuum. It is fully characterised by the parameters mass, charge, and angular momentum. Invoking Penrose's cosmic censor conjecture, the end-product of any gravitational collaps has to be a Kerr-Newman black hole ("No-Hair-Theorem").

Chapter 7

Discussion of the Thermal Time Hypothesis

"An analysis of the concept of time was my solution."
Albert Einstein, German physicist (1879-1955)

7.1 Introduction

In chapter 5 we presented the thermal time hypothesis as it was postulated by Connes and Rovelli [42]. We explicated their idea to relate the physical time flow of a given system in a certain state to the equilibrium dynamics of that state. The TTH is supposed to explain why a particular flow on the algebra of observables is distinguished and causes our perception of time in a fundamentally timeless world. As the authors write themselves, the formulation of the TTH remains vague, since there is no rigorous definition of the term *physical time* available; the problem is precisely to understand the notion of time in generally covariant quantistic or statistical theories. From their viewpoint thermal time is the appropriate generalisation of non-generally covariant notions of time for a hypothetical generally covariant theory of quantum gravity.

The TTH is realised by attaching a physical and this way some sort of real meaning to the modular flow. In fact, the physical meaning of the modular objects appearing in physics is still a mystery. Although the behaviour of a state ω as a KMS equilibrium state under the action of the modular flow $\sigma_s^{\omega\restriction\mathcal{R}(\mathcal{O})}$ seems to suggest that it is some kind of (perturbed or distorted) local dynamics of an observer confined to the spacetime region $\mathcal{O}$, all attempts to justify such an interpretation in general have failed so far. The TTH postulates a physical meaning and thereby essentially shifts the problem to an understanding of the physical meaning of the thermal time concept. The core of the issue remains the same, one would like to gain more insights into the nature of the modular structure and the hidden symmetry coming along with the modular flow. A better, comprehensive knowledge about modular theory is a prerequisite towards a full understanding of the thermal time concept, necessary to decide about its validity. Unfortunately, there is still a long way to go to achieve this aim. What actually can be done is to analyse the TTH from a general qualitative perspective, to investigate its compatibility with certain natural expectations on a physical time flow, and to study its meaning in the sparse number of examples, which unfortunately turn out to be of a very special nature.

This is the point where we want to start our discussion. It is our intention to explore the TTH from various viewpoints. Our goal is to give an objective analysis of the thermal time concept, which, in addition, brings together all the previous chapters of this thesis. We continue to pay attention mainly to non-generally covariant systems. Moreover, it will be useful to distinguish in the following between a *physical time* (cf. chapter 1), which we think of as a time which in some sense is measurable or perceivable by an observer, and a *time with physical properties*, which behaves very similar to a physical time and may provide a useful tool to formally evolve the system and extract certain pieces of information; but which by itself does not need to be directly or indirectly related to the experience of time of an observer in the relevant spacetime region and to represent the local dynamics for him.

First of all we want to discuss the examples of the preceding chapter w.r.t. their significance in supporting the validity of the TTH. Furthermore, we investigate in which situations thermal time can be expected to reduce to non-generally covariant notions of time or to explain the emergence of (certain aspects of) common sense time. We also want to look into the quantum and classical mechanical limit, where the TTH originally comes from. In this limit its formulation is somewhat more natural, since one deals with true density matrices, and, in addition, the action of the modular flow is well-understood. In particular we would like to understand the mechanism which distinguishes the symmetry disclosed by thermal time in this limit.

We will further discuss the physical input which enters the TTH, and if it really defines a complete and consistent time concept which reflects our temporal experience of the world. As we shall see there are a couple of problems impeding the acceptance of thermal time as the origin of time, and we will think about possible restrictions e.g. on the state, guaranteeing that a physical thermal time does emerge.

In the second part starting with section 7.11, we shall take a look at what properties are actually desirable in order to obtain a flow which can be justified at least as a flow with manifestly physical and thermal properties, and discuss to what extent these properties are already incorporated in the thermal time concept, or, on the other hand, if they might be in conflict with it. Above all this concerns the issues if the KMS condition is per se sufficient to ensure the typical dispersive behaviour of a true equilibrium flow, and if thermal time is consistent as a state-dependent flow. We shall do it first on type I factors and add some mainly heuristic considerations on local algebras.

In the classification scheme of chapter 1 thermal time clearly is an internal time. Once the restriction on a specific local algebra took place (on a certain class of observers which interact with the same spacetime region), such that one can forget about the exterior structure of the theory, the thermal time flow is global and unique (disregarding for the time being possibly necessary inequivalent metrizations for the various observers, see below).

7.2 Compatibility between Thermal Time and Relativity

We start our analysis by considering again those cases in which the thermal time flow acts geometrically. Then, thermal time gains a physical meaning be reducing locally to the proper time of observers moving along the modular flow lines. In this section we investigate a bit more detailed in which way thermal time can be regarded as a globally meaningful time, i.e. gives rise to a sensitive evolution on the whole local algebra from an observer's point of view.

Wedge Region

In section 6.4 it was argued that the Hamiltonian H of a uniformly accelerated observer O with acceleration a generating his physical time flow has to be aL, L the generator of the implementation of the Lorentz boosts, because it locally reduces to O's proper time. One can even go a step further to justify this identification. To do so, we describe the wedge region via *Rindler coordinates* [132],

$$\mathrm{d}s^2 = -x_R^2\mathrm{d}t_R^2 + \mathrm{d}x_R^2 + \mathrm{d}y_R^2 + \mathrm{d}z_R^2, \quad t_R, y_R, z_R \in \mathbb{R}, \quad x_R \in \mathbb{R}_{>0}. \tag{7.1}$$

In terms of the ordinary Minkowski coordinates (t, x, y, z) the Rindler coordinates, which cover solely the wedge region, read

$$t_R = \mathrm{Artanh}(t/x), \quad x_R = \sqrt{x^2 - t^2}, \quad y_R = y, \quad z_R = z. \tag{7.2}$$

(7.1) clearly is a static coordinate system. Thus, in the rest frame of Rindler observers satisfying $(x_R, y_R, z_R) = \text{const.}$, which are precisely the uniformly accelerated ones, an unambiguous consistent synchronisation of all static clocks via light signals in the sense of Einstein is possible (which requires rate-synchronisation as well as synchronising their zero-settings). t_R is a preferred, globally defined world time in the Rindler wedge (cf. [78, 103]). Considering again the uniformly accelerated observer O, his proper time can be extended to a global time on the whole wedge when replacing t_R by $t_R^{(a)} := t_R/a$. Along any other orbit, characterised by the acceleration a' say, the relation between proper time $\tau^{(a')}$ and $t_R^{(a)}$ is

$$t_R^{(a)} = \frac{a'}{a}\tau^{(a')}. \tag{7.3}$$

Exactly the same relation to proper time holds for the geometrical time generated by the boost Hamiltonian $H = aL$ along the boost orbits, which therefore coincides with the world time $t_R^{(a)}$. On a geometrical level the (rescaled) thermal time induced by the vacuum state on the wedge algebra obtains a physical meaning not just locally along the modular flow lines, but globally on the whole wedge. It corresponds to the global world time a uniformly accelerated observer would use from the relativistic point of view according to Einstein synchronisation. This strongly suggests that on the quantum level the thermal Hamiltonian is in fact the most appropriate physical Hamiltonian of a uniformly accelerated observer to describe globally the dynamical evolution of all the wedge localised observables.

Another support comes from an alternative quantisation scheme on the wedge. The construction of a free QFT confined to the Rindler wedge is based on the boost Killing field ∂_{t_R}. t_R thus gives rise to a natural time evolution of the quantum fields for uniformly accelerated observers.

Conformal Diamond

Next we investigate if the thermal time flow on the double-cone algebra (in the conformal case) is also globally interpretable from an observer's point of view. In particular this concerns the question why the non-eternal inertial observer should use the (rescaled) thermal Hamiltonian K instead of the the ordinary Hamiltonian H. Of course, the ordinary time translations do not define an automorphism group on the double-cone algebra. However, given an observable localised in a sub-double-cone it can be evolved by H for a certain amount of Newtonian time until the localisation region of the observable is about to leave the double-cone. For the non-eternal inertial observer the (geometrical) evolution then looks a bit like what a static observer in Schwarzschild space would see when an object passes the event horizon and falls into the black hole: The light signals which reach him until the end of his life come from a localisation region which approaches the horizon, while in fact (w.r.t. Newtonian time) the localisation region of the observable has already left the double-cone. Clearly for the eternal inertial observer nothing particular happens in the diamond region, the horizon does not exist for him. He continues his motion beyond the top of the diamond.

Let us do analogous considerations for the diamond we did for the wedge before. For convenience we restrict ourselves to one spatial dimension and a diamond of radius 1 centred around the origin. The metric has the form[1]

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}t^2 + \mathrm{d}x^2, \quad |t| + |x| < 1. \tag{7.4}$$

We introduce coordinates (s, r) such that observers and objects moving on the trajectories of the modular flow become stationary (cf. equation (4.77)),

$$t = \frac{(1-r^2)\sinh(2\pi s)}{(1-r^2)\cosh(2\pi s) + 1 + r^2}, \quad x = \frac{2r}{(1-r^2)\cosh(2\pi s) + 1 + r^2}, \quad s \in \mathbb{R}, \quad |r| < 1 \tag{7.5}$$

$$\Rightarrow \quad \mathrm{d}\tilde{s}^2 = \frac{1}{\left(\cosh^2(\pi s) - r^2\sinh^2(\pi s)\right)^2}\left[-\pi^2(1-r^2)^2\mathrm{d}s^2 + \mathrm{d}r^2\right] \tag{7.6}$$

By construction s is precisely the thermal time. Although these coordinate system is not static (it is merely conformal to a static metric), the components g_{0i} vanish and hence coordinate time again gives rise to a global time on the diamond in accordance with Einstein synchronisation (cf. [78]); in this reference frame events are happening simultaneously if they happen at the same coordinate time, that is thermal time s. Rescaling s such that it reduces on one specific flow line to the proper time of the corresponding observer, on a geometrical level the rescaled thermal time is a natural extension of proper time to a physical, global time in the whole double-cone. This indicates again that on a quantum level the thermal time flow defines globally relevant dynamics for an observer moving along the modular flow lines. For the vacuum state restricted to the double-cone one can find a reference frame w.r.t. which thermal time can be interpreted. The crucial point is that the relevant coordinate system is not given by the ordinary Minkowski coordinates (t, x) but by the "diamond coordinates" (s, r). In this reference frame the state looks like an equilibrium state, and all observers at rest feel to be immersed in thermal equilibrium at temperature (6.49).

[1] For reasons of clarity we prefer here the symbol $\mathrm{d}\tilde{s}$ rather than $\mathrm{d}s$.

If we reconsider the non-eternal inertial observer it becomes clearer what is really going on. The different possible time evolutions of an observable localised in a sub-double-cone (generated by H or K) are due to different choices of the coordinate system, i.e. to different foliations of spacetime into space and time. The one-parameter group of isometries generated by the globally timelike Killing field $(\partial_t)^\mu$ on the whole Minkowski space does not reduce to an isometry group on the double-cone, in particular $(\partial_t)^\mu$ is neither a Killing field nor a conformal Killing field on the double-cone. Since $(\partial_s)^\mu$ *is* a conformal Killing field, the coordinate system (s, r) is somehow preferred and one might even say that it is best suitable to describe the time evolution – especially when taking into account that we are in the conformal case, where it provides a natural basis to do QFT. It is guaranteed that the evolution in s is unitarily implemented. This explains why for the non-eternal inertial observers (as well as for all the non-eternal uniformly accelerated ones) the thermal time evolution is the most natural one on the double-cone; it is not the ordinary one generated by H which we are used to from eternal inertial observers.

Other Cases

Similar arguments can be put forward for future-cones, static de Sitter space, de Sitter diamonds or Schwarzschild space. We will not elaborate on that. From a geometrical viewpoint thermal time does not merely reduce spatially locally to proper time, it gives rise to a global time evolution, which is interpretable in a sensible manner by observers in appropriate reference frames. The reason why this interpretation works is related to the fact that in all these examples the geometrical realisation of the modular flow corresponds to a one-parameter group of (conformal) isometries of the underlying spacetime, whose associated timelike (conformal) Killing field is hypersurface orthogonal.[2] That means that there exists a spacelike hypersurface Σ which is orthogonal to the orbits of the (conformal) isometry, and which preserves this orthogonality property when evolved in thermal time. A coordinate system which uses thermal time (the Killing parameter) as time coordinate as well as spatial coordinates on Σ is (conformal to) a static frame and necessarily has vanishing g_{0i}-terms, crucial for the above global interpretation.

In the general case of an arbitrary (timelike) geometrical action in the sense of definition 4.5.1 thermal time does not need to define a global geometrical time on the whole spacetime region under consideration. The coordinate system constructed from the modular flow lines does not need to permit a consistent synchronisation of all clocks. Then, from the relativistic point of view thermal time is just a spatially local physical time along the flow lines.

Conclusion *In the known cases in which the thermal time flow acts geometrically, its geometrical realisation coincides with the global time in the relevant spacetime region suitable observers will use in appropriately chosen coordinate systems according to Einstein's synchronisation of clocks.*

7.3 Significance of the Examples of Chapter 6

In the previous chapter we discussed several examples on which the TTH was probed. Let us come back once again to these examples. We want to discuss their relevance for really representing cogent and convincing confirmations of the TTH.

The first example concerned an application to KMS equilibrium states, where the TTH works just by definition; equilibrium states contain the full information about the dynamics. A similar conclusion can be drawn from the example where the Tolman-Ehrenfest effect came out. The system is assumed to be in equilibrium w.r.t. the flow corresponding to the globally timelike Killing field. The proper time of observers moving along the Killing flow lines is necessarily just a rescaling of thermal time both parametrising the isometry group defined by the Killing field. Let us underline the main insight: An application to ordinary equilibrium states can by no means be rated as an indication for the TTH to be true, because it might be possible just as much that thermal time and Hamiltonian time coincide accidentally because of the artificiality of the example. We shall argue that the problem with all the other examples presented in chapter 6 concerning their significance lies in the peculiarity that they are either of this type as well, namely that they are constructed in such a way that it is known from the beginning that the system appears to be

[2] A vector field ξ^μ is called *hypersurface orthogonal* if it satisfies the equation $\xi_{[\mu}\nabla_\nu\xi_{\sigma]} = 0$ [132].

in equilibrium for a class of observers, or that the thermal time flow has a geometrical realisation in spacetime which actually can be interpreted just the same. Either way, a physical interpretation of thermal time becomes easily possible, it is the physical time of this class of observers.

For an arbitrary non-equilibrium state (w.r.t. ordinary time on the total von Neumann algebra) one cannot expect that there exists a local observer for whom the system appears to be in equilibrium (see below). Above all these states should be investigated to decide whether thermal time has a physical meaning. For that reason none of the examples is a genuine confirmation, or just indication, of the correctness of the TTH (we focus on the time part, not on the temperature prediction). One has to be careful to avoid jumping to conclusions. To quote Requardt [102], "*physical intuition as to what one has to expect in the general situation is mostly drawn from this meager sample of models, which might be dangerous as one might easily regard a feature as being truely fundamental which turns out [...] to be only an 'epiphenomenon'*".

Robertson-Walker Space

Let us consider the Robertson-Walker space filled with a Maxwell field. One starts with a generally covariant system, but eventually all gauge degrees of freedom are fixed. The coordinate time gets a physical meaning as the proper time of the isotropic observers. The statistical state of the cosmic background radiation is fixed as a Gibbs state at a temperature which is defined in the standard Robertson-Walker time, and thereby constructed in such a way that it appears as an equilibrium state for isotropic observers. Thus, there is no surprise that the TTH directly applied to this state singles out a time variable which is proportional to the Robertson-Walker time.

In [108] it is argued that the distinction of the Robertson-Walker time as the correct physical time is trivial only because of the specification of the state. Before this is done all variables should be equally well suitable candidates as (local) internal times. Our argument is that the whole example becomes somewhat trivial due to the fact that we, as isotropic observers, perceive a time flow w.r.t. which the background radiation is in equilibrium. But what is cause and what is effect? The crucial question which cannot be answered is if the cosmic background radiation is in a Gibbs state because the TTH works and hence determines the time flow we perceive, or, conversely, if the TTH works because the background radiation is "accidentally" in such a state. The fact that thermal time supplies a physical time is again independent of its validity. For the special relativistic system one can bring forward the same argument.

Let us say a few more words concerning the statistical state of the universe which, according to the TTH and following the respective paper by Rovelli [108], determines our perception of time, and is besides regarded as the most important application of the TTH. The cosmic microwave background (CMB) is just a remnant of a very early period of our universe, namely the period when matter and radiation decoupled at $\sim 4 \times 10^5$ years ("time of last scattering"); matter recombined and radiation began a free expansion [132]. It would be astonishing if the state of this background radiation determines a time flow which dominates the tiny world of our everyday-life. Our experience of time would sensitively depend on the presence of the thermalised CMB.[3] If e.g. matter was not so remarkably homogenously distributed when the decoupling took place, we might have a fundamentally different perception of time just because of resulting inhomogeneities in the CMB. And what happens if the CMB disappears from one moment to another? Does it mean that our perception of time stops? Undoubtedly a very radical viewpoint.

In principle the TTH should be applied to the whole universe, including gravity. W.r.t. thermal time it would appear to be in equilibrium. But this is not the case, in particular life cannot exist in equilibrium. Regions with entropy gradient are needed. Only from a very gross, averaged point of view the universe may appear to be in equilibrium. To explain the fact that our universe is not in equilibrium, it is suggested by Rovelli [107] to separate certain degrees of freedom from the system and apply the TTH only to the remaining system, such that the CMB could explain our time, though the entropy could still grow. However, such a separation would be unnatural and veil the mathematical elegance on which the TTH is based. Moreover, it would lead to the conceptual issue how to decide for an arbitrary system which degrees of freedom are relevant for the emergence of thermal time.

[3] Note that the CMB shows certain anisotropies and is in fact not perfectly thermalised.

Unruh Effect

The next example was about the Unruh effect. The TTH is applied to the vacuum state of Minkowski space restricted to the algebra associated to the Rindler wedge. In that case one does not immediately start with an equilibrium state. However, by the Unruh effect it is well-known, that this state is perceived by a certain class of observers, namely the uniformly accelerated ones, as a thermal equilibrium state. It is least of all clear in this example that the TTH predicts precisely the time flow of these observers. On the contrary, this relation is only established by the highly non-trivial Bisognano-Wichmann theorem. A priori it would have been possible as well that the associated modular flow had a non-geometrical action where it might have been much harder to find a physical interpretation. But the fact that the modular flow admits a simple geometrical interpretation (which besides corresponds to a hypersurface orthogonal Killing field) makes the example trivial in view of a test of the TTH. It defines a meaningful evolution on the wedge in the rest frame of uniformly accelerated observers.

A similar argument works in all other cases in which the modular group has a geometrical meaning realised by timelike orbits in the underlying space (cf. section 4.5 and chapter 6), thermal time can be interpreted as (a rescaling of) the unitary implementation of the proper time of observers whose worldlines coincide with the modular flow lines, and may even represent a natural global time evolution for them. If there is an approximate geometrical meaning in certain subregions (see section 4.5), an identification with proper time is partly possible. In all these cases the TTH works perfectly well,[4] though essentially independently of its correctness.

It would be of paramount interest to study the thermal time flow on states generating a modular flow with no geometrical meaning, and to understand its physical meaning or rather if there is any. In such a case thermal time cannot be a Hamiltonian time which implements time translations in the underlying spacetime, and thus one would have a challenging test of the TTH.

Conclusion *The examples do not really help to gain insights into the TTH, in particular they cannot be rated as confirmations of its validity.*

7.4 Thermal Time as the Origin of Time

Using the example of the wedge let us point out an issue which arises even if the modular group has a geometrical meaning when taking the idea of the TTH seriously. The thermal time induced by the vacuum state on a wedge region reduces to geometrical Lorentz boosts. As elucidated in detail, one obtains a physically meaningful interpretation of thermal time, which will be used by all uniformly accelerated observers to describe the evolution of the wedge-localised observables. This was rated by Connes and Rovelli as a confirmation of the TTH. We think that it is the wrong approach, at least in view of its main goal which is to provide an explanation for the origin of time penetrating our world. Let us explicate this point.

The TTH says that for an observer confined to a spacetime region $\mathcal{O}$, the partial state on the local algebra $\mathcal{R}(\mathcal{O})$, supposed to be faithful and normal, generates a time flow which is best suitable to describe the evolution of the system. The TTH does not make any restrictions concerning the class of observers which perceives thermal time. In fact, the TTH is claimed to explain the origin of time, and, thus, should be applicable to all observers in the corresponding spacetime region; each observer should perceive a flow of time and not just a special class. The issue is that there are not only uniformly accelerated observers which are confined to the wedge, also for the other observers thermal time should be the relevant time.[5]

However, there is no reason why the thermal time flow, i.e. the unitary implementation of the Lorentz boosts, should give the relevant dynamics for them. The geometrical meaning of thermal time has nothing to do with their proper times and their separation of spacetime into space and time. Of course, thermal time is in a sense still distinguished for these observers, because one can generally not expect an automorphism group on the algebra whose geometrical meaning contains

[4] Note that even if thermal time and proper time can be identified with each other in certain situations, both time concepts have a totally different origin. Proper time is a classical, geometrical time concept, while thermal time arises in a thermodynamical context and is of a statistical origin.

[5] Of course the same issue arises on the conformal diamond, where in fact *each* observer moving from its bottom to its top has a world line whose causal completion is the diamond.

the worldline of the respective observer. So thermal time might be the most promising candidate available. Nonetheless, we do not see any physical reason why also a non-uniformly accelerated observer causally interacting with the wedge should describe the system in terms of thermal time, i.e. via the Lorentz boosts.[6]

The presumably disturbing appearance of two different times, the wedge-globally defined thermal time flow which is used to describe the evolution of the observables and the spatially local, geometrical proper time, are resolved by Rovelli [113]. He distinguishes between a purely mechanical time and the more complex thermal time concept: Mechanical time, like e.g. proper time, which is determined by the spacetime metric, is not suitable as a "flowing time"; while mechanical time governs the behaviour of mechanical clocks, it is thermal time which we perceive.

We shall construct an example where that issue becomes even more severe, and which shows that dismissing proper time as a mechanical non-flowing time might be a bit too simple. For this purpose we consider a vector state $\Psi \neq \Omega_0$ with finite energy content and differing from the Minkowski vacuum state Ω_0. The Reeh-Schlieder theorem applies and makes sure that Ψ is cyclic and separating for the wedge algebra. On factors the modular flows induced by different states cannot coincide but differ by a non-trivial inner automorphism (cf. p. 54), i.e.

$$\sigma^{\Psi} \neq \sigma^{\Omega}. \tag{7.7}$$

Consequently, the modular flow induced by Ψ cannot have a geometrical meaning corresponding to the orbits of uniformly accelerated observers. Nevertheless, the proper time translations induced by the Killing field are unitarily implemented (we are still in the vacuum sector). For a uniformly accelerated observer with acceleration a they read

$$\alpha_\tau(A) = e^{i\tau aL} A e^{-i\tau aL}. \tag{7.8}$$

Proper time gives rise to a corresponding quantum evolution on the observable algebra just the same as thermal time, and, hence, should be "flowing" just as much as thermal time. We have to deal with two "competing" time flows. There is no reason why the statement of Connes and Rovelli that α_τ *can be interpreted as the physical time characteristic of the [uniformly accelerated] observer* O should fail to be true just because the state has changed. Besides, α_τ remains a preferred flow from the geometrical point of view, a uniformly accelerated observer will still base the construction of a (free) QFT in the wedge on the presence of the globally timelike Killing field.

An argument is needed which explains why O should use σ_s^{Ψ}, which is *not* just some rescaling of α_τ, to describe the evolution of the observables, although his "standard" notion of time, α_τ, is available. Basically, one should expect the Lorentz boost time to be still his relevant physical time, though it does not represent KMS equilibrium dynamics anymore. While starting with the vacuum state Ω_0 one has according to Connes and Rovelli "*two independent and compatible definitions of time flow*", the thermal time flow and the proper time flow, this compatibility is violated if the system is in the state Ψ (note that the same argument applies to a dense subset of states in the relevant Hilbert space).[7]

For the construction of this example the state-dependence of thermal time was exploited together with the fact that the notion of time in conventional QFT is not state-dependent.[8] Under the plausible assumption that quantum gravity effects can be neglected, one would expect the Lorentz boosts to approximate in some sense the "true" time for O in the state Ψ. One should still ascribe a "reality" to the unitary implementation of the Lorentz boosts as the most reasonably time flow for uniformly accelerated observers. It can certainly not be expected that thermal time approximates the Lorentz boost time for all the states for which the Reeh-Schlieder property holds (though possibly for sufficiently small energies, cf. also section 7.17). Thermal time does not seem to be the most natural physical time for uniformly accelerated observers immersed in an excited vacuum state of finite energy content.

[6] A similar objection also holds for the Robertson-Walker example. The TTH singles out a flow which is very natural for the given state to describe the time evolution for isotropic observers, while the perception of time of non-isotropic observers remains an unsolved issue.

[7] One might speculate that thermal time can be understood if O uses another coordinate system in which the thermal time flow (though possibly acting non-geometrically) gives rise to the most natural time evolution. Since the modular flow originates from one single algebra and is not related to the net structure, i.e. to the underlying spacetime, we do not see any reason that this can indeed be expected (we take up this point later again).

[8] It thus might be more reasonable to use the canonical flow, such that the troublesome inner automorphism connecting both flows becomes irrelevant, though this flow causes other problems as we shall see later.

Altogether, we have to conclude that either merely a restricted class of local observers perceives thermal time (which depends on the state), or that there are observers for whom thermal time is a rather "exotic" concept, which for some unknown reasons drives out conventional time flows.[9] Preferring the first alternative, at most for a proper subclass of observers the thermal time flow may define physical dynamics (much more work is left to justify just such a restricted version). In the known examples this class most reasonably seems to consist of those observers following the geometrical flow lines of the modular flow. An arbitrary observer will generally not use the thermal Hamiltonian to describe the evolution of the system. If not all observers perceive thermal time, one is tempted to speculate that there exist cases, most likely those where the modular group has no geometrical meaning, where this class is empty. In fact, the time concept provided by the TTH depends on the state of motion of an observer only as far as it affects the spacetime region he is confined to, i.e. on his asymptotic state of motion.[10] With regard to relativity it is questionable if a time concept, which is to a large extent observer independent, makes sense.

Anyway, the perception of time should not be confined to special observers, since otherwise, making the assumption that the perception of a flow of time is not linked to observers in specific states of motion, the TTH would not be suitable to explain the origin of time. The example of the wedge tells us, that if one wants thermal time to explain the origin of time, a geometrical interpretation is per se not a sufficient confirmation and one has to investigate those examples more carefully. If though one is mainly interested in a physical understanding of thermal time, a geometrical interpretation (via timelike orbits) is fully satisfactory.

Conclusion *The thermal time flow does not seem to be an appropriate and natural physical flow for each observer (though it might admit a physically meaningful interpretation). If this is true, the TTH would not explain the perception of time for arbitrary observers.*

7.5 Thermal Time as an Entirely New Concept of Time

According to Connes and Rovelli "*the thermal time hypothesis is [...] the suggestion of taking the modular flow as the relevant generalisation of the non-relativistic time*". Thus, thermal time must permit a meaningful physical interpretation in the non-generally covariant limit as prerequisite to be acceptable as sensible generalisation of the notion of time in the generally covariant case. In the best case thermal time reduces to one of the standard notions of time existing in this limit, the latter ones being well-established in the sense that they can be measured by suitable observers and are known to give rise to a reasonable evolution of the system. As we shall see this generally cannot be expected, thermal time cannot be regarded as a generalisation of non-generally covariant times and one has to justify an entirely new time concept.

Thermal Time Induced by Non-Equilibrium States

Connes and Rovelli argue that equilibrium states are preferred states and that physics is well-described by small excitations around those states, which is why they probe the TTH in the non-generally covariant limit on KMS equilibrium states. It is certainly true that equilibrium states are distinguished, but it does not entitle to consider exclusively such states. The problem simply is that we know that there are systems in Nature which are not in equilibrium. For instance the whole universe itself is far away from being in thermal equilibrium. There are macroscopic observables whose expectation values do evolve in time. We do not see any reason why the TTH should not be applied in those cases, as well. As mentioned before it is desirable to recover an ordinary notion of time. However, w.r.t. thermal time a non-equilibrium state becomes a KMS equilibrium state. So what happens if the TTH is applied to arbitrary non-equilibrium states w.r.t. the conventional Hamiltonian time? This issue was first pointed out by Requardt [102], even before the formulation of the TTH. Let us recall the argument.

[9] The TTH assigns a kind of universal, global time to each local algebra and thereby to observers living in the underlying spacetime region (possibly up to some rescaling). One may ask if such a global time is really necessary (note that the interpretation of quantum theory is assumed to work without any notion of time), particularly as the theory of relativity says that each observer has his own (locally) meaningful time. There is no reason why the time flow we perceive should have a global, to a certain extent observer independent character.

[10] albeit possibly differing by inequivalent metrizations, see below

For this purpose we consider a non-generally covariant quantum statistical system $(\mathcal{R}, H, \omega)$ with an observable algebra $\mathcal{R}$ and a Hamiltonian H, which is supposed to be in a faithful normal state ω. But now we choose ω as an arbitrary non-equilibrium state, i.e. a state which does not satisfy the KMS condition w.r.t. the time flow α_t generated by H.[11] Even more, we assume ω to be not stationary. The TTH should give rise to a physical time flow. By definition the state ω satisfies the KMS condition w.r.t. the thermal time flow σ_s^ω induced by the modular Hamiltonian $K = -\log \Delta$ and thus is in KMS equilibrium. As a consequence, the measurable expectation values of the observables are stationary when evolved in thermal time

$$\omega(\sigma_s^\omega A) = \omega(A) \text{ for all } s \in \mathbb{R}, \tag{7.9}$$

while they evolve non-trivially in ordinary Hamiltonian time,

$$\omega(\alpha_t A) \neq \omega(A) \text{ in general.} \tag{7.10}$$

Even if the modular structure could have a deeper meaning as a hidden symmetry in the context of non-equilibrium statistical mechanics, Requardt concludes from this result, that the thermal Hamiltonian K cannot be regarded as some sort of perturbed or distorted original dynamics, the thermal time has nothing to do with the real Hamiltonian time.

Possible Notions of Physical Times

Let us get an idea about what the term "physical time" could mean. One main feature of an AQFT is that a class of Hamiltonians is defined via a representation of the Poincaré group generating time flows. These can be measured by suitable observers. A Hamiltonian time of this kind is a physical time. However, such a time flow is defined on the algebra associated with the whole Minkowski space (or the wedge algebra), while the thermal time flow acts typically as an automorphism group on a local algebra associated with a finite, connected spacetime region. Since ordinary time flows do not simply reduce to automorphisms on a local subalgebra, they cannot reduce to the thermal time flow. Even more, if one assumes the time flow to be an automorphism group on the relevant local observable algebra, ordinary time flows are no suitable candidates anymore.

Changing the reference frame, i.e. the foliation of spacetime, may substantially change the time evolution. In another reference frame thermal time might be interpretable e.g. by relating it to coordinate time translations as in the case of the conformal double-cone. Standard QFT as a special relativistic theory does not capture the dynamics in arbitrary reference frames, which, in principle, leaves the opportunity to justify thermal time as a natural evolution in the respective frame in some sense. Altogether, there remains the possibility to relate thermal time to a physical time without identifying it with the ordinary time flow.[12]

Eventually, it is possible that thermal time has to be understood in an entirely new manner. In chapter 1 we have seen that the notion of time does not need to show all the specific properties of common sense time in order to be regarded as a physical time. It is sufficient to show certain physical objective properties. Maybe different physical times emphasize different aspects of common sense time (e.g. we have mentioned that, according to Rovelli, thermal time as a "flowing time" and ordinary time as a rather mechanical time are on differing footings).

Anyway, the above example is undoubtedly an indication that the TTH has to be handled with care, but by no means it can be rated as a counter-example, unless the term "physical time" is specified. The non-coincidence of thermal time and ordinary time does not exclude the former one as a physical time. In the remainder of this section we want to devote ourselves further to

[11] To make sure that both flows act on the same algebra we particularly think of the quantum mechanical limit.

[12] In addition, an internal clock time is physically meaningful and could provide an interpretation of thermal time. It suffices to focus on the classical case. It is measured via a partial observable on the configuration space (appendix B). But there are many problems with it. While thermal time is a Hamiltonian time, one cannot expect the same from a clock time. In general it will be defined only locally. A description in terms of a clock time does not influence the presymplectic structure. The parametrisation is changed, while the physical motions remain the same. In contrast to that, the thermal Hamiltonian will generate completely different physical motions. Lastly, by definition of Newtonian time t, an internal time is expressible as a function of it. The fact that all expectation values are constant in thermal time s but in the non-stationary case non-constant in t underlines that such a relation cannot exist. Internal clock times are generally out of question to emulate thermal time and we shall not consider them any further.

the relation between thermal time and non-generally covariant notions of time. In the quantum mechanical limit both the thermal time flow and the ordinary Hamiltonian time flow act on the same algebra so that these flows can be directly compared with each other. In the algebraic setting one can compare at least general properties of these flows.

Relation to Ordinary Hamiltonian Times in the Quantum Mechanical Limit

Let us consider an ordinary quantum mechanical system consisting of N particles contained in a box. It is defined in terms of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on a separable Hilbert space, supplemented by the giving of a Hamiltonian, H, which generates the physical time flow. If the system is in a faithful normal state ω, which then corresponds to an invertible density matrix ρ, thermal time is defined, too. Since ρ is of trace-class, it is a compact self-adjoint operator. Its spectrum $\sigma(\rho)\setminus\{0\}$ therefore is a purely point spectrum with precisely one limit point $0 \in \sigma(\rho)$, which is not an eigenvalue (cf. appendix A.1). ρ admits an ONB consisting of eigenvectors $\{e_i\}$ with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_i \geq \ldots$ converging to 0. The thermal Hamiltonian reads $K = -\log\rho$ and is a positive, self-adjoint operator with discrete spectrum as it is prevalent in closed quantum mechanical systems. From this viewpoint K is a reasonable Hamiltonian.

The thermal Hamiltonian K depends exclusively on the density matrix ρ, i.e. the statistical state of the system, while the ordinary Hamiltonian H is deduced from the particle content, the interactions between the particles, external fields, suitable chosen boundary conditions, etc. Under the premise that a state can be prepared in quite an arbitrary way, independently of the interactions, K cannot contain any information concerning the ordinary Hamiltonian flow. Conversely, starting from a non-equilibrium state the memory about the initial state is lost in microscopic correlations during the course of its evolution (cf. [139]), and with it every piece of information about K.[13] Hence, we have to infer that K and H, will be independent and uncorrelated to a considerable degree in the general case, they are constructed in a totally different manner. This eventually explains what we already observed before, it is unfeasible to interpret thermal time in terms of usual Hamiltonian times. If one wants to confirm the TTH in the non-generally covariant limit, a new understanding of time is necessary.

We want to illustrate the independence of H and K more concretely. Envisage a system consisting of N particles whose interaction with each other is described by a potential term $\sum_{i\neq j} V(x_i - x_j)$ in the Hamiltonian H. In principle it is possible to prepare a state $\rho = e^{-K}$, where K resembles H up to the interaction term which we assume to be replaced by $\sum_{i\neq j}\tilde{V}(x_i - x_j)$. In that case the thermal Hamiltonian would look like an ordinary physical Hamiltonian. But since the chosen particle content shows a completely different interaction as it is coded in the thermal Hamiltonian, the evolution cannot be related to the "true" one. The evolution is not unphysical in the sense that it has physical properties and one might replace the particles by those which have the "correct" interaction and are otherwise identical. So there is the possibility to "realise" the thermal time flow. Of course, this is not what one would understand under a physical time flow. One can construct many examples of this kind, in which the interaction term is artificially produced via the initial state of the system. A state distinguishes an interaction term which, apart from special cases, has nothing to do with the real dynamics.

In a next step one may think of a thermal Hamiltonian which does not even look like a physical one. We continue considering the finite system consisting of N particles. Preparing an arbitrary state represented by an invertible density matrix, K will usually not take the form of a standard Hamiltonian generating the dynamics of a physical system, which is typically something like

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(x_1, \ldots, x_n). \tag{7.11}$$

For instance K does not need to have the kinematical terms $p_i^2/(2m_i)$. In such a case the system evolved in thermal time behaves in a completely different manner, and one cannot expect that the dynamical system describes an ordinary particle theory from an observer's point of view.

[13] Think for instance of a cup of coffee. If the coffee is hot one knows that it will become cold, if however the coffee is cold, one cannot say if it was hot before, or if it has always been cold.

Albeit the thermal time flow in finite systems is generated by a positive self-adjoint operator, it does not have the form of an ordinary Hamiltonian. There is no reason why an observer who is aware of the particle content should construct a Hamiltonian like K rather than one of the above form, and why he should describe the evolution in terms of the thermal Hamiltonian. Any standard clock will measure the Newtonian time generated by H. So one is tempted to interpret such a situation in the sense that there simply is no observer which accesses thermal time.

