# Models of Scalar Fields in 4D Globally Conformal Invariant Quantum Field Theory

**Master thesis**

submitted by

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# Chapter 1

# Introduction

## 1.1 Motivation

Although the real world is not conformally invariant — the presence of masses and lengths signalizes violation of even the weaker dilation symmetry — people have been motivated to study 4-dimensional quantum field theories with conformal invariance for two main reasons. The physical justification has been that conformal symmetry may be a meaningful approximation at high–energy limits. But, without doubt, the main stimulus for consideration of theories with conformal invariance is that they provide us with a better chance to construct a nontrivial quantum field model. Although theorists have attempted already more than seventy years to establish a model of interacting quantum fields in 4 or in higher dimensions, they have still not succeeded in its formulation. The restricting assumption of higher symmetry allows to obtain more structural information about the theory and simplifies this task. As it was demonstrated in [16], [17], [18], [19], [20], [23] and as we will illustrate later in this thesis, conformal field theories can be analyzed to a remarkable extent by simple methods.

Indeed, if we work within the Wightman axiomatic approach and adopt the notion, that a quantum field theory is globally conformal invariant if its Wightman distributions are invariant under every conformal transformation outside its singularity points, we will end up with a theory with a lot of "nice features" [16] [1]. We first prove that every pair of points in Minkowski space, whose difference vector does not lie on the light cone, can be mapped into any other such pair by a conformal transformation, for instance, any time–like interval can be mapped into any space–like interval. Then this fact, together with local commutativity, leads to the vanishing of the commutator $[\varphi_1(x_1), \varphi_2(x_2)]$ of Bose fields as a distribution for any non-isotropic difference $(x_1-x_2)$ — the Huygens principle. Moreover, this commutator vanishes whenever it is multiplied by a sufficiently large power of the square of the interval $(x_1 - x_2)$. This strong locality property together with energy positivity leads to the condition that Wightman distributions are rational functions in the arguments' differences $(x_i - x_j)^2$. A uniform bound restricting the degrees of the poles of the rational functions is deduced from Hilbert space positivity. All this information obtained about the conformal Bose fields is quite restrictive and allows the explicit construction

[1] Note that the requirement of asymptotic completeness would imply, together with global conformal invariance, the triviality of the S-matrix. However, as conformal field theory is not a particle theory, this assumption may be dropped and another criterium for non-triviality is imposed — a field is considered to be non-trivial if it cannot be expressed as a sum of Wick's products of free fields.

of some correlation functions with a finite number of parameters of more than three fields. Let us recall that in a Wightman theory the Wightman distributions (correlation functions of the local fields) are the main object of interest, as they carry all the information about the fields.

Another big advantage of global conformal invariance is that it allows the derivation in a closed form of the expansion of 4-point functions in partial waves [20]. Every partial wave corresponds to a traceless symmetric representation of the conformal group and is labelled by a number, called a twist, depending on this representation. It is argued that the Wightman positivity at the 4-point level, which is a necessary requirement for Wightman distributions, is equivalent to the non-negativity of all the coefficients in this expansion. These coefficients, in turn, are linear combinations of the parameters of the 4-point functions, therefore we obtain some restrictions on them. Partial wave analysis will be an important tool in our investigation.

Furthermore, global conformal invariance guarantees the existence of the product of two local fields and its explicit construction as an expansion in a series of quasi-primary fields, i.e. of fields with nice transformation properties (the terms of this series will be labelled again by the "twist", as the operator product expansion is related to the partial wave expansion) [23]. Studying the content of this series provides us with important insight for our models and is a powerful method in our analysis. The twist two contribution in the product of two copies of the same field is harmonic in its two arguments and this property has far reaching consequences for the theory.

Finally, in a theory with a unique field $T_{ij}(x)$ with the properties of a stress–energy tensor we can obtain further information about the parameters of a correlation function of the type $\langle \mathcal{A}\mathcal{A}\mathcal{B}\mathcal{B} \rangle$ of two hermitian scalars $\mathcal{A}(x)$ and $\mathcal{B}(x)$ [20]. Conformal invariance fixes the 3-point functions $\langle \mathcal{A}\mathcal{A}T \rangle$ and $\langle T\mathcal{B}\mathcal{B} \rangle$ up to a factor and their proper normalizations, coming from the fact that the stress–energy tensor generates the infinitesimal symmetries in the theory, are called Ward identities. Then, we take into account that the unique stress–energy tensor is isolated in the operator product expansion of any hermitian scalar field with itself by applying a certain differential operator to the twist 2 contribution. This operator is determined by the general properties of $T_{ij}(x)$ and normalized by Ward identities with a coefficient of proportionality depending on the parameters of $\langle \mathcal{A}\mathcal{A}\mathcal{B}\mathcal{B} \rangle$. This allows us to relate some of them to the "central charge", parameterizing the 2-point function of the stress–energy tensor.

A program was set up in [16] and pursued in [16] – [23], [5], [6] for constructing a conformal field theory model in 4 space–time dimensions with rational correlation functions of observable fields by combining global conformal invariance with Wightman axioms and some understandings coming from conformal models in 2 space-time dimensions. This approach is inherently non-perturbative. The main models explored so far are those of a 2-dimensional and of a 4-dimensional scalar. It was shown in [17] that every scalar hermitian field with scaling dimension 2 can be expressed as a sum of normal products of massless scalar fields. The scalar of dimension 4 provides a good candidate for a non-trivial quantum field model and a lot of efforts are invested into its construction [19], [22], [20].

The current task is within the framework of this program. Three different models of hermitian scalar Wightman fields with the property of global conformal invariance will be considered, specified by the scaling dimensions of the fields and by the operator product content. We will show that applying partial wave and operator product content analysis, Ward identities and Cauchy–Schwarz inequalities in the studying of 4- and 5-point functions will be enough to completely determine two of these models, restricting them to the trivial case, and to obtain a lot of information for the third of them.

## 1.2 Organization

This thesis is organized as follows: The second chapter contains short preliminary information which will be necessary in our further investigation. We will discuss briefly the axiomatic background in which our analysis takes place, then we will review the relevant properties of the conformal group and the classification of its representations and last we will pay attention the three fundamental properties of any globally conformal invariant quantum field theory - Huygens principle, rationality of Wightman distributions and the uniform bounds on their singularities. In the third chapter we will introduce the three most important tools in our analysis - operator product expansion, partial wave expansion and Ward identities.

The following three chapters represent the three models that we will consider, the first two of which are the original results of this thesis and the third we know from [26].

The first model is one of a 3-dimensional hermitian scalar $W(x)$ in a theory without a field with the properties of the stress–energy tensor. We prove that its 4-point function consists only of disconnected terms, which means that $W(x)$ is a generalized free field. We accomplish that exploring the 4-point function of $W(x)$ and its mixed 4-point functions with the 4-dimensional scalar $\mathcal{L}(x)$ in the operator product expansion $W \cdot W$, studying their partial wave expansion and applying Cauchy-Schwarz inequalities. We explore also the 5-point function $\langle \mathcal{L}WWWW \rangle$, demonstrating that the 5-point functions analysis may provide an important new information for the theory.

In the second model we assume the existence of a free massless Dirac field $\psi(x)$ together with a 4-dimensional scalar $\mathcal{L}(x)$, which is not available in the Dirac theory itself and we expect to find some contradiction. We have excluded the trivial Yukawa solution $\varphi \cdot \overline{\psi}\psi$ requesting the absence of 1-dimensional scalar $\varphi(x)$ in $\mathcal{L} \cdot \mathcal{W}$ where $\mathcal{W}(x) :=: \overline{\psi}\psi : (x)$, and to strengthen the analysis, we assume the uniqueness of a stress–energy tensor with a central charge $c = 6$ in the theory and that there is no 2-dimensional scalar in $\mathcal{L} \cdot \mathcal{L}$. We will exploit Ward identities, partial wave expansion and studying of the operator product content for the mixed 4-point functions of $\mathcal{L}(x)$ with $\mathcal{W}(x)$, but even pushing our analysis at the 5-point level will not lead us to inconsistency.

In the third model there is a 4-dimensional scalar $\mathcal{L}(x)$ and a 2-dimensional scalar $\phi(x) := \frac{1}{2} : \varphi^2 : (x)$, where $\varphi(x)$ is a massless free scalar. We assume again a unique stress–energy tensor, given by $\varphi(x)$. We prove that $\mathcal{L} \sim: \varphi^4 : (x)$, i.e. we have again a trivial model. This case is an example how Ward identities combined with Wightman positivity and Cauchy–Schwarz inequalites determine the model completely.

# Chapter 2

# GCI and Wightman distributions

In this chapter we will show that the property of global conformal invariance (GCI) combined with the general physical principles has far reaching implications for Bose fields and allows us to obtain a lot of structural information about the models.

Our task is set within the Wightman approach and in the first section of this chapter Wightman axioms are briefly discussed ([9], [11] and [27]). Next, we introduce the conformal group, then we specify the notion of global conformal invariance which we will use in this work and we will make some notes on representation theory of the conformal group in 4 dimensions ([15], [25] and [28]).

The last three sections of this chapter are devoted to the three fundamental features that every GCI Wightman theory of Bose fields in 4 dimensions shares: Huygens principle, rationality of all the correlation functions and their pole bounds [16].

## 2.1 The physical input

Every quantum field theory (QFT) should satisfy some fundamental physical principles. They are mainly: hypothesis of quantum theory, stability properties (existence of vacuum, equilibrium states), energy positivity, assumption of relativistic invariance, absence of acausal influences. In the axiomatic approaches they are counted for by a list of axioms which we request necessarily to be obeyed by the fields in every model. Our fields will be some mathematical constructions which should satisfy this list of postulates. They put a lot of restrictions on the possible field theories and allow us to understand much about their structural properties.

There are several axiomatic approaches which may differ in the technical formulation of the principles (Wightman QFT, Haag–Kastler QFT, Euclidean QFT). Our analysis is in the framework of Wightman axioms [11], [27]. Let us recall them briefly [1]:

- ***Quantum theory***: The states in the theory are represented by unit rays in a separable Hilbert space $\mathcal{H}$, which carries a unitary representation of $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$;
  There is a unique (up to a constant phase) vacuum state $|0\rangle$, invariant under the action of this group;

[1]Here the axioms are formulated for Minkowski space, as it is sufficient for our goals. However, they may be generalized to any even space–time dimensions.

The spectrum of the energy–momentum operator $P^\mu$ (corresponding to the translation subgroup generator) is confined to the forward cone: $p^2 \geq 0$; $p^0 \geq 0$.

- ***Fields***: The fields are operator valued distributions over the Schwartz space of test functions $\mathcal{S}$ (defined on the Minkowski space $M$):
$$\varphi : \mathcal{S}(M) \to Op(\mathcal{H}) \tag{2.1}$$
Together with their adjoints they are defined on a domain $D$ dense in $\mathcal{H}$, containing the vacuum $|0\rangle$;
$D$ is stable under the action $\varphi(f)$, $(f \in \mathcal{S})$ and under the representation of $\mathcal{P}_+^\uparrow$;
If $|\phi\rangle$, $|\psi\rangle \in D$, then $\langle\phi|\varphi_j(f)|\psi\rangle$ is a tempered distribution, regarded as a functional of $f$.

- ***Transformation law of fields***:
$$U(a,A)\varphi_j(f)U(a,A)^{-1} := \sum S_{jk}(A^{-1})\varphi_k\left(f_{a,A}(x)\right) \tag{2.2}$$
with $f_{a,A}(x) = f(A^{-1}(x-a))$ is valid when each side is applied to any vector in $D$, here $a \in \mathbb{R}^4$, $A \in SL(2,\mathbb{C})$.

- ***Locality***: If the supports of the functions $f$ and $g$ are space–like separated, then:
$$[\varphi_j(f), \varphi_k(g)]_\pm \equiv \varphi_j(f)\varphi_k(g) \pm \varphi_k(g)\varphi_j(f) = 0 \tag{2.3}$$
The sign $(-)$ holds whenever one of the fields is a Bose field and it is $+$ otherwise.

- ***Completeness***: The vacuum state is cyclic for the smeared fields:
$$\overline{span\{\varphi(f_1)...\varphi(f_n)|0\rangle : \; f_i \in \mathcal{S}(M),\, n \in \mathbb{N}_0\}} = \mathcal{H} \tag{2.4}$$

$\varphi(f)$ are of the form: $\varphi(f) = \int \varphi(x)f(x)d^4x$. The axioms, together with the Nuclear theorem (see [27]), imply that objects of the type $\mathfrak{W}(x_1,x_2,...,x_n) = \langle 0|\varphi_1(x_1)\varphi_2(x_2)...\varphi_n(x_n)|0\rangle$ exist and that they define tempered distributions on $\mathcal{S}(\mathbb{R}^{4n})$. We will refer to $\mathfrak{W}(x_1,...,x_n)$ as *Wightman distributions.* They will be the central object of our investigation, because they contain all the information about the fields in the theory. This is so due to the existence of the reconstruction theorem, which states that the sequence $\{\mathfrak{W}^n(x_1,...,x_n)\}$, $n \in \mathbb{N}$ of Wightman distributions determines the QFT uniquely up to a unitary equivalence.

We will list here some properties of Wightman distributions, which we will need later [27]:

- ***Spectral condition***: The Fourier transform $\widetilde{W}$ of $W(x_2-x_1,...,x_n-x_{n-1}) = \mathfrak{W}(x_1,x_2,...,x_n)$is a tempered distribution[2] and has support in the product of the future light cones in $M$.

- ***Locality***:
$$\mathfrak{W}(x_1,...,x_i,x_{i+1},...,x_n) = \epsilon_{i,i+1}\mathfrak{W}(x_1,...,x_{i+1},x_i,...,x_n) \tag{2.5}$$
for $(x_i - x_{i+1})^2 > 0$. Here $\epsilon_{ij}$ is a factor depending on whether we deal with Bose or Fermi fields. It is $(-1)$ if both $x_i$ and $x_{i+1}$ refer to Fermi fields and 1 otherwise.

[2]such a function exists due to the fact that Wightman distributions are invariant under any unitary transformations, in particular under translations

- ***Wightman positivity***: For any sequence of test functions $\{f_j\}$, $f_j \in \mathcal{S}(\mathbb{R}^{4j})$ with $f_j = 0$ except for a finite number of $j$, the Wightman distributions satisfy the inequalities:

$$\sum_{j,k=0}^{\infty} \int ... \int \overline{f_j(x_1, ..., x_j)} \mathfrak{W}_{jk}(x_j, x_{j-1}, ..., x_1, y_1, ..., y_k) \times$$
$$\times f_k(y_1, ..., y_k) dx_1 ... dx_j dy_1 ... dy_k \geq 0 \tag{2.6}$$

Here $\mathfrak{W}_{jk} = \langle 0 | \varphi^*_{jj}(x_j) ... \varphi^*_{j1}(x_1), \varphi_{k1}(y_1) ... \varphi_{kk}(y_k) | 0 \rangle$.

- ***Cluster decomposition property***: If $a$ is a space–like vector, then

$$\mathfrak{W}(x_1, ..., x_j, x_{j+1} + \lambda a, x_{j+2} + \lambda a, ..., x_n + \lambda a) \rightarrow \mathfrak{W}(x_1, ..., x_j) \mathfrak{W}(x_{j+1}, ..., x_n) \tag{2.7}$$

as $\lambda \rightarrow \infty$, in the sense of convergence in $\mathcal{S}'$ (here $\mathcal{S}'$ consists of the distributions defined on the space of test functions $\mathcal{S}$).

If the configuration $(x_1, ..., x_n)$ contains several clusters (i.e. subsets of points s.t. all the points in one subset have a large space–like separation from all the points in any other subset), then due to (2.7) we expect that $\mathfrak{W}^n(x_1, ..., x_n) \sim \Pi_r \mathfrak{W}^{n_r}(y_{r,1}, ..., y_{r,n_r})$. Here $r$ labels the cluster, $n_r$ is the number of points in the $r^{th}$ cluster and by $y_{r,k}$ we denote the points $x_i$ which this cluster contains. Therefore, $\mathfrak{W}^n(x_1, ..., x_n)$ can be written as [10]:

$$\mathfrak{W}^n(x_1, ..., x_n) = \sum_P \Pi_r \mathfrak{W}_T^{n_r}(y_{r,1}, ..., y_{r,n_r}) \tag{2.8}$$

where we have a sum over all the possible partitions $P$ of $(x_1, ..., x_n)$ into clusters. We will call $\mathfrak{W}_T^{n_r}(y_{r,1}, ..., y_{r,n_r})$ *truncated functions*. For $\mathfrak{W}_T^1(y_{1,1})$ we know that they are constant because of their translation invariance (Wightman distributions are invariant under all unitary transformations) and we can "redefine" the field s.t. $\mathfrak{W}_T^1(y_{1,1})$ to vanish. Note that in every scale–invariant theory $\mathfrak{W}_T^1(y_{1,1})$ vanishes, as the only constant which does not change under dilations is 0. For free fields all truncated functions with $n \neq 2$ vanish: $\mathfrak{W}_T^{n_r}(y_{r,1}, ..., y_{r,n_r}) = 0$ for $n_r \neq 2$. Thus, for a system of free fields all Wightman distributions are expressed in terms of 2-point functions.

We will mention the two main theorems which will be quite involved in the general results that we will use in our analysis. The first is the Reeh–Schlieder theorem, implying that if the vacuum is cyclic for the polynomial algebra $\mathcal{P}(\mathbb{R}^4)$, then it is also a cyclic vector for $\mathcal{P}(\mathcal{O})$, where $\mathcal{O}$ is an open subset of space–time. A corollary of it relevant for the applications is that if $\mathcal{O}$ is an open set of nonempty interior, then $T \in \mathcal{P}(\mathcal{O})$, $T|0\rangle = 0 \rightarrow T = 0$. In particular, this implies that if the 2-point function $\langle 0 | \varphi^*(x_1) \varphi(x_2) | 0 \rangle$ vanishes, then the field $\varphi(x)$ itself vanishes. The second theorem says that the Wightman distributions $\mathfrak{W}(x_1, ..., x_n)$ are boundary values of analytic functions $\mathfrak{W}(z_1, ..., z_n)$, holomorphic in $\mathcal{G}_n = \{z_1, ..., z_n | \, Im(z_k - z_{k-1}) \in V^+\}$ (we shall call them *Wightman functions*). Then $W(\xi_1, ... \xi_n)$ is the boundary value of an analytic function $W(\zeta_1, ... \zeta_n)$ regular in $\mathcal{I}^n$, where $\mathcal{I} = \{\zeta | \, Im\zeta \in V^+\}$.