Relation to Ordinary Hamiltonian Times in AQFT

Likewise to the quantum mechanical limit, one can argue in the AQFT case. The thermal Hamiltonian $K = -\log \Delta$ induced by a faithful normal state in the GNS representation of a local algebra, which is a type III_1 factor, is a self-adjoint operator whose spectrum is the whole real line (though its negative part is suppressed by proposition 4.1.10), as it naturally happens to be the case e.g. in the thermal sector of AQFTs. Again, up to this point K resembles an ordinary Hamiltonian. To construct the Hamiltonian implementing the ordinary time translations (which form a one-parameter subgroup of the representation of the Poincaré group), one has to know the physical content of the theory, namely the net structure. In special relativity the time evolution of an observer corresponds to a particular separation of spacetime into space and time. The unitary operators implementing the time evolution respect this separation, which is expressed by a geometrical meaning in the underlying spacetime region; evolved observables remain localised. For inertial observers the localisation region is just shifted upwards without any deformation (for uniformly accelerated observers the localisation region is distorted, cf. figure 6.1 on page 77).

In section 4.6 we have seen that such a simple and in the Heisenberg picture typical behaviour can generally not be expected from the thermal time flow. On the contrary, the thermal time flow does neither need to act geometrically nor does it need to respect causality. We have met an example where the modular flow blows up the localisation region of suitable chosen observables arbitrarily in small thermal time intervals, or induces a transformation with a mixing term between spacelike separated double-cones. Although these examples may not provide the typical application domain of the TTH, which generically will be a finite, connected spacetime region in four dimensions, one more time a somewhat atypical behaviour of the thermal time flow is indicated. A geometrical thermal time flow (with timelike orbits) can be associated to the measurable proper times of a suitable class of observers. In general, the proper time translations of observers moving in a Lorentzian spacetime manifold will have nothing to do with the thermal time flow, even if they are related to a Killing field or unitarily implemented on the algebra. The thermal time flow does not need to respect the net structure.

It was mentioned before that thermal time might be interpretable by choosing another coordinate system. Thermal time could give rise to a sensible time flow w.r.t. this frame, particularly as one cannot expect a unitary implementation of the coordinate time translations. It is conceivable that if e.g. the choice of the coordinate system is not based on a timelike (conformal) Killing field, the most natural time evolution is a non-geometrical flow. One might hope to find at least in cases of physical interest a suitable reference frame in which thermal time can be interpreted in such a way. While the dynamics induced by thermal time are simply equilibrium dynamics, the non-trivial part of understanding thermal time would be shifted to its identification with the physical time of suitable chosen reference frames. Anyway, the presence of a spacetime manifold plays no role in modular theory where the thermal time flow comes from. In turn, there is no reason why an understanding of thermal time based on geometrical aspects should generically be possible. Indeed, the interpretation of thermal time should not rely on an underlying Lorentzian manifold and a particular coordinate system. This is integral to the generality of the TTH, which is supposed to be applicable above all in the generally covariant context.

Similarly, a thermal time flow which does not respect causality causes difficulties if one wants to identify it with the time flow perceived by an observer. There is actually no reason why it should respect causality, because the net structure, i.e. the localisation region of the observables, does not enter the TTH. For example it could happen that the thermal time flow acts geometrically in form of spacelike orbits.[14] It seems to be impossible to give a meaningful physical interpretation of a time flow from an observer's point of view which violates causality in such a drastic way.

[14] Note for this that the predictions made by Trebels concerning possible geometrical flow lines refer to the vacuum state, and, moreover, need as prerequisite that the modular flow acts causally geometrically, such that it is essentially assumed from the beginning that causality is respected.

An Example: The Spin-1/2-System

As a specific example we apply the TTH to a simple model, namely to a spin-1/2-system (one may think of silver atoms in their ground state [39]). The relevant Hilbert space is $\mathcal{H}_s = \mathbb{C}^2$. The observables of physical interest, $S_u \equiv \vec{S}\vec{u}$, measure the spin-component in $\vec{u}$-direction. The standard basis of $\mathcal{H}_s$ is $\{|\uparrow\rangle, |\downarrow\rangle\}$, where $|\uparrow\rangle$ is the eigenstate corresponding to spin up in z-direction, while $|\downarrow\rangle$ is related to spin down. W.r.t. this basis one establishes

$$S_{\theta,\varphi} \equiv S_u \equiv \vec{S}\vec{u} = \frac{1}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}. \tag{7.12}$$

Let us assume the system to be initially prepared in such a way that the probability to measure spin up in z-direction is p, and that the probability to measure spin down is $1-p$ (this can always be achieved if the coordinate system is appropriately chosen). The relevant density matrix is then

$$\rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \tag{7.13}$$

and we can determine the thermal time evolution of the observables (supposed that $0 < p < 1$):

$$\sigma_s(S_{\theta,\varphi}) = \rho^{-is} S_{\theta,\varphi} \rho^{is} = \frac{1}{2}\begin{pmatrix} \cos\theta & e^{-is\log\left(\frac{p}{1-p}\right)}e^{-i\varphi}\sin\theta \\ e^{is\log\left(\frac{p}{1-p}\right)}e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \tag{7.14}$$

$$= S_{\theta,\varphi+s\log\left(\frac{p}{1-p}\right)}. \tag{7.15}$$

We find that the spin observables which measure the spin in $\vec{u}$-direction are rotated around the z-axis. What is the intuition behind this observation? If an observer measures the spin in z-direction and finds the probability to measure spin up to be p, he would construct the density matrix (7.13). The observer has a detailed knowledge about the distribution of the spins in z-direction, while the orthogonal directions are on an equal footing for him. That is precisely what the thermal time exploits, which underlines its significance as a preferred symmetry transformation (we shall take up this point in section 7.10). Since the observer cannot distinguish between the directions orthogonal to the z-axis, there is a kind of rotation invariance, and the "geometrical significance" of the thermal time flow is nothing else but a rotation around the z-axis.

While the mixed state causes the emergence of thermal time, a physical time cannot be specified unless the external fields are known. Let us for instance assume that the silver atoms are in a homogeneous magnetic field $\vec{B}_0$ in $\vec{v}$-direction. Only then the physical Hamiltonian describing the observable time evolution of the system can be constructed, it is

$$H = -\gamma B_0 S_v, \tag{7.16}$$

where γ is the gyromagnetic ratio. This emphasizes again, that the thermal time flow describes something very different from ordinary physical time evolutions; it reveals a symmetry of the initial state, and in general there seems to be no chance to relate it to a physically meaningful time flow.

Summary

The considerations of this section underline that in the non-generally covariant limit thermal time does not simply recover one of the common notions of time such as an ordinary Hamiltonian time or proper time (apart from some rather artificial situations), and it seems to be impossible to draw a connection to one of these times. Thermal time and ordinary notions of time are on totally different footing and have in general nothing to do with each other. The spacetime structure on which the latter ones are based is not available, or rather must not be exploited to obtain a time concept which is general enough. Thermal time defines an entirely new time concept, it is state-dependent and has a thermodynamical origin. On the other hand, thermal time is intended to explain our very special perception of time, which *is* related to the non-generally covariant times appearing in physics (even if not all aspects of common sense time are captured).[15] One might draw two

[15] One could claim that the TTH has to work above all when applied to cosmological models, namely to (a subsystem of) our universe, to explain the origin of the flow of time we perceive.

conclusions from this: Either one regards the deviation from ordinary times as disturbing and demands a time concept which in the non-generally covariant limit does reduce to such a time (or is at least related to it in cases of physical interest) and which may be achieved by imposing certain conditions on the state, or one investigates this new notion of time for physical properties and tries to assign a more "abstract" meaning to thermal time. We will follow essentially the second route and look for physical properties of thermal time, although it turns out that additional requirements have to be fulfilled in order to get a meaningful time concept with physical properties.

Even if thermal time is not related to the physical time of an observer, it still can define reasonable local dynamics in a sense to be clarified; e.g. thermal time may be "perceived" by some kind of process rather than by an observer [86]. The KMS equilibrium flow would be deeply hidden in the structure of the system, not accessible for an observer, though it might open the opportunity to formally treat non-equilibrium phenomena by equilibrium flows. The thermal time flow might be a useful tool for studying physical systems and to extract certain pieces of information.

Conclusion *Thermal time can generally not be identified with an ordinary Hamiltonian time in the non-generally covariant limit. It is a completely diverse flow of different origin and with an unfamiliar behaviour. There seems to be no reason why it should be related to the dynamical evolution seen by an observer. In the terminology of the introduction thermal time then is not a physical time in the proper sense. Nonetheless, there remains the opportunity that it can be identified with some yet unknown, reasonable physical time concept.*

7.6 Intrinsic Equilibrium

Whenever the thermal time flow has an (approximate) geometrical meaning corresponding to timelike curves, this is interpreted in the sense that for certain observers, namely those moving along the modular flow lines, the state looks like a equilibrium state whose temperature is the ratio between thermal time and geometrical time. It was explained in section 4.6 that there are strong hints that the modular flow acts non-geometrically in cases of physical interest. The hidden symmetry structure revealed by modular theory has in general nothing to do with a geometrical realisation in spacetime. Invoking the uniqueness of the KMS flow, one may be tempted to speculate that a similar statement also holds for the reversed direction, namely that the absence of a geometrical meaning expresses the impossibility for any observer (in the respective spacetime region) to see the system in equilibrium. This fits into the picture that there is no temperature defined, which is a typical quantity assigned to true thermal equilibrium. The interplay of modular theory and KMS theory just shows that there formally exists a flow which lets the system appear to be in equilibrium (disregarding dispersion for the moment), but as discussed before, there is no reason why it should be accessible for an observer, why it should be a "real" flow.

To obtain a thermal time flow for a physical system in a given state, which as an equilibrium flow has not just a couple of physical properties but which can be expected to be measurable in some sense by an observer whose time is related to thermal time, the system should not be in a completely arbitrary state. To make sure that the treatment of the system as an equilibrium system is physically reasonable, it might be useful to fall back upon the original version of the TTH proposed by Rovelli (see section 5.3) and require the state to be in some kind of intrinsic equilibrium in order for the TTH to be applicable. The state then can be expected to code dynamical information. Accordingly, one has to look for an intrinsic characterisation of the notion of equilibrium in the context of AQFT which does not rely on the notion of time. As a primitive ansatz, this could be the requirement for a geometrical meaning of the modular flow corresponding to timelike curves (this could at least be regarded as a sufficient condition). Anyway, an intrinsic characterisation of equilibrium could be transferred to the generally covariant case in order to get a thermal time concept of restricted validity. It would make the description of the system in terms of equilibrium dynamics physically plausible, and it could also guarantee that in the non-generally covariant limit thermal time does reduce to an ordinary physical time, such as Newtonian time or proper time. Anyway, a suitable characterisation of an intrinsic equilibrium state might be as difficult as the distinction of a physical time flow, whereby the issue is essentially shifted.

Conclusion *Restrictions on the state might ensure a physical thermal time concept.*

7.7 Relation between Thermal Time and Physics

Mathematical Representation of Thermal Time

While the TTH provides an entirely new access to the notion of time, one may question if its assumption about the mathematical representation of a time flow is general enough to cover all cases of physical interest. At first glance it seems to be a very natural assumption to identify the physical time evolution with a σ-weakly continuous group of *-automorphisms on the relevant algebra. Nevertheless, a dynamical law which is homogeneous in time, as well as a continuous evolution of every state of the system, are not necessarily expectable for general systems in which complicated phenomena occur. Thus, one may also take weaker forms of time-evolution into account (see [24] p. 4 and [60] p. 237). We will assume here the system as well as the state to be sufficiently nice, albeit these considerations indicate that the assumption of the TTH concerning the mathematical representation of the time flow might by too simplifying and does not capture the peculiarities of all physical systems.

Physical Content of the Thermal Time Concept

We have seen that the thermal Hamiltonian will in general not recover ordinary physical notions of time. A very fundamental question is to what extent the thermal time flow can be expected at all to incorporate a physical input which is sufficient to justify an interpretation of thermal time as a physical time whatsoever. For this, recall that the thermal time flow is generated from just one single local algebra as soon as the faithful normal partial state on this algebra is known. In section 2.7, however, we have learnt that one single algebra $\mathcal{R}(\mathcal{O})$ does not contain any physical information, it is just a mathematical object, *-isomorphic to the unique hyperfinite type III_1 factor. It is solely the net structure, the way the various local algebras are embedded into each other, which brings in the physics. But the net structure, or rather the subnet with $\mathcal{R}(\mathcal{O})$ as its largest element which is the only relevant part for an observer in $\mathcal{O}$, does not need to be known in order to determine the modular operators, and thereby the thermal time flow.

The TTH claims that an arbitrary faithful normal state distinguishes a physical time flow on the observable algebra at a stage, where the net structure, and hence the physical meaning of the observables, has not entered yet. It is not plausible that one can determine a physical time evolution of an observable whose meaning one does not know (e.g. one does not know whether it is localised in a proper subregion and thereby the exact placement as well as the spacetime extension of the measuring apparatus, which has to be used to carry out the corresponding local operation), in a system whose physical content (interactions, correlations) is not known. Of course, the subnet structure becomes relevant if one tries to interpret the modular flow in physical or geometrical terms. However, from this gross viewpoint it would be very remarkable if the modular flow admits a physical interpretation in any case, i.e. for each physically admissible net.

The second important object which enters the TTH is the state. Also a state as an arbitrary linear functional on the algebra does not contain any physics (the only restrictions are normality and faithfulness), and becomes interpretable only through the net structure. It is known that an abstract local net admits unphysical states with an infinite amount of energy concentrated in finite spacetime regions. However, once an appropriate representation is chosen in form of a net of von Neumann algebras, all (normal) states belong to the same folium and should therefore be of physical relevance. The TTH should be applicable to any faithful normal mathematical state. Certainly there are states which bring in some physics. The vacuum state with a couple of very specific properties is a primary example for that (see below), and indeed the thermal time flow corresponding to the restricted vacuum state seems to contain enough physics to generate a physical time flow, at least this is what the known examples say. Nonetheless, one may doubt if the same holds for an arbitrary chosen state, which does not code a comparable amount of physics.

At this point we want to take a look at the canonical flow, which was alternatively proposed by Connes and Rovelli to represent a physical flow. In that case the (minimal) physical input stemming from the state is cancelled and one has to deal with a purely mathematical flow. The inner automorphisms by which the various modular flows differ and which may contain some physics become irrelevant. Apart from the odd fact that in the quantum mechanical limit all possible thermal time flows are identified with the trivial flow, we do not see any reason why this

flow should do something physical. If we think e.g. of a hypothetical generally covariant theory where the state influences a macroscopic metric field of the underlying manifold, the canonical flow would have to be intepretable for essentially arbitrary metric fields.

The same argument applies in the case of a quantum mechanical system. The thermal time flow emerges from an arbitrary state given by an invertible density matrix on the algebra $\mathcal{B}(\mathcal{H})$. Again, no physics is contained in these objects. Let us think of a system in a box consisting of a finite number of particles. As mentioned before the physics is determined by the interparticle forces, the boundary conditions, and the presence of external fields. But none of these physical aspects is taken into account by the TTH. Again, there is no reason to suspect that thermal time does something physical, something which is related to the "real" dynamical system.

In summary, there is essentially no physical input contained in the thermal time postulate. The thermal time flow does not seem to be more than the mathematical canonical flow of the algebra which is supplemented by a state-dependent inner automorphism which exploits certain peculiarities of the specific state. Additional selection criterions are needed in order to obtain a physical thermal time. Of course, one may reply that the thermal time flow gains thermal properties via the KMS condition, but let us stress afresh that one carefully has to distinguish between a physical time flow and a flow with physical properties. The KMS condition is a purely mathematical condition which obtains its physical relevance for the distinction of equilibrium states only if the automorphism group is identified with time translations which are *a priori* known to be physical. As long as the modular parameter does not represent a physical time, the modular flow has nothing to do with equilibrium statistical mechanics.

Physical Relevance of the Distinguished KMS Flow

To explain that a bit more, let us fall back upon the initial motivation of the TTH. In a non-generally covariant physical system a given time flow selects a preferred class of states, namely the equilibrium states. This was reversed to postulate that a statistical state distinguishes a flow, the thermal time flow. Undoubtedly the thermal time flow is a preferred flow, just the same as an equilibrium state is a preferred state with regard to the conventional viewpoint. But as this does not imply that every non-generally covariant system automatically has to be in (or at least close to) an equilibrium state, there is no reason why the thermal time flow should be (approximately) realised in the system in the sense that it is perceived by certain observers. The relation between KMS theory and modular theory just shows that it is always possible to choose the dynamics, i.e. interparticle forces, external forces, etc., in such a way, that with these forces the system is in KMS equilibrium. If the forces happen to be different, there is a priori no reason why the modular flow should be related to physics. By no means one can conclude without further ado or restrictions that it represents physical time translations. As discussed above, K has some properties which are ascribed to a physical Hamiltonian and it may happen by choosing a suitable particle or field content with appropriate interactions etc. that K can indeed be realised, but in general there is no reason to expect that the evolution of the system is governed by the thermal time flow.

The Thermal Time Flow Induced by the Vacuum State

Most of the examples where the modular group is known stem from the vacuum state restricted to certain spacetime regions. In those cases a physically meaningful interpretation is possible, which may be ascribed to the peculiarities of the vacuum state. One must not forget that it is an extremely distinguished state with very special properties. The vacuum representation is assumed to admit a unitary implementation of the Poincaré group and the vacuum state is invariant under the action of that group, whereby it codes information about the representation of the Poincaré group and a physical time evolution.

That the modular operators induced by the vacuum state and the physical Hamiltonian are not independent is shown off by an intimate relation between their spectral properties, pointed out by Buchholz, D'Antoni and Longo [32]. The modular operators carry information about the energy of the system. There is further the possibility to express the energy nuclearity condition in terms of the modular operators associated with bounded regions in Minkowski space (cf. section

2.6). This modular nuclearity condition makes a statement about the order of the mappings

$$\Xi_{\mathcal{O}_1,\mathcal{O}_2} : \mathcal{R}(\mathcal{O}_1) \to \mathcal{H}, \quad A \mapsto \Delta^{1/4}_{\omega_0 \restriction \mathcal{R}(\mathcal{O}_2)} A\Omega_0, \tag{7.17}$$

where $\mathcal{O}_1 \subset \mathcal{O}_2$ are two bounded regions whose inner distance is sufficiently large (see [32, 60] for the details). Since the energy nuclearity condition is expected to hold in the theories of physical interest, (7.17) shows that the modular operators induced by the vacuum state contain pieces of information about the net structure, i.e. about the physics, expressed via the appearance of the subalgebra $\mathcal{R}(\mathcal{O}_1)$. Of course, this does by no means imply that the modular group associated with the vacuum has to generate a physical time evolution,[16] but at least it underlines its *physical* significance, the modular objects do code physics. In fact, the two equivalent formulations of the nuclearity condition were motivated by the idea that the modular flow describes some form of perturbed dynamics. The authors expected that an observer in a region $\mathcal{O}$ whose time evolution is given by the modular flow comes to a similar conclusion concerning the size of the set of well-localised states as an observer whose time evolution is generated by the ordinary Hamiltonian.

Conclusion *The portion of physics which enters the thermal time concept is too low to justify its physicality without further ado. Only in particular states the thermal time flow might contain enough physical input to generate a physical thermal time flow.*

7.8 Conceptual Issues

While we focused up to now on the physicality of the thermal time concept and possible physical interpretations of thermal time by means of drawing a connection to conventional notions of time, it is our goal in this section to discuss conceptual issues coming along with the TTH and making it (so far) incomplete as a physical time concept.

The Principle of Simplicity and the Metrization of Thermal Time

Following the reasoning of Connes and Rovelli the physical time of a system is chosen in such a way that the system appears to be in thermal equilibrium. Therefore, one just has to deal with equilibrium dynamics. From this point of view one may interpret the thermal time concept as a realisation of Poincaré's principle of simplicity (cf. section 1.4). The physical time of a system is selected in such a way that the dynamics become as simple as possible. Evidently, this does not imply that the physical meaning of the thermal time flow (if it exists at all) is easily obtained. A description how the precise action looks like, e.g. on local subalgebras, is far away from being simple. However, there is an even more fundamental issue.

How does one has to choose a suitable metrization of thermal time, i.e. how does one find the rescaling which relates thermal time and physical time? We do not just perceive a topological time. It was emphasized by Mach (cf. [88]), that there is a feeling of duration which should be reflected in a preferred metric of physical time. If thermal time does indeed correspond to our experience of time, the metrization should be fixed (up to some affine transformation) in order to obtain a well-defined quantity. In Newtonian mechanics and general relativity the metrization is selected by the principle of simplicity. The first idea might be to fall back again upon this principle, that is to identify the modular parameter itself as a physical time. This does not work here: As an application of the TTH to double-cones in the conformal case reveals, one has to expect a highly non-trivial rescaling of thermal time in order to relate it to physical proper time.[17] In general, the modular parameter is *not* a physical time. This fits with our experience, the time of our perception is finite rather than corresponding to the whole real line as the modular parameter does.

In the non-generally covariant case there is a way to obtain the rescaling, f, whenever the modular group has a geometrical meaning, at least for observers moving on the modular flow lines: It has to be chosen in such a way that $\|\partial_{f(s)}\| = 1$. Note that e.g. for the conformal diamond

[16] In the case of a double-cone in a massive or interacting theory this is far away from being clear.

[17] One might claim the modular parameter to define a physical time flow of its own right and to assume that proper time as a mechanical "non-flowing"-time is not really what the observer perceives [113], but as explicated before we think that due to the unitary implementation of the proper time translations it should be on the same footing as thermal time, and to avoid dealing with two time flows, it seems to be reasonable to identify them.

the metrization is observer-dependent, or rather acceleration-dependent. Differently accelerated observers might fall back upon non-equivalent metrizations, which are not related by affine transformations and which are used by them to describe the evolution in their specific relativistic time. For other observers in that spacetime region, or in the non-geometrical case, there seems to be no natural instruction.

If a local algebra $\mathcal{R}(\mathcal{O})$ admits a state ω which generates a modular group with a geometrical realisation (corresponding to timelike orbits), one might consult this state to fix the rescaling and use it for the thermal time flows generated by all the other states [86]. This proceeding might be justified by the cocycle Radon-Nikodým theorem which underlines that there is a strong relation between the various modular flows on a local algebra. Anyway, if $\mathcal{R}(\mathcal{O})$ does not admit a state whose modular flow has a geometrical meaning, or in the generally covariant case the issue remains.

Since the knowledge of a metrization is necessary in order to get a full physical thermal time concept, we have to conclude that the TTH can provide us at most with an incomplete picture of the true physical notion of time

Measurability of Thermal Time

Assuming the TTH to be true in the sense that it indeed defines a physical time, it might be disturbing to ascribe the origin of time to equilibrium dynamics (though this has been the motivation to justify thermal time as a physical thermal time concept), not just because of the existence of irreversible phenomena in our world (see section 7.11), but also because a state-dependent equilibrium flow might impose restrictions on its measurability.

Let us address the issue concerning the measurability of the thermal time flow as a dynamical evolution of the system. One cannot directly measure the time evolution of the observables itself, what experimentally can be verified is the evolution of their expectations values, which are constant w.r.t. thermal time. There remain thermal time correlation functions and fluctuations around the state leading to measurable, non-trivial temporal reactions of the system. However, one has to be a bit careful because of the state-dependence of thermal time.

For instance if the system is prepared in a non-equilibrium state, one cannot observe any "return to equilibrium"-behaviour, because by definition the thermal Hamiltonian is deeply intertwined with the state. A change of the state is necessarily accompanied by a change of the dynamics, such that the system is in equilibrium w.r.t. the new state and an approach to the original state will not be detected (cf. the example in section 7.11). One objection to the measurability of the thermal time correlation functions, which do evolve non-trivially in time, could be as follows. Let us envisage a physical system described by means of an algebra $\mathcal{R}$ and a state ω corresponding to a cyclic and separating vector Ω. A correlation function in thermal time reads

$$F_{AB}(s) = \langle \Omega, \sigma_s^\omega(B)A\Omega \rangle \,. \tag{7.18}$$

As indicated in section 5.2, this function is physically related to the amplitude of detecting $\sigma_s^\omega(B)\Omega = \Delta^{-is}B\Omega \equiv (B\Omega)(s)$ if one prepares $A\Omega$ (which we assume to be cyclic and separating), i.e. to the transition amplitude from $A\Omega$ at thermal time 0 to $B\Omega$ at thermal time s. Accordingly, the starting point is the prepared state $A\Omega$ rather than Ω and one might ask if the system evolves w.r.t. the thermal time induced by $A\Omega$ instead of Ω, so that one actually does not measure the correlation function $F_{AB}(s)$.

We conclude that the state-dependence of the thermal time flow together with the stationarity of the state seems to restrict the possibilities to observe thermal time – supposed that it is something observable at all. And that is precisely the second problem: To measure e.g. $F_{AB}(s)$ one first of all must be able to measure thermal time itself, which is the parameter s. To do that a suitable thermal clock is needed which emulates thermal time. This is delicate, since thermal clocks do not exist in thermal equilibrium.[18] Even more, the parameter s is a remnant of the classical structure, it formally corresponds to the classical Newtonian time used in ordinary quantum mechanics, where one knows that no good quantum clocks exist (cf. section 1.4). Consequently, we do not see any reason why thermal time itself, and with it the evolution induced by thermal time, should be measurable. The observability of the thermal time flow, prerequisite of its physicality, is a problem. If the TTH is correct, methods will have to be clarified how to measure it.

[18] If thermal time has a geometrical meaning it could be measured via mechanical clocks measuring proper time.

To take up section 7.7, it is not clear whether the thermal time concept can be seen as something physical, for, according to MACH, only observable quantities should be regarded as physical (see section 1.4). It is very questionable if a (non-geometrical) thermal time flow is measurable. Even if it has all the properties required from thermal equilibrium states and therefore describes true equilibrium dynamics (see section 7.11), this special flow might be so deeply hidden in the system that there is no chance to perceive it or derive benefit from it. A flow with physical properties does not need to be physical.

Conclusion *An instruction how to choose the physical metrization of thermal time is not provided by the TTH. Moreover, the fact that thermal time represents state-dependent KMS equilibrium dynamics, which can be a deeply hidden symmetry flow, seems to impose restrictions on the observability of the thermal time flow.*

7.9 Thermal Time Hypothesis in the Covariant Formalism

The thermal time flow generates an evolution which acts on the algebra of observables. In the non-generally covariant case (at least in the classical and quantum mechanical limit) there is also the ordinary Hamiltonian time flow acting on the same algebra. We have already risen up the question how to treat these somehow competing and in general different flows, which are both candidates to describe a sensible evolution for an observer. To get a better understanding of the role of both flows, we go over to the covariant formalism (appendix B). The TTH was formulated above all as a proposal for a physical time concept in generally covariant theories. For that reason it appears to be natural to apply the TTH in the covariant formalism, which represents the non-generally covariant system as a constrained system which is much closer to generally covariant theories.

Coming from an ordinary Hamiltonian system, in the covariant approach one forgets about the time variable which is hidden in the presymplectic structure of the system (we restrict ourselves to the more transparent classical case). The solutions of the equations of motion are represented via unparametrised curves defined by the presymplectic structure on the constraint surface $\mathcal{C}$ of the extended phase space Γ_{ex} and form the covariant phase space Γ_c. $\mathcal{C}$ is defined by the covariant Hamiltonian H_c (we neglect other gauge constraints). A state can be defined as a density matrix ρ on $\mathcal{C}$. ρ has to depend on the Hamiltonian time t. To be gauge invariant ρ has to commute with $H_c = p_t + H$, the dependence of ρ on t therefore has to be the standard time evolution. ρ generates a time flow on the constraint surface via the thermal Hamiltonian $K = -\log\rho$. Thereby the covariant constrained system is supplemented by a time flow. Moreover, ρ respects the presymplectic structure and gives rise to a well-defined density ρ_c on the covariant phase space Γ_c, which defines the corresponding thermal time flow on Γ_c.

Of course, applying the TTH in the covariant formalism instead of the conventional one does not change much, but it opens a slight change in perspective which better shows what is going on. One does not have two competing time flows anymore, but rather a thermal time flow which superposes the ordinary Hamiltonian evolution hidden in the presymplectic structure. The thermal Hamiltonian generates a flow which is on an entirely different footing and acts on a different space. $K = -\log\rho$ evolves observables acting on $\mathcal{C}$. Conventional observables, say $A(t)$, are those which measure a partial observable A under the condition that a measurement of a clock T, which we think of as a partial observable emulating Newtonian time, has given the value t. Such an observable is evolved in thermal time s as

$$\frac{\mathrm{d}}{\mathrm{d}s}A(t)(s) = \{A(t)(s), K\}. \tag{7.19}$$

That gives a non-trivial evolution, but clearly we have

$$\langle A(t)(s)\rangle_\rho = \langle A(t)(0)\rangle_\rho \text{ and also } \langle B(s)A(t)(s)\rangle_\rho = \langle B(0)A(t)(0)\rangle_\rho\,, \tag{7.20}$$

because ρ is a Gibbs state w.r.t. thermal time. We see that thermal time is compatible with conventional Hamiltonian time not in the sense that it reconstructs the Hamiltonian time, but in the sense that it respects the structure induced by it on the set of observables. This is necessary, because if one had $\langle A(t)(s_1)\rangle_\rho \neq \langle A(t)(s_2)\rangle_\rho$, one might be in trouble to interpret such an expression with a directly visible and measurable thermal time. In the Schrödinger picture trajectories are

mapped on other trajectories on the constraint surface, this happens not arbitrarily but the flow is chosen in such a way that all expectation functionals remain the same (and even more, that the state becomes a Gibbs state w.r.t. thermal time).

In the special case in which ρ is also a Gibbs state w.r.t. H the flow on the covariant phase space can be interpreted in the following way: A trajectory $(q^i(t), p_i(t), t)$ at thermal time $s = 0$ is mapped on the trajectory $(q^i(t)(t_0), p_i(t)(t_0), t) = (q^i(t + t_0), p_i(t + t_0), t)$ at thermal time $s = t_0$. The physical motions described by both trajectories are the same, with the only difference that the initial values are shifted by t_0. Hence, one can identify the action of thermal time as time evolution in conventional Hamiltonian time. This can also be seen by using the bijection between Γ_c and the ordinary phase space Γ, which arises simply by evaluating the curves at $t = 0$. The ordinary Hamiltonian time can be recovered if the flow preserves the presymplectic structure in the sense that the motion of the curves can be interpreted as a shift of the initial conditions. Since arbitrary states define quite arbitrary flows, this cannot be expected from the thermal time flow. In general it is a flow, which is not related to the "old" dynamics hidden in the phase space structure.

The covariant formalism provides a framework where thermal time and ordinary time are not on an equal footing anymore. This could be a starting point to explain why thermal time is supposed to be the time of our perception – assuming the TTH to be true. But even more, the meaning of the thermal time flow as a symmetry group rather than a physical time flow is underlined.

Thermal Time as a Symmetry

Thermal time defines a foliation on the set of observables. It equips the observables with a "time structure" which superposes the ordinary Hamiltonian time structure. One characteristic feature of those observables which are connected by thermal time is that they have the same expectation value. Even more, thermal time defines classes of observables which are physically indistinguishable. Let us focus on the quantum case now and consider two observables A and B, which are related by thermal time, in particular unitarily equivalent. For simplicity they are supposed to have a discrete spectrum. There exists an s such that

$$\rho^{-is} A \rho^{is} = B, \tag{7.21}$$

Let $B\,|n, j\rangle = b_n\,|n, j\rangle$, $\{|n, j\rangle\}$ being an ONB consisting of eigenvectors. Then

$$\rho^{-is} A \rho^{is}\,|n, j\rangle = b_n\,|n, j\rangle \quad \Rightarrow \quad A\rho^{is}\,|n, j\rangle = b_n \rho^{is}\,|n, j\rangle\,. \tag{7.22}$$

That means $\rho^{is}\,|n, j\rangle$ is an eigenvector of A with eigenvalue b_n, A and B have the same spectrum. The probability $\mathcal{P}_B$ to measure b_n when measuring B is given by

$$\mathcal{P}_B(b_n) = \mathrm{tr}(\rho P_n), \tag{7.23}$$

when P_n is the projector on the eigenspace corresponding to b_n. On the other hand we have

$$\mathcal{P}_A(b_n) = \mathrm{tr}(\rho\, \rho^{is} P_n \rho^{-is}) = \mathrm{tr}(\rho P_n). \tag{7.24}$$

For a given state ρ and a fixed thermal time s, it does not matter if one uses the natural foliation $A(t)$ given by the standard Hamiltonian time, or if one falls back upon the foliation $A(t)(s) =: A_s(t)$ instead. The physics remains the same. A fixed thermal time translation s can be used to define a new time structure on the set of observables. Accordingly, there is a "flow" acting on the set of observables and superposing the ordinary Hamiltonian time structure. It generates equivalence classes of observables. However, in this situation one would speak of a one-parameter group of symmetry transformations rather than a time flow.

For these considerations one only needs that the thermal Hamiltonian commutes with the density matrix ρ, whence they hold for any symmetry flow generated by a Hamiltonian commuting with ρ. What is so special about the symmetry flow distinguished by the TTH? Of course, it is a KMS flow, and we shall try next to exploit peculiarities coming along with this symmetry.

Conclusion *Albeit the covariant formalism treats thermal time and ordinary time on differing footings, it does not seem to permit a physically sensitive interpretation of the thermal time flow, as well, apart from the fact that it distinguishes a group of symmetries.*

7.10 Symmetry Properties Related to Thermal Time

Thermal Time as a Preferred Symmetry

The thermal time flow is constructed from the statistical state. All information coded in the thermal Hamiltonian must exclusively come from the state. As we have seen in the case of equilibrium states it may happen that the state contains further pieces of information like the full dynamical information of the ordinary dynamics, but that is an exception. The modular flow in the first reflects certain peculiarities of the state. It reveals a hidden symmetry of the state in general not visualisable by some transformation in spacetime. Certainly this is a more natural and conservative view than postulating that it distinguishes some meaningful concept of physical time whatsoever. In the quantum mechanical limit this symmetry reduces to an obvious and pretty trivial symmetry group. Besides, one can find much more symmetries, so what is so special about the symmetry transformation singled out by the thermal Hamiltonian in comparison with other symmetry flows which leave the expectations values of all observables invariant? To understand that we consider a type I factor $\mathcal{B}(\mathcal{H})$ which for simplicity we assume to be defined over a *finite* dimensional Hilbert space $\mathcal{H}$. Let ω be a faithful normal state on $\mathcal{B}(\mathcal{H})$ represented by an invertible density matrix ρ,

$$\rho = \sum_n \sum_j \rho_n \left|n,j\right\rangle \left\langle n,j\right|, \quad \rho_n \in \mathbb{R}_{\geq 0}, \tag{7.25}$$

where the index j describes degenerations of the respective eigenvalues. Accordingly, the thermal Hamiltonian reads

$$K = -\log \rho = -\sum_n \sum_j \log \rho_n \left|n,j\right\rangle \left\langle n,j\right|. \tag{7.26}$$

Any other symmetry transformation is generated by a hermitean matrix L which commutes with ρ, i.e. we may assume w.l.o.g. that the ONB $\{|n,j\rangle\}_{n,j}$ diagonalises L as well,

$$L = \sum_n \sum_j L_n^{(j)} \left|n,j\right\rangle \left\langle n,j\right|, \quad L_n^{(j)} \in \mathbb{R}. \tag{7.27}$$

What distinguishes K from a general L? To see that let us assume the state ρ to be perturbed, instead of ρ we consider

$$\rho_\mu = e^{-(K+\mu\tilde{K})}, \quad \mu \in \mathbb{R}, \tag{7.28}$$

with some hermitean matrix $\tilde{K}$. Clearly, the new state possesses new symmetries, in particular it defines a new thermal Hamiltonian $K_\mu = -\log \rho_\mu = K + \mu\tilde{K}$. On the other hand, for sufficiently small μ the old symmetry transformations (such as K and L) are approximate symmetries of the new state, for $\|\rho_\mu - \rho\| \overset{\mu\to 0}{\longrightarrow} 0$ implies

$$\|[S,\rho_\mu]\| \leq \|S(\rho_\mu - \rho)\| + \|[S,\rho]\| + \|(\rho_\mu - \rho)S\| \leq 2\|S\|\|\rho_\mu - \rho\|, \quad S \in \{K, L\}. \tag{7.29}$$

We want to investigate, to what extent the old symmetries, additionally, approximate one of the new, exact symmetry transformations. This is clearly the case for K, which approximates K_μ,

$$\|K_\mu - K\| = |\mu|\|\tilde{K}\| \overset{\mu\to 0}{\longrightarrow} 0. \tag{7.30}$$

For sufficiently small perturbations of the state ρ, $0 < |\mu| \ll 1$, the flow generated by K approximates the symmetry flow generated by K_μ up to any given accuracy on compact intervals (cf. theorem 7.14.4). We will come back to this in a more general context and on the basis of a different motivation in section 7.17. Here, let us investigate if we can do the same for L, namely can we find any exact symmetry transformation L_μ which is approximated by L? In that case there would exist an $\tilde{L}(\mu)$, such that $L_\mu = L + \tilde{L}(\mu)$ and $\lim_{\mu\to 0} \tilde{L}(\mu) = 0$. Let us pay attention to those transformations which are generated by an operator L which is not a function of ρ (and hence of K), i.e. there does not exist an f such that

$$L = \sum_n \sum_j f(\rho_n) \left|n,j\right\rangle \left\langle n,j\right|. \tag{7.31}$$

This can only happen if ρ has a degenerated spectrum (otherwise any operator compatible with ρ would be a function of ρ), and if at least one of the degenerated eigenspaces is not an eigenspace of L. Suppose w.l.o.g. that ρ_1 is degenerated and that $L_1^{(1)} \neq L_1^{(2)}$ are two eigenvalues of L corresponding to $|1,1\rangle$ and $|1,2\rangle$, respectively. It is sufficient to focus on the projections of the relevant operators onto the 2-dim. subspace, $\mathcal{H}_2$, generated by $|1,1\rangle$ and $|1,2\rangle$. Denote by P the projection operator onto this space,

$$P\rho P = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_1 \end{pmatrix}, \quad PLP = \begin{pmatrix} L_1^{(1)} & 0 \\ 0 & L_1^{(2)} \end{pmatrix}. \tag{7.32}$$

Consider a perturbation $\tilde{K}$ of the state which only acts on the previously mentioned subspace,

$$\tilde{K} \doteq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{7.33}$$

We determine the state ρ_μ resulting from this perturbation on $\mathcal{H}_2$:

$$P\rho_\mu P = Pe^{-K}e^{-\mu\tilde{K}}P = P\,\rho \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!}\tilde{K}^n P = \frac{\rho_1}{2}\begin{pmatrix} e^{-2\mu}+1 & e^{-2\mu}-1 \\ e^{-2\mu}-1 & e^{-2\mu}+1 \end{pmatrix} \tag{7.34}$$

Let $L_\mu = L + \tilde{L}_\mu$ be a hermitean matrix generating an exact symmetry transformation of the perturbed state ρ_μ,

$$P\tilde{L}_\mu P = \begin{pmatrix} \lambda_1 & \lambda_2 + i\lambda_3 \\ \lambda_2 - i\lambda_3 & \lambda_4 \end{pmatrix}, \quad \lambda_i = \lambda_i(\mu) \in \mathbb{R}. \tag{7.35}$$

By definition we have $[L_\mu, \rho_\mu] = 0$. Since $P\rho_\mu P = P\rho_\mu = \rho_\mu P$, this implies

$$P[L_\mu,\rho_\mu]P = [PL_\mu P, P\rho_\mu P] = \frac{\rho_1}{2}\begin{pmatrix} 2i\lambda_3(e^{-2\mu}+1) & -c.c. \\ (L_1^{(2)} - L_1^{(1)} + \lambda_4 - \lambda_1)(e^{-2\mu}-1) & -2i\lambda_3(e^{-2\mu}+1) \end{pmatrix} \stackrel{!}{=} 0$$

Clearly this enforces $\lambda_3 = 0$, which leaves the condition

$$\lambda_1(\mu) - \lambda_4(\mu) \stackrel{!}{=} L_1^{(2)} - L_1^{(1)} \neq 0. \tag{7.36}$$

That means that at least one of λ_1 and λ_4 cannot vanish when μ goes to zero, hence $\lim_{\mu\to\infty} L_\mu \neq 0$. For a symmetry transformation which is not a function of the original state one can find a perturbation of the state such that it does not approximate a symmetry of the perturbed state. From this point of view those symmetry transformations which are functions of the state represent stronger symmetries which are more stable. The thermal time flow belongs to them.