## 2.2 The conformal group and the notion of global conformal invariance

In our model we impose the special assumption of a larger symmetry group — *the conformal group*. In this subsection we will discuss the question of conformal invariance in QFT referring mainly to

[25] and [28]. We will restrict to the case of 4-dimensional Minkowski space, although the results can be generalized to any dimensions higher than 2.

**Definition 2.1.** *In $D = 4$ space–time dimensions we define the group of conformal transformations to be the set of all the transformations that preserve the angles locally, or equivalently, that leave the metric tensor invariant up to a scale:*

$$x^\mu \mapsto f^\mu(x), \quad df_\mu df^\mu = \omega(x)^2 dx_\mu dx^\mu \tag{2.9}$$

Obviously, it contains the Poincare transformations and dilations. There is also one other class, the so-called special conformal transformations:

$$x^\mu \mapsto f^\mu, \quad f^\mu = \frac{x^\mu - b^\mu x^2}{1 - b.x + b^2 x^2} \tag{2.10}$$

The neighborhood of the identity of the conformal group is isomorphic to $SO(4,2)$.

***Remark:*** Special conformal transformations may map points from Minkowski space $M$ to infinity and space–like intervals onto time–like intervals. This leads to some paradoxes in quantum field theory: compact local field operators are transformed into non-compact ones and space–like commutativity implies time–like commutativity. Luescher and Mack [13] have shown that a universal covering group of $\widetilde{SO(4,2)}$ acts on the covering of the Minkowski space $\widetilde{M} = S^3 \times \mathbb{R}$ without violation of causality. The causality paradox is understood as an attempt to identify the different layers (copies of $M$) in the covering space. As a rule, the fields are defined on $\widetilde{M}$. The notion of GCI these authors propose is the invariance of Green functions of the Wightman QFT theory under the Euclidean conformal group $G = SO_e(5,1)$. (Euclidean Green functions are obtained by analytical continuation of Wightman functions of the fields to imaginary times.) The implications of such a definition are as follows. The Hilbert space of the theory will carry a unitary representation of the universal covering group of the Minkowskian conformal group $SO(4,2)$. Wightman functions can be analytically continued to a domain of holomorphy which has as a real boundary an $\infty$–sheeted covering $\widetilde{M}$ of $M$. $\widetilde{G}$ (the universal covering group of $G$) can act on $\widetilde{M}$ and $\widetilde{M}$ admits a globally $\widetilde{G}$-invariant causal ordering. In such a theory $\widetilde{M}$ is the space on which a GCI local QFT lives.

In our analysis we will work directly on the physical space, the Minkowski space $M$. We will say that a conformal transformation is defined on a domain in $M$ if it does not move points from this domain to infinity. We adopt the following notion of global conformal invariance, proposed in [16]:

**Definition 2.2.** *A quantum field theory, which satisfies Wightman axioms, is said to be globally conformal invariant (GCI) if for the domain of definition of any conformal transformation $g$ Wightman functions $\mathfrak{W}(z_1, ..., z_n)$ stay invariant under the representation of $g$ in the space of fields.*

Note that this requirement is stronger than the one suggested by Luescher and Mack, which implies invariance in $M$ only under infinitesimal conformal transformations.

Our motivation to work with such a definition is that it gives rise to a class of models with nice structural properties, as we will convince ourselves later, and makes their construction feasible. On the other hand, it makes also applicable purely algebraic techniques from 2-dimensional conformal field theory, like vertex algebras and infinite dimensional Lie algebras [21], [1]. Moreover, in 2-dimensional chiral conformal field theory the observable fields are the GCI ones, which could happen to be the case in higher space–time dimensions, as well.

## 2.3 Elements of representation theory of $\widetilde{SO(4,2)}$

In this section we will give elements of representation theory of the conformal group in 4 dimensions. It is based mainly on information from [15]. The fields in our theory should belong to some representation space.

1. ***The group:***

   The conformal group of 4-dimensional space–time is locally isomorphic to $G = SU(2,2)$, which is:

   $$SU(2,2) = \{u \in SL(4,\mathbb{C}), \quad u\beta u^* = \beta\}; \quad \beta := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.11}$$

   Its universal covering $\widetilde{G}$ is an infinite sheeted covering of $SO(4,2)$. (The universal covering group $\widetilde{G}$ is obtained the following way. We choose as its elements the equivalence classes of the directed paths on $G$, starting at the group identity and ending at the same point, which can be continuously deformed one into another. The group multiplication will be juxtaposition of paths and clearly all the group axioms will be fulfilled with respect to it.)

   We shall assume that the space of states in our theory carries a unitary unit ray representation of $SU(2,2)$ which guarantees a representation of its covering group.

2. ***Positive energy representations:***

   We will be interested to obtain a complete list of unitary irreducible representations of $\widetilde{G}$ which are positive energy representations, i.e. $P^0 \geq 0$. This condition coincides with the conformal hamiltonian $H^0 = \frac{1}{2}(P^0 + K^0)$ having a positive spectrum since $K^0$ is obtained from $P^0$ by unitary transformation Here $K^0$ is the generator of special conformal transformations and $H^0$ is a generator of the Cartan subalgebra of $\widetilde{G}$. Our criterium for energy positivity from now on will be $H^0 \geq 0$.

3. ***Representation spaces:***

   The representation spaces are finite component field representations, labeled by $d$ — the conformal weight, accounting for the representation of the dilation subgroup (related to the transformation of the field by: $x \mapsto \rho x$, $\varphi(x) \mapsto \rho^d \varphi(\rho x)$) and by a finite–dimensional representation of the quantum mechanical Lorentz subgroup $SL(2,\mathbb{C})$, $(j_1, j_2)$ [2]. They consist of vector–valued functions $\varphi_a(x)$ on Minkowski space $M$ with values in the finite–dimensional irreducible representation of $SL(2,\mathbb{C})$. They transform under $g$ in $G$ like an induced representation:

   $$(T(g)\varphi)(x) = S(g,x)\varphi(g^{-1}x), \quad g \in \widetilde{G}, x \in M \tag{2.12}$$

   where $S(g,x)$ is a matrix, depending on the representation.

4. ***Lowest weight representations:***

   It can be proven that every unitary irreducible representation of $\widetilde{G}$ with positive energy is a lowest weight representation. Moreover, every such representation possesses a unique lowest weight $\lambda = (d; -j_1, -j_2)$ and any two such representations with the same lowest weight are unitarily equivalent.

Let us remind (see [4]) that a lowest weight representation is a representation with a vector $\mathcal{V}$ which is cyclic with reference to the group action. This vector is an eigenvector of the generators of the Cartan subalgebra of $\mathcal{G}$ and their eigenvalues in our case are exactly the list $\lambda = (d; -j_1, -j_2)$. $\mathcal{V}$ is annihilated by $T(\mathcal{G}_-)$, which is the representation of the negative root sector from the standard decomposition of the Lie algebra: $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{H} \oplus \mathcal{G}_+$.

According to the spectral theorem for a self–adjoint operator, $H^0$ will have a discrete spectrum in this representation of the form $d+m$, $m$ is some positive integer. The lowest spectral value is $d$.

5. ***Classification:***

   Hilbert space positivity and energy positivity impose some restrictions on the candidates $\lambda = (d; -j_1, -j_2)$ for lowest weight vectors of some unitary irreducible representation of $\widetilde{G}$. All possible representations are contained in the following list:

   (a) trivial representation $d = j_1 = j_2 = 0$;
   (b) $j_1 \neq 0,\ j_2 \neq 0,\ d > j_1 + j_2 + 2$;
   (c) $j_1 j_2 = 0,\ d > j_1 + j_2 + 1$;
   (d) $j_1 \neq 0,\ j_2 \neq 0, d = j_1 + j_2 + 2$;
   (e) $j_1 j_2 = 0,\ d = j_1 + j_2 + 1$.

   It is shown in [16] that under the assumptions of global conformal invariance $d+j_1+j_2$ should be an integer. Moreover, $d$ should be a positive integer and $j_1$ and $j_2$ should be half–integers.

## 2.4 The Huygens principle

An important consequence coming from the definition of GCI above is the fact that any space–like interval in $M$ can be transformed onto a time–like interval and vice–versa by a conformal mapping, which in turn implies the Huygens principle [16].

To observe how the argumentation works, we will "move" to the conformally compactified Minkowski space $\overline{M}$. It is a homogeneous space of the connected conformal group $C_0$, which means that any two points in it can be connected by the action of this group. We can show also that mutually non-isotropic pairs of points form a single conformal orbit in $\overline{M}$, which is related to the statement above. (We recall that two points are said to be mutually isotropic whenever their difference vector is light–like.)

**Definition 4.1.** *The compactified Minkowski space is the following subset of* $\mathbb{C}^4$*:*

$$\overline{M} = \{z \in \mathbb{C}^4;\ z = \frac{\bar{z}}{\bar{z}^2},\ z^2 = \sum_{i=1}^{4}(z_\alpha)^2\} \quad \cong \quad \frac{(S^1 \times S^{D-1})}{\mathbb{Z}_2} \tag{2.13}$$

The Minkowski space $M$ is embedded as a dense open subset in $\overline{M}$ in such a way so the isotropy relation in $M \times M$ extends to an isotropy relation on $\overline{M} \times \overline{M}$ and this relation remains invariant under conformal transformations:

$$z = \frac{\mathbf{x}}{\omega(x)}, \quad z_4 = \frac{1 - x^2}{2\omega(x)}, \quad 2\omega(x) = 1 + x^2 - 2ix^0, \quad x^2 = \mathbf{x}^2 - (x^0)^2 \tag{2.14}$$

We observe that the complement of $M$ to $\overline{M}$, the added points at infinity, form an isotropic $(D-1)$-dimensional cone $K_\infty := \overline{M} \setminus M$. Its tip $p_\infty$ is the only point in this cone s.t. all points isotropic to it in $\overline{M}$ are contained in $K_\infty$. The stabilizer $C_\infty$ of $p_\infty$ in $C_0$ is the Weyl group (Poincare group + dilations). As the isotropy relation in $\overline{M}$ is conformally invariant, $C_\infty$ leaves $M$ and $K_\infty$ invariant.

Now we are ready to prove the following proposition:

**Proposition 4.1.** *Any pair $(p_0, p_1)$ of mutually non–isotropic points of $\overline{M}$ can be mapped into any other such pair $(p_0', p_1')$ by a conformal transformation.*

*Proof:* Due to the transitivity of the action of $C_0$, there are elements $g_0$ and $g_0'$ in it that move $p_0$ and $p_0'$ into the point $p_\infty$.

Since the initial pairs are non–isotropic and the conformal maps preserve the isotropy property, the images $g_0 p_1$ and $g_0' p_1'$ of the two other points will both belong to $M$. They can be moved one into another by a translation $t$ (in $C_\infty$) which leaves $p_\infty$ invariant. Then the element $g \in C_0$ which transforms the pair $(p_0, p_1)$ into $(p_0', p_1')$ will be $g = (g_0')^{-1} t g_0$. $\diamond$

It is clear that this statement would be valid if restricted to $M$, in particular for two pairs, one of which having a space–like difference vector and the other — a time–like one.

In combination with local commutativity, this proposition implies *the Huygens principle* for Bose fields, namely that $[\varphi_j(x_j), \varphi_k(x_k)] = 0$ whenever $(x_j - x_k)^2$ is non-isotropic.

If we multiply the commutator with a factor $[(x_1 - x_2)^2]^N$, where $N$ is a sufficiently large number, given by the condition (2.22), then the following equality will be valid everywhere in $M^{\otimes 2}$:

$$[(x_1 - x_2)^2]^N [\varphi_1(x_1), \varphi_2(x_2)] = 0 \tag{2.15}$$

(We can remark here that this property of GCI fields makes the theory incompatible with scattering theory.)

In the next section we will need the following lemma, which is a generalization of the proposition above [16]:

**Lemma 4.2.** *For each set of points $(x_1, ..., x_m, y_1, y_2)$ in $M$ s.t. $(y_1 - y_2)^2 \neq 0$ and a pair of mutually non-isotropic $(y_1', y_2')$ there exists $g \in C$ s.t. $gx_i \in M$ for $1 \leq i \leq m$ and $y_1' = gy_1,\ y_2' = gy_2$.*

*Proof of the Lemma:* The proof is based on induction in the number of the $x$ points $m$. For $m = 0$ we get the proposition we stated before. If we assume that the Lemma is valid for some positive $m$, we can prove that it is also valid for arbitrary $m + 1$ points. Due to the induction assumption there is a transformation $g'$ s.t. it leaves all the first $m$ points in $M$, $g'x_i \in M$ for $1 \leq i \leq m$, and it maps the first pair of points into the second $y_1' = g'y_1,\ y_2' = g'y_2$. It may happen that $x_{m+1}$ is mapped by g into another point in $M$ and then we are done. In case that it is moved into $K_\infty$ s.t $p := g'x_{m+1}$ it is proven in [16] that there is an element $h$, arbitrary close to the group unit in the stabilizer $C_{y_1', y_2'}$ of the pair $y_1',\ y_2'$ s.t. $hp \in M$. It is also shown that $hg'$ does not move the first $n$ points out of $M$. So, the transformation $g = hg'$ is the transformation that we wanted to find. $\diamond$

## 2.5 Rationality of Wightman distributions

In this section we will consider the following theorem from [16] which determines to a great extent the form of the Wightman distributions $\mathfrak{W}(x_1, ..., x_n)$ of $\varphi(x)$ ($\varphi(x) \in F$, where $F$ is a finite dimensional complex vector space, $\varphi(x)$ is a Bose field) [3]:

**Theorem 5.3.** *Let $\mathfrak{W}(x_1, ..., x_n)$ be a tempered distribution. Then if it satisfies energy positivity, locality and global conformal invariance, it is a boundary value of a rational function of the type:*

$$\mathfrak{W}(x_1, ..., x_n) = \mathfrak{P}(x_1, ..., x_n)\Pi_{1\leq j<k\leq n}(x_{jk}^2 + i0x_{jk}^0)^{-\mu}. \tag{2.16}$$

*Here $x_{jk} := x_j - x_k$, $\mu \geq 0$ is an integer and $\mathfrak{P}(x_1, ..., x_n)$ is a polynomial with values in $F^{\bigotimes n}$.*

*The associated rational function*

$$\mathfrak{R}(x_1, ..., x_n) = \mathfrak{P}(x_1, ..., x_n)\Pi_{1\leq j<k\leq n}(x_{jk}^2)^{-\mu} \tag{2.17}$$

*is fully symmetric and satisfies the conformal invariance condition as a rational function.*

*Sketch of the Proof:*

Let us assume that there is a tempered distribution $\mathfrak{W}(x_1, ..., x_n)$ satisfying the spectrum condition, locality and GCI.

1. Lemma 4.2 and locality imply that $\mathfrak{W}(x_1, ..., x_n)$ is symmetric in the subset of all points $(x_1, ..., x_n)$ in $M^{\otimes n}$ where for every pair $(x_i, x_j)$ the points are mutually non-isotropic;

2. Being a tempered distribution, $\mathfrak{W}(x_1, ..., x_n)$ has singularities of a finite order and together with the strong locality property this implies that $\mu$ exists s.t. the distribution:

$$\mathfrak{P}(x_1, ..., x_n) = \left(\Pi_{1\leq i<j\leq n}(x_{ij}^2)^{\mu}\right) \mathfrak{W}(x_1, ..., x_n) \tag{2.18}$$

   is translation invariant and symmetric everywhere in $M^{\otimes n}$;

3. The Fourier transform $\widetilde{P}(q_1, ..., q_{n-1})$ of $P(x_{12}, ..., x_{n-1,n}) = \mathfrak{P}(x_1, ..., x_n)$ is obtained from $\widetilde{W}(q_1, ..., q_{n-1})$ by the action of a differential operator in $q_1, ..., q_{n-1}$ with constant coefficients. Hence, the support of $\widetilde{P}(q_1, ..., q_{n-1})$ is contained in the support of $\widetilde{W}(q_1, ..., q_{n-1})$, *supp* $\widetilde{P} \subseteq (\overline{V}^+)^{\times(n-1)}$;

4. The total symmetry implies:

$$\begin{aligned} P(y_1, ..., y_{n-1}) &= P(-y_{n-1}, ..., -y_1) \\ \Rightarrow \widetilde{P}(q_1, ..., q_{n-1}) &= \widetilde{P}(-q_{n-1} ... - q_1) \end{aligned} \tag{2.19}$$

   which leads to *supp* $\widetilde{P}(q_1, ..., q_{n-1}) \subseteq (\overline{V}^-)^{\times(n-1)}$;

5. We conclude that: *supp* $\widetilde{P}(q_1, ..., q_{n-1}) \subseteq (\overline{V}^-)^{\times(n-1)} \cap (\overline{V}^+)^{\times(n-1)} = \{0\}$. It implies that $P(y_1, ..., y_{n-1})$ is a polynomial and consequently $\mathfrak{P}(x_1, ..., x_n)$ is a polynomial as well;

[3]This result holds for any space–time dimensions. Here we will restrict the analysis to Bose fields, although it can be generalized for Fermi fields, as well.

6. $W(q_1, ..., q_{n-1})$ admits an analytic continuation $W(\zeta_1, ..., \zeta_{n-1})$ in the backward tube $(T_-^{n-1})$ as a consequence of energy positivity. It leads to the result (2.16);

7. The rational function $\mathfrak{R}(x_1, ..., x_n)$ obtained this way is fully symmetric and conformally invariant. $\diamond$

*For our purposes it will be sufficient to regard Wightman distributions in any GCI theory as conformal rational functions of the form* $\mathfrak{W}^n(x_1, ..., x_n) = \sum_n A_n \Pi_{i \neq j=1}^n (x_{ij}^2)^{n_{ij}}$*, where* $n$ *is a multiindex.*

## 2.6 Pole degrees

In the previous subsection we observed that Wightman functions are rational functions in squares of the argument differences. Taking into account Wightman positivity we can restrict them further, according to the following theorem [16]:

**Theorem 6.4.** *If* $\varphi(x)$ *is a Wightman Bose field in a GCI theory, then the orders of the poles of the rational functions* $\mathfrak{W}(x_1, ...x_n)$ *are uniformly bounded, i.e. the integer* $\mu$ *can be chosen independent of* $n$*.*

*Proof:*

The following vector valued distributions define a map of the test function space $\mathcal{S}(M^{\times k})$ into the Hilbert space $\mathcal{H}$:

$$\Phi(x_1, ..., x_k) = \varphi_1(x_1)...\varphi_k(x_k)|0\rangle \tag{2.20}$$

For $k = 2$ it follows from proposition 4.1 and from Reeh–Schlieder theorem that $\Phi(x_1, x_2) = \Phi(x_2, x_1)$ for any pair of mutually non-isotropic points $(x_1, x_2)$.