The reason for the unstable symmetry groups stands to reason. If the density matrix has a degenerated spectrum the observer is blind for correlations between the pure microstates of the degenerated eigenspaces. In such an eigenspace the correlations can be mixed up arbitrarily. If a (small) perturbation of the state cancels this blindness the symmetry vanishes. That is the reason why only symmetry flows which do not exploit the degeneration of eigenvalues will approximate a symmetry of the perturbed state.

Emergence of Thermal Time in the Quantum Mechanical Limit

It is stressed by Rovelli [110] that it is the ignorance about the microscopic details of the state which generates the thermal time flow. But, let us assume the state ρ for the time being again to be a finite-dimensional hermitean matrix, and consider the limit in which all microstates show the same probability, which means by the postulate of equal a priori probabilities (cf. section 3.1) that the ignorance about the state is maximal. Then ρ is just a multiple of the identity and the thermal time flow becomes completely trivial and invisible on all observables, i.e. there is no (observable) thermal time flow. This suggests that it is not purely the ignorance, but rather the weighting of the pure states constituting the mixed state which singles out a non-trivial thermal time flow.

To get a better idea of that, let us compute the symmetry flow generated by a Hamiltonian L, which then commutes with the density matrix ρ. Moreover, we assume that L has a discrete spectrum. Let $\{|n\rangle\}$ be an ONB of the relevant (possibly infinite-dimensional) Hilbert space consisting of eigenvectors of the invertible density matrix ρ, i.e.

$$\rho = \sum_n \rho_n |n\rangle \langle n| , \quad \rho_n \in \mathbb{R}_{\geq 0}, \quad \sum_n \rho_n = 1. \tag{7.37}$$

The diagonalised density matrix ρ assigns to the system a class of pure microstates $\{|n\rangle\}$ in which the system is supposed to be with corresponding probabilities ρ_n. Since it is assumed that $[\rho, L] = 0$, we choose the above ONB w.l.o.g. such that

$$L = \sum_n L_n |n\rangle \langle n| , \quad L_n \in \mathbb{R}. \tag{7.38}$$

Let A be an observable, then we compute the symmetry flow induced by L:

$$\langle n|A(s)|m\rangle = \langle n|e^{isL} A e^{-isL}|m\rangle = e^{is(L_n - L_m)} \langle n|A|m\rangle . \tag{7.39}$$

W.r.t. this basis the diagonal elements are left invariant, while the off-diagonal entries rotate in the complex plane. Note that direction and angular velocity of this rotations depend exclusively on the n-th and m-th diagonal entry of L. What is the origin of these "microflows"?

Let us consider a measurement of an observable A in the state ρ. As we have seen, for a given state there exist other observables which are physically equivalent in the sense that the same values can be measured with the same probabilities. A symmetry of the state enables one to replace A by $A(s)$. The existence of physically equivalent observables is due to the fact that only parts of the observable A are relevant when it is just probed on ρ. What for a given density ρ is in fact relevant, are the expectation values in all the microstates $|n\rangle$ in which the system could be, i.e. the diagonal elements of A w.r.t. $\{|n\rangle\}$. The off-diagonal elements code information which would only become relevant if one is interested in measuring A in a (normalized) superposition of these microstates, like e.g. $(e^{i\theta_n}|n\rangle + e^{i\theta_m}|m\rangle)/\sqrt{2}$. Then correlations and relative phases play an important role. Anyway, these microstates do not appear in the density matrix, so it does not matter how the corresponding values in the observable are chosen. For any choice of the off-diagonal phase factors, i.e. however correlations between different microstates $|n\rangle$ and $|m\rangle$ are treated, one does not produce observable effects when measuring A only in the state ρ. One can use this observation to generate rotating phases in the complex plane, and induce microscopic flows which join together to a symmetry flow. A symmetry transformation can be regarded as a flow on the set of observables which consists of several microflows rotating the off-diagonal elements in quite an arbitrary way, supposed that the basis is chosen such that it diagonalises the state together with the generator of the transformation.

Now we come back to the thermal Hamiltonian K. In this special case we have

$$\langle n|A(s)|m\rangle = e^{-is(\log(\rho_n/\rho_m))} \langle n|A|m\rangle , \quad \rho_n \in \mathbb{R}_{>0}. \tag{7.40}$$

The rotations are not arbitrary any more, but fixed by the state. More concretely, the thermal time flow is generated in the following way: The ratio of the weights with which two microstates $|n\rangle$ and $|m\rangle$ appear in the density matrix determines a flow on the two matrix elements $\langle n|A|m\rangle$ and $\langle m|A|n\rangle$ of any observable A by evolving these correlation terms via the aforementioned rotation. The absolute value of $\log(\rho_n/\rho_m)$ fixes the angular velocity, while its sign, which expresses which of the state prevails, gives a preferred direction of this flow. It is a gradient in the concentration difference between the microstates constituting the density matrix which generates a microscopic flow. All these microflows join together to the thermal time flow.

Falling back upon the finite-dimensional case with equally probable microstates, we see why there is no thermal time flow. On the microscopic level no direction can be preferred because everything looks the same, a macroscopic flow cannot emerge. This also happens partly if the eigenvalues of ρ are degenerated. Many microflows vanish, but there are still non-trivial ones which form a macroscopic flow. Note that the degeneracy of eigenvalues increases the number of symmetry transformations, though "weakening" the thermal time flow; there are less microflows. The opposite case occurs if the density matrix approaches a pure state. There is one diagonal

element, say ρ_1, which converges to 1, while the remaining ones go to zero. In this situation the angular velocity of the microflow induced by ρ_1 together with any other element ρ_n, $n \neq 1$, goes to infinity, while there remains a preferred direction.

Conclusion *Thermal time reveals a strong symmetry which has its origin in inhomogeneities in the ratio of the various microstates constituting the density matrix.*

7.11 Thermal Time and Irreversibility

Although it is doubtful if the thermal time flow represents a physical time flow as the considerations we did so far underline, it is still possible that it defines in some yet unknown way reasonable local dynamics whatsoever. Anyway, from now on we shall join the issue concerning the physicality of thermal time the end of the queue and explore physical properties of the thermal time flow. Thermal time has a statistical basis, which is why one should expect particularly thermodynamical properties. First of all we wish to devote ourselves to the issue if the thermal time flow is suitable to describe irreversible phenomena. After that we shall investigate its true equilibrium character and its dependence on changes of the state.

The notion of (common sense) time is deeply associated with change, non-equilibrium situations and irreversibility. Dissipative processes are expected to cause our perception of a flowing time. The basis for such a time is given e.g. by the second law of thermodynamics. It is a quantity w.r.t. which the entropy increases. Such a quantity cannot be the purely mechanical time appearing in the reversible mechanical laws and measured by mechanical clocks. One cannot make sense of a distinguished time variable in mechanics which is perceived as "flowing" (cf. section 7.4), though a mechanical time may still be defined in addition to thermal time [113]. There is no irreversibility on a mechanical level. The physical time of our perception which incorporates such a feature has to be an enriched time concept. The idea of a "flowing time" is ascribed to thermodynamics.

The TTH says that the appropriate candidate for the flowing time of our perception is thermal time. Following [112] thermal time is the quantity "*characterizing the unfolding of irreversible phenomena*". However, by definition the thermal time flow is first of all a KMS equilibrium flow. Let us think about if equilibrium dynamics are able to capture all the non-equilibrium phenomena we observe in our world and which play such a dominant role in the universe.[19] Of course, one can study the evolution of fluctuations around a thermal time equilibrium state. But first of all it is not clear that the system reacts in a way one would expect of a true equilibrium flow in the sense that fluctuations decrease in thermal time, an issue which we take up in the next section, and secondly one may doubt if such a time concept is suitable to treat true irreversible, non-equilibrium processes rather than fluctuations around a thermal state.

According to Rovelli,[20] there are non-thermodynamical quantities in the description of a system, which make sure that the thermal time flow gives rise to a non-trivial evolution of expectation values. Consider a thermal equilibrium state at some finite temperature. In this regime QFT allows for the computation of scattering amplitudes, etc., just the same as in the vacuum. The TTH is intended to make sense for the description of dynamical phenomena when there is an overall thermal bath. One measures dynamical phenomena such as correlations between observable quantities in a thermal background (like the CMB background), which can be explicitly time dependent. We think that the existence of non-trivial time phenomena such as e.g. particle scattering in a thermal background is not sufficient to explain the existence of irreversibility.

Let us come back to the mechanical non-flowing time. It might be reasonable that on a purely mechanical level where reversibility prevails, mechanical time, like a suitably emulated Newtonian time, is not "flowing". Anyway, in order to apply the TTH we are inevitable in a thermodynamical context, where Newtonian time becomes thermodynamical (cf. section 1.6). Moreover, (in quantum mechanics) it is implemented as an automorphism group on the observable algebra, where it defines a time flow which is mathematically on exactly the same footing as thermal time. It thus should be

[19] We have seen that e.g. for the diamond regions in Minkowski space one does not really have equilibrium, because the temperature is not constant along the respective orbits which in turn is due to the non-trivial rescaling of thermal time. Nevertheless, it may still be regarded as a kind of local equilibrium, and as indicated in section 6.5 taking the energy-time uncertainty relation into account the state cannot be distinguished from a true equilibrium state.

[20] www.fqxi.org/community/forum/topic/237

equally "flowing". Starting with a non-equilibrium state one really has to deal with irreversibility, so that Newtonian time should be even more "flowing" than thermal time. This brings us back to the issue how to deal with the appearance of two flows. If one ascribes the origin of time to irreversibility, it is absolutely not clear why thermal time should be the preferred, perceived flow. Newtonian time is better suitable to capture irreversibility and is actually known to be a good time in this setting. Although in the algebraic framework proper time as a geometrical time of spacetime is in general not on the same footing as thermal time, we have met an example where it does extend to a global time flow on the algebra (section 7.4). Since an excited vacuum state restricted to the wedge cannot be a KMS state w.r.t. the implementation of the Lorentz boosts, this flow should again be the favoured one to describe non-trivial time phenomena.

An Example: A Gas in a Box

Let us think of an ideal gas consisting of a finite number of particles contained in a box which is in a Gibbs equilibrium state ω_1 w.r.t. the ordinary Hamiltonian time. Then, ordinary time and thermal time coincide. The entropy of the system will not remain constant, but might decrease a bit to return to its maximum value again afterwards. The state will fluctuate around the most probable configuration representing thermal equilibrium. This way non-trivial time evolutions in both ordinary and (possibly) thermal time are observable. Nevertheless, small fluctuations around ω_1 have nothing to do with fully irreversible processes, and large fluctuations are very unlikely. Most of the time the system will be in equilibrium, anyway.

Next, let us assume that the system is brought into another state ω_2. This state is supposed to be prepared in such a way that most particles are close to one corner of the box. How does the time evolution of ω_2 look like? W.r.t. thermal time ω_2 is a Gibbs equilibrium state and nothing happens macroscopically, apart from small fluctuations around ω_2. Conversely, w.r.t. ordinary time ω_2 is a non-equilibrium state and one expects an irreversible process during which the system approaches ω_1 – and this is certainly the alternative one will observe. Although the thermal time flow might show all the physical properties of a true equilibrium flow, it is not suitable to describe fully irreversible processes. The irreversibility argument suggests to prefer the ordinary time as the relevant time.

Furthermore, this example brings to light one more time the problems arising when the TTH is applied to non-equilibrium states. It illustrates what we found in the first sections, namely why thermal time cannot be expected to be observable. The thermal Hamiltonian assumes interactions, external conditions, etc. in such a way that in equilibrium most particles are close to one corner of the box. The real dynamics are different, which is why the particles will disperse as predicted by the ordinary time evolution.

Conclusion *It is very questionable if a time concept emerging from thermal equilibrium is appropriate to reflect the irreversible processes determining our world.*

7.12 Dispersive Properties of the Thermal Time Flow

Being a KMS flow the thermal time flow is intrinsically equipped with a couple of nice properties such as stationarity, passivity and kinematical stability. However, in chapter 3 we have found that the KMS condition alone is not sufficient to distinguish a flow which represents a thermal equilibrium situation, especially when the physicality of the flow is not a priori known. The stability properties ensured by the KMS condition are not intimately related to a sufficiently strong dispersive behaviour of the dynamical system, which is so characteristic for a thermodynamical system. We have seen that weak asymptotic abelianness controls the dispersive behaviour of the system, and is the appropriate additional criterion to establish the full "return to equilibrium"-property from arbitrary perturbations of the state. It reflects dispersion of local disturbances with time; observations performed at far distances in time do not influence each other mutually. Moreover, we have argued that the consistency condition of Haag et al. gives a kind of minimal requirement to obtain a dispersive behaviour.

If thermal time describes equilibrium dynamics with all its dynamical stability properties, it would represent a time flow with a multitude of physical and thermal properties. Only in that

case the idea of Connes and Rovelli to identify the physical time flow by an equilibrium flow in order to establish a thermodynamical concept of time would be fully realised. Time would be statistically interpretable. One might speak of a "return to equilibrium"-time concept as in the surroundings of thermalised systems.[21] It is further stressed by Rovelli [113] that a "return to equilibrium"-behaviour underwrites the second law whereby it turns thermal time into a physical time. Thermal time is intended to describe how a state out of equilibrium disperses down towards equilibrium. Consequently, the question has to be addressed if the modular flow supplies additional stability properties beyond those related to the KMS condition, and even more fundamental, if it is compatible at all with dispersion.

We investigate in the following properties of the thermal time flow on type I factors and on local algebras, focusing on the dispersive behaviour which is to be modelled by weak asymptotic abelianness or alternatively by the consistency condition. It is a prerequisite to get a time flow of physical significance which can be postulated to represent a thermodynamical time concept.

7.13 Thermal Time Flow on Type I Factors

Since the modular flow can be computed explicitly on them, type I factors provide the best starting point to study properties of the thermal time flow. This is of immediate interest for finite quantum systems, but we shall see later that certain aspects which are not related to the asymptotic behaviour carry over to local algebras, supposed that certain natural conditions hold. To study the thermal time evolution on arbitrary type I factors let us fall back upon the

Proposition 7.13.1 ([70])**.** *Let $\mathcal{M}$ be a type I factor on some Hilbert space $\mathcal{H}$, then there are Hilbert spaces $\mathcal{K}$ and $\mathcal{L}$ and a unitary operator $U : \mathcal{H} \to \mathcal{K} \otimes \mathcal{L}$ such that*

$$U\mathcal{M}U^* = \mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}} \quad \text{and} \quad U\mathcal{M}'U^* = \mathbb{1}_{\mathcal{K}} \otimes \mathcal{B}(\mathcal{L}). \tag{7.41}$$

Lemma 7.13.2. *Consider the type I factor $\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$ and assume that $\Omega \in \mathcal{K} \otimes \mathcal{L}$ is a separating vector. Then, $\rho := \mathrm{tr}_{\mathcal{L}} P_\Omega \in \mathcal{B}(\mathcal{K})$ is invertible.*

Proof. We fall back upon the Schmidt representation of Ω (appendix A.1). Let $\{|e_i\rangle \otimes |f_j\rangle\}_{ij}$ be an ONB for $\mathcal{K} \otimes \mathcal{L}$ for which, omitting the tensor product signs,

$$|\Omega\rangle = \sum_i \phi_i \, |e_{\mu_i}\rangle \, |f_{\nu_i}\rangle \,, \quad \phi_i > 0. \tag{7.42}$$

In this form we can simply exploit the separability of Ω; it enforces $\mu_i = i$ (possibly one has to relabel the indices). That yields

$$\rho = \mathrm{tr}_{\mathcal{L}} P_\Omega = \sum_i \phi_i^2 \, |e_i\rangle \langle e_i| \,. \tag{7.43}$$

Since all $\phi_i \neq 0$, the invertibility of ρ is ensured. □

Theorem 7.13.3. *Let $\mathcal{M}$ be a type I factor on a Hilbert space $\mathcal{H}$, and let $U : \mathcal{H} \to \mathcal{K} \otimes \mathcal{L}$ be a unitary such that $U\mathcal{M}U^* = \mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$. Moreover, suppose that ω is a faithful normal state which is represented by a vector $\Omega \in \mathcal{H}$ which is separating for $\mathcal{M}$. Denote by $\rho := \mathrm{tr}_{\mathcal{L}} P_{U\Omega} \in \mathcal{B}(\mathcal{K})$ the density matrix induced by $U\Omega$ on $\mathcal{K}$. Then, the modular flow associated with ω on $\mathcal{M}$ is given as*

$$\sigma_s^\omega(A) = U^*(\rho^{-is} \otimes \mathbb{1}_{\mathcal{L}})UAU^*(\rho^{is} \otimes \mathbb{1}_{\mathcal{L}})U \quad \text{for all} \quad A \in \mathcal{M}. \tag{7.44}$$

Proof. We merely have to check that the modular flow satisfies the KMS condition. To do that we consult definition 3.3.4. Note that since Ω is separating, the same holds for $U\Omega$, such that ρ is invertible by lemma 7.13.2 and gives rise to a faithful normal state over $\mathcal{B}(\mathcal{K})$,

$$\rho = \mathrm{tr}_{\mathcal{L}} P_\Omega = \sum_i e^{\lambda_i} \, |e_i\rangle \langle e_i| \,, \quad \lambda_i \in \mathbb{R}, \quad \mathrm{tr}\, \rho = 1. \tag{7.45}$$

[21] Though, as discussed before, this does not imply that it is a physical time. Imagine a modular flow which acts spacelike geometrically. Then it might happen very well because of spacelike commutativity that the flow is asymptotically abelian and dispersive, albeit will certainly not be consulted by an observer to describe the evolution.

We define a *-subalgebra of $\mathcal{B}(\mathcal{K})$ with unity, consisting of all operators whose off-diagonal elements w.r.t. the basis $\{e_i\}$ are contained in a finite-dimensional submatrix:

$$\tilde{\mathcal{N}} := \{\tilde{A} \subset \mathcal{B}(\mathcal{K}) : \exists\, N \text{ such that } \langle e_i|\tilde{A}|e_j\rangle = 0 \text{ whenever } i \neq j \text{ and } (i > N \vee j > N)\}. \tag{7.46}$$

Clearly, $\tilde{\mathcal{N}}' = \mathbb{C}\mathbb{1}$ and $\tilde{\mathcal{N}}'' = \mathcal{B}(\mathcal{K})$. By the von Neumann density theorem A.4.7 $\tilde{\mathcal{N}}$ is σ-weakly dense in $\mathcal{B}(\mathcal{K})$, and thus $\mathcal{N} := U^*(\tilde{\mathcal{N}} \otimes \mathbb{1}_{\mathcal{L}})U$ is σ-weakly dense in $\mathcal{M}$. Since $\rho^{-is}\tilde{\mathcal{N}}\rho^{is} = \tilde{\mathcal{N}}$, $\mathcal{N}$ is also σ^ω-invariant. For an arbitrary $z \in \mathbb{C}$ and $\tilde{A} \in \tilde{\mathcal{N}}$ we compute

$$\rho^{-iz}\tilde{A}\rho^{iz} = \sum_{j,k} e^{i(\lambda_k - \lambda_j)z} a_{jk} \,|e_j\rangle \langle e_k|\,, \quad a_{jk} := \langle e_j|\tilde{A}|e_k\rangle\,. \tag{7.47}$$

This can be identified with a bounded operator, in particular

$$z \mapsto \eta(\sigma^\omega_{iz}(A)) = \eta(U^*(\rho^{-iz}\tilde{A}\rho^{iz} \otimes \mathbb{1}_{\mathcal{L}})U), \quad A \equiv U^*(\tilde{A} \otimes \mathbb{1}_{\mathcal{L}})U, \tag{7.48}$$

is a well-defined analytic function for each $\eta \in \mathcal{M}_*$. Altogether this implies that $\mathcal{N}$ is a σ-weakly dense σ^ω-invariant *-subalgebra of $\mathcal{M}_{\sigma^\omega}$. Since $\rho\tilde{A}\rho^{-1}$ represents a bounded operator, the following calculation can be carried out without any difficulties for all $A, B \in \mathcal{N}$:

$$\begin{aligned}
\omega(B\sigma^\omega_i(A)) &= \mathrm{tr}_{\mathcal{K}\otimes\mathcal{L}}\,(P_\Omega \cdot B \cdot U^*(\rho \otimes \mathbb{1}_{\mathcal{L}})UAU^*(\rho^{-1} \otimes \mathbb{1}_{\mathcal{L}})U) && (7.49)\\
&= \mathrm{tr}_{\mathcal{K}\otimes\mathcal{L}}\,(P_{U\Omega} \cdot (\tilde{B} \otimes \mathbb{1}_{\mathcal{L}}) \cdot (\rho \otimes \mathbb{1}_{\mathcal{L}})(\tilde{A} \otimes \mathbb{1}_{\mathcal{L}})(\rho^{-1} \otimes \mathbb{1}_{\mathcal{L}})) && (7.50)\\
&= \mathrm{tr}_{\mathcal{K}\otimes\mathcal{L}}\,(P_{U\Omega} \cdot (\tilde{B}\rho\tilde{A}\rho^{-1} \otimes \mathbb{1}_{\mathcal{L}})) && (7.51)\\
&= \mathrm{tr}_{\mathcal{K}}\,(\rho\tilde{B}\rho\tilde{A}\rho^{-1}) = \mathrm{tr}_{\mathcal{K}}\,(\rho\tilde{A}\tilde{B}) && (7.52)\\
&= \mathrm{tr}_{\mathcal{K}\otimes\mathcal{L}}\,(P_{U\Omega} \cdot UABU^*) && (7.53)\\
&= \omega(AB). && \square
\end{aligned}$$

Remark. If Ω is a separating vector for $\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$, $K = -\log\rho \otimes \mathbb{1}_{\mathcal{L}}$, $\rho := \mathrm{tr}_{\mathcal{L}} P_\Omega \in \mathcal{B}(\mathcal{K})$, is a thermal Hamiltonian for $\mathcal{B}(\mathcal{K})$. However, it is not the modular Hamiltonian, because one e.g. has $K\,|\Omega\rangle \neq 0$. If Ω is cyclic and separating $\tilde{\rho} := \mathrm{tr}_{\mathcal{K}} P_\Omega \in \mathcal{B}(\mathcal{L})$ is an invertible density matrix, as well, and one can show that $\Delta = \rho \otimes \tilde{\rho}^{-1}$ is the modular operator. Thus, $K = -\log(\rho \otimes \tilde{\rho}^{-1})$ is the modular Hamiltonian corresponding to Ω, and with it another thermal Hamiltonian for $\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$.

Next let us look for properties of the thermal time flow on type I factors. First of all we observe as an immediate consequence of theorem 7.13.3 that σ_s has non-trivial fixed points,[22] choose e.g.

$$A_0 = U^*(\rho \otimes \mathbb{1}_{\mathcal{L}})U \in \mathcal{M}, \quad \text{then} \tag{7.54}$$

$$\sigma^\omega_s(A_0) = U^*(\rho^{-is} \otimes \mathbb{1}_{\mathcal{L}})(\rho \otimes \mathbb{1}_{\mathcal{L}})(\rho^{is} \otimes \mathbb{1}_{\mathcal{L}})U = U^*(\rho \otimes \mathbb{1}_{\mathcal{L}})U = A_0 \quad \text{for all} \quad s \in \mathbb{R}. \tag{7.55}$$

Since $\mathcal{M}$ is a factor, one finds a $B \in \mathcal{M}$ such that $\langle\phi, [B, A_0]\psi\rangle \neq 0$ for some $\phi, \psi \in \mathcal{H}$, i.e. the system cannot be asymptotically abelian. With regard to the consistency condition let us investigate the function

$$f_{AB}(s) \equiv \omega([B, \sigma^\omega_s(A)]) = \langle\Omega, [B, \sigma^\omega_s(A)]\Omega\rangle\,. \tag{7.56}$$

As in (7.45), by using the Schmidt representation of $U\Omega$, ρ can be cast into the form

$$\rho = \sum_i e^{\lambda_i} \,|e_i\rangle \langle e_i|\,, \quad \lambda_i \in \mathbb{R}, \quad \mathrm{tr}\,\rho = 1, \tag{7.57}$$

where $\{e_i\}$ is a suitable chosen ONB of $\mathcal{K}$. There exist $\tilde{A}, \tilde{B} \in \mathcal{B}(\mathcal{K})$ such that

$$UAU^* = \tilde{A} \otimes \mathbb{1}_{\mathcal{L}}, \quad \tilde{A} = \sum_{i,j} a_{ij} \,|e_i\rangle \langle e_j|\,, \tag{7.58}$$

$$UBU^* = \tilde{B} \otimes \mathbb{1}_{\mathcal{L}}, \quad \tilde{B} = \sum_{i,j} b_{ij} \,|e_i\rangle \langle e_j|\,. \tag{7.59}$$

[22] Theorem 4.4.7 shows that modular flows on semi-finite algebras necessarily have fixed points (such as T^i). In fact, any modular flow which does not act on type III_1 factors has fixed points, see theorem 7.15.1.

We present some intermediate results of the straightforward computation of f_{AB}:

$$\rho^{-is}\tilde{A}\rho^{is} = \sum_{n,k} e^{-is\lambda_n} a_{nk} e^{is\lambda_k} |e_n\rangle \langle e_k| , \tag{7.60}$$

$$[\tilde{B}, \rho^{-is}\tilde{A}\rho^{is}] = \sum_{m,n,k} (b_{mn}a_{nk}e^{-is\lambda_n}e^{is\lambda_k} - a_{mn}b_{nk}e^{-is\lambda_m}e^{is\lambda_n}) |e_m\rangle \langle e_k| . \tag{7.61}$$

Lastly we end up with

$$f_{AB}(s) = \langle \Omega, [B, \sigma_s^\omega(A)]\Omega\rangle = \langle U\Omega, ([\tilde{B}, \rho^{-is}\tilde{A}\rho^{is}] \otimes \mathbb{1})U\Omega\rangle \tag{7.62}$$

$$= \mathrm{tr}_{\mathcal{K}\otimes\mathcal{L}}(P_{U\Omega}[\tilde{B}, \rho^{-is}\tilde{A}\rho^{is}] \otimes \mathbb{1}) = \mathrm{tr}_{\mathcal{K}}(\rho[\tilde{B}, \rho^{-is}\tilde{A}\rho^{is}]) \tag{7.63}$$

$$= \sum_{m,n} e^{\lambda_m}[a_{nm}b_{mn}e^{is(\lambda_m-\lambda_n)} - a_{mn}b_{nm}e^{-is(\lambda_m-\lambda_n)}]. \tag{7.64}$$

The sum appearing in 7.64 is absolute convergent, since $\rho\,[\tilde{B}, \rho^{-is}\tilde{A}\rho^{is}]$ is a trace-class operator.[23] Thus, $f_{AB}(s)$ is an almost periodic function (cf. appendix A.2). This is a typical situation in closed systems, where the Hamiltonian is an operator with a discrete spectrum. In classical mechanics it is related to the *Poincaré recurrence*. To proceed we need another theorem.

Theorem 7.13.4. *Let $\mathcal{R}$ be a von Neumann algebra and let ω be a faithful normal state. Let $\mathfrak{C}_\omega$ be the centralizer of ω,*

$$\mathfrak{C}_\omega \equiv \{A \in \mathcal{R} : \omega(AB) = \omega(BA) \text{ for all } B \in \mathcal{R}\}. \tag{7.65}$$

Then, $\mathfrak{C}_\omega$ coincides with the fixed point algebra of σ^ω, $\mathfrak{C}_\omega = \mathcal{R}^{\sigma^\omega}$.

Proof. The proof can be found in [11] for cyclic and separating vector states and extends immediately to faithful normal states via the GNS construction. □

The theorem shows that $f_{AB}(s)$ vanishes whenever A or B is a fixed point of σ_s^ω. On the other hand, if A is not a fixed point, we can find a B with $f_{AB}(0) \neq 0$ and vice versa. One can thus expect f_{AB} to be non-vanishing for many pairs $A, B \in \mathcal{M}$. By corollary A.2.4 a non-vanishing almost periodic function cannot be an L^1-function, such that we have to conclude that also the consistency condition, $\int_{-\infty}^{\infty} \mathrm{d}t\, \omega([B, \sigma_s^\omega(A)]) = 0$, is violated on type I factors. Along the way we see that asymptotic abelianness is violated for a large number of pairs of elements and not just for fixed points.

To sum up, on type I factors the thermal time flow fails to show any additional stability properties beyond those already ensured by the KMS condition. In conventional closed systems the lack of asymptotic abelianness is already a problem. It is usually resolved by the expectation that the system will approach the equilibrium state and only after huge time intervals (the *Poincaré recurrence time* as the length of time elapsed until recurrence), not accessible for any observation, the system will significantly leave this state. However, due to the unsolved metrization problem one does not know what huge thermal time intervals actually are,[24] one cannot simply justify the absence of asymptotic abelianness. The incompatibility of the thermal time flow with asymptotic abelianness on type I factors might be a serious problem.

Conclusion *The thermal time flow on type I factor violates weak asymptotic abelianness as well as the consistency conditions.*

[23] To see this explicitly from the formula, note that the bounded operators $\tilde{A}, \tilde{B}$ have to satisfy

$$\sum_i |a_{ij}|^2 \leq \|A\|^2, \quad \sum_j |a_{ij}|^2 \leq \|A\|^2, \quad \sum_i |b_{ij}|^2 \leq \|B\|^2, \quad \sum_j |b_{ij}|^2 \leq \|B\|^2.$$

$$\Rightarrow \quad \sum_{m,n} |e^{\lambda_m}[a_{nm}b_{mn}e^{is(\lambda_m-\lambda_n)} - a_{mn}b_{nm}e^{-is(\lambda_m-\lambda_n)}]| \leq \sum_m e^{\lambda_m} \sum_n (|a_{nm}b_{mn}| + |a_{mn}b_{nm}|)$$

$$\leq 2\|A\|\|B\| \sum_m e^{\lambda_m} = 2\|A\|\|B\| < \infty.$$

[24] Note that we have already met examples, such as the conformal diamond, where even infinite thermal time intervals are physically short.

7.14 Thermal Time Flow on Local Algebras

Equilibrium states appearing in AQFT which are expected to exhibit additional dynamical stability properties are living on the quasi-local C*-algebra. In an unbounded medium the effect of a local operation will spread and dissipate so that it is not felt in a finite region after a sufficiently long time interval. We have to deal with flows acting on local von Neumann algebras, which, due to the finiteness of any realistic observation, generically correspond to finite spacetime regions. There may be less room for excitations to dissipate. Hence, we study the compatibility of the thermal time flow associated to such open systems with dispersion, or if this requirement might be in conflict with other natural physical assumptions.

Asymptotic Abelianness and Statistical Independence

Let us begin with some heuristic considerations. If the theory is statistically independent (see section 2.6), correlations can act only in a limited number of directions. This might be too restrictive for excitations of a state to disperse, and, thus, for a local time flow to define a dynamical system which shows a "return to equilibrium"-behaviour. To make things a bit more concrete, we envisage a local von Neumann factor $\mathcal{R}(\mathcal{O}_1)$ acting on a Hilbert space $\mathcal{H}$, $\mathcal{O}_1$ a bounded spacetime region, and a vector state $\omega = \langle\Omega, \cdot\,\Omega\rangle$ with Ω cyclic and separating for $\mathcal{R}(\mathcal{O}_1)$. Given an observable $A \in \mathcal{R}(\mathcal{O}_1)$ we determine the set of all bounded operators being statistically independent of A,

$$M_A^\omega \equiv \{C \in \mathcal{B}(\mathcal{H}) : \langle\Omega, AC\Omega\rangle = \langle\Omega, A\Omega\rangle\,\langle\Omega, C\Omega\rangle\}. \tag{7.66}$$

Now, we consider a flow τ_t, modelled by some one-parameter *-automorphism group on $\mathcal{R}(\mathcal{O}_1)$. For ω to be strongly clustering w.r.t. τ_t (which is equivalent to weak asymptotic abelianness, cf. theorem 3.5.10), the statistical dependence of A on any element $B \in \mathcal{R}(\mathcal{O}_1)$ has to vanish asymptotically when A is evolved with τ_t. Roughly speaking, the size of $M_A^\omega \cap \mathcal{R}(\mathcal{O}_1)$ in $\mathcal{R}(\mathcal{O}_1)$ for each A gives a kind of measure for the difficulty to define a clustering flow on $\mathcal{R}(\mathcal{O}_1)$ for ω.

Now we take a look at the net of von Neumann factors $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ where $\mathcal{R}(\mathcal{O}_1)$ is part of. Let $\mathcal{O}_2 \supset\supset \mathcal{O}_1$ be a bounded spacetime region which is slightly larger than $\mathcal{O}_1$. Demanding statistical independence for the theory there exists a total set of vectors for which

$$\langle\psi, AB'\psi\rangle = \langle\psi, A\psi\rangle\,\langle\psi, B'\psi\rangle \quad \text{for all} \quad A \in \mathcal{R}(\mathcal{O}_1) \quad \text{and} \quad B' \in \mathcal{R}(\mathcal{O}_2'). \tag{7.67}$$

Let us assume that Ω is contained in this set, then automatically $\mathcal{R}(\mathcal{O}_2') \subset M_A^\omega$ holds. Since $\mathcal{R}(\mathcal{O}_1) \cap \mathcal{R}(\mathcal{O}_2') = \mathbb{C}\mathbb{1}$, the set $M_A^\omega \cap \mathcal{R}(\mathcal{O}_1)$ should be "smaller" in comparison with those cases in which statistical independence is violated, so that only a flow (such as σ_s^ω) carefully adjusted to the peculiarities of the given structure can be clustering and asymptotically abelian. For the thermal time flow σ_s^ω, which does not know and hence cannot exploit the specific structure of the theory (cf. section 7.7), this could be too restrictive. Statistical independence might impede the emergence of a dispersive thermal time flow.