Being a tempered distribution, $\Phi(x_1, ..., x_k)$ has a finite order of singularities and there is a positive integer $\mu$ s.t. the distributions:

$$f(x_1, x_2) = (x_{12}^2)^{\mu} \langle\psi|\varphi_1(x_1)\varphi_2(x_2)|0\rangle = f(x_2, x_1) \quad \forall\langle\psi| \in \mathcal{H},\ (x_1, x_2) \in M^{\otimes 2} \tag{2.21}$$

are symmetric in $M \times M$.

Now substitute $\langle\psi| = \langle 0|\varphi_1(x_1), ..., \varphi_k(x_k)$ and $\Phi(x_1, x_2)$ with $\Phi(x_{k+1}, x_{k+2})$. We conclude that the order of the pole of $\mathfrak{W}(x_1, ..., x_{k+2})$ in $x_{k+1\,k+2}^2$ does not exceed $\mu$ and is hence independent of $k$.

Locality then implies that this holds for any pair of arguments. $\diamond$

This result holds for any space–time dimensions. Following Mack's classification from section 2.3 we can estimate the bounds $\mu$ for the case $D = 4$ [16]:

**Proposition 6.5.**

*Let a system of fields (in* $D = 4$*) satisfy Wightman axioms and GCI. Let* $\varphi(x)$ *and* $\vartheta(x)$ *be two Bose fields in this system which are transformed under unit ray representations of* $SU(2, 2)$ *of weights* $(d'; j_1', j_2')$ *and* $(d''; j_1'', j_2'')$ *respectively. Then the pole degree* $\mu$ *of* $((x - y)^2 + i0(x^0 - y^0))^{-\mu}$ *in any Wightman function* $\langle 0|...\varphi(x)...\vartheta(y)...|0\rangle$ *has the upper limit:*

$$\mu \leq \left[\frac{d' + j_1' + j_2' + d'' + j_1'' + j_2''}{2} - \frac{1 - \delta_{j_1' j_2''}\delta_{j_2' j_1''}\delta_{d' d''}}{2}\right] \tag{2.22}$$

*where $[a]$ stands for the integer part of the real number $a$.*

*Sketch of the Proof:*
To prove the Proposition we perform the following steps:

1. Let $\mu$ be the order of $\varphi(x)\vartheta(y)|0\rangle$, as in theorem 6.4 . Because of the total reducibility of the Hilbert space into irreducible representations of $\widetilde{G}$, we can show that $\mu$ does not exceed the order of $\langle 0|\varphi(x)\vartheta(y)|0\rangle$ or the maximal order of possible 3-point conformally invariant Wightman distributions $\langle 0|\varphi(x)\vartheta(y)\phi^{j_2,j_1;d}(z)\rangle$;

2. We use the expression for the 2-point function from [14]: $\langle 0|\vartheta^*(x)\varphi(y)|0\rangle = \frac{H_{2j_1+2j_2}(x-y)}{((x-y)^2+i0(x^0-y^0))^{d+j_1+j_2}}$, where $H_n(x)$ is a tensor–valued homogeneous harmonic polynomial of degree $n$ that is determined up to a normalization constant from conformal invariance. It can be non-zero only for $(j_1', j_2') = (j_1'', j_2'')$, $d' = d''$;

3. Using the general form of the 3–point function and Lemma 10 from [14] we show that:
$$\mu_{j_1,j_2;d} \leq \frac{1}{2}\left(d' + d'' + L_{j_1 j_2}\right) \tag{2.23}$$
$L_{j_1 j_2}$ is the maximal of the integers $l$ for which the $SL(2,C)$ representation $\left(\frac{l}{2}, \frac{l}{2}\right)$ occurs in the triplet product $(j_1', j_2') \otimes (j_1'', j_2'') \otimes (j_1, j_2)$ (when such $l$ does not exist, the 3–point function must be zero);

4. The maximal weight appearing in the tensor product above is $(j_1' + j_1'' + j_1, j_2' + j_2'' + j_2)$;

5. Using this information we can prove the statement of the theorem. $\diamond$

For the applications very important is the following corollary:

**Corollary 6.6.** *Under the assumptions of the proposition above each truncated Wightman function $\langle 0|...\vartheta^*(x)...\vartheta(y)...|0\rangle^T$ will have a strictly smaller power $\mu$ of the pole in $(x-y)^2 + i0(x^0 - y^0)$ than the 2-point function $\langle 0|\vartheta^*(x)\vartheta(y)|0\rangle$. If the weight of $\vartheta$ is $(j_1, j_2; d)$, then $\mu \leq d + j_1 + j_2 - 1$.*

This will mean that the truncated part of some $n$-point function will have orders of poles strictly smaller than those of the disconnected terms.

* * *

As we will illustrate in our further analysis, the Huygens principle, the rationality and the pole–bounds provide us with very restrictive information about Wightman distributions of Bose fields and often allow us to construct easily 4- and 5-point functions with finite number of parameters.

One can obtain analogous results for Fermi fields.

# Chapter 3

# OPE, PWE and Ward identities

In this chapter we will discuss the three most important tools in our analysis — the operator product expansion (OPE), the partial wave expansion (PWE) and Ward identities.

The operator product expansion is a crucial part of the theoretical structure of conformal field theories and studying the spectrum of this series provides us with new information about the fields, as we will demonstrate in the following three chapters.

Partial wave expansions and Ward identities are powerful methods for 4-point functions analysis, making them the most interesting object for our investigation. The results from the previous chapter help us to obtain explicit expressions for 4-point functions depending on a finite number of parameters. Wightman positivity, addressed via PWE, together with Ward identities impose further restrictions on these parameters, as we will see in sections 3.3 and 3.4.

## 3.1 Operator Product Expansion

One of the big profits we gain from GCI is that it allows the existence and the explicit construction of the operator product expansion of local fields [23]. Throughout this section we will work in D=4 space–time dimensions.

Let us form the following distribution:

$$W(x_1, x_2) = [(x_1 - x_2)^2]^d \varphi(x_1)\varphi(x_2) \tag{3.1}$$

where $\varphi(x)$ is a hermitian scalar field of dimension $d$. Taking into consideration the pole bounds (2.22), we show that $W(x_1, x_2)$ is smooth in the variable $x_{12}$ $(= x_1 - x_2)$ and that it is a Huygens bilocal field[1]:

$$[(x_1 - x)^2(x_2 - x)^2]^N [W(x_1, x_2), \vartheta(x)] = 0 \tag{3.2}$$

Then one can just perform the Taylor expansion of $W(x_1, x_2)$ reaching this way an OPE of $\varphi(x_1)\varphi(x_2)$:

$$W(x_1, x_2) = \sum (x_1^{\mu_1} - x_2^{\mu_1})...(x_1^{\mu_n} - x_2^{\mu_n}) \times X^n_{\mu_1...\mu_n}(x_2) \tag{3.3}$$

Here $X^n_{\mu_1...\mu_n}(x_2)$ will be Huygens local fields.

[1] a field $F(x_1, x_2)$ is called (Huygens) bilocal if it commutes with any local field $\vartheta(x)$ for space–like (and time–like) differences $(x_1 - x_3)$ and $(x_2 - x_3)$

The local fields $X^n_{\mu_1,\ldots,\mu_n}(x)$ are not quasi–primary[2], which means that they do not transform homogeneously under conformal transformations in the desired way

$$g : x \mapsto x', \quad \pi(g) : \varphi(x) \mapsto \varphi'(x') = S(g,x)\varphi(x) \tag{3.4}$$

Here $S(g,x)$ is as in (2.12). If we subtract from them derivatives of fields with lower $n$, we will obtain the quasi–primary fields $O^d_{\mu_1,\ldots,\mu_l}(x)$, which are symmetric and traceless of rank $l$ and dimension $d$. We can reorganize the whole (3.3) series into an expansion of such fields.

We call the difference $2\kappa = (d-l)$ a twist. Mack's classification of the representations of the conformal group from section 2.3 implies that every field $O^d_{\mu_1,\ldots,\mu_l}(x)$ has a non-negative twist and that the only field of twist 0 is the constant field. GCI implies, as we will demonstrate later in explicit constructions, that only even twists appear.

We observe that the twist $2\kappa$ contributions all have singularities $[(x_1-x_2)^2]^{d-\kappa}$. This allows us to collect all of them into a term of the type $(x_{12}^2)^{\kappa-d}V_\kappa(x_1,x_2)$, where $V_\kappa(x_1,x_2)$ is a complicated series in twist $2\kappa+l$ terms and their derivatives:

$$V_\kappa(x_1,x_2) = \sum_{l=0}^{\infty}\sum_{\mu_1\ldots\mu_l} K_{\mu_1\ldots\mu_l}(x_1-x_2,\partial_{x_2})O^{l+2\kappa}_{\mu_1\ldots\mu_l}(x_2) \tag{3.5}$$

$K_{\mu_1\ldots\mu_l}(x_1-x_2,\partial_{x_2})$ are infinite formal power series in $x_1-x_2$ with coefficients that are differential operators in $x_2$ acting on the quasi–primary fields $O^{l+2\kappa}_{\mu_1,\ldots,\mu_l}(x_2)$. They can be fixed universally for any conformal QFT. The explicit form of $K_{\mu_1\ldots\mu_l}(x_1-x_2,\partial_{x_2})$ is given in [5].

We can find all the 4-point functions of $V_\nu(x_1,x_2)$ by using the technique of partial wave expansion, so we conclude that all these series are convergent (in a weak sense on bounded energy states). However, these functions are not rational for $\nu \geq 2$ and we understand that in these cases $V_\nu(x_1,x_2)$ are not Huygens fields. So, we should think of them just as generating series for the fixed twist contribution to the OPE of $\varphi(x)$. In the next section we will see that this is not so in the special case $\nu = 1$.

Then, for a GCI neutral scalar field $\varphi(x)$ of dimension $d$ in a 4-dimensional Minkowski space, we organize systematically OPE in terms of twists, or equivalently, in terms of singularities of the quasy–primary fields [23]:

$$\varphi(x_1)\varphi(x_2) \;=\; C_{\varphi\varphi}(x_{12}^2)^{-d}\left(1+\sum_{\nu=1}^{d-1}(x_{12}^2)^{\nu}V_\nu^{\varphi\varphi}(x_1,x_2)\right) + :\varphi(x_1)\varphi(x_2): \tag{3.6}$$

This expansion can be viewed as a light–cone expansion. $C_{\varphi\varphi}$ parameterizes the 2-point function of $\varphi(x)$ (see (4.8)). The bilocal fields $V_\nu^{\varphi\varphi}(x_1,x_2)$ depend on the scaling dimension of $\varphi(x)$. (As the analysis in this section holds for all fields $\varphi(x)$, we write for simplicity just $V_\mu(x_1,x_2)$)

The fields $V_\nu(x_1x_2)$ and the normal product $:\varphi(x_1)\varphi(x_2):$ are mutually orthogonal [23]:

$$\begin{aligned}\langle 0|V_\nu(x_1,x_2)|0\rangle &= \langle 0|V_\lambda(x_1,x_2)V_\nu(x_3,x_4)|0\rangle = 0 \quad \text{if } \lambda\neq\nu \\ \langle 0|V_\nu(x_1,x_2):\varphi(x_3)\varphi(x_4):|0\rangle &= 0, \quad \lambda,\nu = 1,\ldots,d-1 \end{aligned}\tag{3.7}$$

[2]The term quasi–primary originates from the 2-dimensional case, where the Moebius group $SO(D,2)$ is just a subgroup of the larger conformal group that consists of all diffeomorphisms with a factorial action on the two light–cone variables. There the primary fields are the ones transforming under the whole transforming group and quasi–primary are called all those fields which carry a representation of the Moebius group ([8], [25]).

All these results can be generalized to the case of OPE of two different hermitian fields.

## 3.2 The bilocal field $V_1(x_1, x_2)$

The major difference between the fields from the series $V_\nu(x_1, x_2)$ for $\nu = 1$ and for $\nu \geq 1$ in $D = 4$ is that the former satisfy conservation laws:

$$\partial_{x^{\mu_1}} O^{2+l}_{\mu_1 \dots \mu_l}(x) = 0 \tag{3.8}$$

Every twist 2 field $O^{2+l}_{\mu_1 \dots \mu_l}(x)$ has a conserved 2–point function, which is uniquely fixed by conformal invariance and hence by the Reeh–Schlieder theorem we come to (3.8). The effect of this system of conservation laws on $V_1(x_1, x_2)$ is that it must satisfy the D'Alambert equation in its both arguments:

$$\Box_1 V_1(x_1, x_2) = 0 = \Box_2 V_1(x_1, x_2), \quad \Box_j = \frac{\partial^2}{\partial x_j^\mu \partial x_{j\mu}} \tag{3.9}$$

It can be shown [23] that if any of the following three properties of the harmonic field $V_1(x_1, x_2)$ holds, then it implies the remaining two:

1. $V_1(x_1, x_2)$ is bilocal (in the Huygens sense);
2. The correlation functions $\langle V_1 \varphi_1 ... \varphi_n \rangle$, where $\varphi_i(x)$ are local fields, are conformal rational functions;
3. $\langle V_1 \varphi_1 ... \varphi_n \rangle$ satisfy the "single–pole rule" (Lemma 4.8).

In [23] there is a big progress in proving the bilocality of $V_1(x_1, x_2)$. In our task we shall assume that the bilocality of $V_1(x_1, x_2)$ is established and then the other two properties would be a corrolary of this assumption.

This bilocality of the twist 2 fields has important consequences for the theory and, as we will demonstrate in Model 2, it may give new information about the correlation functions.

It is expected that this statement can be generalized for any even space–time dimension $D$ for the $\kappa = d_0$ series, where $d_0 = \frac{D-2}{2}$. $V^{\varphi\varphi}_{d_0}$ then contains all the conserved tensors $O^{2d_0+l}_{\mu_1 \dots \mu_l}(x) =: T_l(x)$ with $T_2$ being the stress–energy tensor. In particular, the conservation law of the stress–energy tensor is encoded in harmonicity of $V^{\varphi\varphi}_{d_0}(x_1, x_2)$ in both arguments.

## 3.3 Partial Wave Expansion

Another big advantage of GCI is that it enables us to derive partial wave expansion in a closed form for 4-point functions, which allows us to address the issue of Wightman positivity at the 4-point level.

Let us consider functions of the type $\langle \mathcal{ABBA} \rangle$ of the two scalar hermitian fields $\mathcal{A}(x)$ and $\mathcal{B}(x)$ of dimensions respectively $d$ and $d + \delta$:. They can be reorganized into a function of the two conformally invariant variables $s$ and $t$, the so-called cross ratios:

$$\langle 0|\mathcal{A}(x_1)\mathcal{B}(x_2)\mathcal{B}(x_3)\mathcal{A}(x_4)|0\rangle = \frac{(\frac{s}{t})^d f(s,t)}{(x_{12}^2)^d (x_{23}^2)^\delta (x_{34}^2)^d}$$
$$s := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad t := \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{3.10}$$

They admit the following decomposition:

$$\langle 0|\mathcal{A}(x_1)\mathcal{B}(x_2)\mathcal{B}(x_3)\mathcal{A}(x_4)|0\rangle = \sum_{\kappa L} \langle 0|\mathcal{A}(x_1)\mathcal{B}(x_2)\Pi_{\kappa L}\mathcal{B}(x_3)\mathcal{A}(x_4)|0\rangle \tag{3.11}$$

Here $\Pi_{\kappa L}$ is the projection operator onto the irreducible positive energy representation of the conformal group $SU(2,2)$ with label $(2\kappa + L, \frac{1}{2}L, \frac{1}{2}L)$, the representation by a rank $L$ symmetric traceless tensor with scaling dimension $2\kappa + L$. This decomposition is known as the partial wave expansion.

Each partial wave has the form (see [20]):

$$\langle 0|\mathcal{A}(x_1)\mathcal{B}(x_2)\Pi_{\kappa L}\mathcal{B}(x_3)\mathcal{A}(x_4)|0\rangle = \frac{1}{(x_{12}^2)^d (x_{23}^2)^\delta (x_{34}^2)^d} B_{\kappa L} \cdot \beta^\delta_{\kappa L}(s,t) \tag{3.12}$$

Here the functions $\beta^\delta_{\kappa L}(s,t)$ do not depend on the field, but only on the representation of $SU(2,2)$ $(2\kappa + L, \frac{1}{2}L, \frac{1}{2}L)$ and in this sense they are universal. They are known and have the following expression [20]:

$$\beta^\delta_{\kappa L}(s,t) = \frac{uv}{u-v}\left(G^\delta_{\kappa+L-\frac{\delta}{2}}(u) G^\delta_{\kappa-1-\frac{\delta}{2}}(v) - (u \leftrightarrow v)\right) = \frac{uv}{u-v}\gamma^\delta_{\kappa L}(u,v) \tag{3.13}$$
$$\gamma^\delta_{\kappa L}(u,v) := G^\delta_{\kappa+L-\frac{\delta}{2}}(u) G^\delta_{\kappa-1-\frac{\delta}{2}}(v) - (u \leftrightarrow v) \tag{3.14}$$

We have defined $u$ and $v$ the following way:

$$s =: uv, \quad t =: (1-u)(1-v) \tag{3.15}$$

$u$ and $v$ remind very much the two chiral variables, the light–cone variables from the 2-dimensional situation [8], [25].