Thermal Time Flow on Local Algebras Satisfying the Split Property

To do a bit more rigorous considerations we attack the issue from another side: We take a look on local nets satisfying the split property, which is essentially equivalent to statistical independence (see section 2.6). The split property says that local algebras can be approximated by type I factors. Our aim is to understand what kind of properties of the well-known thermal time flow on type I factors can be expected to carry over to type III_1 factors. To do so, we start with some functional analytical considerations.

Definition 7.14.1 ([97])**.** Let $\{A_n\}_{n\in\mathbb{N}}$ and A be self-adjoint operators. Then, A_n is said to converge to A in the *strong resolvent sense* if $R_\lambda(A_n) \to R_\lambda(A)$ strongly for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$.

The convergence in the strong resolvent sense is the natural generalisation of strong convergence to unbounded operators whose domains do not need to have a common vector. We prove a modified version of theorem VIII.20 in [97], cf. appendix A.1 for the functional analytical background.

Theorem 7.14.2. *Let $\{A_n\}_{n\in\mathbb{N}}$ and A be positive, invertible, self-adjoint operators on the Hilbert space $\mathcal{H}$, and assume that $A_n \to A$ in the strong resolvent sense. If $g : \mathbb{R}_{>0} \to \mathbb{C}$ is a bounded continuous function, then $g(A_n) \to g(A)$ in the strong sense.*

Proof. Let us first proof the statement for functions $f : \mathbb{R}_{>0} \to \mathbb{C}$ which are continuous, continuously continuable at 0, and vanishing at ∞. By the Stone-Weierstrass theorem, polynomials in $(x+i)^{-1}$ and $(x-i)^{-1}$ approximate the elements in the set $C^0_\infty(\mathbb{R}_{>0})$ of all continuous functions which are continuously continuable at 0 and vanishing at ∞. Thus, for each $\varepsilon > 0$ there is a polynomial $P(s,t)$ such that

$$\|f(x) - P((x+i)^{-1}, (x-i)^{-1})\|_\infty \leq \varepsilon/(3\,\|\psi\|). \tag{7.68}$$

That yields

$$\|f(A_n) - P((A_n+i)^{-1}, (A_n-i)^{-1})\| \leq \varepsilon/3, \tag{7.69}$$
$$\|f(A) - P((A+i)^{-1}, (A-i)^{-1})\| \leq \varepsilon/3. \tag{7.70}$$

Since $A_n \to A$ in the strong resolvent sense, for sufficiently large n the relation

$$\|P((A_n+i)^{-1}, (A_n-i)^{-1})\psi - P((A+i)^{-1}, (A-i)^{-1})\psi\| \leq \varepsilon/3, \quad \psi \in \mathcal{H}, \tag{7.71}$$

holds. It implies

$$\|f(A_n)\psi - f(A)\psi\| \leq \|f(A_n)\psi - P((A_n+i)^{-1}, (A_n-i)^{-1})\psi\| \tag{7.72}$$
$$+ \|P((A_n+i)^{-1}, (A_n-i)^{-1})\psi - P((A+i)^{-1}, (A-i)^{-1})\psi\| \tag{7.73}$$
$$+ \|f(A)\psi - P((A+i)^{-1}, (A-i)^{-1})\psi\| \leq \epsilon, \tag{7.74}$$

i.e. $f(A_n) \to f(A)$ strongly. Let us now pass to the general case. For this let $\psi \in \mathcal{H}$ and $\varepsilon > 0$ be arbitrary. We define $f_m(x) := \exp(-\frac{1}{m}(\frac{1}{x} + x))$ on $\mathbb{R}_{>0}$. Since $\lim_{m\to\infty} f_m(x) = 1$ pointwise and $\|f_m\|_\infty$ is bounded, by use of the spectral theorem we conclude that $f_m(A) \to \mathbb{1}$ strongly. Hence,

$$\|f_m(A)\psi - \psi\| \leq \varepsilon/(6\|g\|_\infty) \tag{7.75}$$

for a suitable chosen m. Moreover, $f_m \in C^0_\infty(\mathbb{R}_{>0})$ implies by the above considerations that $f_m(A_n)\psi \to f_m(A)\psi$ strongly, i.e. there is an $n_0 \in \mathbb{N}$ such that $\|f_m(A_n)\psi - f_m(A)\psi\| \leq \varepsilon/(6\|g\|_\infty)$ for each $n > n_0$. That gives

$$\|f_m(A_n)\psi - \psi\| \leq \varepsilon/(3\|g\|_\infty). \tag{7.76}$$

$g \cdot f_m \in C^0_\infty(\mathbb{R}_{>0})$ ensures the existence of an $n_1 \in \mathbb{N}$ such that

$$\|g(A_n)f_m(A_n)\psi - g(A)f_m(A)\psi\| = \|(g\cdot f_m)(A_n)\psi - (g\cdot f_m)(A)\psi\| \leq \varepsilon/3 \quad \forall n > n_1. \tag{7.77}$$

Altogether, we have for all $n > \max(n_0, n_1)$

$$\|g(A_n)\psi - g(A)\psi\| \leq \|g(A_n)\|\|\psi - f_m(A_n)\psi\| + \|g(A_n)f_m(A_n)\psi - g(A)f_m(A)\psi\| \tag{7.78}$$
$$+ \|g(A)\|\|f_m(A)\psi - \psi\| \leq \varepsilon, \tag{7.79}$$

and we are done. □

To continue we need a couple of other theorems.

Theorem 7.14.3 (Trotter [97])**.** *Let $\{A_n\}$ and A be self-adjoint operators. Then $A_n \to A$ in the strong resolvent sense iff $\exp(itA_n) \to \exp(itA)$ in the strong sense for each t.*

Theorem 7.14.4 (Trotter-Kato Approximation Theorem, cf. [49])**.** *Let $\{U_n(t)\}$ and $U(t)$ be strongly continuous one-parameter unitary groups with generators A_n and A, respectively. Then, $A_n \to A$ in the strong resolvent sense iff $U_n(t) \to U(t)$ uniformly for t in compact intervals in the strong sense.*

Theorem 7.14.5 (D'Antoni-Doplicher-Fredenhagen-Longo [43]). *Let $\{\mathcal{M}_j\}$ be an increasing family of von Neumann algebras, and let $\mathcal{M} \equiv \vee_j \mathcal{M}_j$. Assume that Ω is a cyclic and separating vector for each $\mathcal{M}_j$ and for $\mathcal{M}$. Let Δ_j, J_j and Δ, J be the modular operators and modular conjugations of $\mathcal{M}_j$ and $\mathcal{M}$, respectively. Then, $\Delta_j \to \Delta$ in the strong resolvent sense, and $J_j \to J$ strongly.*

By theorem 7.14.2 the strong resolvent convergence of modular operators Δ_j implies the strong convergence of Δ_j^{is}, which according to theorem 7.14.3 guarantees in turn the strong resolvent convergence of the modular Hamiltonians $K_j = -\log \Delta_j$ to $K = -\log \Delta$. An application of theorem 7.14.4 then establishes the uniform convergence of Δ_j^{is} on compact intervals. Altogether, as a joint consequence of all these theorems we obtain:

Corollary 7.14.6. *Let $\{\mathcal{M}_j\}$ be an increasing family of von Neumann algebras and let $\mathcal{M} \equiv \vee_j \mathcal{M}_j$. Assume that Ω is a cyclic and separating vector for each $\mathcal{M}_j$ as well as for $\mathcal{M}$. Denote by Δ_j and Δ the modular operators of $\mathcal{M}_j$ and $\mathcal{M}$, respectively. Then, $\Delta_j^{is} \to \Delta^{is}$ strongly for each $s \in \mathbb{R}$, and the convergence is uniform on compact intervals.*

Theorem 7.14.7. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector. Moreover, let $\{\mathcal{M}_j\}$ be an increasing family of von Neumann algebras such that $\mathcal{M} = \vee_j \mathcal{M}_j$ and such that Ω is cyclic and separating also for all $\mathcal{M}_j$ (that is cyclic for $\mathcal{M}_1$). Then, for every $\varepsilon > 0$, for any compact interval $S \subset \mathbb{R}$, and for all normalized vectors $\phi, \psi \in \mathcal{H}$ there exists an j_0 such that*

$$|\langle \phi, (\sigma_s^{\mathcal{M}}(A) - \sigma_s^{\mathcal{M}_j}(A))\psi \rangle| < \varepsilon \tag{7.80}$$

for all $j > j_0$, $A \in \mathcal{M}$ with $\|A\| \leq 1$, and $s \in S$. Since all normal states are vector states, such an approximation is in particular possible for each normal state $\eta \in \mathcal{M}_*$,

$$|\eta(\sigma_s^{\mathcal{M}}(A) - \sigma_s^{\mathcal{M}_j}(A))| < \varepsilon \tag{7.81}$$

for all $j > j_0$, $A \in \mathcal{M}$ with $\|A\| \leq 1$, and $s \in S$.

Proof. With the help of the Cauchy-Schwarz inequality we compute

$$|\langle \phi, (\sigma_s^{\mathcal{M}}(A) - \sigma_s^{\mathcal{M}_j}(A))\psi \rangle| = |\langle \phi, \Delta_{\mathcal{M}}^{-is} A \Delta_{\mathcal{M}}^{is} - \Delta_{\mathcal{M}_j}^{-is} A \Delta_{\mathcal{M}_j}^{is})\psi \rangle| \tag{7.82}$$

$$\leq |\langle \phi, (\Delta_{\mathcal{M}}^{-is} - \Delta_{\mathcal{M}_j}^{-is}) A \Delta_{\mathcal{M}}^{is} \psi \rangle| + |\langle \phi, \Delta_{\mathcal{M}_j}^{-is} A (\Delta_{\mathcal{M}}^{is} - \Delta_{\mathcal{M}_j}^{is})\psi \rangle| \tag{7.83}$$

$$\leq \|A \Delta_{\mathcal{M}}^{is} \psi\| \|(\Delta_{\mathcal{M}}^{is} - \Delta_{\mathcal{M}_j}^{is})\phi\| + \|A^* \Delta_{\mathcal{M}_j}^{is} \phi\| \|(\Delta_{\mathcal{M}}^{is} - \Delta_{\mathcal{M}_j}^{is})\psi\| \tag{7.84}$$

$$\leq \|(\Delta_{\mathcal{M}}^{is} - \Delta_{\mathcal{M}_j}^{is})\phi\| + \|(\Delta_{\mathcal{M}}^{is} - \Delta_{\mathcal{M}_j}^{is})\psi\|. \tag{7.85}$$

Because of corollary 7.14.6 both terms on the r.h.s. become arbitrarily small as $j \to \infty$ for all s taken from the compact interval S, hence we just have to choose a j_0 such that the r.h.s. becomes smaller than ε. □

Let us now come back to physics and consider a local net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ acting on a Hilbert space $\mathcal{H}$ and representing some sector of an abstract net. We would like to apply the previous proposition to local algebras $\mathcal{R}(\mathcal{O})$. One question therefore concerns the existence of an increasing family $\{\mathcal{M}_j\}$ of von Neumann algebras with the property that $\mathcal{R}(\mathcal{O}) = \vee_j \mathcal{M}_j$. Let us assume that the net satisfies the split property, i.e. that for every pair of bounded regions $\hat{\mathcal{O}} \subset\subset \mathcal{O}$ one can find an intermediate type I factor $\mathcal{M}$ such that

$$\mathcal{R}(\hat{\mathcal{O}}) \subset \mathcal{M} \subset \mathcal{R}(\mathcal{O}). \tag{7.86}$$

Taking an increasing family of bounded spacetime regions $\{\mathcal{O}_j\}$ such that $\mathcal{O}_j \subset\subset \mathcal{O}_{j+1}$ and $\mathcal{R}(\mathcal{O}) = \vee_j \mathcal{R}(\mathcal{O}_j)$, which is guaranteed e.g. if an *additivity property* holds as it happens to be the case in free theories [121]), the split property can be used to construct an increasing family of type I factors $\mathcal{M}_j$ satisfying

$$\bigvee_j \mathcal{M}_j = \mathcal{R}(\mathcal{O}). \tag{7.87}$$

To this end one simply selects $\mathcal{M}_j$ in such a way that

$$\mathcal{R}(\mathcal{O}_1) \subset \mathcal{M}_1 \subset \mathcal{R}(\mathcal{O}_2) \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{R}(\mathcal{O}). \tag{7.88}$$

Note that $\mathcal{O}$ does not necessarily need to be bounded for this construction.

Let us take a deeper look at the inclusions in (7.88). The split property gives some hints how the local algebras are contained in each other, because the way a type I factor can be contained in another type I factor is very restricted:

Lemma 7.14.8. *Let $\mathcal{N} \subset \mathcal{M}$ be two type I factors on the Hilbert space $\mathcal{H}$. Then, there are Hilbert spaces $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$, and a unitary operator $U : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathcal{K}_3$, such that $U\mathcal{N}U^* = \mathcal{B}(\mathcal{K}_1) \otimes \mathbb{1}_{\mathcal{K}_2} \otimes \mathbb{1}_{\mathcal{K}_3}$ and $U\mathcal{M}U^* = \mathcal{B}(\mathcal{K}_1 \otimes \mathcal{K}_2) \otimes \mathbb{1}_{\mathcal{K}_3}$.*

Proof. We just have to apply proposition 7.13.1 several times. There exist Hilbert spaces $\mathcal{L}$ and $\mathcal{K}_3$, and a unitary operator $V : \mathcal{H} \to \mathcal{L} \otimes \mathcal{K}_3$ such that $V\mathcal{M}V^* = \mathcal{B}(\mathcal{L}) \otimes \mathbb{1}_{\mathcal{K}_3}$. Then $V\mathcal{N}V^* = \tilde{\mathcal{N}} \otimes \mathbb{1}_{\mathcal{K}_3}$. Clearly, $\tilde{\mathcal{N}}$ is *-isomorphic to $\mathcal{N}$, so it is a type I factor, too. Again, we find for $\tilde{\mathcal{N}}$ a unitary $W : \mathcal{L} \to \mathcal{K}_1 \otimes \mathcal{K}_2$, such that $W\tilde{\mathcal{N}}W^* = \mathcal{B}(\mathcal{K}_1) \otimes \mathbb{1}_{\mathcal{K}_2}$. Putting $U := (W \otimes \mathbb{1}_{\mathcal{K}_3})V : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2 \otimes \mathcal{K}_3$ gives a unitary transformation with the desired properties. □

If one starts off with a vector state $\omega = \langle \Omega, \cdot\, \Omega \rangle$ over $\mathcal{R}$ which corresponds to a vector Ω which is separating for $\mathcal{R}(\mathcal{O})$ and cyclic for $\mathcal{R}(\mathcal{O}_1)$ then Ω is cyclic and separating for the whole family $\{\mathcal{M}_j\}$. This happens for example by the Reeh-Schlieder theorem if we are in the vacuum sector, ω is the vacuum state, or any other state analytic for the energy, and $\mathcal{O}_1$, $\mathcal{O}'$ are non-void open regions. Under these conditions theorem 7.14.7 is applicable.

If the split property is fulfilled, any local algebra can be approximated by type I factors. For the latter ones the modular flow is known explicitly, and theorem 7.14.7 shows that if the Reeh-Schlieder property holds and the net satisfies an additivity property, on *compact* thermal time intervals it approximates the thermal time flow on the local algebra arbitrarily well. The automorphisms describing the thermal time evolution on local algebras can be approximated by inner automorphisms in the weak sense of theorem 7.14.7, which are induced by the reduced density matrices appearing in theorem 7.13.3 on type I factors.

Since each $\mathcal{M}_j$ admits a fixed point under the modular flow, one finds for each arbitrarily large thermal time interval an observable in $\mathcal{M}$ which remains constant up to any accuracy in the weak sense in this interval. The thermal time flow is to a certain extent determined by that of type I factors, but this does not concern the asymptotic behaviour, which may drastically differ and which is actually crucial to decide about its physical interpretation as representing true equilibrium dynamics. The asymptotic behaviour is not governed by type I factors, and it remains an open issue if the modular flow on local algebras shows a sufficiently strong dispersive behaviour. While on type I factors any additional stability criterion beyond those ensured by the KMS condition is violated, one cannot simply infer the same for local algebras which are type III_1 factors. This underlines their significance to describe physically realistic systems. Open systems of an infinitely extended medium (with local dynamics) behave completely different than closed systems described by type I factors. Nevertheless, although there is no proof available, it seems to be that asymptotic abelianness cannot be generally expected to hold on local algebras. Let us give another heuristic argument to underline that. For this purpose we consider again the function

$$f_{AB}(s) \equiv \omega([B, \sigma_s^{\omega\restriction\mathcal{R}(\mathcal{O})}(A)]), \quad A, B \in \mathcal{R}(\mathcal{O}), \tag{7.89}$$

where ω is supposed to be a cyclic and separating vector state on a local von Neumann algebra $\mathcal{R}(\mathcal{O})$. Assuming that all the prerequisites of theorem 7.14.7 are fulfilled, we conclude that

$$f_{AB}^{(j)}(s) \to f_{AB}(s) \quad \text{as} \quad j \to \infty, \tag{7.90}$$

and the convergence is uniform on compact intervals, when

$$f_{AB}^{(j)}(s) \equiv \omega([B, \sigma_s^{\omega\restriction\mathcal{M}_j}(A)]) \tag{7.91}$$

denotes the corresponding function on the approximating type I factor $\mathcal{M}_j$. We know from the previous section that all the functions $f_{AB}^{(j)}$ are almost periodic for $A, B \in \mathcal{M}_j$, but the almost

periodicity necessarily carries over to the limit function f_{AB} only when the convergence is uniform on the whole real line (proposition A.2.5). Howsoever, it cannot be excluded as well, and it would undoubtedly be an utmost remarkable fact if for all $A, B \in \mathcal{M}_1 \subset \mathcal{R}(\mathcal{O})$, for all suitable spacetime regions $\mathcal{O}$, and for all physically realisable faithful normal states the limit function is nice, in the sense that it tends asymptotically to zero, is an L^1-function, and even more has a vanishing integral.[25] Only if all these properties are fulfilled, the modular flow would satisfy the consistency condition, and a necessary criterion for weak asymptotic abelianness would hold. There is no recognisable mechanism why this should be the case. So one is entitled to conjecture that dynamical stability is violated in cases of physical interest, although a formal proof seems to be out of sight.

To conclude, dynamical stability does not seem to be in conflict with the modular flow on type III_1 factors, albeit at the same time it does not seem to be enforced by the modular flow, and there is no reason why it should generally hold. Demanding asymptotic abelianness in order to obtain a true "return to equilibrium"-time concept presumably imposes restrictions on the class of states from which thermal time emerges.

Note for this that even if $(\mathcal{R}, \tau)$ is an asymptotically abelian system, the perturbed system $(\mathcal{R}, \tau^P)$ does not need to be asymptotically abelian [104], i.e. the thermal time flow may represent true equilibrium dynamics only for particular states. This asymmetry has a physical origin, it arises whenever the perturbation isolates a finite subsystem. The isolation by perturbation of finite subsystems is reflected by the lack of asymptotic abelianness of the perturbed system and this is related to the formation of bound states in scattering theory.

This reveals an issue when working with the canonical flow. All the modular flows induced by the various states are inner equivalent. They are identified with each other to give an outer automorphism, the canonical flow. For this to make sense, all modular flows have to be regarded as *physically* equivalent. However, if some of these flows show dispersive properties and others do not, physical properties are not respected by the equivalence relation; inner equivalent flows are not physically equivalent if priority is given to a thermodynamical, dispersive behaviour.

Conclusion *Dynamical stability properties of the thermal time flow on local algebras cannot be excluded, though there are indications which suggest that they do not hold in any case of interest.*

7.15 Fixed Points and the Thermal Time Flow

Implications on the Dynamical Stability Behaviour

Let $\mathcal{R}$ be a von Neumann algebra which is not a factor. Then, there is a non-trivial element $A \notin \mathbb{C}\mathbb{1}$ in $\mathcal{R}$ which commutes with all elements in $\mathcal{R}$; in particular it is in the centralizer of any state ω and hence by theorem 7.13.4 it is necessarily a fixed point of the modular flow induced by any faithful normal state on $\mathcal{R}$. According to theorem 4.4.7 the modular flows on semi-finite von Neumann algebras have fixed points, too. Thus, only type III factors do not need to have fixed points under the modular flow. This result can be strengthened:

Theorem 7.15.1. *Let $\mathcal{R}$ be a von Neumann algebra and let ω be a faithful normal state. Assume that σ_s^ω does not have any fixed points. Then, $\mathcal{R}$ is a type III_1 factor.*

Proof. The proof can be found in [11] for cyclic and separating vector states, and extends to faithful normal states via the GNS construction. □

The local algebras appearing in AQFT are type III_1 factors, and the aforementioned theorem shows that they are actually the only ones which are compatible with the non-existence of fixed points. The existence of fixed points under the thermal time flow would destroy weak asymptotic abelianness and thereby the "return to equilibrium"-property. But even if their existence is not an intrinsic feature on local algebras, they do not seem to be excludable, as well. Nevertheless, according to theorem 7.13.4 the existence of fixed points does not violate the consistency condition

[25] Note that if one indeed finds A and B such that the consistency condition is violated, they are localised in a proper subregion of $\mathcal{O}$, so one cannot simply argue for omitting them by saying that the theory is not to be tested at the horizon, as in the quasi-local case where one sometimes disregards quasi-local observables.

of Haag et al. Taking into account that the consistency condition is intended to indicate dynamical stability w.r.t. infinitesimal perturbations this observation is plausible. There is no approach to equilibrium if one studies the evolution of the expectation value of a fixed point in a perturbed state, which is constant, but at the same it remains forever close to the equilibrium expectation value if the perturbation is small. In spite of the existence of fixed points a minimal dynamical stability behaviour is in principle possible. We will show that the existence of two non-commuting modular fixed points, one of which with a countable spectrum, does have consequences also for the validity of the consistency condition w.r.t. the thermal time flow: By passing to the thermal time flow induced by a suitable perturbed state, it is violated.

Theorem 7.15.2. *Let $\mathcal{R}$ be a von Neumann algebra and $\omega = \langle \Omega, \cdot\, \Omega\rangle$ a state which corresponds to a cyclic and separating vector Ω. Moreover, let $A, B \in \mathcal{R}$ be two non-commuting self-adjoint fixed points of σ_s^ω, such that A has a countable spectrum. Then, there exists a $C \in \mathcal{R}$, which can be chosen as self-adjoint, such that $f(s) \equiv \omega^A([\sigma_s^{\omega^A}(B), C])$ is a non-vanishing almost periodic function, defined by an absolute convergent sum, i.e.*

$$f(s) = \sum_n a_n e^{is\lambda_n}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C}, \quad \sum_n |a_n| < \infty. \tag{7.92}$$

Proof. Recall (p. 51), that the perturbed state $\omega^A = \langle \Omega^A, \cdot\, \Omega^A\rangle$ was defined via

$$\Omega^A \equiv e^{-(K+A)/2}\Omega / \| e^{-(K+A)/2}\Omega \|, \tag{7.93}$$

where $K \equiv -\log \Delta$ is the modular Hamiltonian of the unperturbed state. The modular operator Δ_A of the perturbed state satisfies (cf. proposition 4.2.3)

$$\Delta_A^{is}\psi = \lim_{n\to\infty} \left(\Delta^{is/n} e^{-isA/n} J e^{isA/n} J \right)^n \psi. \tag{7.94}$$

A is a modular fixed point, hence

$$[\Delta^{is}, A] = 0 \quad \Rightarrow \quad [\Delta^{is/n}, e^{-isA/n}] = 0. \tag{7.95}$$

Since we always have $[\Delta^{is}, J] = 0$, this gives

$$[\Delta^{is/n}, J e^{isA/n} J] = 0. \tag{7.96}$$

$A \in \mathcal{R}$ implies $e^{\pm isA/n} \in \mathcal{R}$, which yields

$$[e^{-isA/n}, J e^{isA/n} J] = 0. \tag{7.97}$$

Altogether, we end up with the following expression for the perturbed modular operator:

$$\Delta_A^{is} = \Delta^{is} e^{-isA} J e^{isA} J. \tag{7.98}$$

From this we can determine the modular flow of the perturbed state acting on some $D \in \mathcal{R}$,

$$\sigma_s^{\omega^A}(D) = \Delta_A^{-is} D \Delta_A^{is} = J e^{-isA} J e^{isA} \Delta^{-is} D \Delta^{is} e^{-isA} J e^{isA} J \tag{7.99}$$

$$= e^{isA} J e^{-isA} J \sigma_s^\omega(D) J e^{isA} J e^{-isA} = e^{isA} \sigma_s^\omega(D) e^{-isA} \tag{7.100}$$

$$\Rightarrow \quad \sigma_s^{\omega^A}(B) = e^{isA} B e^{-isA}. \tag{7.101}$$

By assumption $[A, B] \neq 0$, thus B cannot be a fixed point of $\sigma_s^{\omega^A}$. Since the centralizer of ω^A coincides with the fixed point algebra of $(\mathcal{R}, \sigma_s^{\omega^A})$, there exists a $C \in \mathcal{R}$ such that $\omega^A([B, C]) \neq 0$. Exploiting that A has a countable spectrum and thus admits an ONB consisting of eigenvectors (cf. theorem A.1.11), we conclude similar to p. 115, cf. in particular equation (7.61), that

$$f_C(s) \equiv \omega^A([\sigma_s^{\omega^A}(B), C]) = \langle \Omega^A, [e^{isA} B e^{-isA}, C] \Omega^A \rangle \tag{7.102}$$

is a non-vanishing almost periodic function with $f(0) \neq 0$ and defined by an absolute convergent sum. To obtain a self-adjoint $\tilde{C}$ (supposed that C itself is not self-adjoint) one proceeds as follows: First we observe that $f_{C^*} = -\overline{f}_C$, which implies

$$f_{C+C^*} = 2i\,\mathrm{Im}(f_C), \quad f_{i(C-C^*)} = 2i\,\mathrm{Re}(f_C). \tag{7.103}$$

Since $f_C(0) \neq 0$, at least one of the expressions $f_{C+C^*}(0)$ and $f_{i(C-C^*)}(0)$ is non-vanishing, so that one simply has to replace C by the self-adjoint element $C + C^*$ or $i(C - C^*)$, respectively. □

The dynamical system $(\mathcal{R}, \sigma_s^{\omega^A})$ violates the consistency condition, in particular there are pairs (such as B and C) which violate weak asymptotic abelianness w.r.t. ω^A.

Remark. As mentioned above, it is by no means clear if there are cases at all in which the modular flow induced on a local algebra admits non-commuting fixed points. If they do exist it might be possible to obtain a countable spectrum by suitable spectral projections (of course non-commutativity has to be preserved) in order to fulfil all the prerequisites of theorem 7.15.2. In the type I case when the system is described by an invertible density matrix on $\mathcal{B}(\mathcal{H})$, two non-commuting fixed points exist whenever the density matrix has a degenerated spectrum.

Consequences for the Spectrum of the Modular Hamiltonian

A vector Ω which is cyclic and separating for a von Neumann algebra $\mathcal{R}$ is an eigenvector of its modular group Δ_Ω^{is} with eigenvalue 1, and of the modular Hamiltonian $K_\Omega = -\log \Delta_\Omega$ with eigenvalue 0. Let $A \in \mathcal{R}$ be a (non-trivial) fixed point of the modular flow σ_s^Ω. Then $A\,|\Omega\rangle$ is an eigenvector of Δ_Ω^{is},

$$\Delta_\Omega^{is} A\,|\Omega\rangle = A\,|\Omega\rangle\,. \tag{7.104}$$

Since Ω is separating, it cannot be an eigenvector of A, i.e. $A\,|\Omega\rangle \neq \lambda\,|\Omega\rangle$ for $\lambda \in \mathbb{C}$ (otherwise $A = \lambda \mathbb{1}$). Consequently, fixed points imply the existence of additional, linearly independent eigenvectors with eigenvalue 1 (or with eigenvalue 0 for the modular Hamiltonian K). Next, we study what one can say about the eigenvalues of the modular Hamiltonian under the conditions of theorem 7.15.2. To do so we first need the

Proposition 7.15.3. *Let $\mathcal{R}$ be a von Neumann algebra and $\omega = \langle \Omega, \cdot\, \Omega\rangle$ a state which corresponds to a cyclic and separating vector Ω. Moreover, let $A, B \in \mathcal{R}$ be two non-commuting self-adjoint fixed points of σ_s^ω, such that A has a countable spectrum. Then, there exist $\phi, \psi \in \mathcal{H}$ such that $f(s) = \langle \phi, \Delta_A^{is}\psi\rangle$ is a non-vanishing almost periodic function, defined by an absolute convergent sum. If, in addition, $[[A,B],A] \neq 0$, $f(s)$ can be chosen as non-constant.*

Proof. Put $\phi := B\Omega^A$ and define $\psi := C\Omega^A$ for some $C \in \mathcal{R}$ to be specified below. Similar to the cases before one shows that

$$f(s) = \omega^A(B\Delta_A^{is}C) = \langle \Omega^A, \sigma_s^{\omega^A}(B)C\Omega^A\rangle = \langle \Omega^A, e^{isA}Be^{-isA}C\Omega^A\rangle \tag{7.105}$$

is an almost periodic function defined via an absolute convergent sum. The choice of C does of course not affect the almost periodicity of f. By theorem 7.15.2 there exists a self-adjoint $C \in \mathcal{R}$, such that $\omega^A([B,C]) \neq 0$, which implies $\langle \phi, \psi\rangle = \omega^A(BC) \neq 0$ and hence the non-vanishing of f.

For the second statement it might be necessary to choose another C. For the time being let C again be arbitrary. We determine the derivative of f evaluated at 0,

$$f(s) = \omega^A(B\Delta_A^{is}C) = \omega^A(e^{isA}Be^{-isA}C), \tag{7.106}$$

$$f'(s) = i\omega^A([A, e^{isA}Be^{-isA}]C) \quad \Rightarrow \quad f'(0) = i\omega^A([A,B]C). \tag{7.107}$$

The modular flow $\sigma_s^{\omega^A}$ of $[A,B]$, which is a fixed point under σ_s^ω reads

$$\sigma_s^{\omega^A}([A,B]) = e^{isA}\sigma_s^\omega([A,B])e^{-isA} = e^{isA}[A,B]e^{-isA}. \tag{7.108}$$

From the assumption $[[A,B],A] \neq 0$ we conclude that $[A,B]$ is not in the fixed point algebra $\mathcal{R}^{\sigma^{\omega^A}}$, and hence not in the centralizer $\mathfrak{C}_{\omega^A}$. As in the previous proof we can find a $C \in \mathcal{R}$ with $f'(0) \neq 0$, such that the assertion comes out. □

Proposition 7.15.4 ([1])**.** *Let $\mu_1(\lambda)$ and $\mu_2(\lambda)$ be two complex-valued functions of bounded variation, normalized in the sense that*

$$\lim_{\lambda\to-\infty} \mu_i(\lambda) = 0, \quad \lim_{\lambda\to\infty} \mu_i(\lambda) = 1, \quad \lim_{\lambda\nearrow\lambda_0} \mu_i(\lambda) = \mu_i(\lambda_0). \tag{7.109}$$

Assume that for all $s \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-is\lambda}\,\mathrm{d}\mu_1(\lambda) = \int_{-\infty}^{\infty} e^{-is\lambda}\,\mathrm{d}\mu_2(\lambda), \tag{7.110}$$

then $\mu_1 = \mu_2$.

Theorem 7.15.5. *Let $\mathcal{R}$ be a von Neumann algebra and $\omega = \langle \Omega, \cdot\, \Omega\rangle$ a state which corresponds to a cyclic and separating vector Ω. Moreover, let $\phi, \psi \in \mathcal{H}$ be such that such that $f(s) \equiv \langle \psi, \Delta^{is}\phi\rangle$ is a non-constant almost periodic function, defined by an absolute convergent sum. Then, the modular Hamiltonian $K = -\log \Delta$ has additional eigenvalues (in addition to 0).*

Proof. $f(s) = \sum_n a_n e^{is\lambda_n}$ with $\lambda_n \in \mathbb{R}$, $a_n \in \mathbb{C}$ and $\sum_n |a_n| < \infty$; w.l.o.g. let $a_n \neq 0$ for all n and $\lambda_n \neq \lambda_m$ whenever $n \neq m$. We can write

$$f(s) = \int e^{-is\lambda}\, \mathrm{d}\mu(\lambda), \quad \mu(\lambda) = \sum_{n\,:\,\lambda_n < \lambda} a_n. \tag{7.111}$$

Since f is non-constant, we assume w.l.o.g. $f(0) = 1$. This can always be achieved by taking $\tilde{f}(s) := \frac{f(s-s_0)}{f(s_0)}$ instead of $f(s)$, where s_0 is chosen such that $f(s_0) \neq 0$. Then μ fulfils the above normalization conditions. Using the spectral decomposition of the modular Hamiltonian $K \equiv -\log \Delta$ we have

$$f(s) = \int_{-\infty}^{\infty} e^{-is\lambda}\, \mathrm{d}\, \langle \psi, E(\lambda)\phi\rangle\,. \tag{7.112}$$

By theorem 7.15.4 [26]

$$\langle \psi, E(\lambda)\phi\rangle = \mu(\lambda). \tag{7.113}$$

Clearly, $\mu(\lambda)$ has all λ_n as points of discontinuity, and thus the same must be true for $E(\lambda)$. Since f is not constant, at least one of these points, λ_{n_0} say, is non-vanishing,

$$E(\lambda_{n_0} + 0) - E(\lambda_{n_0}) \neq 0 \text{ in the strong sense,} \tag{7.114}$$

which precisely means that $\lambda_{n_0} \neq 0$ is an eigenvalue of K (see appendix A.1). □

Combining these results we conclude this section by the

Corollary 7.15.6. *Let $\mathcal{R}$ be a von Neumann algebra and $\omega = \langle \Omega, \cdot\, \Omega\rangle$ a state corresponding to a cyclic and separating vector Ω. Moreover, let $A, B \in \mathcal{R}$ be two self-adjoint fixed points of σ_s^ω, such that A has a countable spectrum and $[[B, A], A] \neq 0$. Then, the modular Hamiltonian corresponding to ω^A has eigenvalues $\lambda \neq 0$, which are embedded eigenvalues when $\mathcal{R}$ is a type III_1 factor.*

Remark. In the case of a type I factor the modular Hamiltonian, which is constructed from reduced density matrices (cf. the remark after theorem 7.13.3), will have many eigenvalues, so that this result does not give something new.

A closer look at the almost periodic function

$$f(s) = \langle \phi, \Delta_A^{is}\psi\rangle = \langle \Omega^A, e^{isA} B e^{-isA} C \Omega^A\rangle \tag{7.115}$$

reveals, that the eigenvalues of the modular Hamiltonian are differences of eigenvalues of A.

Furthermore, the corollary implies that under the stated conditions there exists, apart from Ω^A, an eigenvector ψ, such that $s \mapsto \langle \psi, \sigma_s^\omega(D)\psi\rangle$ is constant for all D. This underlines that the system does not have the "return to equilibrium"-property; even more ψ is an additional stationary state of the flow $\sigma_s^{\omega^A}$. Taking the initial assumption of the existence of two fixed points of the unperturbed flow into account, this is of course not a surprising result.

Conclusion *The existence of fixed points under the modular flow on type III_1 factors can neither be excluded nor confirmed. However, the existence of non-commuting fixed points (one of which with a countable spectrum) is not compatible with the consistency condition as the weakest form of dynamical stability.*

[26] The spectral decomposition is always of bounded variation, (7.111) is also of bounded variation since there is evidently a decomposition into non-decreasing functions.

7.16 Relevance of the Asymptotic Behaviour and the Role of Type I Factors

On the one hand the asymptotic behaviour of the thermal time flow is important to decide whether the evolution shows a sufficient strong form of dispersion such that the flow represents a sensible equilibrium flow, on the other hand one may ask if the asymptotic behaviour of the thermal time flow is experimentally accessible, and therefore of physical significance.

As an example let us consider the diamond in the conformal case. The action is geometrically and the algebras associated with subdiamonds are shifted under the thermal time evolution towards the top of the diamond (figure 6.2 on page 79), whereby they become smaller and smaller shrinking to a point which represents the top of the diamond in the limit $s \to \infty$. The boundary of the diamond is regarded as horizon, playing the same role as infinity for an inertial observer. In a realistic scenario an observer will study the behaviour of a finite number of observables, $A_1, \ldots, A_n$ say. To make sure that the theory is tested only locally, not at the horizon, we assume that they are localised in a subdiamond $D_1 \subset\subset D$ (see [32] for an analogous assumption). Moreover, we take into account that only spacetime regions with a minimum extension are accessible by realistic measurement devices with finite resolution in space and time.[27] Let us assume that a spacetime region is not accessible if it is contained in the tiny diamond D_ε. One can find a *finite* thermal time s_0 after which the observables $A_1, \ldots, A_n$ are localised in diamonds close to the top of D which are all contained in D_ε. Accordingly, the asymptotic behaviour cannot be measured. We do not know whether a similar conclusion, which in this specific situation originates from the conformal transformation relating wedge and double-cone, can be expected in more general cases; nevertheless the example shows that it is a priori not clear if the asymptotic behaviour is relevant. For local dynamics dispersion may not be of primary significance.