$G^\delta_n(z)$ are expressed in terms of hypergeometric functions:

$$G^\delta_n(z) := z^n \cdot F(n, n, 2n+\delta; z) \tag{3.16}$$

We find the coefficients $B_{\kappa L}$ for a certain 4-point function of the type above of two hermitian scalars according to the following prescription. First, we substitute (3.15) in (3.10) and organize the obtained expression in the form:

$$\frac{1}{(x_{12}^2)^d (x_{23}^2)^\delta (x_{34}^2)^d}\left[\left(\text{powers of } u \text{ or } \frac{u}{1-u}\right) \cdot \left(\text{powers of } v \text{ or } \frac{v}{1-v}\right) - (u \leftrightarrow v)\right] \tag{3.17}$$

Next, we use the expansion formulae [20]:

$$\begin{aligned}
z^p &= \sum_{\nu\in p+\mathbb{N}_0} \frac{(-1)^{\nu-p}}{(\nu-p)!}\frac{(p+\alpha)_{\nu-p}(p+\beta)_{\nu-p}}{(\nu+p+\gamma-1)_{\nu-p}}\cdot z^\nu F(\nu+\alpha,\nu+\beta;2\nu+\gamma;z),\\
\left(\frac{z}{1-z}\right)^p &= (1-z)^\alpha \sum_{\nu\in p+\mathbb{N}_0} \frac{(-1)^{\nu-p}}{(\nu-p)!}\frac{(p+\alpha)_{\nu-p}(p+\gamma-\beta)_{\nu-p}}{(\nu+p+\gamma-1)_{\nu-p}}\cdot z^\nu F(\nu+\alpha,\nu+\beta;2\nu+\gamma;z)\\
(a)_n: &= \frac{(a+n-1)!}{(a-1)!}
\end{aligned}\tag{3.18}$$

which are valid for $2p+\gamma>0$. Setting $\alpha=\beta=0,\ \gamma=\delta$, we organize (3.17) into an expansion of the form:

$$\frac{1}{(x_{12}^2)^d(x_{23}^2)^\delta(x_{34}^2)^d}\sum_{\mu\nu}X_{\mu\nu}\left(G_\mu^\delta(u)G_\nu^\delta(v)-(u\leftrightarrow v)\right)\tag{3.19}$$

Then we relabel $\mu=\kappa+L-\frac{\delta}{2}$ and $\nu=\kappa-1-\frac{\delta}{2}$ if $\mu>\nu$ and $\nu=\kappa+L-\frac{\delta}{2}$, and $\mu=\kappa-1-\frac{\delta}{2}$ if $\mu<\nu$. We obtain the partial wave coefficients:

$$B_{\kappa L}=X_{\kappa+L-\frac{\delta}{2},\kappa-1-\frac{\delta}{2}}-X_{\kappa-1-\frac{\delta}{2},\kappa+L-\frac{\delta}{2}}\quad(2\kappa>\delta).\tag{3.20}$$

If $\mathcal{A}(x)$ coincides with $\mathcal{B}(x)$ in (3.10), then we will have the additional symmetry under the exchange of $x_1$ and $x_2$. It can be easily shown that this leads to vanishing of all coefficients from (3.20) with odd $L$, which is related to the fact that OPE of two copies of the same field contains only terms of even rank.

Analogously, we obtain the partial wave expansion of functions of the type $\langle\mathcal{AABB}\rangle$:

$$\langle 0|\mathcal{A}(x_1)\mathcal{A}(x_2)\mathcal{B}(x_3)\mathcal{B}(x_4)|0\rangle=\langle 0|\mathcal{A}(x_1)\mathcal{A}(x_2)|0\rangle\langle 0|\mathcal{B}(x_3)\mathcal{B}(x_4)|0\rangle\sum_{\kappa L}B_{\kappa L}\cdot\beta_{\kappa L}(s,t)\tag{3.21}$$

where $\beta_{\kappa L}(s,t)$ coincides with $\beta^\delta_{\kappa L}(s,t)$ for $\delta=0$.

Now we are ready for Wightman positivity analysis. We use the method of [20], where Wightman positivity is shown to be equivalent to the non-negativity of all the coefficients in the PWE. These coefficients are linear combinations of the parameters of the 4-point function, so we expect to obtain some restrictions about them. In short, this equivalence follows from several observations. First, Hilbert space positivity implies that each partial wave from the decomposition (as in (3.12)) should satisfy separately Wightman positivity. Studying the PWE in the case of Wick's products of massless scalar free fields, we find that each universal function $\beta_{kL}(s,t)$ appears with a positive coefficient and we conclude that $\beta_{kL}(s,t)$ is positive itself. This implies that in the partial wave expansion of every other field all the coefficients should be non-negative, as well. Hence, partial wave expansions provide us with a method to exploit Wightman positivity at the 4-point level.

## 3.4 Ward Identities

In this section we will discuss Ward identities, arising in a theory with a unique field with the quantum numbers of the stress–energy tensor (SET) and allowing to relate some of the parameters of a function of the type $\langle\mathcal{AABB}\rangle$ to the central charge from the 4-point function of SET.

As it is proposed in [12], in the axiomatic approach to conformal field theory the existence of a stress–energy tensor $T_{ij}(x)$ should be required. (This postulate would exclude the presence of generalized free fields, as we will demonstrate in our first model in chapter 4.) In analogy to the Poincare symmetry case, this tensor is used in the construction of conserved currents $\tau_\mu(x)$ (where $\mu$ is a multi–index) whose integrals generate infinitesimal coordinate symmetries via commutators with fields:

$$\int d\sigma^\mu(y)[\varphi(x), \tau_\mu(y)] = iG\varphi(x) \tag{3.22}$$

Here $G$ is the generator of infinitesimal coordinate symmetry.

The properties of the conformally invariant stress–energy tensor are as follows:

- *conserved*: by definition;
- *symmetric*: in general the canonical stress–energy tensor is not symmetric. However, we can modify it into a symmetric one by adding a divergence in the first index of a tensor $B^{\mu\nu\rho}(x)$ antisymmetric in its first two indices. Such an addition will not affect its integrals (3.22);
- *traceless*: it follows from the condition of time invariance of the dilation generator;
- *rank 2, scaling dimension 4*: Wilson [29] has shown that the dimension of the stress–energy tensor must necessarily coincide with the space–time dimension $D$.

Let us assume now, that there is a unique field with the properties of $T_{ij}(x)$ in the theory. It will be convenient to work with $T(x,v) := T_{ij}(x)v^iv^j$, where $v^i$ is some constant vector. According to [20], it should appear in the expansion of the operator product of any hermitian scalar field $\varphi(x)$ with itself. We isolate it the following way [20]:

$$T(x,v) = \gamma^\varphi \cdot \mathcal{D}_{12}(v)V_1^{\varphi\varphi}(x_1,x_2)|_{x_1=x_2=x} \tag{3.23}$$

The differential operator $\mathcal{D}_{12}(v)$ should be of rank 2 and should produce a symmetric, traceless and conserved tensor field. It is uniquely determined by these conditions:

$$\mathcal{D}_{12}(v) := \frac{1}{6}\Big\{(v\cdot\partial_{x_1})^2 + (v\cdot\partial_{x_2})^2 - 4(v\cdot\partial_{x_1})(v\cdot\partial_{x_2}) + v^2(\partial_{x_1}\cdot\partial_{x_2})\Big\} \tag{3.24}$$

The coefficient of proportionality $\gamma^\varphi$ is related to the parameters of $\langle 0|V_1^{\vartheta\vartheta}(x_1,x_2)V_1^{\varphi\varphi}(x_3,x_4)|0\rangle$, as we will verify in the following calculations.

The 4-point function of $T(x,v)$ in any theory with two copies of the hermitian scalar field $\varphi(x)$ is determined up to a factor just by conformal invariance with normalization fixed by the condition (3.22) [20]:

$$\langle 0|\varphi(x_1)T(x_2,v)\varphi(x_3))|0\rangle = d_\varphi\langle 0|\varphi(z_1)\varphi(x_3)|0\rangle x_{13}^2\omega_3^{(1)}(x_1;x_2,v;x_3) \tag{3.25}$$

with

$$\omega_3^{(1)}(x_1;x_2,v;x_3) = \frac{2}{3x_{12}^2x_{23}^2}\left\{\left(2X_{13}^2.v\right)^2 - v^2\frac{x_{13}^2}{x_{12}^2x_{23}^2}\right\}, \quad X_{13}^2 := \frac{x_{23}}{x_{23}^2} + \frac{x_{12}}{x_{12}^2} \tag{3.26}$$

This relation is referred to as Ward identity. Inserting (3.6) into (3.25), we obtain:

$$\langle 0|V_1^{\vartheta\vartheta}(x_1,x_2)T(x_2,v))|0\rangle = d_\vartheta\omega_3^{(1)}(x_1;x_2;x_3,v) \tag{3.27}$$

Its left hand side is also obtained by applying the operator $\gamma^{\varphi}\mathcal{D}_{34}(v)$ to $\langle 0|V_1^{\vartheta\vartheta}(x_1,x_2)V_1^{\varphi\varphi}(x_3,x_4)|0\rangle$, taking into account (3.23). Then, comparing the result after this manipulation with (3.27), we can find the coefficient $\gamma^{\varphi}$, expressed in terms of the parameters of $\langle 0|V_1^{\vartheta\vartheta}(x_1,x_2)V_1^{\varphi\varphi}(x_3,x_4)|0\rangle$.

The 2-point function of $T(x,v)$ is also determined just by conformal symmetry [20]:

$$\langle 0|T(x_1,v_1)T(x_2,v_2)|0\rangle = c\cdot\omega_2^{(1)}(x_1,v_1;x_2,v_2) \tag{3.28}$$

with

$$\omega_2^{(1)}(x_1,v_1;x_2,v_2) = \frac{4}{3(x_{12}^2)^4}\Big\{(2v_1\cdot r(x_{12})\cdot v_2)^2 - v_1^2v_2^2\Big\}, \quad r(x) := 1 - 2\frac{x\otimes x}{x^2} \tag{3.29}$$

The constant $c$ can be viewed as an analogue to the Virasoro central charge, which labels the representations of the stress–energy tensor in 2-dimensional CFT [8], [25].

Applying $\mathcal{D}_{12}(v)$ to (3.27) and taking into consideration (3.28), we find the relation between $c$ and $\gamma^{\varphi}$:

$$c = d_{\varphi}\cdot\gamma^{\varphi} \tag{3.30}$$

In particular, combining (3.30) with the previous relation between $\gamma^{\varphi}$ and the parameters of the 4-point functions, we relate them to the central charge $c$.

# Chapter 4

# Model 1

## 4.1 The Setting

In this chapter we will consider a globally conformal invariant quantum field theory satisfying Wightman axioms in 4D Minkowski space, in which there is no field with the quantum numbers of the stress–energy tensor (SET), i.e. a field of rank 2 and scaling dimension 4. We will assume that the theory is generated by a 3-dimensional hermitian scalar field[1] and will denote it with $W(x)$. We will prove that under the conditions above $W(x)$ is a generalized free field according to the definition in section 4.2. This definition is shown to be equivalent to the 4-point function of $W(x)$ consisting only of disconnected terms.

Our strategy will be as follows. First, we will explore the 4-point function of a 3-dimensional field in a general GCI Wightman theory in 4D Minkowski space. We will show that its properties in such a theory — rationality, the pole–rule, Huygens principle and the transformation properties — fix the form of its connected term up to three free parameters, the values of which are further restricted by Wightman positivity. Then we will proceed to show that in a theory without SET all these parameters vanish and hence $W(x)$ is a generalized free field. Studying only the 4-point function of $W(x)$ will not be sufficient for our goal as it allows to exclude only two of the three parameters — $a$ and $b$. Therefore, we involve in our analysis also the mixed 4-point function of $W(x)$ with the 4-dimensional scalar field $\mathcal{L}(x)$ which is present in the OPE of $W \cdot W$ if $W(x)$ is not a generalized free field: $\langle \mathcal{L}WW\mathcal{L} \rangle$ [2]. One of its free parameters, which we will denote by $D$, is related to the last surviving parameter $c$ of $\langle WWWW \rangle$ via a Cauchy–Schwartz inequality, which comes from Hilbert space positivity. Furthermore, we move to the 5-point level, exploring the mixed function $\langle \mathcal{L}WWWW \rangle$. Its limit at $x_5 \to x_4 \to 0$ after a multiplication with $x_{45}^2$ produces the mixed 4-point function $\langle \mathcal{L}WW\mathcal{L} \rangle$ and some relations between the parameters of those functions are found. These relations together with the operator product content and Wightman positivity analysis finally lead to the vanishing of the parameter $D$ of $\langle \mathcal{L}WW\mathcal{L} \rangle_T$ and this in turn leads to the vanishing of the last surviving parameter $c$ of $\langle WWWW \rangle_T$.

This simple model shows that going beyond the 4-point level can give decisive new information.

[1] we will call fields of scaling dimension $d$ (as defined in section 2.3) $d$-dimensional

[2] with $\langle ABCD \rangle$ we will denote shortly the function $\langle 0|A(x_1)B(x_2)C(x_3)D(x_4)|0\rangle$ when it will not cause ambiguities

## 4.2 Generalized free fields

Let us remind that a generalized free field $\varphi(x)$ is a field, whose commutator is proportional to the identity operator:

$$[\varphi(x_1), \varphi(x_2)] = c1 \tag{4.1}$$

Due to Hilbert space positivity and Reeh–Schlieder theorem this definition can be straightforward shown to be equivalent to the requirement:

$$\langle 0|\varphi(x_1)\varphi(x_2)...\varphi(x_{n-1})\varphi(x_n)|0\rangle = \sum \Pi \langle 0|\varphi(x_i)\varphi(x_j)|0\rangle \tag{4.2}$$

i.e. every $n$-point function of $\varphi(x)$ can be presented as a sum of products of 2-point functions of $\varphi(x)$. This indicates the lack of interactions and justifies the term "generalized free field".

In fact, the validity of (4.2) for $n = 4$ already implies (4.1) [26]:

**Lemma 2.7.** *If $\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle_T = 0$, then $\varphi(x)$ is a generalized free field.*

*Proof:*
The assumption implies:

$$\langle \varphi_1\varphi_2\varphi_3\varphi_4\rangle = \langle \varphi_1\varphi_2\rangle\langle \varphi_3\varphi_4\rangle + \langle \varphi_1\varphi_3\rangle\langle \varphi_2\varphi_4\rangle + \langle \varphi_1\varphi_4\rangle\langle \varphi_2\varphi_3\rangle \tag{4.3}$$

Here we have denoted $\varphi_i := \varphi(x_i)$. From (4.3) one obtains:

$$\langle ([\varphi_1, \varphi_2] - \langle [\varphi_1, \varphi_2]\rangle) ([\varphi_3, \varphi_4] - \langle [\varphi_3, \varphi_4]\rangle)\rangle = 0 \tag{4.4}$$

Then, using Reeh–Schlieder theorem, we arrive at:

$$[\varphi_1, \varphi_2] = \langle [\varphi_1, \varphi_2]\rangle \tag{4.5}$$

which means that $\varphi(x)$ is a generalized free field.

## 4.3 The 4-point function of a GCI 3-dimensional Wightman field $W(x)$

In section 2.1 we observed that due to its cluster decomposition property (2.7) any $n$-point function in a Wightman field theory has the form (2.8) and the 1-point function of a conformally invariant field is 0. Therefore, the 4-point function of the 3-dimensional field $W(x)$ has the following expression:

$$\langle 1234\rangle = \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle + \langle 1234\rangle_T \tag{4.6}$$

Here and for further use in this chapter we adopt the notation:

$$\langle 1...n\rangle := \langle 0|W(x_1)...W(x_n)|0\rangle \tag{4.7}$$

From [7] we know that the 2-point function of the GCI scalar fields $\varphi(x)$ and $\vartheta(x)$ is nonzero only for coinciding scaling dimensions $d_\varphi$ and $d_\vartheta$ and has the form:

$$\langle 0|\varphi(x_i)\vartheta(x_j)|0\rangle = \frac{C_{\varphi\vartheta}}{(x_{ij}^2)^{d_\varphi}}, \quad x_{ij} := x_i - x_j \tag{4.8}$$

Then we find for $\langle ij \rangle$:

$$\langle ij \rangle = \frac{C_{WW}}{(x_{ij}^2)^3} \tag{4.9}$$

In a GCI theory, according to the last three sections of chapter 2, the connected term $\langle 1234 \rangle_T$ has the following properties:

- *rationality:* $\langle 1234 \rangle_T$ is a rational function of the type

$$\langle 1234 \rangle_T = \sum A_m \Pi (x_{ij}^2)^{m_{ij}} \tag{4.10}$$

  where $m$ is a multi–index (Theorem 5.3);

- *restriction on poles:* Corollary 6.6 tells us that the orders of the poles of $\langle 1234 \rangle_T$ are strictly smaller than those of the disconnected part, which means that $m_{ij} > -3 \; \forall i \neq j$.

- *permutation symmetry:* $\langle 1234 \rangle_T$ is invariant under all permutations of indices of the variables (a corollary of Huygens principle from section 2.4);

There is one more limitation coming from the transformation properties of $\langle 1234 \rangle_T$.

- *conformal invariance:* if we keep all the points fixed except $x_i \to \infty$, then each term in $\langle 1234 \rangle$ behaves as proportional to $|x_i|^{-2d}$. It means that for every term in (4.10) the following equations $\forall\, i$ must be valid: $\sum_k m_{ik} = -d_i$.

By simple arithmetics we find that all the structures satisfying these conditions are:

$$\frac{x_{ij}^2 x_{kl}^2}{(x_{ik}^2)^2 (x_{jl}^2)^2 (x_{il}^2)^2 (x_{jk}^2)^2}, \quad \frac{1}{x_{ik}^2 x_{jl}^2 (x_{il}^2)^2 (x_{jk}^2)^2}, \quad \frac{1}{x_{ij}^2 x_{kl}^2 x_{ik}^2 x_{jl}^2 x_{il}^2 x_{jk}^2} \tag{4.11}$$

Here $(i, j, k, l)$ are all the permutations of $(1, 2, 3, 4)$. All the permutations of every of the three structures above appear in (4.10) with the same coefficient because of the permutation symmetry.

Then, we write the truncated 4-point function of $W(x)$ in the following compact and convenient for the partial wave analysis form. Here $s$ and $t$ are the same as defined in section 3.3:

$$\langle 1234 \rangle_T = \langle 12 \rangle \langle 34 \rangle \left[ a \left( \frac{s^4}{t^2} + \frac{s}{t^2} + st \right) + b \left( \frac{s^3}{t} + \frac{s^3}{t^2} + \frac{s^2}{t^2} + \frac{s}{t} + s^2 + s \right) + c \frac{s^2}{t} \right] \tag{4.12}$$

The full 4-point function will be:

$$\begin{aligned} \langle 1234 \rangle = \langle 12 \rangle \langle 34 \rangle & \left[ 1 + s^d + \left( \frac{s}{t} \right)^d + a \left( \frac{s^4}{t^2} + \frac{s}{t^2} + st \right) + \right. \\ & \left. + b \left( \frac{s^3}{t} + \frac{s^3}{t^2} + \frac{s^2}{t^2} + \frac{s}{t} + s^2 + s \right) + c \frac{s^2}{t} \right] \end{aligned} \tag{4.13}$$

We have calculated the following two examples as realizations of (4.13).

Let us consider the 4-point function of the field $A^3(x)$, where $A(x)$ is a free scalar field. The canonical scaling dimension of free scalar fields is 1, hence their cubes will have scaling dimension

3. The 4-point function of the normal–ordered cube can be calculated using Wick's theorem and is found to be a sum of products of 2-point functions of $A$, which are (see (4.8)) :

$$\langle 0|A(x)A(y)|0\rangle = \frac{C_{AA}}{(x-y)^2} \tag{4.14}$$

It is straightforward shown that the cube coincides with the normal-ordered cube. Then we obtain:

$$\begin{aligned}\langle 0|A^3(x_1)A^3(x_2)A^3(x_3)A^3(x_4)|0\rangle \sim & \frac{1}{(x_{12}^2)^3(x_{34}^2)^3}\left(1+s^3+\frac{s^3}{t^3}+\right.\\ & \left.+9\left(\frac{s^3}{t}+\frac{s^3}{t^2}+\frac{s^2}{t^2}+\frac{s}{t}+s^2+s\right)+36\frac{s^2}{t}\right)\end{aligned} \tag{4.15}$$

We can give a second example with the field $:\overline{\psi}\psi:(x)$ which we construct with the free fermion field $\psi(x)$. Free Dirac fields have canonical scaling dimension $\frac{3}{2}$, thus $:\overline{\psi}\psi:(x)$ is 3-dimensional. As in the previous example, using Wick's theorem, we obtain a sum of products of 2-point functions of the type $\langle 0|\psi^a(x)\overline{\psi}_b(y)|0\rangle$. These functions are related to the 2-point functions of the canonical free scalar $A(x)$ as follows from [28] (in our case $m=0$):

$$\langle 0|\psi^a(x)\overline{\psi}_b(y)|0\rangle = -\left(\gamma^\lambda\frac{\partial}{\partial x^\lambda}-im\right)^a_b(\langle 0|A(x)A(y)|0\rangle) \sim \frac{\left(\gamma^\lambda\right)^a_b(x-y)_\lambda}{(x-y)^4} \tag{4.16}$$

Then, using the standard result for traces of $\gamma$-matrices [24]:

$$\frac{1}{4}tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau\right) = \delta_{\mu\nu}\delta_{\rho\tau}-\delta_{\mu\rho}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\rho} \tag{4.17}$$

we arrive at the following relation:

$$\begin{aligned}\langle 0|:\overline{\psi}\psi:(x_1):\overline{\psi}\psi:(x_2):\overline{\psi}\psi:(x_3):\overline{\psi}\psi:(x_4)|0\rangle \sim & \frac{1}{(x_{12}^2)^3(x_{34}^2)^3}\left[1+s^3+\frac{s^3}{t^3}+\right.\\ & \left.+\frac{1}{4}\left(\left(\frac{s}{t^2}+st+\frac{s^4}{t^2}\right)-\left(\frac{s^3}{t}+\frac{s^3}{t^2}+\frac{s^2}{t^2}+\frac{s}{t}+s^2+s\right)\right)\right]\end{aligned} \tag{4.18}$$

## 4.4 Wightman positivity of $\langle WWWW\rangle$ and absence of SET

As we are working within a Wightman theory, $\langle 0|W(x_1)W(x_2)W(x_3)W(x_4)|0\rangle$ must satisfy also Wightman positivity (2.6). In section 3.3 it is argued that every GCI 4-point function of the type $\langle 0|A(x_1)B(x_2)B(x_3)A(x_4)|0\rangle$ admits a partial wave expansion (PWE) as in (3.11) and that its Wightman positivity property is equivalent to all the coefficients $B_{\kappa L}$ in this expansion being non-negative. As it is clear from the definition of PWE these coefficients will be linear combinations of the free parameters $a$, $b$ and $c$ of the 4-point function and we expect to obtain some bounds on their values. We have performed the PWE of the 4-point function of $W(x)$ according to the prescription from the same section, and the results for the coefficients $B_{\kappa L}$ are listed in Appendix A.1 together with the Wightman positivity analysis. The general conclusion is that there is a region in the 3-dimensional space of the parameters $a$, $b$ and $c$ in which Wightman positivity is satisfied. This region is restricted by the inequalities

$$a \geq 0, \quad a+b \geq 0 \tag{4.19}$$

coming from $B_{\kappa L} \geq 0$ for $\kappa = 1$ and $L \to \infty$, $L = 0$ respectively, and by an infinite system of inequalities of the type:

$$c \geq -(Xa + Yb + Z), \quad c \leq -Ua + Vb + W. \tag{4.20}$$

which come for higher twists s.t. $\kappa$ takes only even values in the first and only odd values in the second case. All the parameters in the last two inequalities are positive functions of $\kappa$.

Next, we demand that in our theory there is no field with the quantum numbers of SET, i.e. $d = 4$ and $L = 2$, hence there is no such field in the OPE of $W \cdot W$. Let us recall now (section 3.3) that every partial wave from the PWE is of the form

$$\langle 12\Pi_{\kappa L}34\rangle = \langle 12\rangle\langle 34\rangle B_{\kappa L}\beta_{\kappa L}(s,t), \quad \beta_{\kappa L}(s,t) > 0$$

where $\Pi_{\kappa L}$ projects onto the dimension $d = 2\kappa + L$ rank $L$ tensor in $W(x_3)W(x_4)|0\rangle$. Then the absence of a field with these quantum numbers in the OPE of $W \cdot W$ will result in vanishing of the corresponding partial wave. Therefore, the absence of a 4-dimensional rank 2 field implies that the coefficient $B_{12}$ should vanish.

If there is no SET in the theory, it means that there is no 2-dimensional scalar too, because this field is a Wick square of massless scalar fields [17] and hence its OPE with itself would produce a SET.

**New information:** *The absence of SET implies that there is no 2-dimensional scalar in the theory.*

Then such a field is absent from the OPE of $W \cdot W$ and hence the coefficient $B_{10}$ should vanish as well.

We use the results from (A.1):

$$B_{12} = \frac{1}{3}(7a + b), \quad B_{10} = 2(a + b) \tag{4.21}$$

Putting these coefficients equal to zero leads to $a = 0$ and $b = 0$.

**New information:** *The absence of SET and a 2-dimensional scalar imply that the parameters $a = 0$ and $b = 0$.*

Now we will show that the vanishing of these two coefficients results in absence of any twist 2 structures in PWE of $\langle WWWW\rangle$. This result will have large application in our further analysis.

By simple arguments we can prove that twist 2 contributions can be produced by and only by terms of (4.13) which contain $s$ to the power 1. For this purpose, observe that the universal functions corresponding to partial waves of twist 2 are $\beta_{1L}(u,v) = \frac{uv}{u-v}\left(G_{1+L}(u)G_0(v) - (u \leftrightarrow v)\right)$, $G_n(x)$ are defined in section 3.3, $G_0(x) = 1$. Then, if we convert (4.13) into a function of the variables $u$ and $v$ using (3.15) and multiply it with $\frac{u-v}{uv}$, the structures in it which have twist two contributions should be proportional to $(G_{1+L}(u) - G_{1+L}(v))$. Taking into consideration the expansions (3.18) of $u$, $v$, $\hat{u} := \frac{u}{u-1}$ and $\hat{v} := \frac{v}{v-1}$ in terms of $G_n(u)$ and $G_n(v)$ we conclude that such structures can be produced by and only by mono-variable functions of $u$, $v$, $\hat{u}$ and $\hat{v}$. It is easy to show that such functions are given by and only by terms of (4.13) in which there is $s$ to the power 1.

If we let $a = b = 0$, then all such terms will vanish, hence the absence of SET leads to the absence of the whole twist two series in the PWE of $\langle 0|W(x_1)W(x_2)W(x_3)W(x_4)|0\rangle$, i.e. $B_{1L} = 0$, $\forall L$. This, in turn, implies the absence of the twist 2 series in the OPE of $W \cdot W$. We can verify this statement taking into consideration (3.7), i.e. that structures of different twists in the OPE are orthogonal to each other, in combination with the Reeh–Schlieder theorem.

**New information:** *$a = b = 0$ imply the absence of the whole twist 2 series in the OPE of $W \cdot W$.*

Up to now we managed to eliminate only two of the three coefficients — $a$ and $b$. Wightman positivity puts some limits on the possible values of $c$, but still not sufficient to exclude it. These bounds are given by the coefficients $B_{20} = c$ and $B_{30} = \frac{1}{6}(12 - c)$ being non-negative, which is equal to $0 \leq c \leq 12$.

**New information:** *Wightman positivity restricts $c$ to the region $0 \leq c \leq 12$.*

Suppose that the parameter $c$ is non-zero. Then the PWE coefficient $B_{20}$ will be non-zero, as we can see above and this indicates the presence of a 4-dimensional scalar field $\mathcal{L}$ in the OPE of $W \cdot W$. We isolate it the following way:

$$\mathcal{L}(x) := x_{12}^2 \left[W(x_1)W(x_2) - \langle W(x_1)W(x_2)\rangle\right]|_{x_1=x_2=x} \tag{4.22}$$

To summarize, Wightman positivity analysis requires that the parameters $a$ and $b$ of $\langle WWWW\rangle$ vanish, implying the absence of the whole twist 2 series in the OPE of $W \cdot W$. Furthermore, it restricts $c$ to the region $[0, 12]$.

So, the $\langle WWWW\rangle$ analysis itself is not sufficient to prove that $W(x)$ is generalized free field. Therefore, we involve in our analysis also the mixed 4-point function $\langle W\mathcal{L}\mathcal{L}W\rangle$

## 4.5 The mixed 4-point function $\langle W\mathcal{L}\mathcal{L}W\rangle$

In this section we construct the mixed 4-point function $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle$ in analogy to how we proceeded in section 4.3, exploiting the requirement for GCI together with Wightman axioms. We have again the expression in terms of truncated functions:

$$\begin{aligned}\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle &= \langle 0|W(x_1)W(x_4)|0\rangle\langle 0|\mathcal{L}(x_2)\mathcal{L}(x_3)|0\rangle + \\ &+\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle_T \end{aligned}\tag{4.23}$$

Here the disconnected part consists only of $\langle 0|W(x_1)W(x_4)|0\rangle\langle 0|\mathcal{L}(x_2)\mathcal{L}(x_3)|0\rangle$ because any mixed 2-point functions between $W(x)$ and $\mathcal{L}(x)$ vanish due to their different scaling dimensions, as we know from section 4.3.

$\langle W\mathcal{L}\mathcal{L}W\rangle_T$ has to obey similar requirements to those for $\langle WWWW\rangle_T$, however slightly modified because of the different scaling dimensions of $\mathcal{L}(x)$ and $W(x)$:

- *rationality:*
$$\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle_T = \sum A_m \Pi(x_{ij}^2)^{m_{ij}} \tag{4.24}$$
- *restriction on poles:* $m_{ij}$ in (4.24) should be greater than -3 for $\{i, j\} = \{1, 4\}$ and greater than -4 otherwise;

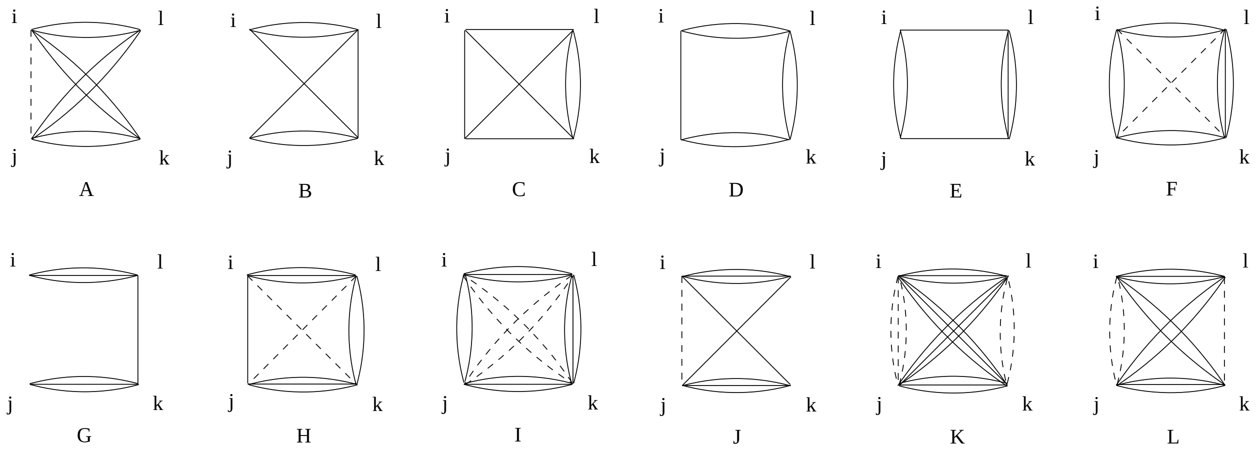


Figure 4.1: Terms of $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle_T$

- *permutation symmetry:* $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle_T$ is invariant under permutations of indices $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ of the variables;

- *conformal invariance:* if we keep all the points fixed except $x_i \to \infty$ then each term in $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle$ behaves as proportional to $|x_i|^{-2d_W}$ if $i = 1,4$ and $|x_i|^{-2d_\mathcal{L}}$ if $i = 2,3$. It follows that for every term in (4.10) the following equations will be valid: $\sum_k m_{ik} = -3$ if $i = 1,4$ and $\sum_k m_{ik} = -4$ if $i = 2,3$.

These would be the restrictions on the 4-point function $\langle 0|W(x_1)W(x_2)\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle_T$ as well after a proper rearrangement of the indices.

It will be very convenient for current use and for further applications and mathematical manipulations to present all these structures which satisfy the conditions above in terms of graphs and to work with them. The correspondence is as follows:

1. to every term of $\langle W\mathcal{L}\mathcal{L}W\rangle_T$ of the form $\Pi(x_{ij}^2)^{m_{ij}}$ corresponds a graph;

2. every graph has 4 vertices, to each of the 4 points $x_r$ in which $\langle W\mathcal{L}\mathcal{L}W\rangle_T$ is taken corresponds a vertex of the graph, and we label it by $r$;

3. the negative degree of $x_{rs}^2$ in a certain term is counted for by the number of normal edges connecting the vertices $r$ and $s$ of the corresponding graph;

4. the positive degree of $x_{tp}^2$ in a certain term is counted for by the number of dashed edges connecting the vertices $t$ and $p$ of the corresponding graph;

5. all permutations of the same graph that are in agreement with the conditions above should appear in $\langle W\mathcal{L}\mathcal{L}W\rangle_T$ with the same coefficient.

The graphs that stand for possible terms of $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle_T$ are restricted from the pole-bounds and conformal invariance to have the following structure: the total number of edges coming out of a certain vertex should be three for the vertices 1 and 4 and four for the

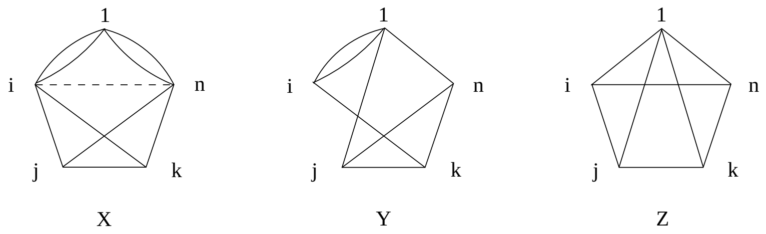


Figure 4.2: Terms of $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$

vertices 2 and 3; moreover, the number of edges connecting any two vertices must not exceed three and there can be at most two edges between the vertices 1 and 4.

All the possible such graphs are presented on figure 4.1. Here $\{i, j\}$ take values $\{1, 4\}$ and $\{k, l\}$ take values $\{2, 3\}$.

## 4.6 The 5-point function $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$

To reduce further the list of possible structures, we will make use of the fact that the 4-point function $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)\mathcal{L}(x_4)|0\rangle$ is related to the 5-point function $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle$. Namely, the definition of $\mathcal{L}(x)$ (4.22) implies directly the following connection:

$$\begin{aligned} \lim_{x_5\to x_4} x_{45}^2 \langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)\left[W(x_4)W(x_5) - \langle 0|W(x_4)W(x_5)|0\rangle\right]|0\rangle = \\ = \langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)\mathcal{L}(x_4)|0\rangle \end{aligned} \tag{4.25}$$

Then it is immediately seen that the truncated mixed 4-point function will be reproduced by the truncated 5-point function at the limit $x_5 \to x_4$ after a multiplication by $x_{45}^2$.

Following the strategy used before, we show that its GCI properties together with Wightman axioms require that $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$ is a rational function of the type $\sum A_m \Pi(x_{ij}^2)^{m_{ij}}$, where $m_{ij}$ is greater than -4 if any of $i$ or $j$ is equal to 1 and greater than -3 otherwise. Moreover, $\sum_k m_{ik} = -4$ for $i = 1$ and $\sum_k m_{ik} = -3$ for $i = 2, 3, 4, 5$. Structures, which can be transformed into each other by a permutation of the indices $(2, 3, 4, 5)$, appear in $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$ with the same coefficient.

We will use the same graph representation as explained in the previous section to display the list of possible structures. All graphs with 5 vertices, for which there are 4 edges coming out of vertex 1 and 3 edges coming out of every other vertex s.t. there are at most three edges connecting any two vertices and if none of these vertices is 1 the edges are not more than 2, are candidates for terms of $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$.

Let us remind the result from section 4.4 that the absence of SET in the theory leads to the absence of the whole twist two series in the OPE of $W \cdot W$, which will help us to reduce drastically the list of potential terms of $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$. Using (3.6) we show that

$$\lim_{x_{12}^2\to 0}(x_{12}^2)^2\left(W(x_1)W(x_2) - \langle W(x_1)W(x_2)\rangle\right) = 0 \tag{4.26}$$

if there is no twist two contribution, as we have subtracted the singular part and as all the higher twist contributions will appear with some positive power of $x_{12}^2$ which is zero at the limit. As a consequence, we find:

$$\lim_{x_{45}^2\to 0}(x_{45}^2)^2\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T = 0 \tag{4.27}$$

In terms of graphs, to multiply a graph with $(x_{45}^2)^2$ would mean to subtract two edges connecting its vertices 4 and 5, a "negative" edge (or a factor $x_{45}^2$ in the numerator) will be counted for by a dashed edge. If the number of these edges in the initial structure is smaller or equal to 1, after the multiplication we will have a factor $(x_{45}^2)^n$ in the numerator and the limit $x_{45}^2 \to 0$ will "kill" the structure.