An issue which is closely related to this point concerns the metrization of thermal time (cf. section 7.8). We have seen that a non-trivial rescaling of the modular parameter might be necessary in order to relate it to a physically reasonable time. In particular, it does happen that an infinite thermal time interval is identified with a finite physical time interval. In that situation an infinite tail end of thermal time corresponds to an arbitrarily small interval of physical time, and, one more time, the asymptotic behaviour of thermal time would not be accessible for an observer, for any realistic measurement has a finite resolution in physical time. An infinite thermal time interval would be perceived as an infinitesimal physical time interval. Since one should naturally expect the physical time perceived by an observer living in a finite spacetime region to be finite, a non-trivial rescaling could be generic in all the cases in which thermal time corresponds to a physical time. This emphasizes that the problem of the metrization of thermal time is of great physical relevance.

Another question is whether dispersion expressed in terms of asymptotic abelianness is really desirable. If the thermal time flow associated with a finite spacetime region admits a geometrical meaning corresponding to timelike orbits (or respects the causal structure in the sense that the localisation region of an observable is shifted upwards in a timelike direction), asymptotic abelianness in thermal time essentially expresses abelianness in proper time or coordinate time after finite intervals which might be an unwanted feature.[28] If the thermal time has a non-geometrical meaning which is completely mixing the localisation regions of the various observables things might be different, but in that case it is questionable if thermal time can be regarded as an observable, physical flow.

Consider a local algebra $\mathcal{R}(\mathcal{O})$, and assume that all the prerequisites of theorem 7.14.7 hold. By the restriction on finite thermal time intervals, i.e. disregarding the asymptotic behaviour, and on observables which are contained in some subalgebra $\mathcal{R}(\mathcal{O}_1)$, $\mathcal{O}_1 \subset\subset \mathcal{O}$, the approximation of the system via a type I factor, including an approximation of the thermal time flow, becomes possible. For this one chooses a slightly smaller intermediate type I factor $\mathcal{R}(\mathcal{O}_1) \subset \mathcal{M} \subset \mathcal{R}(\mathcal{O})$ which approximates $\mathcal{R}(\mathcal{O})$ sufficiently well. Applying an appropriate unitary transformation U it

[27] An observable, or rather an operation procedure, specifies the intrinsic construction of the measuring apparatus as well as its placement in spacetime. The time of the measurement completes this definition. A dynamical law, that is a time flow, defines an equivalence between procedures at different times [60]. In our case that means that the measuring apparatus corresponding to an observable does not shrink in thermal time, instead each of the observables related by thermal time defines its own apparatus, whose construction eventually requires a non-realisable employment of hardware.

[28] In the case of a conformal Minkowski diamond one has timelike comutativity anyway, so this is not a problem.

is possible to study $\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$ in order to make all physical predictions. In that case a vector state Ω, which we assume to be cyclic and separating for both $\mathcal{R}(\mathcal{O})$ and $\mathcal{M}$, can be replaced by an invertible density matrix $\rho = \mathrm{tr}_{\mathcal{L}}(P_{U\Omega})$ on $\mathcal{B}(\mathcal{K})$ and everything is the same as for closed systems. The expectation values of the observables remain the same,

$$\omega(A) = \langle \Omega, A\Omega \rangle = \mathrm{tr}(\rho \tilde{A}), \quad A \in \mathcal{M}, \quad \tilde{A} \in \mathcal{B}(\mathcal{K}), \quad A = U^*(\tilde{A} \otimes \mathbb{1}_{\mathcal{L}})U. \tag{7.116}$$

Theorem 7.14.7 says that knowing the thermal time flow induced by Ω on the approximating type I factor is physically sufficient. It is generated by the reduced density matrix, which, as an invertible operator, represents a properly statistical mixture of states. Local algebras are "practically" of type I.

This is related to a paper by Requardt [102] who ascribes the true origin of the modular structure to the quantum decoherence of the vacuum. There are stochastic quantum fluctuations passing through the region $\mathcal{O}$. On the full scale these fluctuation patterns would display a certain coherence which can be seen by taking all local observations in Minkowski space into account. An observer confined to a spacetime region $\mathcal{O}$ is forced to make observations only within $\mathcal{O}$. He can acquire only restricted, less coherent and fragmented pieces of information about the vacuum state and its excitations. The full quantum coherence of the vacuum is lost for him and cannot be reconstructed locally. The vacuum appears as a mixed state, which is conjectured by Requardt to consist of a statistical mixture of a complete sample of pure states, so called *pieces of the vacuum* and thereby to be a kind of "trace state".

If one regards $\mathcal{M}$ rather than $\mathcal{R}(\mathcal{O})$ as the relevant algebra, this "trace state" is precisely induced by the density matrix ρ, which, as we have seen, is indeed related to the modular flow on the local algebra. Requardt concludes from this that the modular structure is a consequence of the destruction of quantum coherence of the full Lorentz invariant vacuum state when observations are bound to restricted spacetime domains, and that the mixed state formed by the "pieces of the vacuum" has generally nothing to do with a thermodynamical behaviour, a point of view we have also argued for in the course of this chapter. Only if the coherence across the boundary is sufficiently weak, a true equilibrium behaviour might prevail (such as in the example of the Rindler wedge).

Conclusion *There are reasons to suspect that the asymptotic behaviour of the thermal time flow is physically irrelevant. Practically then it might be satisfactory to deal with type I factors.*

7.17 State-Dependence of Thermal Time

Requirements on State-Dependent Notions of Time

The notion of a state-dependent time flow may give rise to another conceptual issue. As described in section 3.1 given a physical system an observer makes a couple of measurements of some macroscopic observables from which he constructs a state which is best suitable to reflect his measurement results. We have also mentioned that only a weak*-neighbourhood in the set of all states is experimentally accessible. In other words there is a whole neighbourhood of states which is physically equivalent for the observer.

As a natural requirement to obtain a consistent state-dependent notion of time, which has some predictive power, the time flow should not depend too sensitively on the state. Choosing a (faithful normal) state in a neighbourhood of a given state, one should get an evolution which in some sense (to be specified below) is close to the original one. In fact, we have seen in section 4.3 that on von Neumann factors two states never share the same modular automorphism group. The thermal time flow does necessarily change non-trivially with the state, so that in principle the state-dependence of the thermal time flow could cause problems. In the following we first explore the state-dependence of the thermal time flow on type I factors, while we take a look at local algebras afterwards.

State-Dependence of the Thermal Time Flow on Type I Factors

We focus on type I factors which admit a separating vector. Then, every faithful normal state is represented by a separating vector. In the following we study the behaviour of the thermal time flow if a sequence of separating vectors approximates a given separating vector in norm.

Lemma 7.17.1. *Let Ω be a normalized vector on the type I factor $\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$. Moreover, let $\{\Omega_i\}$ be a sequence of normalized vectors converging to Ω in norm. Denote by $\rho := \mathrm{tr}_{\mathcal{L}} P_{\Omega}$ and $\rho_i := \mathrm{tr}_{\mathcal{L}} P_{\Omega_i}$ the density matrices induced by Ω and Ω_i, respectively. Then, $\rho_i \otimes \mathbb{1}_{\mathcal{L}} \to \rho \otimes \mathbb{1}_{\mathcal{L}}$ in the norm sense.*

Proof. Let $\tilde{\Omega}_i := \Omega_i - \Omega$, then $\|\tilde{\Omega}_i\| \to 0$, and let $\{e_k \otimes f_l\}$ be an ONB of $\mathcal{K} \otimes \mathcal{L}$ w.r.t. which Ω is represented in the Schmidt representation (cf. appendix A.1),

$$\Omega = \sum_j a_j \, e_{\mu_j} \otimes f_{\nu_j}, \quad \sum_j |a_j|^2 = 1, \tag{7.117}$$

$$\tilde{\Omega}_i = \sum_{k,l} b_{kl}^{(i)} \, e_k \otimes f_l, \quad \sum_{k,l} |b_{kl}^{(i)}|^2 \overset{i\to\infty}{\longrightarrow} 0. \tag{7.118}$$

Furthermore let

$$\psi = \sum_{m,n} c_{mn} \, e_m \otimes f_n, \quad \sum_{m,n} |c_{mn}|^2 \equiv \|\psi\|^2 \tag{7.119}$$

be an arbitrary vector in $\mathcal{K} \otimes \mathcal{L}$.

$$\rho_i = \mathrm{tr}_{\mathcal{L}} P_{\Omega_i} = \rho + \mathrm{tr}_{\mathcal{L}}(|\tilde{\Omega}_i\rangle \langle\Omega| + |\Omega\rangle \langle\tilde{\Omega}_i| + |\tilde{\Omega}_i\rangle \langle\tilde{\Omega}_i|), \tag{7.120}$$

and we show that the second summand (tensored by $\mathbb{1}_{\mathcal{L}}$) converges to zero in norm. For convenience we elaborate on that only for one of the terms appearing in the trace:

$$\Big((\mathrm{tr}_{\mathcal{L}}(|\tilde{\Omega}_i\rangle \langle\Omega|) \otimes \mathbb{1}_{\mathcal{L}}\Big) |\psi\rangle = \sum_{m,n} c_{mn} \left[\mathrm{tr}_{\mathcal{L}} \left(\sum_{j,k,l} a_j^* b_{kl}^{(i)} \, |e_k\rangle \, |f_l\rangle \langle e_{\mu_j}| \langle f_{\nu_j}| \right) \otimes \mathbb{1}_{\mathcal{L}} \right] |e_m\rangle \, |f_n\rangle \tag{7.121}$$

$$= \sum_{m,n} c_{mn} \left[\sum_{j,k} a_j^* b_{k\nu_j}^{(i)} \, |e_k\rangle \langle e_{\mu_j}| \otimes \mathbb{1}_{\mathcal{L}} \right] |e_m\rangle \, |f_n\rangle = \sum_{j,k,n} c_{\mu_j n} a_j^* b_{k\nu_j}^{(i)} \, |e_k\rangle \, |f_n\rangle \,. \tag{7.122}$$

That yields with the help of the Cauchy-Schwarz inequality

$$\| \Big((\mathrm{tr}_{\mathcal{L}}(|\tilde{\Omega}_i\rangle \langle\Omega|) \otimes \mathbb{1}_{\mathcal{L}}\Big) |\psi\rangle \|^2 = | \sum_{k,l,m,n} c_{\mu_m n} a_m^* b_{k\nu_m}^{(i)} c_{\mu_l n}^* a_l b_{k\nu_l}^{(i)*} | \tag{7.123}$$

$$\leq \sum_{l,m} |(\sum_n c_{\mu_m n} c_{\mu_l n}^*) a_m^* a_l (\sum_k b_{k\nu_m}^{(i)} b_{k\nu_l}^{(i)*})| \tag{7.124}$$

$$\leq \sum_{l,m} |a_m^* a_l| \sqrt{\sum_n |c_{\mu_m n}|^2} \sqrt{\sum_n |c_{\mu_l n}|^2} \sqrt{\sum_k |b_{k\nu_m}^{(i)}|^2} \sqrt{\sum_k |b_{k\nu_l}^{(i)}|^2} \tag{7.125}$$

$$\leq \|\psi\|^2 \left(\sum_m |a_m| \sqrt{\sum_k |b_{k\nu_m}^{(i)}|^2} \right) \left(\sum_l |a_l| \sqrt{\sum_k |b_{k\nu_l}^{(i)*}|^2} \right) \tag{7.126}$$

$$\leq \|\psi\|^2 \left(\sum_m |a_m|^2 \right) \left(\sum_{k,m} |b_{k\nu_m}^{(i)}|^2 \right) = \|\psi\|^2 \left(\sum_{k,m} |b_{k\nu_m}^{(i)}|^2 \right) \leq \|\psi\|^2 \|\tilde{\Omega}_i\|^2 \to 0. \tag{7.127}$$

A similar calculation reveals

$$\| \Big((\mathrm{tr}_{\mathcal{L}}(|\Omega\rangle \langle\tilde{\Omega}_i|) \otimes \mathbb{1}_{\mathcal{L}}\Big) |\psi\rangle \|^2 \leq \|\psi\|^2 \|\tilde{\Omega}_i\|^2, \tag{7.128}$$

$$\| \Big((\mathrm{tr}_{\mathcal{L}}(|\tilde{\Omega}_i\rangle \langle\tilde{\Omega}_i|) \otimes \mathbb{1}_{\mathcal{L}}\Big) |\psi\rangle \|^2 \leq \|\psi\|^2 \|\tilde{\Omega}_i\|^4. \tag{7.129}$$

Altogether we have

$$\|\rho_i \otimes \mathbb{1}_{\mathcal{L}} - \rho \otimes \mathbb{1}_{\mathcal{L}}\| \leq 2\|\Omega_i - \Omega\|^2 + \|\Omega_i - \Omega\|^4, \tag{7.130}$$

which ensures the norm convergence. □

Next, we fall back upon the

Proposition 7.17.2 ([97]). *Let $\{A_n\}$ and A be uniformly bounded self-adjoint operators, then $A_n \to A$ in the strong resolvent sense iff $A_n \to A$ strongly.*

Corollary 7.17.3. *Let Ω be a normalized separating vector on the type I factor $\mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$. Moreover, let $\{\Omega_i\}$ be a sequence of normalized separating vectors converging to Ω in norm. Denote by $\rho := \mathrm{tr}_{\mathcal{L}} P_{\Omega}$ and $\rho_i := \mathrm{tr}_{\mathcal{L}} P_{\Omega_i}$ the reduced density matrices induced by Ω and Ω_i, respectively. Then $\rho_i^{is} \otimes \mathbb{1}_{\mathcal{L}} \to \rho^{is} \otimes \mathbb{1}_{\mathcal{L}}$ in the strong sense, and the convergence is uniformly on compact intervals.*

Proof. By lemma 7.13.2 $\rho_j \otimes \mathbb{1}_{\mathcal{L}}$ and $\rho \otimes \mathbb{1}_{\mathcal{L}}$ are positive, invertible, self-adjoint operators, which are altogether bounded by 1. As a joint consequence of lemma 7.17.1, proposition 7.17.2 and theorem 7.14.2 $\rho_j^{is} \otimes \mathbb{1}_{\mathcal{L}} \to \rho^{is} \otimes \mathbb{1}_{\mathcal{L}}$ in the strong sense. Proceeding in exactly the same way as in the remark before corollary 7.14.6, the assertion comes out. □

Theorem 7.17.4. *Let $\mathcal{M}$ be a type I factor on the Hilbert space $\mathcal{H}$. Let ω be a faithful normal state which corresponds to a separating vector Ω. Moreover, let $\omega_j = \langle \Omega_j, \cdot\, \Omega_j \rangle$ be an arbitrary sequence of faithful normal states, such that the associated sequence of separating vectors Ω_j converges to Ω in norm. Then, for each $\varepsilon > 0$, any compact interval $S \subset \mathbb{R}$, and any pair of state vectors $\phi, \psi \in \mathcal{H}$ there exists an j_0 such that*

$$| \langle \phi, (\sigma_s^{\omega}(A) - \sigma_s^{\omega_j}(A))\psi \rangle | < \varepsilon \tag{7.131}$$

for all $j > j_0$, $s \in S$, and $A \in \mathcal{R}$ with $\|A\| \leq 1$. Since all normal states are vector states, such an approximation is in particular possible for all normal states (cf. theorem 7.14.7).

Proof. Denote by $U : \mathcal{H} \to \mathcal{K} \otimes \mathcal{L}$ the unitary transformation for which $U\mathcal{M}U^* = \mathcal{B}(\mathcal{K}) \otimes \mathbb{1}_{\mathcal{L}}$, and consider any pair of normalized vectors $\phi, \psi \in \mathcal{H}$.

$$\begin{aligned}
| \langle \phi, (\sigma_s^{\omega}(A) - \sigma_s^{\omega_j}(A))\psi \rangle | &= | \langle U\phi, \left[(\rho^{-is} \otimes \mathbb{1}_{\mathcal{L}}) UAU^* (\rho^{is} \otimes \mathbb{1}_{\mathcal{L}}) - (\rho_j^{-is} \otimes \mathbb{1}_{\mathcal{L}}) UAU^* (\rho_j^{is} \otimes \mathbb{1}_{\mathcal{L}}) \right] U\psi \rangle | \\
&\leq | \langle U\phi, \left[(\rho^{-is} \otimes \mathbb{1}_{\mathcal{L}} - \rho_j^{-is} \otimes \mathbb{1}_{\mathcal{L}}) UAU^* (\rho^{is} \otimes \mathbb{1}_{\mathcal{L}}) \right] U\psi \rangle | && (7.132) \\
&+ | \langle U\phi, \left[(\rho_j^{-is} \otimes \mathbb{1}_{\mathcal{L}}) UAU^* (\rho^{is} \otimes \mathbb{1}_{\mathcal{L}} - \rho_j^{is} \otimes \mathbb{1}_{\mathcal{L}}) \right] U\psi \rangle | && (7.133) \\
&\leq \|A\| \left(\|(\rho^{is} \otimes \mathbb{1}_{\mathcal{L}} - \rho_j^{is} \otimes \mathbb{1}_{\mathcal{L}}) U\phi\| + \|(\rho^{is} \otimes \mathbb{1}_{\mathcal{L}} - \rho_j^{is} \otimes \mathbb{1}_{\mathcal{L}}) U\psi\| \right) && (7.134)
\end{aligned}$$

The assertion follows as a direct consequence of corollary 7.17.3. □

State-Dependence of the Thermal Time Flow on Local Algebras

Now, let us pass to local algebras where things are much more involved. Again, we consider a converging sequence of separating vectors. It would be nice to have a similar result also in this context. The problem is that the modular operators are defined in quite an abstract way via the polar decomposition of the Tomita operator, which makes it difficult to predict in which way a change in the state influences the modular flow without the exact knowledge of the flow. One exception is when the separating vectors are of the form

$$\Omega^{A_j} = e^{(\log \Delta_{\Omega} - A_j)/2} \Omega / \| e^{(\log \Delta_{\Omega} - A_j)/2} \Omega \|, \quad A_j = A_j^* \in \mathcal{R} \quad \text{with} \quad \|A_j\| \to 0, \tag{7.135}$$

such that proposition 3.5.1 and 4.2.3 apply.

Since the convergence derived for type I factors holds on compact intervals, one might hope that this kind of consistency of the thermal time flow is a property which carries over from the approximating type I factors, supposed that the split property holds. From section 2.5 we know that the Reeh-Schlieder theorem holds for a dense set of vectors in the vacuum sector. It therefore is possible to construct a converging sequence of cyclic and separating vectors which preserve these properties also w.r.t. the approximating type I factors, such that theorem 7.14.7 is applicable.

Unfortunately, to gain insights into the state-dependence of the thermal time flow on local algebras by exploiting this method, the thermal time flows induced by Ω and Ω_j, respectively must be close to each other in the sense of (7.131) for all j larger than some j_0, and this j_0 must be selectable independently of the type I factor. While the convergence of the reduced density matrices ρ_j is indeed completely independent of the type I factor (cf. the proof of proposition 7.17.1, equation (7.130)), a similar result for ρ_j^{is} or the modular group is not available.

Conclusion *One should require the thermal time flow to be consistent with its state-dependence in the sense that it does not react too sensitive on changes of the state. At least for type I factors this can be shown to be the case.*

7.18 Concluding Remarks

We postpone the conclusions of this analysis to the final chapter. Let us add a remark concerning the mathematical accessibility of the thermal time flow on local algebras, instead. We have seen that on compact intervals the modular flow on local algebras can be approximated by the modular flow induced on suitable type I factors, supposed that the split property and some other mild conditions hold. Although we have used this fact to speculate about asymptotic properties, it is merely the finite behaviour which one can make rigorous statements about. To decide about the existence of fixed points under the modular flow and to study its sensitiveness on changes of the state, it is precisely the behaviour on finite intervals which is relevant, though the presence of fixed points would allow to make statements about the asymptotic behaviour. Anyway, the validity of these two important, potential properties could not have been clarified by using the split property. The source of the difficulty to do that can be named as follows: For a given separating vector state one can immediately write down the modular flow induced by this state on a type I factor. Because of lemma 7.14.8 it is also possible to compare the modular flows induced by a given state on a finite increasing sequence of type I factors. Unfortunately, performing the partial trace when passing from one factor to another, smaller one, changes the reduced density matrix and even more the modular flow in an uncontrollable way and thereby prevents predictions concerning these properties on local algebras.

Chapter 8

Conclusions and Outlook

"Being is real. Becoming is an illusion"
PLATO, Greek philosopher (427 - 347 B.C.)

Due to an incompatible treatment of the notion of time in quantum theory and general relativity it is a reasonable idea to assume a fundamental timelessness of Nature, which means that the notion of time is meaningless, or at least completely irrelevant on a fundamental level. To explain why, nevertheless, our macroscopic world is apparently penetrated by a distinguished flow of time, CONNES and ROVELLI developed the thermal time hypothesis according to which it is the thermodynamical context where this flow emerges from. The distinguished physical time flow of our perception is postulated to coincide with the equilibrium dynamics of the statistical state in which the system is found to be. Mathematically the TTH is implemented in an elegant way by exploiting the Tomita-Takesaki theory and its relation to the KMS theory. The expectation of CONNES and ROVELLI is that thereby the thermal time flow automatically gains thermal and with it physical properties. Indeed, we have seen that the TTH works well in a couple of important examples. It leads to natural evolutions, though all the examples are of a rather artificial nature, which is why they cannot be rated as convincing hints that the TTH is true.

Apart from that, there are well-founded doubts to accept thermal time as a physical time concept representing some sort of real dynamics. In the course of the previous chapter we have met various open questions and unsolved problems which are accompanied by the TTH and come to the fore particularly in the non-generally covariant limit. First of all, thermal time is an entirely new notion of time. It has a statistical origin, while ordinary notions of time are related to geometry. Thermal time shows an unfamiliar behaviour (e.g. it can act non-geometrically) and, disregarding specially constructed cases, will be completely unrelated to the well-established ordinary times. Since we found examples where the TTH does not single out a natural evolution for an observer, one may call into question if he really perceives thermal time, or if there are observers at all using thermal time to describe the dynamical evolution.

A potential way out is to give a more abstract meaning to thermal time, still to be clarified, or to restrict the application domain of the TTH, which then is presumably not suitable to fully explain the origin of time. Unfortunately, there are more problems coming along with a physical thermal time concept. The amount of physics entering the TTH seems to be too small to justify thermal time as a physical time whatsoever, and it is not clear if and how thermal time can be measured at all – prerequisite to be a *physical* quantity. A procedure, or first off all a clock is needed, by which (effects of) thermal time can be measured, concerning especially the modular parameter s itself. One has to be careful when assigning a physical reality to the modular flow. In order to make sure that the thermal time flow is not just a mathematical flow with a couple of nice properties, but something which can be expected to be perceivable by certain observers, one could require the system to be in an intrinsically characterised equilibrium state (whatever it looks like) before the TTH is applicable. Another problem is that the known examples suggest that a non-trivial rescaling of thermal time is necessary to obtain a suitable metrization (which e.g. leads to finite physical time intervals in finite spacetime regions). In general there is no guidance how to choose this metrization, such that the TTH is incomplete so far.

Even if thermal time cannot be identified with a physical, observable notion of time, it is a preferred flow which reveals a hidden, particularly stable symmetry structure. In addition, it could be a useful tool to study the physical system, and to extract certain properties from it. With regard to a more abstract interpretation of thermal time, it is also important to look for physical properties of the thermal time flow. This concerns above all its true equilibrium character on which the motivations of Connes and Rovelli to formulate the TTH crucially rely.

The KMS condition is first of all a mathematical condition about certain analyticity properties which becomes a physical criterion for the distinction of equilibrium states only when the automorphism group is known to be a physical time flow. Moreover, we have seen that the stability properties coded in the KMS condition are not sufficient to ensure a dispersive equilibrium flow. Suitable conditions which do that turn out to be weak asymptotic abelianness or the consistency condition. Both are violated on type I factors and there is no mechanism recognisable due to which one of these conditions should be expectable to be generically satisfied on local algebras. The true thermal character of the thermal time KMS flow was therefore conjectured to be violated in situations of physical interest, though a formal proof is out of sight. It seems to be necessary to impose additional conditions on the state to give reasons for a dispersive behaviour. In that case it would be interesting to know whether it is possible to formulate a condition (at least a sufficient one), which implies asymptotic abelianness or the consistency condition, directly in terms of the state, not relying on the explicit knowledge of the modular objects. Conversely, the asymptotic behaviour, which decides about dispersive properties of the flow, may not be relevant practically. For instance, it will be not measurable if an infinite tail end of thermal time is, due to rescaling, identified with an infinitesimal physical time interval.

Thermal time is intended to be a properly "flowing" time, a property which is usually ascribed to irreversibility and the second law. Since the thermal time flow is mathematically identical with ordinary notions of times (supposed they are implemented on the algebra), and since in addition irreversible processes are better captured by the latter ones, it is not clear what the origin of this "flowing" should be. Even more, on the basis of irreversibility other time flows are much more natural to explain our perception of a flowing time. Eventually the thermal time flow should not depend too sensitively on the state, which would reduce the predictability of its evolution (practically a state can be determined only up to a weak*-neighbourhood). While this can be verified on type I factors, it is again the algebraic setting where a proof is lacking.

All these issues, which were explained in chapter 7, suggest that the TTH (or at least the TTH alone) is not appropriate to determine a physical time flow, or just a time flow with physical or thermal properties. Nonetheless, we emphasize that the possibility is not excluded that thermal time is related to some still unknown "real" physical time whatsoever. However, postulating thermal time as a physical time requires a deeper analysis and cannot be justified without much further ado; all the above mentioned problems have to be resolved. A really striking test of the TTH would be to investigate how it works when applied to a state far away from equilibrium, or when no geometrical interpretation is possible. For example, it would be extremely fruitful to know the thermal time flow on the Minkowski double-cone algebra in the massive or interacting case. This way one would come closer to an answer of many questions around the modular structure and the thermal time concept. One can neither expect the modular action to be geometrical nor that there is an observer for whom the state appears to be thermalised. It would be interesting if and to what extent one can attach a physical interpretation to the thermal time flow.

Albeit the thermal time flow might be a flow with a couple of physical properties (possibly supplied by restrictions on the state), whereby it gains some abstract meaning, the main goal has to be an explanation of the origin of at least certain features of common sense time. Otherwise, since the basic assumption we started with was a fundamental timeless world in which all physical interpretations can be done without falling back upon a notion of time, thermal time would not be needed. Of course, one may reply that in the end the TTH has to work merely when applied to (a subsystem of) our universe in order to explain our specific origin of time. Anyway, all these issues do not change the fact that the thermal time hypothesis is undoubtedly an outermost interesting and fascinating statement, that it is highly desirable to understand what this special flow does, and it does not lessen the importance to gain a better understanding of the still mysterious modular objects. In particular the last point leaves much space for future investigations.

Appendix A

Mathematical Preliminaries

"Mathematics is the queen of sciences [...]. She often condescends to render service to astronomy and other natural sciences [...]."

Carl Friedrich Gauss, German mathematician (1777-1855)

A.1 Functional Analysis

In the following we present some striking results from functional analysis. The stated definitions and theorems are taken from [47, 97, 114]. Proofs are omitted altogether, they can be found in the cited references. Throughout appendix A.1 (if not stated otherwise) by an operator, say A, we mean a linear map acting on a dense subspace $\mathcal{D}(A)$ of some complex Hilbert space $\mathcal{H}$, *the domain of* A, whose range lies in $\mathcal{H}$. The inner product of the Hilbert space $\mathcal{H}$ is denoted by $\langle \cdot, \cdot \rangle$.

Closedness and Self-Adjointness

Definition A.1.1 ([97])**.** An operator A is called *closed* if its graph $\Gamma(A) = \{(\xi, A\xi) : \xi \in \mathcal{D}(A)\}$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. A is said to be *closable* if it has a closed extension, i.e. $\exists\, A'$ such that $\Gamma(A') \supset \Gamma(A)$. Every closable operator A has a smallest closed extension $\bar{A}$, called its *closure*.

Definition A.1.2 ([97])**.** Let A be an operator on $\mathcal{H}$ and define $\mathcal{D}(A^*)$ to be the set of all $\xi \in \mathcal{H}$ for which there is an $\eta \in \mathcal{H}$ with

$$\langle \xi, A\zeta \rangle = \langle \eta, \zeta \rangle \quad \text{for all} \quad \zeta \in \mathcal{D}(A). \tag{A.1}$$

For each such $\xi \in \mathcal{D}(A^*)$ define $A^*\xi := \eta$. A^* is called the *adjoint* of A.

Definition A.1.3 ([114])**.** An operator A is called *self-adjoint* if $A = A^*$.

Theorem A.1.4 ([97])**.** *Let A be an operator on $\mathcal{H}$.*

(i) The adjoint, A^*, is closed. In particular, self-adjoint operators are closed.

(ii) A is closable iff $\mathcal{D}(A^*)$ is dense. In that case $\bar{A} = A^{**}$.

(iii) If A is closable, then $(\bar{A})^* = A^*$.

Bounded Operators

We define the *operator norm* of an operator A acting on a Hilbert space $\mathcal{H}$ to be

$$\|A\| := \sup_{\xi \in \mathcal{D}(A),\, \|\xi\| \leq 1} \|A\xi\| \tag{A.2}$$

An operator is *bounded* if its operator norm is finite.

Theorem A.1.5 ([114])**.** *An operator A is bounded iff $\mathcal{D}(A) = \mathcal{H}$ and A is closed.*

Spectral Theory of Self-Adjoint Operators

It is possible to develop the spectral theory for normal operators. However, with regard to physical applications we prefer to focus our considerations from the beginning on self-adjoint operators.

Definition A.1.6 ([97])**.** Let A be a closed operator on $\mathcal{H}$. A complex number λ is in the *resolvent set*, $\rho(A)$, if $\lambda - A$ is a bijection of $\mathcal{D}(A)$ onto $\mathcal{H}$. For $\lambda \in \rho(A)$, $R_\lambda(A) \equiv (\lambda - A)^{-1}$, which is bounded, is called the *resolvent* of A at λ. If $\lambda \notin \rho(A)$, λ is said to be in the *spectrum* $\sigma(A)$ of A.

Theorem A.1.7 ([47])**.** *The spectrum of a self-adjoint operator A is real.*

Theorem A.1.8 (Spectral Theorem [114])**.** *To every self-adjoint operator A on $\mathcal{H}$ corresponds a unique resolution E of the identity (see e.g. [1, 114] for the definition), the so-called spectral decomposition of A, defined on the Borel subsets of the real line, which satisfies*

$$\langle \xi, A\eta \rangle = \int_{-\infty}^{\infty} \lambda \, \mathrm{d} \, \langle \xi, E(\lambda)\eta \rangle \,, \quad \eta \in \mathcal{D}(A), \quad \xi \in \mathcal{H}. \tag{A.3}$$

The integral is a Lebesgue-Stieltjes integral. Moreover, E is concentrated on $\sigma(A) \subset \mathbb{R}$ in the sense that $E(\sigma(A)) = \mathbb{1}$. The above expression is usually written in the form $A = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E(\lambda)$, when equality is thought in the sense of bilinear forms.

Let E be a resolution of the identity. By definition E is left-continuous in the sense of strong convergence. A point λ is said to be a point of *constancy* if there is an $\varepsilon > 0$ such that

$$E(\lambda + \varepsilon) - E(\lambda - \varepsilon) = 0, \tag{A.4}$$

otherwise it is a point of *growth*. A point λ is called a *point of discontinuity* if

$$E(\lambda + 0) - E(\lambda) \neq 0, \tag{A.5}$$

and a point of *continuity* if

$$E(\lambda + 0) - E(\lambda) = 0. \tag{A.6}$$

Definition A.1.9 ([97])**.** Let A be a closed operator on $\mathcal{H}$. A vector $\psi \in \mathcal{D}(A)$ which satisfies $A\psi = \lambda\psi$ for some $\lambda \in \mathbb{C}$ is called an *eigenvector* of A, λ is called the corresponding *eigenvalue*. If λ is an eigenvalue, then $\lambda - A$ is not injective so λ is in the spectrum of A. The set of all eigenvalues is called the *point spectrum* of A. If λ is not an eigenvalue and if $\mathrm{ran}(\lambda - A)$ is not dense, then λ is said to be in the *residual spectrum*.

Proposition A.1.10 ([1])**.** *Let E be the spectral decomposition of a self-adjoint operator A and let λ be a real number. Then,*

(i) $\lambda \in \rho(A)$ iff λ is a point of constancy of E;

(ii) λ is an eigenvalue of the operator A iff λ is a point of discontinuity of E.

If λ is an eigenvalue of the operator A acting on a Hilbert space $\mathcal{H}$, then $(E(\lambda + 0) - E(\lambda))\mathcal{H}$ is the corresponding eigenspace.

Orthonormal Bases Consisting of Eigenvectors

Theorem A.1.11 ([114])**.** *Let $A \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ as separable Hilbert space, and assume the spectrum of A to be countable. Then, there exists an ONB consisting of eigenvectors.*

Definition A.1.12 ([97])**.** An operator $A \in \mathcal{B}(\mathcal{H})$ is called *compact* if A takes bounded sets in $\mathcal{H}$ into precompact sets in $\mathcal{H}$. Equivalently, A is compact iff for every bounded sequence $\{\psi_n\} \subset \mathcal{H}$, $\{A\psi_n\}$ has a convergent subsequence in $\mathcal{H}$.

In particular trace-class operators and Hilbert-Schmidt operators are compact.

Theorem A.1.13 (Hilbert-Schmidt Theorem [97])**.** *Let A be a self-adjoint compact operator on a separable Hilbert space $\mathcal{H}$. Then, there is an ONB $\{e_n\}$ consisting of eigenvectors, $Ae_n = \lambda_n e_n$, and $\lambda_n \to 0$ as $n \to \infty$.*

Polar Decomposition for Closed Operators

Definition A.1.14 ([97]). An operator A is called *positive*, written $A \geq 0$, if $\langle \xi, A\xi \rangle \geq 0$ for all $\xi \in \mathcal{D}(A)$.

Theorem A.1.15 (Square Root Lemma [114]). *Let A be a self-adjoint operator. Then, the following statements are satisfied:*

(i) A is positive iff the spectrum of A is non-negative, i.e. $\sigma(A) \subset [0, \infty)$.

(ii) If, in addition, A is positive, then there exists a unique positive, self-adjoint operator B, written as $\sqrt{A}$, such that $B^2 = A$.

Proposition A.1.16 ([97]). *Every bounded, positive operator on a complex Hilbert space $\mathcal{H}$ is self-adjoint.*

Theorem A.1.17 ([97]). *Let A be an arbitrary closed operator. Then, A^*A is a densely defined positive, self-adjoint operator.*

Definition A.1.18 ([97]). Let A be a closed operator. Guided by the previous theorems, we introduce the notation $|A| \equiv \sqrt{A^*A}$. Note that $|A|$ is a positive, self-adjoint operator.

Definition A.1.19 ([97]). An operator U from $\mathcal{H}_1$ into $\mathcal{H}_2$ with $\mathcal{D}(U) = \mathcal{H}_1$ is called an *isometry*, if $U^*U = \mathbb{1}_{\mathcal{H}_1}$, or, equivalently, if $\|U\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}$ (which then implies $\langle U\xi, U\eta \rangle_{\mathcal{H}_2} = \langle \xi, \eta \rangle_{\mathcal{H}_1}$ for all $\xi, \eta \in \mathcal{H}_1$ by the polarization identity). If, in addition, U is onto, or, equivalently, if $UU^* = \mathbb{1}_{\mathcal{H}_2}$, U is said to be a *unitary operator*. Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be *isomorphic* if such an unitary operator exists. An operator U is called a *partial isometry* if U is an isometry when restricted to the closed subspace $(\ker U)^\perp$.

Theorem A.1.20 (Polar Decomposition Theorem [97]). *Let A be a closed operator. Then, there exists a partial isometry $U : (\ker A)^\perp \to \overline{\operatorname{ran} A}$ such that $A = U|A|$. U is uniquely determined by these properties and the additional condition that $\ker |A| = \ker A$.*

Stone's Theorem

Definition A.1.21 ([97]). An operator-valued function $U : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ is called a *strongly continuous one-parameter unitary group* if:

(i) $U(t)$ is unitary for all $t \in \mathbb{R}$;

(ii) $U(t + s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$;

(iii) $\lim_{t \to t_0} U(t)\xi = U(t_0)\xi$ for all $t_0 \in \mathbb{R}$ and $\xi \in \mathcal{H}$.