If there is a double edge between any two of the vertices $i = 2,3,4,5$, then the permutation of the corresponding structure in which this edge connects the points $x_4$ and $x_5$ will "survive" through the limit (4.27). However, the sum of all such surviving structures should vanish according to the same formula. There could arise no cancelation between any two of these structures. An argument for this statement will be that the 10 intervals of the 5 points in the 4D Minkowski space are algebraically independent variables and if $x_{45}^2 \to 0$ vanishes, it would not affect the free choice of the other 9 variables. Then all these structures will remain different from each other after the limit and their sum is zero iff all of them are separately zero.

Hence, any terms of $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T$ corresponding to graphs with at least one double edge between a couple of vertices, none of which is 1, must vanish. This, together with the pole-bound which does not allow existence of structures with more than two edges between any two vertices $i$ and $j$ for $i, j = 2...5$, implies that the maximal number of edges between such vertices is 1.

There are three structures obeying all necessary conditions and they are shown in figure 4.2, where $(i, j, k, l)$ are allowed to be all the permutations of (2,3,4,5).

Next, we observe that if we perform the transition limit of all these permutations to terms of $\langle\mathcal{L}WW\mathcal{L}\rangle$ as in (4.25) only structures from figure 4.1 with coefficients $A$, $B$, $C$, and $D$ are reproduced as demonstrated in figure 4.3, the arrow on the figure denotes the transition from all permutations of the indices $(i, j, k, n)$ of the 5-point structure.

**New information:** *The 5-point function analysis leads to the vanishing of the parameters $E$, $F$, $G$, $H$, $I$, $J$, $K$ and $L$ from figure 4.1.*

## 4.7 Wightman positivity of $\langle W\mathcal{L}\mathcal{L}W\rangle$

In this section we will exploit the Wightman positivity condition for $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle$ and it will exclude two more structures from the list of candidates for terms of the truncated function $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle_T$. Without lost of generality, for simplicity, we assume that $C_{WW} = 1$ from (4.8), which together with the definition of $\mathcal{L}(x)$ (4.22), implies $C_{\mathcal{L}\mathcal{L}} = 2 + c$.

As earlier, we use for this purpose the partial wave expansion of $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle$ requesting all the coefficients to be non-negative. The results for the coefficients $B_{\kappa L}$ are listed in Appendix A.2.

Consider the coefficients $B_{\frac{3}{2}L}$ and $B_{\frac{5}{2}L}$:

$$B_{\frac{3}{2}L} = \frac{(L!)^2(L+1)}{(2L+1)!}\left[D + (-1)^L(B + A - AL(L+2))\right] \tag{4.28}$$

$$B_{\frac{5}{2}L} = \frac{(L+1)!(L+2)!}{(2L+3)!}\left[C((-1)^L + 1) + B(-1)^L(L+1)(L+3)\right] \tag{4.29}$$

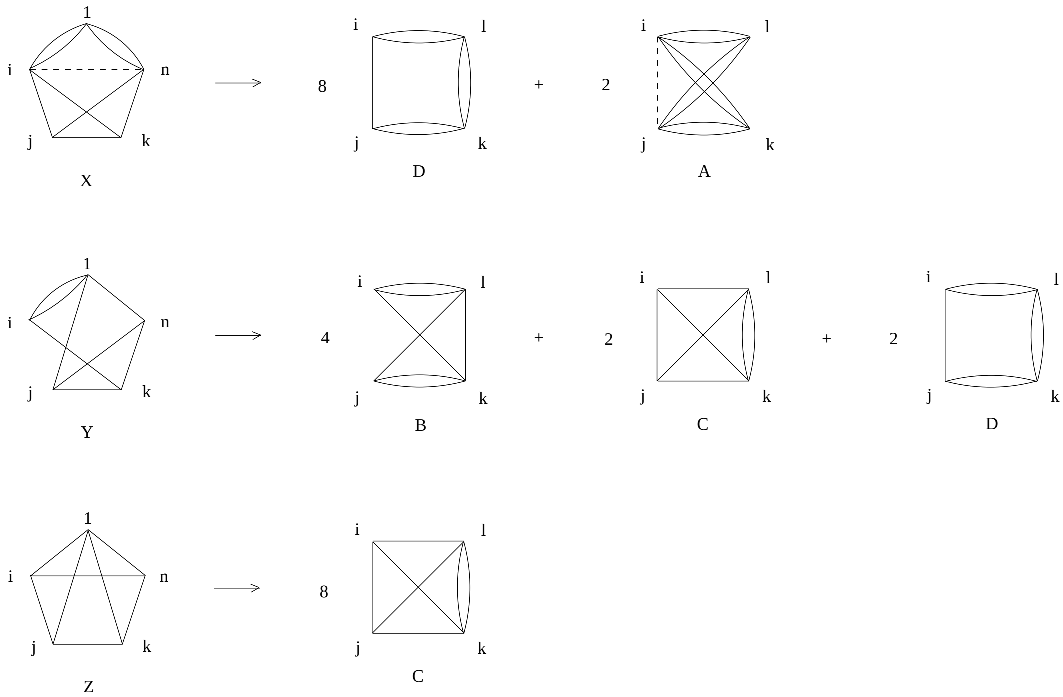


Figure 4.3: Transition $\langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)W(x_4)W(x_5)|0\rangle_T \to \langle 0|\mathcal{L}(x_1)W(x_2)W(x_3)\mathcal{L}(x_4)|0\rangle_T$

We observe that the leading contributions are with alternating signs in the limit $L \to \infty$ for odd and even values of $L$. This would result in violation of Wightman positivity unless the coefficients $A$ and $B$ are equal to zero.

So, we are left with two structures for $\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle$ — the structures in figure 4.3 with coefficients $C$ and $D$. Further analysis gives the following restrictions for them:

$$C \geq 0, \quad \frac{C}{6} - (2+c) \leq D, \quad 6(2+c) - \frac{C}{15} \geq D \tag{4.30}$$

These conditions come from the non-negativity of the coefficients $B_{\kappa L}$ respectively for $\kappa = \frac{5}{2},\ L = 0$; $\kappa = \frac{7}{2},\ L = 0$ and $\kappa = \frac{7}{2},\ L = 1$. These are the strongest bounds that we can "extract" from the conditions $B_{\kappa L} \geq 0$.

**New information:** *Wightman positivity analysis of $\langle W\mathcal{L}\mathcal{L}W\rangle$ excludes the parameters $A$ and $B$ and gives the bounds (4.30) for the parameters $C$ and $D$.*

## 4.8 A relation between $c$ and $D$

In this subsection we will show that Hilbert space positivity yields a Cauchy–Schwartz inequality which gives a relation between the values of $c$ from $\langle WWWW\rangle$ and of $D$ from $\langle W\mathcal{L}\mathcal{L}W\rangle$. For this purpose we construct the field $\widetilde{W}(x)$:

$$\widetilde{W}(x) := (x_{12}^2)^2 W(x_1)\mathcal{L}(x_2)|_{x_1=x_2=x} \tag{4.31}$$

This field represents the 3-dimensional scalar in the OPE of $\mathcal{L} \cdot W$. Then we form the following product, that should be non-negative, $\lambda$ is an arbitrary real number:

$$\begin{aligned}\langle (W(x_1) - \lambda\widetilde{W}(x_1))(W(x_2) - \lambda\widetilde{W}(x_2))\rangle &= \langle (W(x_1)W(x_2)\rangle - \lambda\langle\widetilde{W}(x_1)W(x_2)\rangle - \\ &\quad -\lambda\langle W(x_1)\widetilde{W}(x_2)\rangle + \lambda^2\langle\widetilde{W}(x_1)\widetilde{W}(x_2)\rangle\end{aligned} \tag{4.32}$$

Using the explicit form of $\mathcal{L}(x)$ (4.22), the expressions for the 4-point functions $\langle WWWW\rangle$ (4.13) and $\langle W\mathcal{L}\mathcal{L}W\rangle$ (figure 4.1) and performing the limits (4.31), we prove the following equalities:

$$\begin{aligned}\langle\widetilde{W}(x_1)W(x_2)\rangle &= \langle W(x_1)W(x_2)\rangle c \\ \langle W(x_1)\widetilde{W}(x_2)\rangle &= \langle W(x_1)W(x_2)\rangle c \\ \langle\widetilde{W}(x_1)\widetilde{W}(x_2)\rangle &= \langle W(x_1)W(x_2)\rangle c \cdot D\end{aligned} \tag{4.33}$$

Combining (4.32) and (4.33) we reach the inequality:

$$1 - 2\lambda c + \lambda^2 c \cdot D \geq 0 \tag{4.34}$$

It must be fulfilled for every value of $\lambda$, hence it has a negative discriminant:

$$D = c^2 - c \cdot D \leq 0 \tag{4.35}$$

So, we arrive at the relation:

$$c^2 \leq c \cdot D \tag{4.36}$$

## 4.9 Relations of $C$ and $D$ with $X$, $Y$ and $Z$

As we discussed earlier, the limit at $x_5 \to x_4$ of $\langle\mathcal{L}WWWW\rangle_T$ after multiplication with $x_{45}^2$ reproduces the mixed 4-point function $\langle\mathcal{L}WW\mathcal{L}\rangle_T$, so we expect that there are linear relations between their parameters. These relations can be deduced from figure 4.3 and they are:

$$\begin{aligned}A &= 2X \\ B &= 4Y \\ C &= 2Y + 8Z \\ D &= 8X + 2Y\end{aligned} \tag{4.37}$$

In section 4.7 we showed that Wightman positivity requires that the coefficients $A$ and $B$ should be zero. The structure corresponding to the parameter $A$ of the 4-point function is reproduced only by the $X$-structure of the 5-point function, hence the vanishing of $A$ implies the vanishing of $X$. Similarly, the vanishing of $B$ implies the vanishing of $Y$. Then we observe that to structure $D$ contribute only the $X$ and the $Y$ structures, and if they are absent the structure $D$ must be absent as well.

The Cauchy-Schwartz inequality from the previous section gives the relation: $c \cdot A \geq c^2$ and now we showed that $D = 0$. This, together with Wightman positivity bounds of $c$ (section 4.4) $0 \leq c \leq 12$, leads us to the conclusion that $c = 0$, which we aimed to show.

* * *

So, exploring $\langle WWWW\rangle$, $\langle W\mathcal{L}\mathcal{L}W\rangle$ and $\langle \mathcal{L}WWWW\rangle$ with the methods of partial wave and operator product content analysis and Cauchy–Schwarz inequalities, we proved $\langle WWWW\rangle_T = 0$, implying (as explained in section 4.2) that $W(x)$ is a generalized free field.

# Chapter 5

# Model 2

## 5.1 The Setting

It has been shown [26] that it is not possible to couple a GCI scalar field of dimension 3 to a massive free scalar such that both fields share the same SET, except the trivial solution $\varphi^3(x)$ ($\varphi(x)$ is a massless free scalar). This argument exploited Wightman positivity and Ward identities. Analogous case with a field of dimension 4 is considered in Model 3. In this chapter we explore the analogous possibility to couple a hermitian dimension 4 scalar $\mathcal{L}(x)$ to the free massless Dirac field $\psi(x)$ (looking for a contradiction). Such a field does not exist *within* the Dirac theory. To exclude the trivial Yukawa solution $\varphi \cdot \overline{\psi}\psi$ we assume the absence of a dimension 1 scalar in the OPE of $\mathcal{L}(x)$ with $\mathcal{W} :=: \overline{\psi}\psi : (x)$ (and assume absence of dimension 2 scalar in $\mathcal{L} \cdot \mathcal{L}$). To strengthen the analysis, we assume that the field with the quantum numbers of the stress–energy tensor is unique and is given by a free Dirac spinor field $\psi(x)$, giving rise to Ward identities.

Our strategy will be to search for a contradictory information about the parameters of the mixed 4-point functions of $\mathcal{L}(x)$ and $\mathcal{W}(x)$. We use the familiar techniques from Model 1 of operator product content and Wightman positivity analysis. That will leave us with two free parameters for $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L} \rangle$ and $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W} \rangle$ together with some positivity restrictions on them, and one of the parameters is fixed by Ward identities to a value allowed by these bounds. Then, as before, we proceed to the 5-point level, but even there we will not find an inconsistency.

## 5.2 Partial wave analysis of $\langle \mathcal{W}\mathcal{W}\mathcal{W}\mathcal{W} \rangle$, $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L} \rangle$ and $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W} \rangle$

In this section we will use partial wave analysis, encompassing operator product content analysis and Wightman positivity analysis under the assumptions of our task, to eliminate 10 of the free parameters of $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L} \rangle$ and $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W} \rangle$ (see figure 4.1) and to find some bounds on the remaining two — $F$ and $A$. Studying the partial wave expansion of $\langle \mathcal{W}\mathcal{W}\mathcal{W}\mathcal{W} \rangle$ also gives us some information about the mixed 4-point functions, as the coefficients from the PWE of $\langle \mathcal{W}\mathcal{W}\mathcal{W}\mathcal{W} \rangle$ and of $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L} \rangle$ are related through Cauchy–Schwartz inequalities.

Partial wave analysis will allow us to obtain some new information about the operator product content of $\mathcal{W} \cdot \mathcal{W}$ and of $\mathcal{L} \cdot \mathcal{W}$ as well, which will be crucial in our further investigation at the 5-point level.

### 5.2.1 Expressions for $\langle \mathcal{WWWW} \rangle$, $\langle \mathcal{WWLL} \rangle$ and $\langle \mathcal{WLLW} \rangle$

In our analysis we will need the expressions for the 4-point functions $\langle \mathcal{WWWW} \rangle$, $\langle \mathcal{WWLL} \rangle$ and $\langle \mathcal{WLLW} \rangle$ which we found previously[1].

The general 4-point function of a 3-dimensional scalar field $W(x)$ in a GCI Wightman theory was constructed in section 4.3 and we will display here our result:

$$\langle 0|W(x_1)W(x_2)W(x_3)W(x_4)|0\rangle = \langle 0|W(x_1)W(x_2)|0\rangle\langle 0|W(x_3)W(x_4)|0\rangle \left\{ 1 + s^3 + \left(\frac{s^3}{t^3}\right) + \right.$$
$$\left. +a\left(\frac{s^4}{t^2} + \frac{s}{t^2} + st\right) + b\left(\frac{s^3}{t} + \frac{s^3}{t^2} + \frac{s^2}{t^2} + \frac{s}{t} + s^2 + s\right) + c\frac{s^2}{t} \right\} \tag{5.1}$$

with $s$ and $t$ as in section 3.3.

We also explored the general forms of $\langle WW\mathcal{LL} \rangle$ and of $\langle W\mathcal{LL}W \rangle$ in such a theory with $\mathcal{L}(x)$ being a 4-dimensional scalar. The possible structures of their connected parts obey the same requirements up to rearrangements of indices and are presented on figure 4.1 in terms of graphs as explained in the relevant section. The corresponding expressions will be:

$$\langle 0|W(x_1)W(x_2)\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle = \langle 0|W(x_1)W(x_2)|0\rangle\langle 0|\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle \left\{ 1 + A\frac{s^4}{t^2} + \right.$$
$$+B\left(\frac{s^3}{t^2} + \frac{s^3}{t}\right) + C\frac{s^2}{t} + D\left(s^2 + \frac{s^2}{t^2}\right) + E\left(\frac{s}{t} + s\right) +$$
$$+F\left(\frac{s}{t^2} + st\right) + G\left(s^3 + \frac{s^3}{t^3}\right) + H\left(s^2 t + \frac{s^2}{t^3}\right) +$$
$$\left. +I\left(st^2 + \frac{s}{t^3}\right) + J\left(\frac{s^4}{t} + \frac{s^4}{t^3}\right) + K\frac{s^6}{t^3} + L\left(\frac{s^5}{t^2} + \frac{s^5}{t^3}\right) \right\} \tag{5.2}$$

$$\langle 0|W(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)W(x_4)|0\rangle = \frac{1}{(x_{12}^2)^3 x_{23}^2 (x_{34}^2)^3} \left\{ \frac{s^3}{t^3} + Ast + \right.$$
$$+B\left(s + s^2\right) + C\frac{s^2}{t} + D\left(\frac{s}{t} + \frac{s^3}{t}\right) + E\left(\frac{s^2}{t^2} + \frac{s^3}{t^2}\right) +$$
$$+F\left(\frac{s}{t^2} + \frac{s^4}{t^2}\right) + G\left(1 + s^3\right) + H\left(\frac{s^4}{t} + \frac{1}{t}\right) +$$
$$\left. +I\left(\frac{1}{t^2} + \frac{s^5}{t^2}\right) + J\left(t + s^2 t\right) + Kt^3 + L\left(t^2 + st^2\right) \right\} \tag{5.3}$$

### 5.2.2 PWE analysis for $\langle \mathcal{WWWW} \rangle$

Let us first recall some general results for the special case of $\mathcal{W}(x) =: \overline{\psi}\psi : (x)$, where $\psi(x)$ is a free Dirac spinor field. We discussed in section 4.3 that $\mathcal{W}(x)$ is a 3-dimensional scalar field. It was also shown (formula (4.18)) that $a = \frac{1}{4}$, $b = -\frac{1}{4}$ and $c = 0$.

[1]Without lost of generality, we will assume in our analysis for simplicity that the parameters $C_{\mathcal{WW}}$ and $C_{\mathcal{LL}}$ of the 2-point functions $\langle \mathcal{WW} \rangle$ and $\langle \mathcal{LL} \rangle$ are equal to 1.

Then we observe (A.1) that in this case the partial wave coefficient $B_{10} = 2(a + b) = 0$, hence there is no twist 2 scalar field in the PWE of $\langle\mathcal{WWWW}\rangle$. Moreover, under the current conditions the coefficients $B_{2L}$, which we know from (A.2), will vanish for all $L$, which means that the whole twist 4 series is absent from the PWE of $\langle\mathcal{WWWW}\rangle$.

**New information:** *The OPE of $\mathcal{W}\cdot\mathcal{W}$ in the free Dirac theory does not contain a scalar field dimension 2 and the whole twist 4 series is absent from it.*

### 5.2.3 Cauchy–Schwarz inequalities

Let us denote for the current section with $B_{\kappa L}^{\langle\mathcal{WWLL}\rangle}$ the coefficients in the partial wave expansion of $\langle\mathcal{WWLL}\rangle$, with $B_{\kappa L}^{\langle\mathcal{WWWW}\rangle}$ — those for $\langle\mathcal{WWWW}\rangle$ and with $B_{\kappa L}^{\langle\mathcal{LLLL}\rangle}$ — those for $\langle\mathcal{LLLL}\rangle$. We will show that the vanishing of $B_{2L}^{\langle\mathcal{WWWW}\rangle}$ implies the vanishing of $B_{2L}^{\langle\mathcal{WWLL}\rangle}$ as well.

For this purpose we will prove the following inequality:

$$B_{\kappa L}^{\langle\mathcal{LLLL}\rangle} B_{\kappa L}^{\langle\mathcal{WWWW}\rangle} \geq \left(B_{\kappa L}^{\langle\mathcal{WWLL}\rangle}\right)^2 \tag{5.4}$$

Let us form the product:

$$\langle 0|((x_{12})^2\mathcal{L}(x_1)\mathcal{L}(x_2) - \lambda\mathcal{W}(x_1)\mathcal{W}(x_2))\Pi_{\kappa L}((x_{34})^2\mathcal{L}(x_3)\mathcal{L}(x_4) - \lambda\mathcal{W}(x_3)\mathcal{W}(x_4))|0\rangle \tag{5.5}$$

Here $\Pi_{\kappa L}$ is a projector as in section 3.3. Hilbert space positivity requires that (5.5) is non-negative and after some further development we arrive at the following inequality:

$$\left(B_{\kappa L}^{\langle\mathcal{LLLL}\rangle} - 2\widetilde{\lambda} B_{\kappa L}^{\langle\mathcal{WWLL}\rangle} + \widetilde{\lambda}^2 B_{\kappa L}^{\langle\mathcal{WWWW}\rangle}\right)\beta_{\kappa L} \geq 0 \tag{5.6}$$

where $\widetilde{\lambda} := \frac{C_{\mathcal{WW}}}{C_{\mathcal{LL}}}\lambda$. We know that the universal functions $\beta_{\kappa L}$ are positive, hence we have:

$$B_{\kappa L}^{\langle\mathcal{LLLL}\rangle} - 2\widetilde{\lambda} B_{\kappa L}^{\langle\mathcal{WWLL}\rangle} + \widetilde{\lambda}^2 B_{\kappa L}^{\langle\mathcal{WWWW}\rangle} \geq 0 \tag{5.7}$$

The inequality should be fulfilled for any value of $\widetilde{\lambda}$. This means that its discriminant is negative, which leads us to (5.4). It has the structure of a Cauchy–Schwarz inequality.

Then, we find as a consequence that:

$$B_{2L}^{\langle\mathcal{WWWW}\rangle} = 0 \to B_{2L}^{\langle\mathcal{WWLL}\rangle} = 0 \tag{5.8}$$

It means that the absence of twist 4 structures in the PWE of $\langle\mathcal{WWWW}\rangle$ implies vanishing of this series in the PWE of $\langle\mathcal{WWLL}\rangle$ as well.

### 5.2.4 PWE analysis for $\langle\mathcal{WWLL}\rangle$

Let us use in advance a result from the partial wave analysis of $\langle\mathcal{WLLW}\rangle$, which we will discuss in more details in the next section: all structures with a triple edge between two vertices one of which refers to an argument of $\mathcal{L}(x)$ and the other — to an argument of $\mathcal{W}(x)$, must be excluded due to the absence of 1-dimensional scalar in the OPE of $\mathcal{L}\cdot\mathcal{W}$. This allows us to eliminate the graphs from figure 4.1 with coefficients $G$, $H$, $I$, $J$, $K$ and $L$.

At the end of the previous section we showed that under the assumptions of our task there are no twist 4 structures in the PWE of $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L}\rangle$. Note that this series is represented by partial waves of the form (see section 3.3):

$$\langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)\Pi_{2L}\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle = \langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)|0\rangle\langle 0|\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle B_{2L}\beta_{2L} \tag{5.9}$$

$$\beta_{2L} = \left(\frac{uv}{u-v}\right)(G_{2+L}(u)G_1(v) - G_1(u)G_{2+L}(v)) \tag{5.10}$$

Then let us convert (5.2) into a function of the variables $u$ and $v$ (using (3.15)). We obtain:

$$\langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle_T = \langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)|0\rangle\langle 0|\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle \left(\frac{uv}{u-v}\right)\Big\{Cu\hat{v} +$$
$$+(E+F)u - Fu^2 + \frac{1}{2}(A+B+D+F)u^2v + (A+B)u^2\hat{v} + \frac{1}{2}Au^2\hat{v}^2 - (u \leftrightarrow v)\Big\} - \left({}^{u\leftrightarrow v}_{\hat{u}\leftrightarrow\hat{v}}\right) \tag{5.11}$$

Formulae (3.18) for the expansion of $u$, $v$, $\hat{u}$ and $\hat{v}$ in terms of $G_\mu(u)$ and $G_\nu(v)$ imply that structures as in (5.9) are produced by and only by the part of (5.11) containing terms of the type below and all their versions with one or both of the variables with hats, where $\hat{x} := -\frac{x}{1-x}$:

$$uv, \quad u^2v, \quad uv^2 \tag{5.12}$$

So, the terms of (5.11) which contribute to the twist 4 series are $\left(\frac{1}{2}Cu\hat{v} + (A+B+D+F)u^2v + +(A+B)u^2\hat{v} - (u \leftrightarrow v)\right) - \left({}^{u\leftrightarrow v}_{\hat{u}\leftrightarrow\hat{v}}\right)$. Using formulae (3.18) we show that the twist 4 contributions of $(A+B)u^2v$ and $(A+B)u^2\hat{v}$ cancel. The remaining two terms $\frac{1}{2}Cu\hat{v}$ and $(D+F)u^2v$ cannot cancel each other, hence we conclude that $C = 0$ and $D + F = 0$.

Next, we take into consideration the absence of a 2-dimensional scalar field contribution (which follows from the absence of such field in the PWE of $\langle \mathcal{W}\mathcal{W}\mathcal{W}\mathcal{W}\rangle$ and Cauchy–Schwarz inequality, or simply from the fact that this field is absent from the OPE of $\mathcal{W}\cdot\mathcal{W}$). It is related to the universal function $\beta_{10} = \left(\frac{uv}{u-v}\right)(G_2(u)G_0(v) - G_0(u)G_2(v))$ in the PWE with $G_0(z) = 1$. Such structure is given only by the part $(E+F)(u - v - \hat{u} + \hat{v})$ and we conclude that $E + F = 0$.

Let us summarize the results that we obtained in this section:

**New information:** *The absence of twist 4 structures and of a 2-dimensional scalar contribution in the PWE of $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L}\rangle$ plus absence of 1-dimensional scalar in the OPE of $\mathcal{L}\cdot\mathcal{W}$ results in $D = E = -F$, $C = 0$ and vanishing of the parameters from $G$ to $L$. So, we are left with three parameters for $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L}\rangle$ and for its permutation $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W}\rangle$: $F$, $A$ and $B$.*

### 5.2.5 Wightman positivity of $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W}\rangle$

Let us first concentrate on the twist 1 series in the PWE of $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W}\rangle$. One of the assumptions of our task is that the scalar contribution to this series is absent. Note that higher rank contributions are not allowed because of unitarity bounds, coming from Mack's classification (see section 2.3), then we conclude:

**New information:** *The absence of twist 1 scalar together with unitarity bounds imply the vanishing of the whole twist 1 series in the OPE of $\mathcal{L}\cdot\mathcal{W}$.*

This result leads directly to:

$$\lim_{x_{34}^2\to 0}(x_{34}^2)^3\langle 0|\mathcal{W}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)|0\rangle = 0 \tag{5.13}$$

By similar arguments as in section 4.6, we show that this limit requires that all graphs from figure 4.1 with a triple edge between two vertices, one of which referring to $\mathcal{L}(x)$ and the other — to $\mathcal{W}(x)$, must be excluded. Then, the coefficients $G$, $H$, $I$, $J$, $K$ and $L$ must vanish.

Further in this section we will address the issue of Wightman positivity of $\langle\mathcal{WLLW}\rangle$ via partial wave analysis to find more restrictions for the three remaining parameters from the last section[2]. The partial wave coefficients are calculated according to the procedure described in section 3.3 and are listed in Appendix A.3.

We observe that in our special case:

$$B_{\frac{5}{2}L} = \frac{(L+1)!(L+2)!}{(2L+3)!}(-1)^L(A+B)(L+1)(L+3) \tag{5.14}$$

Note that then Wightman positivity will be violated either for even or for odd values of $L$, because of the alternating sign $(-1)^L$, unless $A+B=0$. Then all the coefficients $B_{\frac{5}{2}L}$ will be zero and hence the whole twist 5 series will be absent from the PWE of $\langle\mathcal{WLLW}\rangle$.

**New information:** *When 4-dimensional scalar field is coupled to Dirac field, then, the OPE of $\mathcal{L}\cdot\mathcal{W}$ does not contain twist 5 fields (assuming that the twist 1 scalar is absent from it).*

Next, consider the coefficient $B_{\frac{3}{2}L}$:

$$B_{\frac{3}{2}L} = \frac{L!(L+1)!}{(2L+1)!}\left\{A(-1)^L L(L+2) + \frac{1}{2}FL(L+1)(L+2)\right\} \tag{5.15}$$

For $L=0$ this coefficient is zero, which indicates the lack of twist 3 scalar contribution in the PWE of $\langle\mathcal{WLLW}\rangle$.

**New information:** *Under the conditions of our task there is no 3-dimensional scalar in the OPE of $\mathcal{L}\cdot\mathcal{W}$.*

At the limit $L\to\infty$ the leading term of (5.15) is $\frac{L!(L+1)!}{(2L+1)!}\frac{1}{2}FL(L+1)(L+2)$ and Wightman positivity requires that $F\geq 0$. Further analysis would show that this is the strongest Wightman positivity bound from below.

The upper bound for $F$ we get from the twist 7 coefficients:

$$\begin{aligned} B_{\frac{7}{2}L} \quad &= \quad \frac{(L+2)!(L+3!)}{(2L+5)!}\left\{\frac{1}{12}(L+1)(L+2)(L+3)(L+4)(L+5)+\right. \\ &\left. +F(L+2)(L+4)\left[(-1)^{L+1}-\frac{1}{2}(L+3)\right]\right\} \end{aligned} \tag{5.16}$$

The condition $B_{\frac{7}{2}0}\geq 0$ leads to $F\leq\frac{1}{2}$, which is the strongest bound from above that Wightman positivity analysis gives.

[2]remember that Wightman positivity is equivalent to all the coefficients in the PWE being non-negative and those coefficients are linear combinations of the three parameters.

We obtain the strongest restrictions for the parameter $A$ from the twist three series (5.15) for $L = 1$ and $L = 2$ respectively and they are $A \geq -F$ and $A \leq \frac{3}{2}F$.

To resume:

**New information:** *The Wightman positivity analysis of $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W}\rangle$ tells us that $A = -B$ and yields the bounds for the remaining two parameters:*

$$0 \leq F \leq \frac{1}{2}, \quad -F \leq A \leq \frac{3}{2}F \tag{5.17}$$

### 5.2.6 Twist 2 series of $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L}\rangle$

The partial wave analysis leaves $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L}\rangle$ with the following terms:

$$\begin{aligned}\langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle_T &= \langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)|0\rangle\langle 0|\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle\Bigg\{A\left(\frac{s^4}{t^2}-\frac{s^3}{t}-\frac{s^3}{t^2}\right)+\\ &\quad +F\left(st+\frac{s}{t^2}-s-\frac{s}{t}-s^2-\frac{s^2}{t^2}\right)\Bigg\}\\ &= \langle 0|\mathcal{W}(x_1)\mathcal{W}(x_2)|0\rangle\langle 0|\mathcal{L}(x_3)\mathcal{L}(x_4)|0\rangle\left(\frac{uv}{u-v}\right)\left\{A(\hat{u}^2v^2-u^2\hat{v}^2)+\right.\\ &\quad \left.+F(-u^2+v^2+\hat{u}^2-\hat{v}^2)\right\}\end{aligned} \tag{5.18}$$

The part with coefficient $F$ consists only of mono-variable terms in $u$ and $v$ whose expansions in terms of hypergeometric functions produce only structures of the form $G_\mu(u) - G_\mu(v)$ and hence only twist two contributions in the PWE. The part with coefficient $A$ produces structures of twist strictly higher than 4 as it contains no mono-variable terms and no terms of the form (5.12). So the twist 2 series is produced exclusively by the $F$-terms. Then we will have:

$$\langle 0|V_1^{\mathcal{W}\mathcal{W}}(x_1,x_2)V_1^{\mathcal{L}\mathcal{L}}(x_3,x_4)|0\rangle = (x_{12})^2(x_{34})^2F\left(st+\frac{s}{t^2}-s-\frac{s}{t}-s^2-\frac{s^2}{t^2}\right) \tag{5.19}$$

Here we have denoted with $V_1^{\phi\phi}$ the twist 2 bilocal field in the OPE of $\phi\cdot\phi$.

This result will be necessary for the application of Ward identities.

## 5.3 Ward Identities

In this section we will exploit the assumption of the uniqueness of SET. For this purpose we will use the Ward identities which we discussed in section 3.4. They will prove to be a powerful tool in our analysis and will help us to fix the value of $F$ — one of the parameters of our mixed 4-point functions.

According to section 3.4, if SET is unique, then it is isolated in the OPE of $\mathcal{W}\cdot\mathcal{W}$ and of $\mathcal{L}\cdot\mathcal{L}$ the following way:

$$T(x,v) = \gamma^{\mathcal{W}}\mathcal{D}_{12}(v)V_1^{\mathcal{W}\mathcal{W}}(x_1,x_2)|_{x_1=x_2=x} = \gamma^{\mathcal{L}}\mathcal{D}_{12}(v)V_1^{\mathcal{L}\mathcal{L}}(x_1,x_2)|_{x_1=x_2=x} \tag{5.20}$$

The differential operator is:

$$\mathcal{D}_{12}(v) := \frac{1}{6}\{(v.\partial_{x_1})^2 + (v.\partial_{x_2})^2 - 4(v.\partial_{x_1})(v.\partial_{x_2}) + v^2(\partial_{x_1}.\partial_{x_2})\} \tag{5.21}$$

$\gamma^{\mathcal{W}}$ and $\gamma^{\mathcal{L}}$ are coefficients of proportionality. We know from the same section that $d_{\mathcal{W}} \cdot \gamma^{\mathcal{W}} = d_{\mathcal{L}} \cdot \gamma^{\mathcal{L}} = c$ where $c$ is the "central charge" from the 2-point function of the SET (3.28). It is shown in [20] that if the SET is given by a free Dirac spinor, then $c = 6$. So, we find:

$$\gamma^{\mathcal{W}} = 2, \quad \gamma^{\mathcal{L}} = \frac{3}{2} \tag{5.22}$$

Next, we will find the coefficients $\gamma^{\mathcal{L}}$ and $\gamma^{\mathcal{W}}$ as functions of the parameter $F$ and thus we can fix its value. We remind that Ward identities yield the following formula (section 3.4):

$$\langle 0|V_1^{\phi\phi}(x_1,x_2)T(x_3,v)|0\rangle = d_\phi \omega_3^{(1)}(x_1,x_2,x_3;v) \tag{5.23}$$

Here $d_\phi$ is the scaling dimension of the field $\phi$, $\omega_3^{(1)}(x_1,x_2,x_3;v)$ is as in section 3.4:

$$\omega_3^{(1)}(x_1,x_2,x_3;v) = \frac{2}{3}\frac{1}{x_{13}^2 x_{23}^2}\left\{\left[2\left(\frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}\right)\cdot v\right]^2 - v^2\frac{x_{12}^2}{x_{13}^2 x_{23}^2}\right\} \tag{5.24}$$

Then, applying the differential operator $\mathcal{D}(v)$ to the 4-point function $\langle 0|V_1^{\mathcal{WW}}(x_1,x_2)V_1^{\mathcal{LL}}(x_3,x_4)|0\rangle$ found in (5.19), we obtain:

$$\begin{aligned}
\langle 0|V_1^{\mathcal{WW}}(x_1,x_2)T(x_3,v)|0\rangle &= \gamma^{\mathcal{L}}\mathcal{D}_{34}(v)\langle 0|V_1^{\mathcal{WW}}(x_1,x_2)V_1^{\mathcal{LL}}(x_3,x_4)|0\rangle|_{x_3=x_4} \\
&= \gamma^{\mathcal{L}}\frac{1}{x_{13}^2x_{23}^2}4F\frac{1}{x_{13}^2x_{23}^2}\left\{\left[2\left(\frac{x_{13}}{x_{13}^2}-\frac{x_{23}}{x_{23}^2}\right)\cdot v\right]^2 - v^2\frac{x_{12}^2}{x_{13}^2x_{23}^2}\right\} = \\
&= \gamma^{\mathcal{L}}6F\omega_3^{(1)}(x_1,x_2,x_3;v) \qquad (5.25)\\
\langle 0|V_1^{\mathcal{LL}}(x_3,x_4)T(x_1,v)|0\rangle &= \gamma^{\mathcal{W}}\mathcal{D}_{12}(v)\langle 0|V_1^{\mathcal{WW}}(x_1,x_2)V_1^{\mathcal{LL}}(x3,x_4)|0\rangle|_{x_1=x_2} \\
&= \gamma^{\mathcal{W}}\frac{1}{x_{31}^2x_{41}^2}4F\frac{1}{x_{13}^2x_{23}^2}\left\{\left[2\left(\frac{x_{13}}{x_{13}^2}-\frac{x_{23}}{x_{23}^2}\right)\cdot v\right]^2 - v^2\frac{x_{12}^2}{x_{13}^2x_{23}^2}\right\} = \\
&= \gamma^{\mathcal{W}}6F\omega_3^{(1)}(x_3,x_4,x_1;v) \qquad (5.26)
\end{aligned}$$

A comparison with (5.23) shows that $\gamma^{\mathcal{W}} = \frac{2}{3F}$ and $\gamma^{\mathcal{L}} = \frac{1}{2F}$. Then, using (5.22), we find that $F = \frac{1}{3}$, which is consistent with the bounds $0 \leq F \leq \frac{1}{2}$ found in section 5.2.5. It follows that $-\frac{1}{3} \leq A \leq \frac{1}{2}$.

**New information:** *The Ward identity requires that $F = \frac{1}{3}$ and yield the following bounds for $A$: $-\frac{1}{3} \leq A \leq \frac{1}{2}$.*

In [20] it is shown that Ward identities provide us with a relation between two parameters of $\langle\mathcal{LLLL}\rangle$: $c = \frac{16}{6a_1+6a_2}$. However, this result does not give further information about the mixed functions.