Theorem A.1.22 (Stone's Theorem [97]). *$U(t)$ is a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$ iff there exists a self-adjoint operator H such that $U(t) = e^{itH}$. H is called the infinitesimal generator of $U(t)$.*

Schmidt Representation

Consider a vector $\Omega \in \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two separable Hilbert spaces. Taking ONBs $\{e_i\}$ of $\mathcal{H}_1$ and $\{f_j\}$ of $\mathcal{H}_2$, Ω can be expanded in the following way:

$$\Omega = \sum_{i,j} a_{ij} e_i \otimes f_j, \quad a_{ij} \in \mathbb{C}, \quad \sum_{i,j} |a_{ij}|^2 = \|\Omega\|^2. \tag{A.7}$$

In fact, one can always find ONBs $\{\tilde{e}_i\}$ of $\mathcal{H}_1$ and $\{\tilde{f}_j\}$ of $\mathcal{H}_2$, such that

$$\Omega = \sum_i \phi_i \tilde{e}_{\mu_i} \otimes \tilde{f}_{\nu_i}, \quad \phi_i > 0, \quad \sum_i \phi_i^2 = \|\Omega\|^2, \tag{A.8}$$

where $\{\mu_i\}$ as well as $\{\nu_i\}$ are pairwise disjoint (see e.g. [91]). This representation of Ω is called *Schmidt representation*. From the physical point of view this is interesting, because it establishes the existence of operators $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes B$ such that a measurement of one of them in the state Ω uniquely determines the value of the other one.

Borel-Functional Calculus for Self-Adjoint Operators

The spectral theorem can be used to define functions of self-adjoint operators (see [47] for the details). To this end, let A be some self-adjoint operator, and denote by E its spectral decomposition. If f is a complex Borel function defined E-almost everywhere on $\mathbb{R}$,[1] then

$$f(A) \equiv \int_{\sigma(A)} f(\lambda)\,\mathrm{d}E(\lambda), \tag{A.9}$$

is a well-defined, closed operator on the dense domain

$$\mathcal{D}(f(A)) = \{\eta \in \mathcal{D}(A) : \int_{-\infty}^{\infty} |f(\lambda)|^2\,\mathrm{d}\,\langle \eta, E(\lambda)\eta\rangle < \infty\}. \tag{A.10}$$

$f(A)$ has the following properties:

(i) $\|f(A)\eta\|^2 = \int |f(\lambda)|^2\,\mathrm{d}\,\langle \eta, E(\lambda)\eta\rangle$ for all $\eta \in \mathcal{D}(f(A))$;

(ii) $f(A)^* = \overline{f}(A)$, in particular $f(A)$ is self-adjoint whenever f is real-valued;

(iii) $R_\mu(A) = \int \frac{1}{\mu-\lambda}\,\mathrm{d}E(\lambda)$ for all $\mu \in \rho(A)$.

The next two propositions show that, supposed one takes care of domains, functions of operators satisfy the relations one is used to from ordinary functions.

Proposition A.1.23 ([47]). *Let A be a self-adjoint operator, and let f and g be two Borel functions defined E-almost everywhere on $\mathbb{R}$. Then, the following relations hold for $f(A)$ and $g(A)$:*

(i) $(\alpha f)(A) = \alpha f(A)$ *for all* $\alpha \in \mathbb{C}$;

(ii) $f(A) + g(A) \subset (f+g)(A)$;

(iii) $f(A)g(A) \subset (f\cdot g)(A)$,[2] *in particular for bounded functions* f, g *one has* $(f\cdot g)(A) = f(A)g(A) = g(A)f(A)$.

Proposition A.1.24 ([71]). *If A is a self-adjoint operator and f and g are Borel functions on $\mathbb{R}$, then $(f\circ g)(A) = f(g(A))$.*

We focus our considerations now on a positive, invertible, self-adjoint operator A [72].[3] We have $\sigma(A) \subset \mathbb{R}_{\geq 0}$. Clearly, the functional calculus applies. We are interested in Borel functions of A defined on $\mathbb{R}_{>0}$. In fact, the coincidence of two Borel functions f, g on $\sigma(A)\setminus\{0\}$ already implies $f(A) = g(A)$, such that we do not have to care about 0. We consider the special case in which the Borel function is given as $f_z(x) = e^{iz\log x} = x^{iz}$, $z \in \mathbb{C}$, where log is the complex logarithm on $\mathbb{C}$, defined in the usual fashion. This function is of paramount interest in modular theory. A^s is a positive self-adjoint operator, while A^{it} is unitary, $s, t \in \mathbb{R}$. As a consequence of proposition A.1.24

$$A^z \equiv (\exp\circ\, z\log)(A) = \exp(z\log A). \tag{A.11}$$

Since log is real-valued function on $\mathbb{R}_{>0}$, $\log A$ is self-adjoint. Hence, by Stone's theorem A.1.22, $t \mapsto A^{it} = e^{it\log A}$ is a strongly continuous one-parameter unitary group. Moreover, it can be shown that the inverse of A agrees with $A^{-1} \equiv f_{-1}(A)$, where $f_{-1}(x) = x^{-1}$, and that

$$(A^{-1})^z = A^{-z} \quad \forall\, z \in \mathbb{C} \quad \text{and} \quad (A^{-1})^t = A^{-t} = (A^t)^{-1} \quad \forall\, t \in \mathbb{R}. \tag{A.12}$$

It is established e.g. in [72] that

$$A^z A^{it} = A^{it} A^z = A^{z+it} \quad \text{for all } t \in \mathbb{R} \text{ and } z \in \mathbb{C}. \tag{A.13}$$

To sum up, one can verify essentially all the rules one would expect from the symbolic expression A^z. This is crucial for calculations with the modular operator.

[1] A real function f is said to be a *Borel function* if for each $\alpha \in \mathbb{R}$ the set $\{t : f(t) > \alpha\}$ is a Borel set. A complex function is a Borel function if its real and imaginary part are Borel functions.

[2] $(f\cdot g)(x) \equiv f(x)g(x)$.

[3] An operator is called *invertible* if it is one-to-one and has dense range.

Conjugate-Linear Operators

Since the Tomita-Takesaki theory deals with the possibly rather unfamiliar concept of conjugate-linear operators we give a brief overview of the main features of this type of operator. For more details we refer to [72]. An operator A acting on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ is called *conjugate-linear* or *anti-linear* if its domain $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$ and

$$A(a\xi + b\eta) = \bar{a}A\xi + \bar{b}A\eta \quad \text{for all } \xi, \eta \in \mathcal{D}(A) \text{ and } a, b \in \mathbb{C}. \tag{A.14}$$

We denote by $\overline{\mathcal{H}}$ the *conjugate Hilbert space* of $\mathcal{H}$. $\overline{\mathcal{H}}$ and $\mathcal{H}$ are equal in the category of sets, however, $\overline{\mathcal{H}}$ has a different Hilbert space structure which is derived from that of $\mathcal{H}$. The scalar multiplication is defined by

$$\lambda \circ \xi := \bar{\lambda} \cdot \xi \quad \text{for each } \xi \in \overline{\mathcal{H}} \text{ and for all } \lambda \in \mathbb{C}, \tag{A.15}$$

where "$\cdot$" is the usual scalar multiplication in $\mathcal{H}$. Moreover, the inner product on $\overline{\mathcal{H}}$ is given by

$$\langle \xi, \eta \rangle_{\overline{\mathcal{H}}} := \overline{\langle \xi, \eta \rangle}_{\mathcal{H}} = \langle \eta, \xi \rangle_{\mathcal{H}} \,. \tag{A.16}$$

Norm and weak topology on $\mathcal{H}$ and $\overline{\mathcal{H}}$ coincide. The key observation is the following: Any conjugate-linear operator A on $\mathcal{D}(A) \subset \mathcal{H}$ can be identified as a linear operator between $\mathcal{D}(A)$ and $\overline{\mathcal{H}}$. Clearly, an analogous result holds for the reverse direction. That is the reason why much of the theory of linear operators can be adapted so as to apply also in the conjugate-linear case. For instance, notions like "closedness", "preclosedness", or "self-adjointness" turn out to be independent of the interpretation of the map.[4]

Let A be a conjugate-linear operator. We define $\mathcal{D}(A^*)$ as the set of all $\xi \in \mathcal{H}$ for which there is an $\eta \in \mathcal{H}$ such that

$$\langle \xi, A\zeta \rangle = \overline{\langle \eta, \zeta \rangle} = \langle \zeta, \eta \rangle \quad \text{for all} \quad \zeta \in \mathcal{D}(A). \tag{A.17}$$

For each such $\xi \in \mathcal{D}(A^*)$ we set $A^*\xi := \eta$. As in the case of linear operators, A^* is called the *adjoint of* A. The appearance of the complex conjugation in this formula is due to the above identification of linear and conjugate-linear maps.

A conjugate-linear operator A is called an *anti-isometry*, if $\mathcal{D}(A) = \mathcal{H}$ and $\|A\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}$. Note, however, that for an anti-isometry the relation

$$\langle A\eta, A\xi \rangle = \langle \xi, \eta \rangle \tag{A.18}$$

holds, as can be checked by using the polarization identity.

Finally, let us consider a closed conjugate-linear operator $A : \mathcal{H} \supset \mathcal{D}(A) \to \mathcal{H}$. When viewed as linear operator from $\mathcal{D}(A)$ to $\overline{\mathcal{H}}$, A admits a polar decomposition $A = U|A|$, where $U : \mathcal{H} \to \overline{\mathcal{H}}$ is a partial isometry, and $|A|$ a positive, self-adjoint operator acting on $\mathcal{H}$.[5] Thus, a polar decomposition theorem also holds for an anti-linear operator $A = U|A|$. $U : \mathcal{H} \to \mathcal{H}$ is now a conjugate-linear partial isometry, while $|A| : \mathcal{H} \to \mathcal{H}$ remains linear.

A.2 Almost Periodic Functions

The theory of almost periodic function was developed to a large extent by H. Bohr [16] in close analogy to the theory of periodic functions. We state here the central definitions and results. For this purpose let us consider an arbitrary function $f : \mathbb{R} \to \mathbb{C}$. The real number $\tau(\varepsilon)$ is called *translation number of* f *corresponding to* $\varepsilon > 0$, whenever

$$|f(x + \tau) - f(x)| \leq \varepsilon \quad \text{for all} \quad x \in \mathbb{R}. \tag{A.19}$$

The sum or difference of translation numbers corresponding to ε_1 and ε_2 is a translations number corresponding to $\varepsilon_1 + \varepsilon_2$. A set $E \subset \mathbb{R}$ is said to be *relatively dense* if there exists an $L > 0$ such that every interval $[a, a + L]$ contains at least one element of E.

[4] One has to be careful when applying the functional calculus, cf. [72].

[5] In the linear case (as well as in the anti-linear case) the polar decomposition theorem is extendable to operators linking two distinct Hilbert spaces.

Definition A.2.1 ([16]). A continuous function $f : \mathbb{R} \to \mathbb{C}$ is called *almost periodic* when for each $\varepsilon > 0$ there exists a relatively dense set of translation numbers of f corresponding to ε.

The next theorem summarises some basic properties of almost periodic functions.

Theorem A.2.2 ([16]). *If f and g are almost periodic functions, the following assertions are true:*

(i) *f is bounded;*

(ii) *f is uniformly continuous;*

(iii) *$f \pm g$ are almost periodic functions;*

(iv) *$f \cdot g$ is an almost periodic function.*

Proposition A.2.3 ([16]). *Let f be an almost periodic function. Then, for each $\varepsilon > 0$ there exist L and δ, such that each interval $[a, a + L]$, $a \in \mathbb{R}$, contains an entire interval of length δ whose points are translation numbers $\tau(\varepsilon)$.*

From this we immediately obtain the next

Corollary A.2.4. *Let $f \not\equiv 0$ be an almost periodic function. Then, f cannot be an L^1-function.*

Proof. With regard to the definition of Lebesgue integrals, we assume w.l.o.g. that f is a positive function with $a := f(0) > 0$. Choose $\varepsilon = a/2$. By proposition A.2.3 there exists a δ such that $f(x) \geq a/2$ for infinitely many disjoint intervals of length δ, which immediately implies the assertion. □

It can be shown that every sum of a finite number of continuous periodic functions is almost periodic. In particular, every *trigonometric polynomial*

$$s(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x}, \quad a_n \in \mathbb{C}, \quad \lambda_n \in \mathbb{R}, \tag{A.20}$$

is an almost periodic function. Consider now the set $\{s(x)\}$ of all these trigonometric polynomials. Denote by $H\{s(x)\}$ the set of functions which can be approximated uniformly on the whole of $\mathbb{R}$ by $\{s(x)\}$.

Proposition A.2.5 ([16]). *Let $\{f_n\}$ be a sequence of almost periodic functions which converges uniformly on $\mathbb{R}$ to a function f. Then, f is also almost periodic.*

The proposition immediately implies that all functions in $H\{s(x)\}$ are almost periodic, in particular each function of the form

$$f(x) = \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C}, \quad \sum_n |a_n| < \infty, \tag{A.21}$$

is almost periodic. One of the most important results in the theory of almost periodic functions is that $H\{s(x)\}$ already contains all almost periodic functions:

Theorem A.2.6 (Fundamental Theorem [16]). *The set $H\{s(x)\}$ is identical with the class of almost periodic functions, in particular every almost periodic function can be approximated uniformly by trigonometric polynomials of the form (A.20).*

A.3 Algebras, Representations and States

In this part we throw a glance at C*-algebras being central objects in AQFT. We go into the most important definitions leading to the notion of abstract C*-algebras, consider some fruitful properties of states over C*-algebras, and introduce the GNS construction which gives rise to a preferred representation of a C*-algebra for a given state. For the details we refer the reader to [23, 71].

Abstract C*-Algebras

A vector space $\mathcal{A}$ over the field of complex numbers $\mathbb{C}$ is called an *algebra* if $\mathcal{A}$ is equipped with a product which is assumed to be associative and distributive. If there exists an identity element $\mathbb{1}$ in the algebra $\mathcal{A}$ for which $A = A \cdot \mathbb{1} = \mathbb{1} \cdot A$ for all $A \in \mathcal{A}$, then $\mathcal{A}$ is called an *algebra with unit*, or a *unital algebra*. The unit element is unique whenever it exists.

A mapping $\mathcal{A} \ni A \mapsto A^* \in \mathcal{A}$ is called an *involution of the algebra* $\mathcal{A}$ if for all $A, B \in \mathcal{A}$ the following conditions hold:

(i) $A^{**} = A$;

(ii) $(AB)^* = B^* A^*$;

(iii) $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$ for $\alpha, \beta \in \mathbb{C}$.

An algebra $\mathcal{A}$ with an involution is said to be a **-algebra*. A subset $\mathcal{B}$ of a *-algebra $\mathcal{A}$ is called *self-adjoint* if $A \in \mathcal{B}$ implies $A^* \in \mathcal{B}$. In particular, note that with this definition an element $A \in \mathcal{A}$ is self-adjoint iff $A = A^*$. An element A of a *-algebra is said to be *normal* if $AA^* = A^*A$.

Let $\mathcal{A}$ be an algebra with identity. The *resolvent set* $\rho(A)$ of an element $A \in \mathcal{A}$ is the set of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is invertible. The *spectrum* $\sigma(A)$ of A is the complement of $\rho(A)$ in $\mathbb{C}$. An element A of a *-algebra $\mathcal{A}$ is defined to be *positive*, if it is self-adjoint and its spectrum $\sigma(A)$ is contained in $\mathbb{R}_{\geq 0}$. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$.

If an algebra $\mathcal{A}$ is equipped with a norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}_{\geq 0}$, satisfying the assumptions

(i) $\|A\| = 0$ iff $A = 0$,

(ii) $\|\alpha A\| = |\alpha| \|A\|$, $\alpha \in \mathbb{C}$,

(iii) *triangle inequality*: $\|A + B\| \leq \|A\| + \|B\|$,

(iv) *product inequality*: $\|AB\| \leq \|A\| \|B\|$,

then, $\mathcal{A}$ is called a *normed algebra*. A normed algebra which is complete w.r.t. the topology induced by the norm is a *Banach algebra*. Moreover, a Banach algebra with an involution for which $\|A\| = \|A^*\|$ is said to be a *Banach* **-algebra*.

Definition A.3.1 ([23])**.** An *(abstract) C*-algebra* is a Banach *-algebra $\mathfrak{A}$ with the property

$$\|A^* A\| = \|A\|^2 \quad \text{for all} \quad A \in \mathfrak{A}. \tag{A.22}$$

A C*-algebra does not necessarily need to contain an identity; nevertheless it may be embedded into a larger C*-algebra which contains an identity. For this reason one may assume for convenience that a C*-algebra does contain an identity element.[6]

An element A of a unital C*-algebra $\mathfrak{A}$ is called *isometry* if $A^* A = \mathbb{1}$, it is said to be *unitary* if $A^* A = \mathbb{1} = AA^*$.

Theorem A.3.2 ([60])**.** *Let $\mathfrak{A}$ be a C*-algebra. Then $\mathfrak{A}^+$ is a convex cone, i.e. $\alpha A + \beta B \in \mathfrak{A}^+$ for all $A, B \in \mathfrak{A}^+$ and $\alpha, \beta \in \mathbb{R}^+$.*

The following results are the abstract pendants to the square root lemma, the polar decomposition theorem, etc.

Theorem A.3.3 ([71])**.** *If $\mathfrak{A}$ is a C*-algebra and $A \in \mathfrak{A}$, then the following conditions are equivalent:*

(i) $A \in \mathfrak{A}^+$;

(ii) $A = H^2$ for some $H \in \mathfrak{A}^+$;

(iii) $A = B^* B$ for some $B \in \mathfrak{A}$.

When these conditions are satisfied, H is unique.

Hence, for positive elements A of a C*-algebra $\mathfrak{A}$ one can define the square root, $\sqrt{A} := H$. For any element A its modulus is defined as $|A| = \sqrt{A^* A}$. Let $\mathfrak{A}$ be a unital C*-algebra, and let $A \in \mathfrak{A}$ be an invertible element, then A can be decomposed as $A = U|A|$, where $U = A|A|^{-1}$ is a unitary element of $\mathfrak{A}$.

[6] Note that states, which will also be introduced in this section, on C*-algebras without identity have a canonical extension to the enlarged algebra with unit.

*-Homomorphisms and *-Isomorphisms

A *-*homomorphism* between two *-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is a homomorphism $\tau : \mathcal{A}_1 \to \mathcal{A}_2$ between $\mathcal{A}_1$ and $\mathcal{A}_2$ in the category of algebras respecting the *-structure, i.e. $\tau(A^*) = \tau(A)^*$ for all $A \in \mathcal{A}_1$. A *-homomorphism is called a *-*isomorphism* if it is one-to-one and onto.

Theorem A.3.4 ([71]). *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two C*-algebras, and let $\tau : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a *-homomorphism. Then, the following conditions hold:*

(i) τ is positive, i.e. preserves positivity;

(ii) $\|\tau(A)\| \leq \|A\|$ for all $A \in \mathfrak{A}_1$, in particular τ is continuous;

(iii) $\tau(\mathfrak{A}_1)$ is a C*-algebra.

Proposition A.3.5 ([71]). *Each *-isomorphism τ between two C*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is norm-preserving, i.e. $\|\tau(A)\| = \|A\|$ for all $A \in \mathfrak{A}_1$.*

Representations

A *representation* of a C*-algebra $\mathfrak{A}$ is defined to be a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space an π a *-homomorphism of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$. The representation $(\mathcal{H}, \pi)$ is said to be *faithful* if π is a *-isomorphism between $\mathfrak{A}$ and $\pi(\mathfrak{A})$. Faithful representations are characterised by norm-preserving representations. Naturally, these are the most important ones. One can show that each C*-algebra admits a faithful representation [71].

A representation $(\mathcal{H}, \pi)$ of a C*-algebra $\mathfrak{A}$ is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces which are invariant under the action of $\pi(\mathfrak{A})$.[7]

Proposition A.3.6 ([23]). *Let $(\mathcal{H}, \pi)$ be a representation of the C*-algebra $\mathfrak{A}$. Then, the following conditions are equivalent:*

(i) $(\mathcal{H}, \pi)$ is irreducible;

(ii) Multiples of the identity are the only bounded operators commuting with $\pi(\mathfrak{A})$;

(iii) For every $0 \neq \psi \in \mathcal{H}$ the set $\pi(\mathfrak{A})\psi$ is dense in $\mathcal{H}$, or $\pi(\mathfrak{A}) = 0$ and $\mathcal{H} = \mathbb{C}$.

A C*-algebra $\mathfrak{A}$ must be expected to admit a huge variety of representations. The distinction between certain types of representations is not advantageous. Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are said to be *(unitarily) equivalent*, whenever there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_1(A) = U\pi_2(A)U^*$ for all $A \in \mathfrak{A}$. A weaker form of equivalence is introduced below.

A *-automorphism τ of the C*-algebra $\mathfrak{A}$ is called *implementable in the representation* π if there exists a unitary operator U_τ acting in the representation space such that

$$\pi(\tau(A)) = U_\tau \pi(A) U_\tau^*. \tag{A.23}$$

An important class of representations is the class of cyclic representations. A *cyclic representation* of a C*-algebra $\mathfrak{A}$ is defined to be a triple $(\mathcal{H}, \pi, \Omega)$, where $(\mathcal{H}, \pi)$ is a representation of $\mathfrak{A}$ and Ω is a vector in $\mathcal{H}$ which is *cyclic* for $\pi(\mathfrak{A})$ in $\mathcal{H}$, which means that $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$.

States

We introduce two important topologies on the dual space $\mathfrak{A}^*$ of a C*-algebra $\mathfrak{A}$.

- The *uniform topology* is the metric topology induced by the norm,

$$\|\omega\| = \sup_{A \in \mathfrak{A},\, \|A\|=1} |\omega(A)|. \tag{A.24}$$

- For $A \in \mathfrak{A}$, $\omega \mapsto |\omega(A)|$ is a semi-norm on $\mathfrak{A}^*$. The *weak* topology* is the topology induced by these semi-norms. It is of great physical relevance (see chapter 2).

[7] In that case $\{0\}$ and $\mathcal{H}$ are the only invariant subspaces.

States on C*-algebras are of fundamental importance for the construction of representations. We precede by giving the mathematical definition of a state.

Definition A.3.7 ([23]). A linear form ω over a C*-algebra $\mathfrak{A}$ is called a *state* if

(i) ω is positive, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$, and if

(ii) ω is normalized, i.e. $\|\omega\| = 1$.

The collection of all positive linear functionals over $\mathfrak{A}$ is the positive cone of the dual space $\mathfrak{A}^*$, denoted by $\mathfrak{A}^{*+}$. A state is a normalized element of $\mathfrak{A}^{*+}$. In the next proposition we have collected the most important properties of states.

Proposition A.3.8 ([23, 60]). *Let ω be a state on a C*-algebra $\mathfrak{A}$ (or, more general, a positive linear functional). Then, the following conditions hold for all $A, B \in \mathfrak{A}$:*

(i) ω is continuous and bounded;

(ii) $\omega(A^*B) = \overline{\omega(B^*A)}$, and $\omega(A^*) = \overline{\omega(A)}$ (i.e. ω is real);

(iii) Cauchy-Schwarz inequality: $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$;

(iv) $|\omega(A^*BA)| \leq \omega(A^*A)\|B\|$;

(v) $|\omega(A)|^2 \leq \omega(A^*A)\|\omega\|$;

The set of all states, $E_{\mathfrak{A}}$, on a unital C^*-algebra $\mathfrak{A}$ forms a convex, weakly* compact subset of its dual space $\mathfrak{A}^*$. Hence, convex combinations of states are again states. This mathematical operation corresponds to the physical procedure of mixing (*mixed states*). Conversely, $E_{\mathfrak{A}}$ contains extremal points, the *extremal states*, and the whole set of states is generated from the extremal states by convex combination and weak* closure. Moreover, the property of positivity enables one to order positive linear functionals in a natural manner. This can be used to introduce the notion of *pure states*. A state ω is called *pure*, if $\omega - \tilde{\omega}$, $\tilde{\omega} \in \mathfrak{A}^{*+}$, can only be positive for $\tilde{\omega} = \lambda\omega$, $0 \leq \lambda \leq 1$.

All these peculiarities of the set of states turn out to be very fruitful from the physical point of view when studying e.g. thermodynamical mixed and pure phases. In fact, the set of pure states coincides with the set of extremal states. Since we are essentially dealing with von Neumann factors (see below), which are less suitable to study such phenomena, we do not pursue this direction any further, which is why we refer to the literature for more details [23, 24, 60]. Some additional properties of states are discussed in the next section in the context of von Neumann algebras. A bit more about the physical meaning of states is included in the main text, chapter 2.

Gelfand-Naimark-Segal Construction

Let $(\mathcal{H}, \pi)$ be a representation of a C*-algebra $\mathfrak{A}$ and $\Omega \in \mathcal{H}$ an arbitrary vector satisfying $\|\Omega\| = 1$, a *state vector*. Every state ω on a C*-algebra determines a representation π_ω over some Hilbert space $\mathcal{H}_\omega$ where it is represented as a cyclic state vector $\Omega_\omega \in \mathcal{H}_\omega$, in the sense that $\omega(A) = \langle\Omega_\omega, \pi(A)\Omega_\omega\rangle$. The construction of this canonical representation associated with a given state goes back to Gelfand and Naimark (1943), and to Segal (1947). In their honours it is called *Gelfand-Naimark-Segal construction* or simply *GNS construction*. The GNS representation is of fundamental importance in physics.

Theorem A.3.9 (GNS Construction [23]). *Let ω be a state over the C*-algebra $\mathfrak{A}$. Then, there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathfrak{A}$, the GNS representation, such that*

$$\omega(A) = \langle\Omega_\omega, \pi_\omega(A)\Omega_\omega\rangle \quad \text{for all} \quad A \in \mathfrak{A}. \tag{A.25}$$

Moreover, the representation $(\mathcal{H}_\omega, \pi_\omega)$ is unique up to unitary equivalence.

Corollary A.3.10 ([23]). *Let ω be a state over a C*-algebra $\mathfrak{A}$ and τ a *-automorphism of $\mathfrak{A}$ which leaves ω invariant. Then, there exists a uniquely determined unitary operator U_ω acting on $\mathcal{H}_\omega$, which leaves Ω_ω invariant, $U_\omega\Omega_\omega = \Omega_\omega$, and which implements τ,*

$$\pi_\omega(\tau(A)) = U_\omega \pi_\omega(A) U_\omega^* \quad \text{for all} \quad A \in \mathfrak{A}. \tag{A.26}$$

Theorem A.3.11 ([23]). *Let ω be a state over a C*-algebra $\mathfrak{A}$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the associated GNS representation. Then, $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible iff ω is pure, or equivalently, iff ω is extremal.*

A.4 Von Neumann Algebras

Besides C*-algebras, von Neumann algebras are central objects in the algebraic setting of quantum field theory. Furthermore, they are crucial ingredients of modular theory. In this section we intend to present some basic definitions together with a rough survey of important properties and characteristics of von Neumann algebras. Detailed supplements can be found e.g. in [23, 70, 72].

Topologies

The starting point for the study of von Neumann algebras is given by the set of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$, which we denote by $\mathcal{B}(\mathcal{H})$. First of all one has to fix a topology. For this purpose we introduce five important operator topologies on $\mathcal{B}(\mathcal{H})$. All these topologies are locally convex.[8] They are introduced in terms of families of (semi-)norms. More generally, one can show [71] that each family of semi-norms on a vector space which separates points defines a locally convex topology w.r.t. which all these semi-norms become continuous functions. Conversely, every locally convex topology arises from a suitable family of semi-norms.

- The *uniform topology* is the metric topology induced by the operator norm,
$$\|A\| = \sup_{\xi\in\mathcal{H},\,\|\xi\|=1} \|A\xi\|. \tag{A.27}$$
- For $\xi \in \mathcal{H}$, $A \mapsto \|A\xi\|$ is a semi-norm on $\mathcal{B}(\mathcal{H})$. The *strong topology* is the topology on $\mathcal{B}(\mathcal{H})$ defined by these semi-norms.
- Consider all sequences $\{\xi_n\}$ in $\mathcal{H}$ such that $\sum_n \|\xi_n\|^2 < \infty$. Then $A \mapsto \left(\sum_n \|A\xi_n\|^2\right)^{1/2} < \infty$ is a semi-norm on $\mathcal{B}(\mathcal{H})$. The set of these semi-norms defines the *σ-strong topology*.
- For $\xi, \eta \in \mathcal{H}$, $A \mapsto |\langle \xi, A\eta\rangle|$ is a semi-norm on $\mathcal{B}(\mathcal{H})$. The *weak topology* is the topology on $\mathcal{B}(\mathcal{H})$ defined by all these semi-norms. Indeed, due to the polarization identity, it suffices to restrict oneself to semi-norms with $\xi = \eta$.
- Let $\{\xi_n\}$, $\{\eta_n\}$ be two sequences in $\mathcal{H}$ such that $\sum_n \|\xi_n\|^2 < \infty$ and $\sum_n \|\eta_n\|^2 < \infty$. Then $A \mapsto \sum_n |\langle \xi_n, A\eta_n\rangle| < \infty$ is a semi-norm on $\mathcal{B}(\mathcal{H})$. The topology on $\mathcal{B}(\mathcal{H})$ induced by these semi-norms is called *σ-weak topology*.

If "$<$" means "finer than", the relations between the various topologies on $\mathcal{B}(\mathcal{H})$ can be summarised as follows:

$$\begin{array}{ccccc} \text{uniform} & < & \sigma\text{-strong} & < & \sigma\text{-weak} \\ & & \wedge & & \wedge \\ & & \text{strong} & < & \text{weak} \end{array}$$

Concrete C*-Algebras

Definition A.4.1 ([60])**.** A *concrete C*-algebra* is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the uniform topology.

Every concrete C*-algebra is an abstract C*-algebra. Conversely, as a consequence of theorem A.3.9 every abstract C*-algebra can be represented by a concrete C*-algebra.

For concrete C*-algebras we have two definitions of positivity. However, as a joint consequence of theorem A.1.15 and proposition A.1.16 we have the result that an element A of a concrete C*-algebra is positive in the sense of definition A.1.14 iff it is positive according to the definition for abstract C*-algebras, namely that it is self-adjoint and has a non-negative spectrum.

Theorem A.4.2 (Kadison Transitivity Theorem [23, 71])**.** *Let $\mathfrak{A}$ be an irreducible concrete C*-algebra over the Hilbert space $\mathcal{H}$, and denote by $\{\psi_1, \ldots, \psi_n\}$ and $\{\varphi_1, \ldots, \varphi_n\}$ two families of vectors in $\mathcal{H}$. If there exists a $T \in \mathcal{B}(\mathcal{H})$ such that $T\varphi_i = \psi_i$ for $i = 1, \ldots, n$, then it follows that there exists an $A \in \mathfrak{A}$ with the same property and with $\|A\| = \|T\|$. If T is self-adjoint, A may be chosen self-adjoint, and if T is unitary, A may be chosen unitary.*

[8] A *locally convex space* is a topological vector space in which the topology has a base consisting of convex sets. A *locally convex topology* on a vector space V is a topology w.r.t. which V becomes a locally convex space.

Definition of von Neumann Algebras

Definition A.4.3 ([60]). A unital *-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the weak topology is said to be a *von Neumann algebra*.

Since the weak topology is coarser than the uniform topology, every von Neumann algebra is a C*-algebra, whence all the result about C*-algebras also hold for von Neumann algebras. Let S be an arbitrary subset of $\mathcal{B}(\mathcal{H})$. The *commutant of* S, denoted by S', is defined to be

$$S' \equiv \{A \in \mathcal{B}(\mathcal{H}) : [A, S] = 0\}. \tag{A.28}$$

Theorem A.4.4 ([60]). *Let $S \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set. Then*

(i) S' is a von Neumann algebra;

(ii) S'' is the smallest von Neumann algebra containing S;

(iii) $S''' = S'$.

Statement (ii) immediately leads to an alternative characterisation of von Neumann algebras:

Corollary A.4.5 (Von Neumann Double Commutant Theorem [23]). *Instead of the topological definition A.4.3, a von Neumann algebra may be equally well defined as a self-adjoint subset $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ with unit, satisfying $\mathcal{R}'' = \mathcal{R}$.*

Theorem A.4.6 ([23]). *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a *-subalgebra of operators on a Hilbert space $\mathcal{H}$ containing the identity. Then, $\mathcal{A}'' = \mathcal{A}$ iff $\mathcal{A}$ is weakly / strongly / σ-weakly / σ-strongly closed.*

Corollary A.4.7 (Von Neumann Density Theorem [23]). *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a *-subalgebra of operators on a Hilbert space $\mathcal{H}$ containing the identity. It follows that $\mathcal{A}$ is dense in $\mathcal{A}''$ in the weak, strong, σ-weak, and σ-strong topology.*

Studying the representations of abstract C*-algebras, it is fruitful to complete the operator algebra in some topology which is weaker than the uniform topology, but which, nonetheless, has some form of uniformity on the finite-dimensional subspaces. Von Neumann's density theorem, which also holds for some other topologies we did not mention here, shows that the closure is independent of the particular choice of the topology. This leads to the conclusion that there are only two kinds of topological *-subalgebras of $\mathcal{B}(\mathcal{H})$, concrete C*-algebras and von Neumann algebras. We shall focus on the latter ones (of course the former belong to them).

Proposition A.4.8 ([10]). *Let $\mathcal{R}$ be a von Neumann algebra. Then, the weak*-topology (stemming from the dual of the predual of $\mathcal{R}$) and the σ-weak topology coincide on $\mathcal{R}$.*

Operators Affiliated with a von Neumann Algebra

Definition A.4.9 ([71]). A closed operator A is *affiliated* with a von Neumann algebra $\mathcal{R}$, written $A\eta\mathcal{R}$, whenever $U^*AU = A$ for each unitary operator U commuting with $\mathcal{R}$.

Assume that A is a self-adjoint operator with spectral decomposition E, and S a Borel subset of $\mathbb{C}$. Denote by χ_S the characteristic function of S. The projection operator $E(S) = \chi_S(A)$ is called the *spectral projection for A corresponding to S*.

Proposition A.4.10 ([23]). *Let A be a closed operator affiliated with a von Neumann algebra $\mathcal{R}$. If $A = U|A|$ is the polar decomposition of A, then U and the spectral projections of $|A|$ lie in $\mathcal{R}$.*

Cyclic and Separating Subsets

Let us introduce two properties which can be assigned to a subset S of a Hilbert space $\mathcal{H}$ on which a von Neumann algebra $\mathcal{R}$ acts, and which can be met quite often in the course of this thesis.

Definition A.4.11 ([23]). Let $\mathcal{R}$ be a von Neumann algebra on $\mathcal{H}$ and let $S \subset \mathcal{H}$ be a subset.

(i) S is *cyclic* for $\mathcal{R}$ if $\overline{\mathcal{R}\mathcal{K}} = \mathcal{H}$.[9]

(ii) S is *separating* for $\mathcal{R}$ if, for any $A \in \mathcal{R}$, $A\psi = 0$ for all $\psi \in \mathcal{K}$ implies $A = 0$.

We fall back upon this definition mainly in cases in which S consists of just one single vector, or rather a state vector. Note that if two operators $A, B \in \mathcal{R}$ have the same action on a separating vector, they must necessarily coincide.

Lemma A.4.12 ([23]). *Let $\mathcal{R}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then, a subset $S \subset \mathcal{H}$ is separating for $\mathcal{R}$ iff S is cyclic for $\mathcal{R}'$ and vice versa.*

Normal States and Unitary Implementation Theorem

Let us now study to what extent the action of a von Neumann algebra on a Hilbert space is governed by its algebraic structure. The main tool to do that is the theory of normal states on von Neumann algebras.

Definition A.4.13 ([72]). A state ω on a von Neumann algebra $\mathcal{R}$ is said to be *normal* if for each monotone increasing net A_α in $\mathcal{R}^+$ with an upper bound

$$\omega(\text{l.u.b.}_\alpha A_\alpha) = \text{l.u.b.}_\alpha\, \omega(A_\alpha). \tag{A.29}$$

It turns out that this sort of continuity condition is equivalent to σ-weak continuity. Actually this may be regarded as a motivation to introduce the σ-weak topology. It implies that normal states are preserved by *-isomorphisms. Fortunately, there exists a more accessible description of normal states via density matrices,[10] which already indicates their physical importance. They provide a generalisation of classical probability theory to non-commutative probability theory. The various characterisations of normal states are collected in the next theorem.

Theorem A.4.14 ([72]). *Let ω be a state over a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$. Then, the following conditions are equivalent:*

(i) ω *is normal.*

(ii) There exists a density matrix ρ*, such that* $\omega(A) = \text{tr}(\rho A)$.

(iii) ω *is* σ*-weakly continuous.*

(iv) ω *is weakly continuous on the unit ball of* $\mathcal{R}$.

(v) ω *is strongly continuous on the unit ball of* $\mathcal{R}$.

(vi) $\omega(\sum_\alpha E_\alpha) = \sum_\alpha \omega(E_\alpha)$ *for each family* $\{E_\alpha\}$ *of orthogonal projections in* $\mathcal{R}$.

(vii) $\omega = \sum_n \langle \phi_n, \cdot\, \phi_n \rangle$ *and* $\sum_n \|\phi_n\|^2 = 1$.

(viii) $\omega = \sum_n \langle \phi_n, \cdot\, \phi_n \rangle$ *and* $\sum_n \|\phi_n\|^2 = 1$*, where* $\{\phi_n\}$ *is an orthogonal family of vectors in* $\mathcal{H}$.

Corollary A.4.15 ([72]). *The set of normal states of a von Neumann algebra is norm closed.*

Theorem A.4.14 implies that the σ-weak topology does only depend on the order structure of a von Neumann algebras and not on the particular Hilbert space representation. This observation leads to the next theorem.

Theorem A.4.16 ([23]). *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two von Neumann algebras and τ a *-homomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$. Then, beside the predictions of theorem A.3.4, τ is σ-weakly and σ-strongly continuous. Thus, τ also preserves the topological structure.*

Definition A.4.17 ([60]). Two von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, are called *spatially isomorphic*, or *unitarily equivalent*, if there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\mathcal{R}_2 = U\mathcal{R}_1 U^*. \tag{A.30}$$

[9] The bar denotes the closed linear span.

[10] A *density matrix* ρ is a positive trace-class operator on $\mathcal{H}$ with $\text{tr}\,\rho = 1$.

Combining the assertion of the preceding theorem with the fact that *-isomorphisms preserve normal states, one ends up with the following key result.

Theorem A.4.18 (Unitary Implementation Theorem [72, 124])**.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two von Neumann algebras acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Moreover, $\mathcal{R}_1$ and $\mathcal{R}_2$ are supposed to admit a cyclic and separating vector. Then, every *-isomorphism π of $\mathcal{R}_1$ onto $\mathcal{R}_2$ is spatial, i.e. π is implemented by a unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$,*

$$\pi(A) = UAU^* \quad \textit{for all} \quad A \in \mathcal{R}_1. \tag{A.31}$$

The theorem is very powerful to understand how the action of a von Neumann algebra on a Hilbert space is governed by its algebraic structure. From this point of view the theorem states that for von Neumann algebras with cyclic and separating vector the spatial action is completely determined by its algebraic structure.

Folium of a Representation

In a typical situation in AQFT one deals with the (weak closure) of the GNS representation π_ω induced by a state ω on an (abstract) C*-algebra $\mathfrak{A}$ (or, alternatively, on a von Neumann algebra). In the representation space, ω corresponds to a cyclic vector Ω_ω, $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$. It can easily be checked that $\omega_\psi(A) := \langle \psi, \pi_\omega(A)\psi \rangle$ defines a state for each $\psi \in \mathcal{H}_\omega$, i.e. states over C*-algebras (von Neumann algebras) come in families. States of this type are usually referred to as *vector states of the representation* $(\mathcal{H}_\omega, \pi_\omega)$ or *vector states of the state* ω. Even more generally, one may consider states of the form

$$\omega_\rho(A) = \operatorname{tr} \rho \pi_\omega(A), \tag{A.32}$$

where ρ is a density matrix on $\mathcal{B}(\mathcal{H}_\omega)$. ω_ρ is said to be a *normal state of* ω. The set of all these states is called the *folium of the state* ω or *folium of the representation* π_ω. It coincides with the set of normal states of the von Neumann algebra $\pi_\omega(\mathfrak{A})''$. The states which belong to the same folium are pictured physically as states which differ from each other only by quasi-local excitations.

We already introduced the unitary equivalence of two representations. A slightly weaker but more natural notion of equivalence, closer related to physics, is given by the concept of quasi-equivalence.

Definition A.4.19 ([72])**.** Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of a C*-algebra $\mathfrak{A}$ are said to be *quasi-equivalent* if there is a *-isomorphism τ from $\pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A})''$, such that

$$\tau(\pi_1(A)) = \pi_2(A) \quad \text{for all} \quad A \in \mathfrak{A}. \tag{A.33}$$

The quasi-equivalence of two representations is equivalent to the coincidence of their folia [23]. The folium of a representation uniquely determines its quasi-equivalence class.

Proposition A.4.20 ([60])**.** *The folium of a representation and the set of vector states of a representation are norm closed subsets in the positive cone of the dual space, $\mathfrak{A}^{*+}$, of the corresponding C*-algebra $\mathfrak{A}$.*

GNS Representation

Let us explore now the GNS representation induced by normal states on von Neumann algebras.

Proposition A.4.21 ([23])**.** *Let ω be a normal state on a von Neumann algebra $\mathcal{R}$, and let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the associated GNS representation. Then, $\pi_\omega(\mathcal{R})$ is a von Neumann algebra and π_ω is normal in the sense that $\pi_\omega(\text{l.u.b.}_\alpha A_\alpha) = \text{l.u.b.}_\alpha \pi_\omega(A_\alpha)$ for any bounded, increasing net $\{A_\alpha\}$ in $\mathcal{R}^+$.*

We continue by introducing another useful property that can be assigned to a state.

Definition A.4.22 ([23])**.** A state ω on a von Neumann algebra $\mathcal{R}$ is called *faithful* if $\omega(A) > 0$ for all non-zero $A \in \mathcal{R}^+$.

Proposition A.4.23 ([72]). *Let $\mathcal{R}$ be a von Neumann algebra, and ω a faithful normal state on $\mathcal{R}$. Then, the associated GNS construction $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ yields a faithful representation π_ω of $\mathcal{R}$, and $\pi_\omega(\mathcal{R}) \subset \mathcal{B}(\mathcal{H}_\omega)$ is a von Neumann algebra with a cyclic and separating vector Ω_ω*

By the unitary implementation theorem A.4.18 this means that the representations π_{ω_i}, $i = 1, 2$, induced by two faithful normal states ω_1 and ω_2 are unitary equivalent. Thus, a representation of a faithful normal state is uniquely determined up to unitary equivalence, and is, in that sense, independent of the choice of the state.

Theorem A.4.24 ([72]). *Let ω be a normal state on a von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$, which admits a separating vector for $\mathcal{R}$. Then, there exists a $\psi \in \mathcal{H}$ such that $\omega = \langle \psi, \cdot\, \psi \rangle$, i.e. each normal state is a vector state. If $\mathcal{R}$ admits a cyclic and separating vector and ω is a faithful normal state, $\psi \in \mathcal{H}$ can be chosen as cyclic and separating.*

As a joint consequence of proposition A.4.23 and theorem A.4.24 we have the

Corollary A.4.25. *The folium of a faithful normal state is completely specified by its vector states.*

A.5 Classification Theory of von Neumann Algebras

The last section of appendix A is devoted to the classification theory of von Neumann algebras. The study of this subject was initiated by Murray and von Neumann in a couple of papers between 1936 and 1943. Apart from mathematical questions their work was motivated by physical reasons, namely by the question of possible decompositions of quantum systems into independent subsystems. A subclassification of type III factors, based on modular theory, was pioneered by Connes in 1973. Much more detailed accounts on this subject are contained e.g. in [40, 10, 23, 72].

Projection Operators

An element P of a Banach *-algebra is called *projection* whenever $P^2 = P = P^*$. Two projections P and Q on a von Neumann algebra $\mathcal{R}$ are said to be *equivalent*, written $P \sim Q$, if there exists a $V \in \mathcal{R}$, such that $P = V^*V$ and $Q = VV^*$. A *central projection* Z is a projection from $\mathcal{R} \cap \mathcal{R}'$. The orthogonal complement of the union of all central projections in $\mathcal{R}$ which are orthogonal to P is called *central carrier of* P (cf. [10, 71, 72] for the details).

Now, let P be a projection of an arbitrary von Neumann algebra $\mathcal{R}$.

- P is called *finite* if $Q \leq P$ and $Q \sim P$ imply $Q = P$ for all $Q \in \mathcal{R}$, otherwise it is called *infinite*.
- If P is infinite and ZP is either 0 or infinite for each central projection $Z \in \mathcal{R} \cap \mathcal{R}'$, P is said to be *properly infinite*.
- If there does not exist a non-zero projection $Q \in \mathcal{R}$ such that $Q \leq P$, then P is called *purely infinite*.
- P is said to be *abelian*, if $P\mathcal{R}P$ is a commutative subalgebra of $\mathcal{R}$.

The distinction between all these types of projections suggests an algebraical description of von Neumann algebras. For this purpose let $\mathcal{R}$ denote again a von Neumann algebra.

- $\mathcal{R}$ is *finite* / *properly infinite* if $\mathbb{1}$ is finite / properly infinite.
- $\mathcal{R}$ is called *semi-finite*, if $\mathbb{1}$ is not purely infinite.
- $\mathcal{R}$ is of *type I*, if there is a non-zero abelian projection with central carrier $\mathbb{1}$.
- $\mathcal{R}$ is of *type* I_n, $n \in \mathbb{N} \cup \{\infty\}$, if $\mathbb{1}$ is the sum of n equivalent abelian projections.
- $\mathcal{R}$ is of *type II*, if it has a finite projection with central carrier $\mathbb{1}$, but there are no non-zero abelian projections.

- $\mathcal{R}$ is of *type* II_1, if it is of type II and finite.
- $\mathcal{R}$ is of *type* II_∞, if it is of type II and properly infinite.
- $\mathcal{R}$ is of *type III*, if $\mathbb{1}$ is purely infinite.

Theorem A.5.1 (Type Decomposition Theorem [72])**.** *Every von Neumann algebra $\mathcal{R}$ is uniquely decomposable into a direct sum of von Neumann algebras of the various types,*

$$\mathcal{R} = \mathcal{R}_I \oplus \mathcal{R}_{II_1} \oplus \mathcal{R}_{II_\infty} \oplus \mathcal{R}_{III}. \tag{A.34}$$

The type $\mathrm{I}_n/\mathrm{I}_\infty/\mathrm{II}_n/\mathrm{II}_\infty/\mathrm{III}$ property of a von Neumann algebra is invariant under *-isomorphisms. The classification of von Neumann algebras can thus be done up to *-isomorphisms.

Definition A.5.2 ([71])**.** A von Neumann algebra $\mathcal{R}$ whose centre $\mathcal{R} \cap \mathcal{R}'$ is trivial is called a *factor*.

A von Neumann algebra $\mathcal{R}$ is a factor iff

$$\mathcal{R} \vee \mathcal{R}' \equiv (\mathcal{R} \cup \mathcal{R}')'' = \mathcal{B}(\mathcal{H}). \tag{A.35}$$

A (σ-finite) von Neumann algebra splits in a generalised direct sum, or direct integral, of factors. Hence the question of classifying von Neumann algebras reduces to the classification of factors [23].

Tracial Weights

An alternative approach to the classification of von Neumann algebras is obtained by considering *tracial weights*, mappings from $\mathcal{R}^+$ to $[0, +\infty]$ with otherwise analogous properties as traces, which generalise the concept of tracial states. Tracial weights contain information about the type of a von Neumann algebra: An algebra of type III is characterised by the non-existence of a non-zero tracial weight which shows certain properties, namely normality and semi-finiteness.[11] Assuming that such a tracial weight does exist, one obtains a finer type distinction from its range. To this end, take a factor (with a separable predual) which is not of type III. In that case there exists an essentially unique non-zero normal semi-finite weight. The restriction of this weight on the set of projections gives the *Murray and von Neumann dimension function*. The type of the factor is deducible from the range of this mapping. For further details we refer to the literature [41, 72]

Finer Classification of Type III Factors

From the physical point of view factors of type I and III are of particular importance (cf. section 2.7). Factors of type I_n are *-isomorphic to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ has dimension n, $n \in \mathbb{N} \cup \{\infty\}$, which completes their classification. We exploit this crucial property of type I factors in section 7.13, where also a stronger statement may be found. For a finer classification of type III factors one has to look for suitable invariants. This was achieved by Connes [40, 41] with the help of modular theory. He defined the *spectral invariant* of a von Neumann algebra $\mathcal{R}$ (with a separable predual)[12]

$$S(\mathcal{R}) \equiv \bigcap_{\omega \in \mathfrak{S}_0} \operatorname{spec} \Delta_\omega, \quad \mathfrak{S}_0 \equiv \{\omega : \omega \text{ a faithful normal state on } \mathcal{R}\}. \tag{A.36}$$

It is established by the cocycle Radon-Nikodým theorem 4.4.5 that the set

$$T(\mathcal{R}) \equiv \{s : \sigma_s^\omega \text{ is an inner automorphism}\} \subset \mathbb{R} \tag{A.37}$$

is independent of the choice of the faithful normal state ω, it is a property of $\mathcal{R}$. Next, let us define the Arveson spectrum, which turns out to be a closed subset of the real line [10, 11, 73]:

[11] A tracial weight τ is *semi-finite* if every non-zero $A \in \mathcal{R}^+$ majorizes some non-zero $B \geq 0$ with $\tau(B) < \infty$, and *normal* if $\tau(\sup A_i) = \sup \tau(A_i)$ for every bounded increasing net $\{A_i\}$ in $\mathcal{R}^+$.

[12] To capture the general case states has to be replaced by weights in the following. Some care is needed in the non-semi-finite case where both spectral invariants do not need to coincide [73].

Definition A.5.3 ([73]). Let $(\mathcal{R},\tau)$ be a W*-dynamical system, then the *Arveson spectrum of* τ, denoted by $\operatorname{spec}\tau$,[13] is defined to be the set of all $\lambda\in\mathbb{R}$, such that $\hat{f}(\lambda)=0$ ($\hat{f}$ the Fourier transform) for all L^1-functions f for which

$$\tau_f(A):=\int f(t)\tau_t(A)\,\mathrm{d}t=0\quad\text{for all}\quad A\in\mathcal{R}. \tag{A.38}$$

The integral is understood in the weak sense.

Proposition A.5.4 ([10]). *Assume that τ is implemented by a strongly continuous unitary group $U(t)$ on $\mathcal{H}$ with generator H, and assume there exists a cyclic and separating vector Ω in $\mathcal{R}$ such that $U(t)\Omega=\Omega$ for all $t\in\mathbb{R}$, then* $\operatorname{spec}\tau=\operatorname{spec}H$.

In particular, if a state ω corresponds to a cyclic and separating vector, $\operatorname{spec}\sigma^\omega=\operatorname{spec}(-\log\Delta_\omega)$. Eventually we can define the *Connes spectrum*,

$$\Gamma(\mathcal{R})\equiv\bigcap_{P\in\mathcal{P}^{\mathcal{R}}_{\sigma^\omega}}\operatorname{spec}\sigma^\omega\big|_{P\mathcal{R}P},\quad \mathcal{P}^{\mathcal{R}}_{\sigma^\omega}\equiv\{P:P\text{ a non-zero projection in }\mathcal{R}^{\sigma^\omega}\}, \tag{A.39}$$

where $\mathcal{R}^{\sigma^\omega}\equiv\{A\in\mathcal{R}:\sigma^\omega_s(A)=A\text{ for all }s\in\mathbb{R}\}$ is the *fixed point algebra* of σ^ω. Again, it is the cocycle Radon-Nikodým theorem which makes sure that the Connes spectrum is independent of ω. The following theorem, which is due to Connes, discloses the relations between the various invariants.

Theorem A.5.5 (Connes [23]). *Let $\mathcal{R}$ be a factor.*

(i) $S(\mathcal{R})$ *is a closed subset of* $\mathbb{R}_{\geq 0}$.

(ii) $S(\mathcal{R})\setminus\{0\}$ *is a subgroup of* $(\mathbb{R}_{>0},\cdot)$.

(iii) $\Gamma(\mathcal{R})=\log(S(\mathcal{R})\setminus\{0\})$.

(iv) If $S(\mathcal{R})\neq\{0,1\}$, *then* $T(\mathcal{R})=\{t\in\mathbb{R}:e^{it\lambda}=1\text{ for all }\lambda\in\Gamma(\mathcal{R})\}$.

Since $S(\mathcal{R})\setminus\{0\}$ is a closed subgroup of the multiplicative group of positive reals, there remain only few possibilities for $S(\mathcal{R})$ which are used for a subclassification of type III factors [73].

- $S(\mathcal{R})=\{1\}\ \Rightarrow\ \mathcal{R}$ is finite.[14]
- $S(\mathcal{R})=\{0,1\}\ \Rightarrow\ \mathcal{R}$ is semi-finite or of *type* III_0.[15]
- If $S(\mathcal{R})=\{0\}\cup\lambda^{\mathbb{Z}}$, $\lambda\in(0,1)$, $\mathcal{R}$ is of *type* III_λ.
- If $S(\mathcal{R})=\mathbb{R}_{\geq 0}$, $\mathcal{R}$ is of *type* III_1.

There is a continuum of non-equivalent types. The special feature of III_1 factors is that the spectrum of each Δ_ω is maximal, the whole of $\mathbb{R}_{\geq 0}$, the spectrum of the modular Hamiltonian $K_\omega\equiv-\log\Delta_\omega$ is the whole real line. Type III_1 factors are of great interest in AQFT. We close this section by throwing a glance at so-called hyperfinite factors.

Definition A.5.6 ([23]). A factor, which is the weak closure of the union of an ascending sequence of finite-dimensional von Neumann algebras, is called *hyperfinite*.

For hyperfinite factors a complete classification is known. It was shown by Connes [41] that up to *-isomorphisms there exist only one hyperfinite factor of type II_1, one of type II_∞, and one of type III_λ for each $0<\lambda<1$. He also classified the hyperfinite factors of type III_0. The investigations could eventually be completed by Haagerup [61] who proved that in the type III_1 case there exists up to *-isomorphisms a unique hyperfinite type III_1 factor, a result which is of paramount physical importance (cf. section 2.7).

[13] For reasons of clarity the spectrum is denoted in this section by spec (instead of σ) in order to distinguish it from the modular group.

[14] If $S(\mathcal{R})$ is defined via weigths, $S(\mathcal{R})=\{1\}$ implies that $\mathcal{R}$ is semi-finite.

[15] If $S(\mathcal{R})$ is defined via weigths, the semi-finite case can be excluded.

Appendix B

Covariant Formalism in Classical and Quantum Mechanics

"Tempus item per se non est."
Titus Lucretius Caro, Roman philosopher (99 - 55 B.C.)

The conventional formulations of classical mechanics and quantum mechanics are deeply intertwined with the notion of an a priori given external time. The time evolution is generated by a Hamiltonian, which, in the case of a special relativistic theory, is included in the representation of the Poincaré group. Time is a primary concept. In contrast to that, the viewpoint we have adopted throughout this thesis assumes a fundamental timelessness of Nature. To continue regarding both theories, classical and quantum mechanics, not just a pure epiphenomena, an external Newtonian time cannot be justified (apart from a phenomenological approximate time), and one has to make sense of the basic features of these theories without the need to make a specific identification of time at any stage (such a formulation is also desirable for the description of a truly closed system like the universe itself). The purpose of this appendix is to present *covariant approaches* to classical and quantum mechanics which satisfy this task and which, in addition, may be regarded as generalisations of the conventional approaches. One way to realise the covariant formalism is obtained by describing the systems as constrained systems.

While the core of the conventional approaches is to capture the evolution of states or observables in time, the covariant formulations deal with correlations between (partial) observables. Our intention is above all to underline that as a matter of principle there is the possibility of such timeless formulations, which is why we focus on the basic ideas and refer to the literature cited below for further discussions and detailed examples. We want to outline that the notion of time might be not as crucial for these theories as it is suggested by their standard presentations. Moreover, the covariant formalism is relevant for an alternative formulation of the time hypothesis (section 5.2), and is taken up in our discussion in chapter 7.

B.1 Covariant Formulation of Classical Mechanics

A covariant formulation of classical mechanics was pursued among others by Rovelli [105, 110], based on related accesses which were studied for instance by Arnold [6] or Souriau [119], and builds on the idea to construct the phase space from entire histories without any reference to an instant of time, which most originally goes back to Lagrange. There are other approaches like the Hamilton-Jacobi approach or the path-integral formalism, for a brief overview see [110].

From Symplectic to Presymplectic Mechanics

Standard Hamiltonian mechanics is usually described in the *symplectic formalism* [6]. The system is defined by a triple (Γ, σ, H). Herein Γ denotes the phase space of the system, i.e. the cotangent bundle T^*V of the configuration space V. It is assumed to be a $2N$-dimensional manifold given

by the canonical coordinates q^i (which span V) and their conjugate momenta p_i, $i = 1, \dots, N$. $\sigma = \mathrm{d}p_i \wedge \mathrm{d}q^i$ is the standard symplectic form, i.e. a closed non-degenerate differential 2-form,[1] equipping the cotangent bundle Γ with a natural symplectic structure. The definition of the system is completed by the Hamiltonian $H = H(p_i, q^i) : \Gamma \to \mathbb{R}$, which is assumed to be a smooth function on phase space. The time evolution is realised as a flow $t \mapsto (q^i(t), p_i(t))$ on Γ. The Hamiltonian vector field X_H, defined by the equation $\sigma(X_H) = -\mathrm{d}H$, specifies this time evolution,[2] it is the tangent field to the time flow. This equation is simply the geometrical form of the Hamilton equations of motion, in coordinates X_H reads

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{B.1}$$

The conventional form of the Hamilton equations can be restored via

$$\dot{x} = X_H(x), \quad x \equiv (q^i, p_i). \tag{B.2}$$

Now, we go over to the *presymplectic formulation* of mechanics [6, 105, 110], which turns out to be applicable to a larger class of mechanical systems. For reasons of motivation we start off by the above system (Γ, σ, H) and consider the symplectic manifold (Γ_{ex}, σ_c), where $\Gamma_{ex} = \Gamma \times \mathbb{R}^2$ is the *extended phase space* with coordinates q^i, p_i, t, $p_t (\equiv -E)$. $\sigma_c = \sigma + \mathrm{d}p_t \wedge \mathrm{d}t$ is the natural symplectic 2-form on the cotangent bundle $\Gamma_{ex} = T^*(V_c) = T^*(V) \times \mathbb{R}^2$, where $V_c = \Gamma \times \mathbb{R}$ is the *covariant configuration space*. Moreover, we introduce the Hamiltonian constraint

$$H_c(p_i, q^i, p_t, t) \equiv p_t + H(p_i, q^i) \approx 0, \tag{B.3}$$

weakly vanishing in the sense of Dirac [45]. H_c denotes the *covariant Hamiltonian*. The constraint defines a $(2N+1)$-dimensional constraint surface $\mathcal{C}$ in Γ_{ex}. The closed degenerate 2-form induced on $\mathcal{C}$ by σ_c is the presymplectic form

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}H(p_i, q^i) \wedge \mathrm{d}t. \tag{B.4}$$

A presymplectic form is *non-singular*, which means that it admits an up to rescaling uniquely defined vector field Y such that $\omega(Y) = Y^i\omega_{ij} = 0$, the *null vector field of* ω. The solutions of the equations of motion are the *integral lines*, or *orbits*, of Y. Since Y is merely determined up to rescaling, the integral lines are defined only up to reparametrisation. In coordinates we have

$$Y \propto \frac{\partial}{\partial t} + X = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{B.5}$$

The equations of motion take on the form

$$\frac{\mathrm{d}q^i}{\mathrm{d}\tau} = N(\tau)\frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -N(\tau)\frac{\partial H}{\partial q^i}, \tag{B.6}$$

and are invariant under reparametrisation of τ. The parameter τ is artificial and has no physical meaning. The function $N(\tau)$ is called *lapse function*. Different choices of the lapse function determine different parameters τ. The motion is not described by a flow $t \mapsto (q^i(t), p_i(t))$ anymore, but by a graph $(t, q^i(t), p_i(t))$, which describes correlations between the variables of the system. Therefore, the presymplectic system $(\mathcal{C}, \omega)$, and also the constrained system $(\Gamma_{ex}, \sigma_c, H_c)$, incorporate all the physical information contained in the original system (Γ, σ, H), except for the specification which one is the time variable.[3] While the covariant Hamiltonian H_c describes correlations, the ordinary Hamiltonian H generates the evolution w.r.t. a distinguished time variable.

The space Γ_c of all orbits of ω in the presymplectic space $\mathcal{C}$ inherits naturally a symplectic structure of $\mathcal{C}$, whereby it becomes a symplectic manifold (such a system is e.g. studied in [119]). Γ_c is said to be the *covariant phase space*, the elements of Γ_c are called *(covariant) states*. A covariant state is defined in the sense of Heisenberg as an orbit, it does not evolve. The phase

[1] Closed means $\mathrm{d}\sigma = 0$, non-degenerate means that $\sigma(X) = 0$ implies $X = 0$.

[2] A symplectic space (Γ, σ) has the property that any smooth real function f on Γ determines a vector field X_f by the relation $\sigma(X_f) = -\mathrm{d}f$.

[3] Though it is somehow still coded in H_c which depends linearly on p_t and quadratically on the momenta p_i.

space of the symplectic system, Γ, and Γ_c are related by a (highly non-unique) bijection from the state of the symplectic system at some time t_0 to the solution it generates uniquely.

The covariant formalism provides an observer-independent description, expressed by the disappearance of a preferred time variable. Conversely, the very definition of e.g. the conventional phase space assumes that a choice of a reference frame has been made (a distinction of a class of observers). Special relativistic systems are typical examples of systems where the covariant formalism is much more natural, because it provides a manifestly Lorentz invariant approach.

Covariant Formalism

It is possible to describe a system in the covariant language initially given by a constrained system, not necessarily emerging from a symplectic one. To do so, we start from the triple $(\Gamma_{ex}, \sigma_c, H_c)$, from which we deduce the presymplectic tuple $(\mathcal{C}, \omega)$. Note that there is no necessity anymore for Γ_{ex} to be of the form $\Gamma \times \mathbb{R}^2$. That is the reason why the covariant approach is more general than the usual one.

One does not have any preferred time variable. Nevertheless, one can pick a variable q_t of the configuration space (a partial observable as we shall see soon) and deparametrise the system, at least locally, by using q_t, i.e. the other variables are evolved w.r.t. q_t. In that case q_t represents an internal clock time. However, an internal time will in general not have the properties that characterise a Hamiltonian time. The solutions of the equations of motion are still given by the integral lines of the null vector field Y of the presymplectic form. The dynamics may be expressed via the correlation of the variables w.r.t. such an internal time q_t.

Observables

According to Rovelli [109], for the consistency of the covariant approach, changes in the definition of observables are necessary. One has to distinguish carefully between two types of observables.

Definition B.1.1 ([109])**.** A *partial observable* is a physical quantity to which one can associate a measuring procedure leading to a number.
A *(complete) observable* is a quantity whose value can be predicted by the theory if the state is known, or whose probability distribution can be computed (to anticipate the quantum case).

In classical mechanics partial observables are smooth real functions on the configuration space V_c, in particular q^i is a partial observable. They are candidates which may be used as internal time. A complete observable on the other hand is a smooth real function on the space of solutions of the equations of motion, i.e. on the covariant phase space Γ_c. Equivalently, it can be expressed as a function A on the presymplectic space $\mathcal{C}$ which is constant along the orbits, $Y(A) \equiv Y^i \partial_i A = 0$, and also as a function A on the extended phase space Γ_{ex} which has vanishing Poisson brackets with the covariant Hamiltonian H_c, $\{H_c, A\} = 0$. Complete observables are composed of partial observables. Since they commute with the covariant Hamiltonian, they are constants of motion. We shall come back to this point in the context of quantum mechanics.

Correlations between Partial Observables

The covariant approach captures dynamics by correlations between partial observables, or relative evolution of partial observables w.r.t. each other. A complete observation yields a point $q = (q^i, t)$ in V_c.[4] Performing a sequence of measurements of q gives a curve γ in V_c, the *physical motion* of the system. γ describes a correlation between the partial observables expressed in the equation

$$\tilde{f} : V_c \to \mathbb{R}, \quad \tilde{f}(q^i, t) = 0. \tag{B.7}$$

Repeating the experiment results in different curves γ and, thus, different correlations. However, the experimental experience says that every curve can be uniquely labelled by a covariant state

[4] Assuming that the system admits an external time. But since we shall not fall upon the special role of the external time t, that does not influence these considerations and generalises in the obvious way to the case in which no time is preferred. Our goal is simply to establish the relation to the conventional notions.

in Γ_c. Each such state determines a correlation between the partial observables in V_c. All the correlations are contained in the *evolution equation* defined by the vanishing of a suitable function

$$f : \Gamma_c \times V_c \to \mathbb{R}. \tag{B.8}$$

If the state is known, the evolution equation predicts all possible observations. Each covariant state φ determines via $f(\varphi, \cdot) = \tilde{f}_\varphi(\cdot) = 0$ a physical motion of the system. Conversely, for each point in V_c the surface $f = 0$ determines the set of all states compatible with these values of the partial observables. How does f look like? A given point in Γ_c selects via the presymplectic form ω an orbit in $\mathcal{C}$ which projects down to a curve in V_c where it is precisely a physical motion and defines the correlations in question.

Gauge Symmetries

Let us say a few words concerning gauge symmetries. One can easily incorporate systems with gauge degrees of freedom into this formalism. In that case the Hamiltonian constraint is supplemented by gauge constraints, combined in the function $C : \Gamma_{ex} \to W$. W is a k-dimensional vector space, k the number of constraints.[5] The system in view is (V_c, σ_c, C). There is no necessity of drawing a distinction between the constraints generated by gauge symmetries and the Hamiltonian constraint. The presymplectic space $\mathcal{C}$ is specified by $C \approx 0$, and the proceeding is the same as above: The symplectic structure on $\Gamma_{ex} = T^*V_c$ defines a presymplectic form ω on the constraint surface $\mathcal{C}$. $\mathcal{C}$ has dimension $2N - k$. The kernel of ω is k-dimensional and the physical motions are k-dimensional surfaces in V_c. Y is still the k-dimensional multi-tangent to the motion and satisfies $\omega(Y) = 0$. The equations of motion, parametrised with τ_j, $j = 1, \ldots, k$, read

$$\frac{\mathrm{d}q^i(\vec{\tau})}{\mathrm{d}\tau_j} = N_j(\vec{\tau})\frac{\partial C(q^k, p_k)^j}{\partial p_i}, \quad \frac{\mathrm{d}p_i(\vec{\tau})}{\mathrm{d}\tau_j} = -N_j(\vec{\tau})\frac{\partial C(q^k, p_k)^j}{\partial q^i}. \tag{B.9}$$

Choosing any curve $\vec{\tau}(\tau)$ on the surface of motion, τ an arbitrary parameter, the motion may be represented by the one-dimensional curve $q^i(\tau) = q^i(\vec{\tau}(\tau))$ in V_c. Different choices of the functions N_j describe different, but gauge equivalent curves on the surface of motion.

A covariant state is a gauge equivalence class of solutions of the equations of motion. Because a choice of initial data determines uniquely a gauge equivalence class of solutions the one-to-one correspondence between Γ_c and Γ persists. Complete observables remain arbitrary functions on Γ_c. Defining them on Γ_{ex}, one has to demand not only the commutation with H_c but also vanishing Poisson brackets with all gauge constraints. Observables are gauge invariant quantities, they can be predicted uniquely up to all gauges once the solutions of the equations of motion are known.

Classical Field Theory

What we have seen is that it is indeed possible to express classical mechanics in a time-independent language. The *mechanical* structure of our world does not require the concept of a preferred variable representing a physical time. One can extend the covariant formulation to classical field theories (cf. [7, 110]). Let us just briefly sketch how the implementation works. One employs the presymplectic formulation of classical mechanics. Again, the presymplectic space $\mathcal{C}$ is defined by $C \approx 0$. It is assumed that the initial value problem is well-posed such that each dynamical trajectory in the configuration space V_c is completely specified by its initial data. Then $\mathcal{C}$ carries a natural presymplectic structure defined by the solutions of the field equations, it is a presymplectic manifold, which in general does not have a natural cotangent bundle structure. In the common case of gauge theories C includes the Hamiltonian constraint as well as all the gauge constraints. The orbits determined by the presymplectic structure coincide with the Hamiltonian or gauge transformations. All solutions of the field equations which are in the same gauge equivalence class have to be identified physically. The gauge equivalence classes span the physical phase space which inherits a natural symplectic structure. It is the covariant phase space Γ_c. Actually things are a bit more subtle in field theory, because the integrals defining the symplectic structure may not converge. This issue can be avoided when the boundary conditions are adjusted carefully.

[5] For $k = 1$, $C = H_c$ is the covariant Hamiltonian and everything is supplied as before.

The covariant formulation of classical theories becomes really necessary, and is actually quite natural, if one wants to incorporate generally covariant systems, since the notion of a state or observable at a given instant of time, requiring the concept of a simultaneity surface, does not make any physical sense because of the diffeomorphism covariance. Remarkably, all fundamental (classical) dynamical systems in Nature seem to exhibit a presymplectic structure, while there are systems that can be formulated this way, but which do not admit a Hamiltonian symplectic formulation. The most prominent example of a system which permits a covariant formulation but (presumably) no symplectic one, is general relativity. The existence of presymplectic non-Hamiltonian systems may serve as a profound indication for the physical importance of the covariant approach. In the case of general relativity Γ_c is isomorphic to the reduced ADM phase space, but the covariant formulations emphasizes that general covariance is preserved. For general relativity the adjustment of the boundary conditions is done e.g. in [7] for globally hyperbolic spacetimes and gravitational fields which are asymptotically flat at spatial infinity.

B.2 Covariant Formulation of Quantum Mechanics

Conventional quantum mechanics is usually understood in terms of the *Copenhagen interpretation* of quantum theory, either in the Schrödinger picture or in the Heisenberg picture. It is founded on well-known postulates [39]. A covariant ansatz to cast quantum mechanics into a form which does not require a (non-weakly vanishing) Hamiltonian was treated among others by Hartle [62]. A geometrical quantisation procedure was proposed by Souriau [119]. Other promising timeless approaches are discussed in [66, 69], using e.g. internal clock variables to describe the dynamics of the system by means of conditional probabilities. Our description follows Rovelli [105, 110] and Reisenberger and Rovelli [101], and is in close analogy with the classical covariant approach.

Spacetime States

In standard quantum mechanics it is assumed that measurements are instantaneous. Dropping this idealization a covariant formulation becomes possible, as quantum version of the classical covariant counterpart. The conventional Hamiltonian formulation is based on a Hilbert space $\mathcal{H}$, or, more rigorously, on a rigged Hilbert space $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}'$,[6] which is identified as the state space, and self-adjoint operators acting on $\mathcal{H}$, which represent the observables of the theory. Moreover, one has a Hamiltonian H which generates the time evolution. The Hamiltonian is a self-adjoint operator on $\mathcal{H}$ with a spectrum bounded from below. In the Heisenberg picture the time-dependence is carried by the observables while states are time-independent,

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i[A(t), H] \quad \Leftrightarrow \quad A(t) = e^{itH}A(0)e^{-itH}. \tag{B.10}$$

To make things concrete, let us consider for the time being a non-relativistic particle without spin, then we have $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}^3x)$. Every state $|\Psi\rangle$ may be expanded in terms of the generalised eigenstates $|x\rangle \in \mathcal{S}$ of the position operator X,

$$|\Psi\rangle = \int \mathrm{d}^3x\, \psi(x)\, |x\rangle\,. \tag{B.11}$$

$\psi(x) = \langle x|\Psi\rangle$ is the wavefunction of the particle. $X\,|x\rangle = |x\rangle$ implies $X(t)\,|x;t\rangle = x\,|x;t\rangle$ when $|x;t\rangle \equiv e^{itH}\,|x\rangle$. Clearly $\psi(x,t) \equiv \langle x;t|\Psi\rangle$ satisfies the Schrödinger equation. Now, given a suitable space of complex test functions $\mathcal{E}$ acting on spacetime, one defines for a function $f \in \mathcal{E}$

$$|f\rangle := \int \mathrm{d}^3x\, \mathrm{d}t\, f(x,t)\, |x;t\rangle\,. \tag{B.12}$$

Whenever $|f\rangle$ is normalizable it is called a *spacetime state*. Spacetime states are generalisations of conventional states where $f(x,t) = \psi(x)\delta(t)$, $\psi(x) \in L^2(\mathbb{R}^3)$. However, they are reinterpreted,

[6] A *rigged Hilbert space* is a triple $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}'$, the *Gelfand triple*, where $\mathcal{H}$ is a Hilbert space, $\mathcal{S}$ a proper subset which is dense in $\mathcal{H}$, and $\mathcal{S}'$ the dual of $\mathcal{S}$ (see [15] for more details). A manifold M with measure $\mathrm{d}\mu$ determines a rigged Hilbert space $\mathcal{S}_M \subset \mathcal{H}_M \subset \mathcal{S}'_M$, where $\mathcal{S}_M$ is the space of smooth functions on M with fast decrease, $\mathcal{H}_M = L^2(M, \mathrm{d}\mu)$, and $\mathcal{S}'_M$ is the space of tempered distributions on M.

ordinary quantum states in $\mathcal{H}$ which form a dense subset. Note further, that the identification $f \mapsto |f\rangle$ is highly degenerated. One can associate a wave function ψ_f satisfying the Schrödinger equation to each spacetime state $|f\rangle$ via

$$\psi_f(x,t) = \langle x;t|f\rangle = \int \mathrm{d}^3x'\,\mathrm{d}t'\ \langle x|e^{-i(t-t')H}|x'\rangle\, f(x',t'). \tag{B.13}$$

An important class of spacetime states is given for $f_{\mathcal{O}}(x,t) := \chi_{\mathcal{O}}(x,t)$, where $\mathcal{O}$ is assumed to be a compact region in spacetime, and χ is the characteristic function. Let us denote the state $|f_{\mathcal{O}}\rangle$ after normalization by $|\mathcal{O}\rangle$.

What is the meaning of spacetime states? To understand that, let us envisage a position measurement of the particle. While conventional states may be thought of as being associated with results of instantaneous position measurements with finite resolution in space,[7] spacetime states correspond to the results of realistic measurements, where the measuring device also has a finite resolution in time. If the particle is e.g. detected in x_0 at time t_0 by a measuring apparatus with space resolution of order Δx and time resolution of order Δt the state prepared by the measurement is a spacetime state $|f\rangle$, with f localised in $[x_0-\Delta x, x_0+\Delta x]\times[t_0-\Delta t, t_0+\Delta t]$. Moreover, using the usual probabilistic interpretation of the wave function one can show that for sufficiently small[8] regions $\mathcal{O}$ the probability that an ideal detector detects the particle in $\mathcal{O}$ is

$$\mathcal{P}_{\mathcal{O}} = |\,\langle\mathcal{O}|\Psi\rangle\,|^2. \tag{B.14}$$

In turn, one can establish that the probabilistic interpretation in terms of detection probabilities in spatial regions can be derived from (B.14), i.e. both interpretations are physically equivalent, though (B.14) does not refer to a notion of time.

To sum up, spacetime states become natural objects if one drops the unrealistic idealization that measurements are instantaneous. They are a necessary technical ingredients for a spacetime symmetric formulation of quantum mechanics.

Wheeler-DeWitt Equation

Similar to the classical case, the covariant approach requires a reformulation of a Hamiltonian quantum system as a constrained system. Again, one may incorporate gauge symmetries, which we forego here for simplicity. Studying the quantisation of constrained systems dates back to Dirac [45], who described a general procedure to do that. Certain subtleties in Dirac's quantisation scheme are resolved in a modern technique called *group averaging*, which supplies a constructive method for obtaining an inner product on the physical state space. For an overview see [82]. The general idea is to quantise the unconstrained system in the usual way and obtain an auxiliary so-called *kinematical state space* $\mathcal{K}$, on which the constraints are to be formulated as operators. Those states which are annihilated by the constraints form the physical state space $\mathcal{H}_c$. Group averaging uses the Hilbert space structure of $\mathcal{K}$ to solve the constraints and define a physical inner product on $\mathcal{H}_c$ in such a way that self-adjoint operators on $\mathcal{K}$ that commute with all the constraints are self-adjoint operators on $\mathcal{H}_c$, as well (which is important for the concept of observables). This construction will be used for the covariant formulation. Therein the dynamics of a standard quantum system is defined by the *Wheeler-DeWitt equation*

$$H_c\psi(x,t) = 0, \tag{B.15}$$

where the Wheeler-DeWitt operator H_c is supposed to be the covariant Hamiltonian,

$$H_c(T,X,P_t,P) = P_t + H \doteq -i\partial_t - \sum_{j=1}^{n}\frac{\partial_j^2}{2m_j} + V(x_1,\dots x_n). \tag{B.16}$$

Using the Wheeler-DeWitt constraint equation the special role of time has disappeared. If one starts from the classical covariant configuration space $V_c = V\times\mathbb{R}$, the standard quantisation

[7] Note that for the preparation of the state $|x\rangle \notin \mathcal{H}$ an infinite resolution in space would be needed, as well.

[8] That means smaller than any other physically relevant dimension involved in the problem. The probabilities corresponding to larger regions depend on the relative phase of the wavefunction at different spacetime points.

scheme leads to a rigged Hilbert space $\mathcal{S} \subset \mathcal{K} \subset \mathcal{S}'$. $\mathcal{S}$ is the state of smooth functions on V_c with fast decrease, $\mathcal{K} = L^2(V_c, \mathrm{d}^n x\, \mathrm{d}t)$, and $\mathcal{S}'$ contains the tempered distributions on V_c. The solutions of the Wheeler-DeWitt equation form a linear space $\mathcal{H}_c$. Let us construct a natural inner product on $\mathcal{H}_c$ via the group averaging method [101, 110].[9] We first introduce the densely defined operator

$$P : \mathcal{S}' \to \mathcal{H}_c, \quad f(x,t) \mapsto \int \mathrm{d}\tau\, e^{-i\tau H_c} f(x,t). \tag{B.17}$$

By Fourier-expanding $f(x,t)$ w.r.t. t one shows that [101]

$$(Pf)(x,t) = \int \mathrm{d}^n x'\, \mathrm{d}t'\ \langle x|e^{-i(t-t')H}|x'\rangle\, f(x',t') = \psi_f(x,t), \tag{B.18}$$

and, thus, Pf is indeed a solution of the Wheeler-DeWitt equation. Roughly speaking the highly degenerate mapping P projects down on the space of solutions. However, in general P is not a projector, because its domain is a proper subset of $\mathcal{S}'$. Nevertheless, one has $\mathcal{S} \subset \mathcal{D}(P)$. Defining

$$\langle f|f'\rangle_{\mathcal{S}} := \langle f|P|f'\rangle_{\mathcal{K}} \tag{B.19}$$

gives a degenerate inner product on $\mathcal{S}$. Dividing $\mathcal{S}$ by the kernel of this inner product, that is identifying f and f' whenever $Pf = Pf'$ and completing in norm, one obtains a Hilbert space. But if $Pf = Pf'$, f and f' define the same solution of the Wheeler-DeWitt equation, it is the solution which corresponds to the spacetime state $|f\rangle$. Consequently, the Hilbert space can be identified with $\mathcal{H}_c$ in the category of linear spaces. Thus H_c (or P, respectively) equips $\mathcal{H}_c$ with a Hilbert space structure. Given two solutions $\psi_f = Pf$ and $\psi_{f'} = Pf'$, the inner product is

$$\langle \psi_f|\psi_{f'}\rangle_{\mathcal{H}_c} \equiv \langle f|P|f'\rangle_{\mathcal{K}}\,. \tag{B.20}$$

$P\mathcal{S}$ is dense in $\mathcal{H}_c$, whence the latter one is completely determined by $\mathcal{S}$ and H_c. The covariant state space $\mathcal{H}_c$ is canonically isomorphic to the conventional Hilbert space $\mathcal{H}$ containing states at the fixed time $t = t_0$. Under this identification $\langle\cdot|\cdot\rangle_{\mathcal{H}_c}$ is precisely the usual inner product $\langle\cdot|\cdot\rangle_{\mathcal{H}}$. The main conclusion however is that the mapping

$$P : \mathcal{S} \to \mathcal{H}_c, \quad f \mapsto |f\rangle\,, \quad \langle x;t|f\rangle = \psi_f(x,t) \tag{B.21}$$

has been expressed in terms of the covariant Hamiltonian instead of the ordinary one, i.e. the scalar product on $\mathcal{H}_c$ is definable just from $\mathcal{S}$ and H_c without identifying a time variable or falling back upon the Hilbert space structure of the instantaneous state space $\mathcal{H}$.

Covariant Formulation

These results give the opportunity to extend quantum mechanics to a more general class of systems. Let us start with a classical constrained system (V_c, σ_c, H_c). By the canonical quantisation procedure one constructs a rigged Hilbert space $\mathcal{S} \subset \mathcal{K} = L^2(V_c, \mathrm{d}^n x) \subset \mathcal{S}'$ like the one in the previous subsection, the kinematical state space, where now V_c does not need to be of the form $V \times \mathbb{R}$ anymore. Furthermore the quantisation of the covariant Hamilton function leads to a self-adjoint operator $H_c = H_c(X,P)$ acting on $\mathcal{K}$, the covariant Hamiltonian. The *covariant state space* $\mathcal{H}_c$ consists of all $\psi \in \mathcal{K}$ which are solutions of the Wheeler-DeWitt equation,

$$H_c(x, -i\partial_x)\psi = 0. \tag{B.22}$$

Like in presymplectic mechanics these Heisenberg states represent histories of the system. $\mathcal{H}_c$ is a linear space. To define a Hilbert space structure, one uses the improper projector

$$P : \mathcal{S} \to \mathcal{H}_c, \quad f(x) \mapsto \int \mathrm{d}\tau\, e^{-i\tau H_c} f(x) \equiv \langle x|f\rangle \tag{B.23}$$

[9] One cannot use the inner product of $\mathcal{K}$, because the continuous nature of the spectrum of the constraint operator makes the solutions of the Wheeler-DeWitt equation to infinite-norm states in $\mathcal{K}$. One could pull back the inner product from $\mathcal{H}$ to $\mathcal{H}_c$ by evaluating the elements in $\mathcal{H}_c$ at $t = t_0$, but we do not want to use this time structure.

and proceeds as above. To each finite region $\mathcal{O} \subset V_c$ one associates the normalized spacetime state $|\mathcal{O}\rangle$. Now, one *postulates* that for sufficiently small regions the probability to find a quantum system in the state $|\Psi\rangle$ in the region $\mathcal{O}$ to be

$$\mathcal{P}_{\mathcal{O}} := |\langle \mathcal{O}|\Psi\rangle|^2. \tag{B.24}$$

If the system has been detected to be in $\mathcal{O}$, the state after the measurement is given by $|\mathcal{O}\rangle$. The theory is defined by the same postulates as conventional quantum mechanics in the Heisenberg picture (excluding the time axiom), which all immediately carry over to the covariant case. According to Reisenberger and Rovelli [101] dropping the time axiom does not compromise the other axioms or the probabilistic interpretation of the theory.

Observables

We use definition B.1.1 to distinguish between (complete) observables and partial observables [105, 109]. The latter ones are the self-adjoint operators acting on the kinematical state space $\mathcal{K}$. Note that also time is treated as an operator T (supposed that the system stems from a Hamiltonian system) which acts in the position representation by multiplication, $T|x,t\rangle = t|x,t\rangle$. Clearly this is necessary because time and position have to be on the same footing. As usual, a complete observable is a self-adjoint operator on the covariant state space $\mathcal{H}_c$. It is a conjunction of partial observables. Equivalently, a self-adjoint operator A defined on $\mathcal{K}$ is an observable whenever[10]

$$[A, H_c] = 0. \tag{B.25}$$

Quantum mechanics is about correlations between partial observables, an ordinary time variable possibly among them. What an observable actually measures, is the value of a partial observable assuming that a measurement of another partial observable has given a certain value. The differentiation between these two types of observables is important. For instance we emphasize that according to the covariant approach neither position nor time are observables but only position at a given time. There exists no observable corresponding to the classical variables t or x, $[X, H_c] \neq 0$, $[T, H_c] \neq 0$. Nevertheless, for both position and time exist measuring procedures that produce a number, they are partial observables.

In conventional quantum mechanics in the Heisenberg picture observables are time-dependent. One may interpret an expression like $A(t)$ by saying it is the observable which measures the physical quantity $\mathcal{A}$ at time t. Alternatively, the reading of a clock can be included in the list of operations that define an observable. In that case $A(t)$ and $A(t')$ have to be regarded as different observables, which are related by the time axiom

$$A(t') = e^{i(t'-t)H} A(t) e^{-i(t'-t)H}. \tag{B.26}$$

However, if no time is available, one can use some partial observable X as internal time operator (for the use of internal times to describe a dynamical evolution see also [92]). Clearly, $A(s)$ will in general not satisfy (B.26). s is supposed to be the result of the measurement of X, it reflects a clock time. The evolution of A in s is implicitly contained in (B.25). One can neither expect s to be a Hamiltonian time nor the evolution to be unitary, so that the total probability does not need to be conserved. Only partial observables with suitable properties generate a unitary evolution. Those partial observables may not exist. Moreover, it may happen that the set of all observables for a fixed s does not form a complete set. If so, the outcome of measurements of all observables at clock time s does not uniquely determine the state.

In the covariant approach time is interpreted as a particular structure on the set of observables. This structure codes the fact that observables are defined in terms of correspondences of partial observables. Generically, observables of physical interest are constructed out of two partial observables, one of which is the reading of a clock. After fixing a suitable clock time, the set of all observables can be foliated into one-parameter families that are given by the same partial observables at different clock readings. This interpretation captures the idea that the passage of time should be identified with correlations inside the system.

[10] This condition makes sure, that the range of A is indeed in $\mathcal{H}_c$. Observables defined in this way correspond to the classical ones which are constant along the orbits of the presymplectic system.

"When" a measurement of an observable $A(t)$ or $A(s)$ takes place is meaningless, because the states do not evolve. The reformulation of a Hamiltonian quantum mechanical system in the covariant formalism makes dynamics trivial. On the other hand, the kinematics become non-trivial. One has to solve the problem of identifying which observable measures a given physical quantity at a given internal time, i.e. one has to determine a time structure on the set of all observables. In a presymplectic quantum system which corresponds to a Hamiltonian system, the observables are precisely the Heisenberg observables, and the vanishing commutator between the observables and the covariant Hamiltonian simply reduces to the time axiom.

There remains a conceptual difficulty regarding the possibility of associating probabilities to sequences of measurements. A measurement of an observable $\mathcal{A}$ is accompanied by the collapse of the wavefunction. The state is projected on an eigenstate of the corresponding observable A. As said before, the time of the measurement has no meaning, but, according to the Copenhagen interpretation, the order in which two different measurements are performed plays a crucial role in the theory. Even if the observables describe measurements of certain quantities at a given clock time s, the problem may be that s does not need to define an ordering relation for the measurement procedures. However, the ordering is not necessarily related to time. In conventional quantum mechanics one may e.g. ask for the conditional probability that the outcome of a measurement of $A(t_1)$ was a, assuming that at time $t_2 > t_1$ one has measured the eigenvalue b of $B(t_2)$, in the same way one would do it for $t_2 < t_1$. The ordering is really a matter of question and does neither depend on the presence of time nor on an ordering relation defined by that time.

In this section it was shown how to put quantum mechanics into a form where the time variable is treated on an equal footing with the other dynamical variables in the extended configuration space. The special role of time essentially depends on idealizations in the measurement.

B.3 Concluding Remarks

It seems reasonable to complete this appendix with some remarks concerning the covariant reformulations presented here. In classical mechanics the covariant formalism is a rather unproblematic generalisation of the standard approach, which can be used to treat a larger class of dynamical systems. Physically one essentially has to drop the notion of the idealized distinguished Newtonian notion of time. The covariant approach does not distinguish any internal variable as time, but once a Hamiltonian time can be chosen, both approaches are perfectly equivalent.

Nevertheless, one problem could be that the observables are required to commute with the Hamiltonian constraint. For certain dynamical systems this requirement might be extremely restrictive [131] (the same is expected to be true in the associated quantum theory). Time merely emerges as a phenomenological concept in the covariant approach associated with internal times. In order to avoid inconsistent predictions, restrictions on the time structure have to be imposed. A severe challenge is to characterise those partial observables which are suitable for playing the role of an internal time and give rise to a well-defined, consistent clock time [131]. Since our intention has mainly been to get rid of the notion of time, this aspect is rather secondary for us.

In the quantum case things are more involved. Apart from the modifications which were done in analogy with the classical case, the introduction of spacetime states was necessary, and the probabilistic interpretation concerning measurements procedures had to be adapted to the covariant framework, source of a couple of additional problems (see also [69, 76]). The quantum systems which have been explicitly studied in the covariant formalism are of a rather elementary nature, and only for these systems it has been shown that the predictions reduce to those of ordinary quantum mechanics in suitable limits (e.g. that passing from spacetime states to the usual temporally sharp localised states gives back the ordinary probability interpretation). It may happen that one has to struggle with problems when dealing with more complex quantum systems.

A crucial aspect which has to be clarified concerns the correct treatment of measurement procedures. Let us consider e.g. a quantum system whose state is prepared by certain non-commensurable measurements in two disjoint spacetime regions, such that parts of the first lie in the future of the second and vice versa. Since there is no natural ordering there is the issue about the final state of the system, i.e. the order in which the corresponding projections has to be applied to the initial state.[11]

[11] A possible solution could be to reduce both measurements to one single measurement (cf. [101]).

Further typical issues concern operator-ordering problems in the quantisation procedure of covariant systems, the question how to interpret the observables and how to measure them practically. Eventually one has to expect serious technical difficulties (in both, the classical and the quantum case), because it is generally a hard challenge to construct e.g. the covariant phase space in concrete models, which requires determining the space of solutions to the equations of motion. In addition, observables in constrained systems are extremely difficult to construct (see [118]).

Howsoever, the covariant formulations of classical and quantum mechanics, even if not yet fully exploited and understood, do suggest that there is in principle the possibility to abandon the concept of time for defining the basic concepts of these theories. There is no incompatibility between classical and quantum mechanics and the notion of a fundamentally timeless world.

Bibliography

[1] Akhiezer N. I., and Glazman I. M.: *Theory of Linear Operators – Volume I*, Boston: Pitman Advanced Publishing Program (1981)

[2] Araki H.: *Einführung in die axiomatische Quantenfeldtheorie I*, Vorlesung an der Eidgenössischen Technischen Hochschule, Zürich (1962)

[3] Araki H.: *Type of von Neumann Algebras Associated to the Free Field*, Prog. Theor. Phys. **32**, 956-961 (1964)

[4] Araki H.: *Relative Hamiltonian for Faithful Normal States of a von Neumann Algebra*, Pub. Res. Inst. Math. Sci. Kyoto Univ. **9**, 165-209 (1973)

[5] Araki H.: *Mathematical Theory of Quantum Fields*, Oxford: Oxford University Press (1999)

[6] Arnold V. I.: *Mathematical Methods of Classical Mechanics*, Berlin: Springer (1989)

[7] Ashtekar A., Bombelli L., and Reula O.: *The Covariant Phase Space of Asymptotically Flat Gravitational Fields*, in: *Mechanics, Analysis and Geometry: 200 years after Lagrange*, ed. Francaviglia, Amsterdam: North-Holland, 417-450 (1991)

[8] Barbour J. B.: *The End of Time – The Next Revolution in Physics*, Oxford: Oxford University Press (1999)

[9] Barbour J. B.: *The Nature of Time*, FQXi essay contest on *The Nature of Time* (2008)

[10] Baumgärtel H., and Wollenberg M.: *Causal Nets of Operator Algebras*, Berlin: Akademie Verlag (1992)

[11] Baumgärtel H.: *Operatoralgebraic Methods in Quantum Field Theory*, Berlin: Akademie Verlag (1995)

[12] Birrell N. D., and Davies P. C. W.: *Quantum Fields in Curved Space*, Cambridge: Cambridge University Press (1982)

[13] Bisognano J. J., and Wichmann E. H.: *On the Duality Condition for a Hermitian Scalar Field*, J. Math. Phys. **16**, 985-1007 (1975)

[14] Bisognano J. J., and Wichmann E. H.: *On the Duality Condition for Quantum Fields*, J. Math. Phys. **17**, 303-321 (1976)

[15] Böhm, A.: *The Rigged Hilbert Space and Quantum Mechanics*, Berlin: Springer (1978)

[16] Bohr H.: *Almost Periodic Functions*, New York: Chelsea Publishing Company (1951)

[17] Borchers H.-J.: *On the Vacuum State in Quantum Field Theory II*, Commun. Math. Phys. **1**, 57-79 (1965)

[18] Borchers H.-J., and Schumann R.: *A Nuclearity Condition for Charged States*, Lett. Math. Phys. **23**, 65-77 (1991)

[19] Borchers H.-J.: *The CPT Theorem in Two-Dimensional Theories of Local Observables*, Commun. Math. Phys. **143**, 315-332 (1992)

[20] Borchers H.-J., and Yngvason J.: *From Quantum Fields to Local von Neumann Algebras*, Rev. Math. Phys. **4**, 15-47 (1992)

[21] Borchers H.-J., and Yngvason J.: *Modular Groups of Quantum Fields in Thermal States*, J. Math. Phys. **40**, 601-624 (1999)

[22] Borchers H.-J.: *On Revolutionizing Quantum Field Theory with Tomita's Modular Theory*, J. Math. Phys. **41**, 3604-3673 (2000)

[23] Bratteli O., and Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics 1*, Berlin: Springer (1987)

[24] Bratteli O., and Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics 2*, Berlin: Springer (1996)

[25] Bros J., and Moschella U.: *Two-Point Functions and de Sitter Quantum Fields*, Rev. Math. Phys. **8**, 327-391 (1996)

[26] Bros J., Epstein H., and Moschella U.: *Analyticity Properties and Thermal Effects for General Quantum Field Theory on de Sitter Space-Time*, Commun. Math. Phys. **196**, 535-570 (1998)

[27] Brunetti R., Guido D., and Longo R.: *Modular Structure and Duality in Conformal Quantum Field Theory*, Commun. Math. Phys. **156**, 201-219 (1993)

[28] Buchholz D.:*On the Structure of Local Quantum Fields with Non-Trivial Interaction*, in: *Proc. of the Int. Conf. on Operator Algebras, Ideals and their Applications in Theoretical Physics*, eds. Baumgärtel, Lassner, Pietsch and Uhlmann, Leipzig: Teubner Verlagsgesellschaft 146-153 (1978)

[29] Buchholz D., and Wichmann E. H.: *Causal Independence and the Energy-Level Density of States in Local Quantum Field Theory*, Commun. Math. Phys. **106**, 321-344 (1986)

[30] Buchholz D., D'Antoni C., and Fredenhagen K.: *The Universal Structure of Local Algebras*, Commun. Math. Phys. **111**, 123-135 (1987)

[31] Buchholz D., and Jacobi P.: *On the Nuclearity Condition for Massless Fields* Lett. Math. Phys. **13**, 313-323 (1987)

[32] Buchholz D., D'Antoni C., and Longo R.: *Nuclear Maps and Modular Structure II: Applications to Quantum Field Theory*, Commun. Math. Phys. **129**, 115-138 (1990)

[33] Buchholz D., and Verch R.: *Scaling Algebras and Renormalization Group in Algebraic Quantum Field Theory*, Rev. Math. Phys. **7**, 1195-1239 (1996)

[34] Buchholz D., and Haag R.: *The Quest for Understanding in Relativistic Quantum Physics*, J. Math. Phys. **41**, 3674-3697 (2000)

[35] Butterfield J., and Isham C. J.: *On the Emergence of Time in Quantum Gravity*, in: *The Arguments of Time*, ed. Butterfield, Oxford: Oxford University Press (1999)

[36] Candelas P., and Dowker J. S.: *Field Theories on Conformally Related Spacetimes: Some Global Considerations*, Phys. Rev. D **19**, 2902-2907 (1979)

[37] Carroll S. M.: *What if Time Really Exists?*, FQXi essay contest on *The Nature of Time* (2008)

[38] Casini H., and Huerta M.: *Reduced Density Matrix and Internal Dynamics for Multicomponent Regions*, Class. Quantum Grav. **26**, 185005 (2009)

[39] Cohen-Tannoudji C., Diu B., and Laloë F.: *Quantenmechanik Teil 1*, Berlin: Walter de Gruyter (1999)

[40] Connes, A.: *Une classification des facteurs de type III*, Ann. Sci. École Norm. Sup. **6**, 133-252 (1973)

[41] Connes, A.: *Noncommutative Geometry*, New York: Academic Press (1994)

[42] Connes A., and Rovelli C.: *Von Neumann Algebra Automorphisms and Time-Thermodynamics Relation in Generally Covariant Quantum Theories*, Class. Quantum Grav. **11**, 2899-2917 (1994)

[43] D'Antoni C., Doplicher S., Fredenhagen K., and Longo R.: *Convergence of Local Charges and Continuity Properties of W^*-Inclusions*, Commun. Math. Phys. **110**, 325-348 (1987)

[44] Davies P. C. W.: *About Time*, London: Viking (1995)

[45] Dirac P. A. M.: *Lectures on Quantum Mechanics*, New York: Belfer Graduate School of Science (1964)

[46] Driessler W.: *On the Type of Local Algebras in Quantum Field Theory*, Commun. Math. Phys. **53**, 295-297 (1977)

[47] Dunford N., and Schwartz J. T.: *Linear Operators – Part II: Spectral Theory*, New York: Interscience Publishers (1963)

[48] Ellis G. F. R.: *On the Flow of Time*, FQXi essay contest on *The Nature of Time* (2008)

[49] Engel K.-J., and Nagel R.: *A Short Course on Operator Semigroups*, Berlin: Springer (2005)

[50] Fredenhagen K.: *On the Modular Structure of Local Algebras of Observables*, Commun. Math. Phys. **97**, 79-89 (1985)

[51] Fredenhagen K., and Haag R.: *Generally Covariant Quantum Field Theory and Scaling Limits*, Commun. Math. Phys. **108**, 91-115 (1987)

[52] Gallavotti G., and Verboven E.: *On the Classical KMS Boundary Condition*, Il Nuovo Cimento **28** B, 274-286 (1975)

[53] Gibbons G. W., and Hawking S. W.: *Cosmological Event Horizons, Thermodynamics, and Particle Creation*, Phys. Rev. D **15**, 2738-2751 (1977)

[54] Grünbaum A.: *Philosophical Problems of Space and Time*, Dordrecht: D. Reidel Publishing Company (1973)

[55] Guido D., and Longo R.: *A Converse Hawking-Unruh Effect and dS^2/CFT Correspondence*, Ann. Henri Poincaré **4**, 1169-1218 (2003)

[56] Guido D.: *Modular Theory for the von Neumann Algebras of Local Quantum Physics*, arXiv:0812.1511v1 [math.OA] (2008)

[57] Haag R., and Kastler D.: *An Algebraic Approach to Quantum Field Theory*, J. Math. Phys. **5**, 848-861 (1964)

[58] Haag R., Hugenholtz N. M., and Winnink M.: *On the Equilibrium States in Quantum Statistical Mechanics*, Commun. Math. Phys. **5**, 215-236 (1967)

[59] Haag R., Kastler D., and Trych-Pohlmeyer E. B.: *Stability and Equilibrium States*, Commun. Math. Phys. **38**, 173-193 (1974)

[60] Haag R.: *Local Quantum Physics*, Berlin: Springer (1996)

[61] Haagerup U.: *Connes' Bicentralizer Problem and Uniqueness of the Injective Factor of Type III_1*, Acta. Math. **158**, 95-148 (1987)

[62] Hartle J. B.: *Spacetime Quantum Mechanics and the Quantum Mechanics of Spacetime*, in: *Proceedings 1992 Les Houches School, Gravitation and Quantisation*, eds. Julia and Zinn-Justin, Paris: Elsevier Science (1995)

[63] Hawking S. W., and Ellis G. F. R.: *The Large Scale Structure of Spacetime*, Cambridge: Cambridge University Press (1973)

[64] Hawking S. W.: *Particle Creation by Black Holes*, Commun. Math. Phys. **43**, 199-220 (1975)

[65] Hislop P. D., and Longo R.: *Modular Structure of the Local Algebras Associated with the Free Massless Scalar Field Theory*, Commun. Math. Phys. **84**, 71-85 (1982)

[66] Hoge M. O.: *Relationale Zeit in der Quantenphysik*, diploma thesis, Hamburg (2008)

[67] Hugenholtz N. M.: *On the Factor Type of Equilibrium States in Quantum Statistical Mechanics*, Commun. Math. Phys. **6**, 189-193 (1967)

[68] Inoue A.: *Tomita-Takesaki Theory in Algebras of Unbounded Operators*, Berlin: Springer (1998)

[69] Isham C. J.: *Canonical Quantum Gravity and the Problem of Time*, in: *Integrable Systems, Quantum Groups, and Quantum Field Theory*, eds. Ibort and Rodriguez, Dordrecht: Kluwer, 157-287 (1993)

[70] Jones, V. F. R.: *Von Neumann Algebras*, lecture notes, math.berkeley.edu/∼vfr (2009)

[71] Kadison R. V., and Ringrose J. R.: *Fundamentals of the Theory of Operator Algebras Volume I: Elementary Theory*, SanDiego: Academic Press (1983)

[72] Kadison R. V., and Ringrose J. R.: *Fundamentals of the Theory of Operator Algebras Volume II Advanced Theory*, Orlando: Academic Press (1986)

[73] Kastler D.: *Equilibrium States of Matter and Operator Algebras*, Symposia Mathematica vol. XX 49-107 (1976)

[74] Kiefer C.: *Quantum Gravity*, Oxford: Oxford University Press (2007)

[75] Kiefer C.: *Does Time Exist in Quantum Gravity?*, FQXi essay contest on *The Nature of Time* (2008)

[76] Kuchař K. V.: *Time and Interpretations of Quantum Gravity*, in: *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, eds. Kunstatter and Williams, Singapore: World Scientific, 211-314 (1993)

[77] Kuckert B.: *Borchers' Commutation Relations and Modular Symmetries in Quantum Field Theory*, Lett. Math. Phys. **41**, 307-320 (1997)

[78] Landau L. D., and Lifschitz E. M.: *Lehrbuch der theoretischen Physik Band II – Klassische Feldtheorie*, Thun und Frankfurt am Main: Verlag Harri Deutsch (1997)

[79] Lledó F.: *Modular Theory by Example*, arXiv:0901.1004v1 [math.OA] (2009)

[80] Longo R.: *Notes on Algebraic Invariants for Non-Commutative Dynamical Systems* Commun. Math. Phys. **69**, 195-207 (1979)

[81] Longo R., Martinetti P., and Rehren K.-H.: *Geometric Modular Action for Disjoint Intervals and Boundary Conformal Field Theory*, Rev. Math. Phys. **22**, 331-354 (2010)

[82] Marolf D.: *Group Averaging and Refined Algebraic Quantization: Where Are We?*, in: *The Ninth Marcel Grossmann Meeting: Proceedings (Rome 2000)*, eds. Gurzadyan, Jantzen and Ruffini, Singapore: World Scientific (2002)

[83] Martinetti P., and Rovelli C.: *Diamond's Temperature: Unruh Effect for Bounded Trajectories and Thermal Time Hypothesis*, Class. Quantum Grav. **20**, 4919-4931 (2003)

[84] Martinetti P.: *A Brief Remark on Unruh Effect and Causality*, J. Phys.: Conf. Ser. **67**, 012027 (2007)

[85] Martinetti P.: *Conformal Mapping of Unruh Temperature*, Mod. Phys. Lett. **A24**, 1473-1483 (2009)

[86] Martinetti P.: private communication

[87] McTaggart J. M. E.: *The Unreality of Time*, Mind **17**, 456-473 (1908)

[88] Mittelstaedt P.: *Der Zeitbegriff in der Physik*, Heidelberg: Spektrum Akademischer Verlag (1996)

[89] Mund J.: *The Bisognano-Wichmann Theorem for Massive Theories*, Ann. Henri Poincaré **2**, 907-926 (2001)

[90] Narnhofer H., Peter I., and Thirring W.: *How Hot Is de Sitter Space?*, Internat. J. Modern Phys. **10**, 1507-1520 (1996)

[91] von Neumann J.: *Mathematische Grundlagen der Quantenmechanik*, Berlin: Springer (1996)

[92] Page D. N., and Wootters W. K.: *Evolution without Evolution: Dynamics Described by Stationary Observables*, Phys. Rev. D **27**, 2885-2892 (1983)

[93] Paul H.: *Über quantenmechanische Zeitoperatoren*, Annalen der Physik **464**, 252-261 (1962)

[94] Penrose R.: *Computerdenken*, Heidelberg: Spektrum Akademischer Verlag (2002)

[95] Prigogine I.: *Vom Sein zum Werden: Zeit und Komplexität in den Naturwissenschaften*, München: Piper (1982)

[96] Pusz W., and Woronowicz S. L.: *Passive States and KMS States for General Quantum Systems*, Commun. Math. Phys. **58**, 273-290 (1978)

[97] Reed M., amd Simon B.: *Methods of Modern Mathematical Physics I: Functional Analysis*, San Diego: Academic Press (1980)

[98] Reeh H., and Schlieder S.: *Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern*, Il Nuovo Cimento **22**, 1051-1068 (1961)

[99] Reichenbach H.: *The Direction of Time*, Berkeley: University of California Press (1956)

[100] Reichenbach H.: *Philosophie der Raum-Zeit-Lehre*, Braunschweig: Vieweg (1977)

[101] Reisenberger M., and Rovelli C.: *Spacetime States and Covariant Quantum Theory*, Phys. Rev. D **65**, 125016 (2002)

[102] Requardt M.: *Quantum Field Theory in Restricted Domains of Minkovski Space. The 'Thermalisation' of the Physical Vacuum as the Physics of certain 'Open Systems'*, unpublished (1991)

[103] Rindler W.: *Essential Relativity*, New York: Springer (1977)

[104] Robinson D. W.: *Return to Equilibrium*, Commun. Math. Phys. **31**, 171-189 (1973)

[105] Rovelli C.: *Time in Quantum Gravity: An Hypothesis*, Phys. Rev. D **43**, 442-456 (1990)

[106] Rovelli C.: *What Is Observable in Classical and Quantum Gravity?*, Class. Quantum Grav. **8**, 297-316 (1991)

[107] Rovelli C.: *Statistical Mechanics of Gravity and the Thermodynamical Origin of Time*, Class. Quantum Grav. **10**, 1549-1566 (1993)

[108] Rovelli C.: *The Statistical State of the Universe*, Class. Quantum Grav. **10**, 1567-1578 (1993)

[109] Rovelli C.: *Partial Observables*, Phys. Rev. D **65**, 124013 (2002)

[110] Rovelli C.: *Quantum Gravity*, Cambridge: Cambridge University Press (2004)

[111] Rovelli C.: *Forget Time*, FQXi essay contest on *The Nature of Time* (2008)

[112] Rovelli C., and Smerlak M.: *Thermal Time and the Tolman-Ehrenfest Effect: Temperature as the "Speed of Time"*, arXiv:1005.2985v2 [gr-qc] (2010)

[113] Rovelli C.: private communication

[114] Rudin W.: *Functional Analysis*, Boston: McGraw-Hill (1991)

[115] Saffary T.: *Modular Action on the Massive Algebra*, PhD thesis, Hamburg (2005)

[116] Sewell G. L.: *Relativity of Temperature and the Hawking Effect*, Phys. Lett. **79A**, 23-24 (1980)

[117] Sewell G. L.: *Quantum Fields on Manifolds: PCT and Gravitationally Induced Thermal States*, Annals of Physics **141**, 201-224 (1982)

[118] Smolin L.: *The Present Moment in Quantum Cosmology: Challenges to the Arguments for the Elimination of Time*, arXiv:gr-qc/0104097v1 (2000)

[119] Souriau J.-M.: *Structure of Dynamical Systems*, Boston: Birkhäuser (1997)

[120] Streater R. F., and Wightman A. S.: *PCT, Spin and Statistics and all that*, New York: Benjamin (1964)

[121] Summers S. J., and Wichmann E. H.: *Concerning the Condition of Additivity in Quantum Field Theory*, Ann. Inst. H. Poincaré **47**, 113-124 (1987)

[122] Summers S. J.: *Yet more Ado about Nothing: The Remarkable Relativistic Vacuum State*, arXiv:0802.185v2 [math-ph] (2008)

[123] Swieca J. A., and Völkel A. H.: *Remarks on Conformal Invariance*, Commun. Math. Phys. **29**, 319-342 (1973)

[124] Takesaki M.: *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Berlin: Springer (1970)

[125] Takesaki M.: *Theory of Operator Algebras II*, Berlin: Springer (2000)

[126] Tian Y.: *De Sitter Thermodynamics from Diamonds's Temperature*, JHEP **06** (2005) 045

[127] Tolman R. C., and Ehrenfest P.: *Temperature Equilibrium in a Static Gravitational Field*, Phys. Rev. **36**, 1791-1798 (1930)

[128] Tolman R. C.: *The Principles of Statistical Mechanics*, Oxford: Oxford University Press (1950)

[129] Trebels S.: *Über die geometrische Wirkung modularer Automorphismen*, PhD thesis, Göttingen (1997)

[130] Unruh W. G.: *Notes on Black-Hole Evaporation*, Phys. Rev. D **14**, 870-891 (1976)

[131] Unruh W. G., and Wald R. M.: *Time and the Interpretation of Canonical Quantum Gravity*, Phys. Rev. D **40**, 2598-2614 (1989)

[132] Wald, R. M.: *General Relativity*, Chicago: University of Chicago Press (1984).

[133] Wald, R. M.: *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, Chicago: University of Chicago Press (1994)

[134] Whitrow G. J.: *The Natural Philosophy of Time*, Oxford: Clarendon Press (1980)

[135] DeWitt B. S.: *Quantum Gravity: The New Synthesis*, in: *General Relativity, an Einstein Centenary Survey*, eds. Hawking and Israel, Cambridge: Cambridge University Press (1979)

[136] Yngvason J.: *A Note on Essential Duality*, Lett. Math. Phys. **31**, 127-141 (1994)

[137] Yngvason J.: *The Role of Type III Factors in Quantum Field Theory*, lecture notes, von Neumann Centennial Conference, Budapest (2003)

[138] Zeh H.-D.: *The Physical Basis of the Direction of Time*, Berlin: Springer (1989)

[139] Zubarev D., Morozov V., and Röpke G.: *Statistical Mechanics of Nonequilibrium Processes, Volume I*, Berlin: Akademie Verlag (1996)

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