## 5.4 The 5-point function $\langle\widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle$

In the previous chapter we demonstrated that 5-point functions consideration may give decisive new information about 4-point functions. Indeed, if we project onto the scalar contribution in the OPE of two fields composing a 5-point function, we will end up with a 4-point function. Such a projection is possible in the limit of coinciding parameters of these fields (after a multiplication with the appropriate degree of the relevant arguments' difference and subtracting the singular terms) and it will provide us with some linear relations between the parameters of the two functions.

For this reason we extend our analysis in the current task to the 5-point level, but even then we will not find an inconsistency in our model.

Note that, as there is no 4-dimensional scalar in the OPE of $\mathcal{L}\cdot\mathcal{W}$ (section 5.2.2) and no 3-dimensional scalar in the OPE of $\mathcal{L}\cdot\mathcal{W}$ (section 5.2.5), our only possibility to construct a 5-point function out of which we can obtain the mixed 4-point function remains [3] $\langle\widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle$ with $\widetilde{\mathcal{L}}(x)$ such that $\mathcal{L}(x)$ is the 4-dimensional scalar in the OPE of $\widetilde{\mathcal{L}}\cdot\mathcal{L}$. The two functions are related through the limit:

$$\lim_{x_2\to x_1}(x_{12}^2)^2\langle 0|\widetilde{\mathcal{L}}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T = \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T \tag{5.27}$$

We will first explore the 5-point function $\langle\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ out of which we will reach $\langle\widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ by distinguishing one of the $\mathcal{L}$-vertices of the graphs which account for its terms. The only difference that will occur will be that some of the terms of the new function will not have coinciding parameters anymore. Then we will perform the limit (5.27) and it will allow us to establish a correspondence between the parameters of $\langle\widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ and of $\langle\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$.

### 5.4.1 Constructing the 5-point function $\langle\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$

In this subsection we will construct the 5-point function $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$. We know that, due to GCI and Wightman axioms, it is a rational function of the type $\sum_n A_n\Pi((x_{ij})^2)^{n_{ij}}$ and we will use again the graph representation of its different terms as introduced in section 4.5. It will be convenient to call the vertices corresponding to arguments of $\mathcal{L}(x)$ in the 5-point function $\mathcal{L}$-vertices and those corresponding to arguments of $\mathcal{W}(x)$ — $\mathcal{W}$-vertices. With the same argumentation as before we show that each graph consists of 5 vertices with 4 edges going out of every $\mathcal{L}$-vertex and 3 edges going out of every $\mathcal{W}$-vertex and that the maximal number of edges connecting any two vertices must be equal to 3 if any of them is an $\mathcal{L}$-vertex and to 2 otherwise. Exchanging two vertices of the same type should produce two structures accounting for terms in the sum above with the same coefficient because of the permutation symmetry.

We will take into consideration one extra restriction for these graphs related to the conditions of the present model. Namely, due to the lack of twist 1 scalar in the OPE of $\mathcal{L}\cdot\mathcal{W}$ no triple edges between a $\mathcal{W}$- and an $\mathcal{L}$-vertex are allowed (the reasoning for such a statement can be found in section 5.2.5).

Further restrictions on the possible structures will come from the following Lemma, which is proven in [20] on the basis of the assumption that $V_1^{\phi\phi}$ is bilocal (see section 3.2):

[3]Note that it is possible to prove the existence of such a field $\widetilde{\mathcal{L}}$ [26], provided the OPE of $\mathcal{L}\cdot\mathcal{L}$ contains a dimension 4 scalar, i.e $b'\neq 0$ in [20]

**Lemma 4.8.** *If $f = \sum_\mu C_\mu \Pi(x_{ij}^2)^{\mu_{ij}}$ is harmonic in the argument $x_l$, then $C_\mu = 0$ if $\exists\, k \neq m$, both different from $l$, such that $\mu_{lk} < 0$ and $\mu_{lm} < 0$.*

To make use of this Lemma, let us consider the following limits, at which we arrive using (3.6):

$$\lim_{x_{45}^2\to 0}(x_{45}^2)^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T \prec \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{W}\mathcal{W}}(x_4,x_5)|0\rangle \quad (5.28)$$

$$\lim_{x_{12}^2\to 0}(x_{12}^2)^3\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T \prec \langle 0|V_1^{\mathcal{L}\mathcal{L}}(x_1,x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle \quad (5.29)$$

$$\lim_{x_{34}^2\to 0}(x_{34}^2)^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T \prec \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)V_{\frac{3}{2}}^{\mathcal{L}\mathcal{W}}(x_3,x_4)\mathcal{W}(x_5)|0\rangle \quad (5.30)$$

By the sign $\prec$ we mean that the terms of the function from the left hand side are contained among the terms of the function from the right hand side. Moreover, these limits obviously produce the terms with leading orders of singularities respectively in $(x_{45}^2)^0$, $(x_{12}^2)^0$ and $(x_{34}^2)^0$.

As it was discussed in section 3.1, $V_1^{\mathcal{L}\mathcal{L}}(x_1,x_2)$ and $V_1^{\mathcal{W}\mathcal{W}}(x_1,x_2)$ are harmonic in both arguments. We argue in Appendix B that $V_{\frac{3}{2}}^{\mathcal{L}\mathcal{W}}(x_1,x_2)$ is harmonic in its second argument. Then the 5-point functions from the right hand sides of the limits above will be harmonic in the relevant variables. Hence, Lemma 4.8 holds for them and also for the terms from the left hand side.

Let us concentrate on the first limit. Applied to the graphs, it will lead to subtracting a double edge between the vertices 4 and 5. It is obvious that all graphs with a number of such edges smaller than 2 will be eliminated by the limit. There cannot occur singularities because the pole–rule (Proposition 6.5) does not allow more than 2 edges between the two $\mathcal{W}$-vertices. The surviving structures will not have an edge between those vertices. Then, using Lemma 4.8 and after similar considerations of the second and the third limit, we deduce the following rule:

**Rule:** *If there is a double edge between vertices 4 and 5, then each of those vertices is connected by one or more normal edges only to one of the remaining vertices. The same holds for any two $\mathcal{L}$-vertices if there is a triple edge connecting them, and for any $\mathcal{W}$-vertex if it is connected by a double edge to an $\mathcal{L}$-vertex.*

All the graphs obeying the requirements considered up to now are listed in figure 5.1. We have denoted by $j$, $k$ the $\mathcal{W}$-vertices and by $i$, $n$, $m$ — the $\mathcal{L}$-vertices.

### 5.4.2 OPE analysis

In this subsection we will use the information that we obtained from the partial wave analysis in section 5.2 to eliminate many of the candidates for terms of $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$. Let us make here the following agreement: the structures, corresponding to the twelve permutations of every graph from figure 5.1 which are obtained by exchanging vertices of the same type, appear in $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$ multiplied by the same number (because of permutation symmetry) and we will refer to this number as "the parameter of the graph".

1. *The lack of scalar field dimension 2 in the OPE of $\mathcal{W}\cdot\mathcal{W}$* (section 5.2.2) means that:

$$\lim_{x_4\to x_5}(x_{45}^2)^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T = 0 \quad (5.31)$$

Performing this limit will be equivalent to subtracting a double edge between the vertices $j$ and $k$ of the graphs and merging them. The surviving structures after this procedure are displayed on figure 5.2, while all structures 7) — 22) obviously give zero. According to (5.31) their linear combination

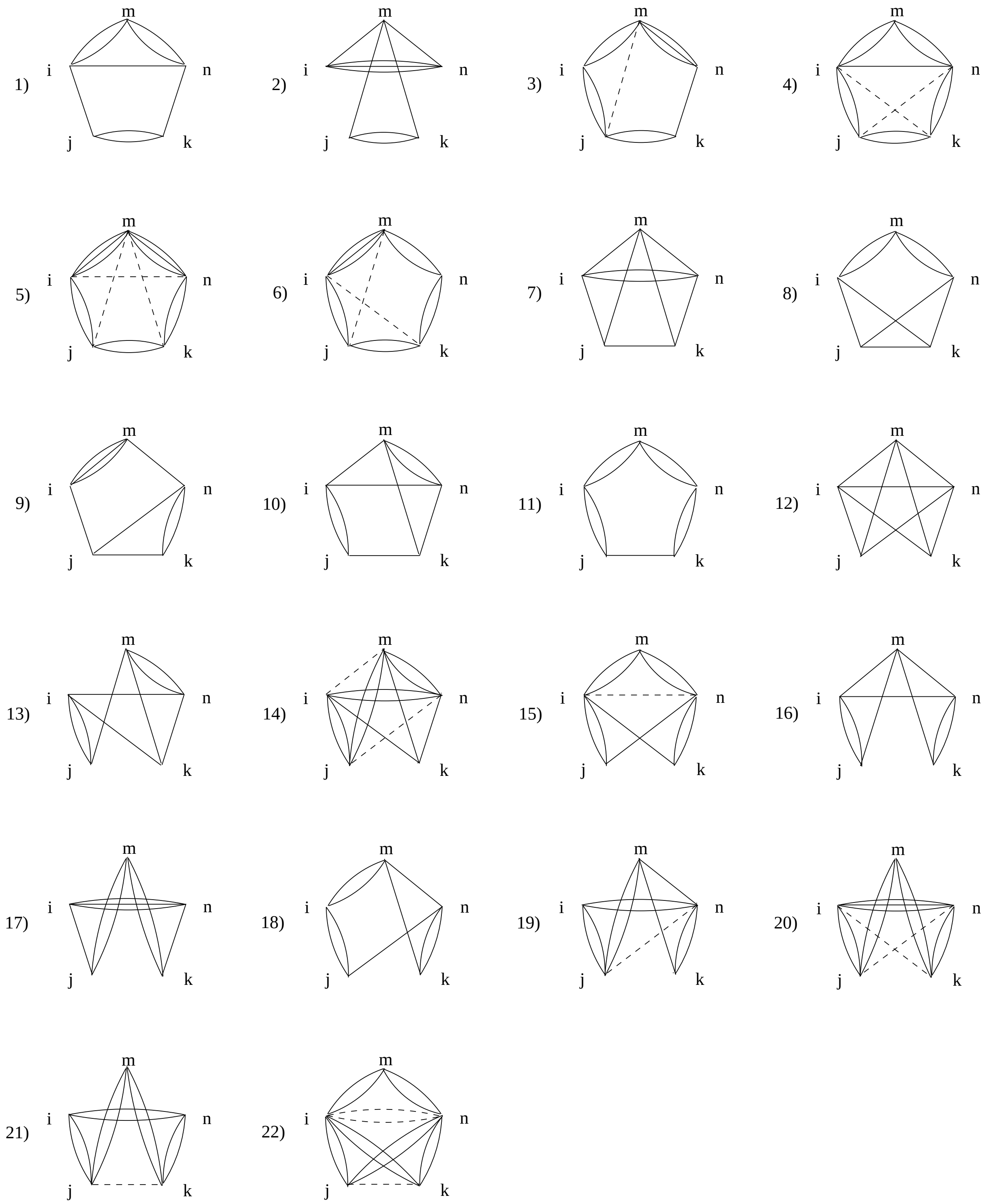


Figure 5.1: Terms of $\langle \mathcal{LLLWW} \rangle_T$

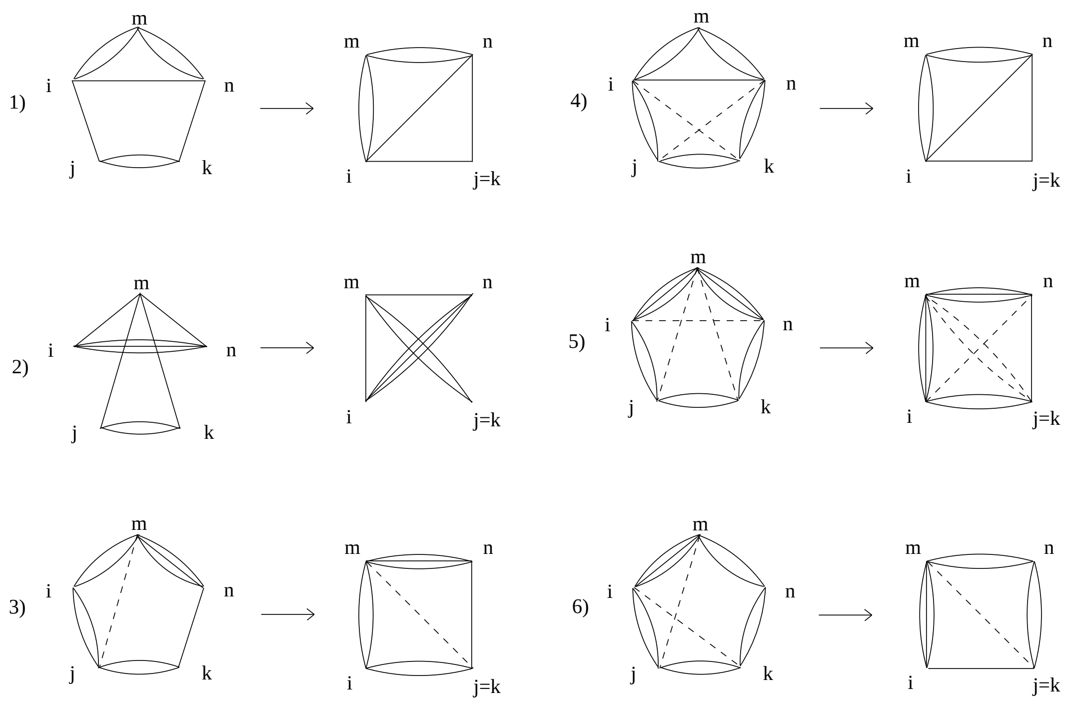


Figure 5.2: Performing the limit $\lim_{x_4 \to x_5} (x_{45}^2)^2 \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$

should vanish. We observe that graphs 2) and 5) produce structures that cannot cancel with any other, hence their corresponding parameters in $\langle\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle$ must be zero. We notice the *two couples of coinciding structures* $1)-4)$ *and* $3)-6)$. They must cancel each other in the sum, so *they should appear with opposite parameters. We call them respectively* $\alpha,\ -\alpha,\ \beta,\ -\beta$.

2. We treat the condition of a *lack of scalar of dimension 2 in the OPE of* $\mathcal{L}\cdot\mathcal{L}$ (assumption of the task) in a similar way. The corresponding limit is:

$$\lim_{x_1 \to x_2} (x_{12}^2)^3 \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T = 0 \tag{5.32}$$

We show that *structure* 9) *should vanish and that the couple* $17)-20)$ *should come up with opposite parameters. We call them* $\gamma$ *and* $-\gamma$.

3. Using (3.6) we verify that *the lack of twist 4 structures in the OPE of* $\mathcal{W}\cdot\mathcal{W}$ (section 5.2.5) yields the following equation:

$$\begin{aligned}&\lim_{x_{45}^2 \to 0} x_{45}^2 \big[\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T - \\ &-(x_{45}^2)^{-2}\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{W}\mathcal{W}}(x_4,x_5)|0\rangle\big] = 0\end{aligned} \tag{5.33}$$

As we already showed, the leading terms[4] of $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{W}\mathcal{W}}(x_4,x_5)|0\rangle$ are obtained through the limit (5.28) and they are represented by those graphs that have initially a double edge

[4] Note that the assumption of bilocality of $V_1^{\phi\phi}(x_1,x_2)$ (section 3.2) requires that its correlation functions with other local fields are rational.

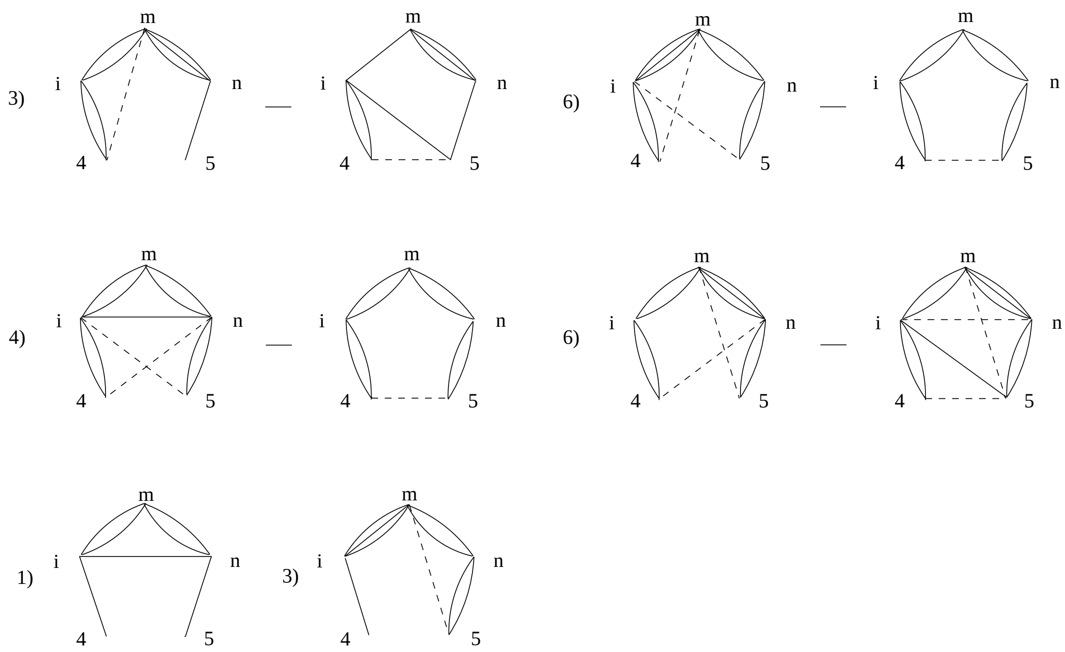


Figure 5.3: Terms of $\lim_{x_{45}\to 0}(x_{45}^2)^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$ and their complement to harmonic structures in $x_4$

between the $\mathcal{W}$-vertices. To obtain the full 5-point function, we must complete those terms to harmonic functions in both arguments $x_4$ and $x_5$ by adding further structures of order $x_{45}^2$ ($\langle \mathcal{LLL}V_1^{\mathcal{WW}}\rangle$ is harmonic in the arguments of $V_1^{\mathcal{WW}}$). We achieve this using the following information [26]:

- if the $x_j$-dependent factor of a certain term of $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{WW}}(x_j,x_k)|0\rangle$ is $\frac{1}{x_{jl}^2}$ for some $l\in\{1,2,3\}$, then this term is harmonic in the argument $x_j$;
- if the $x_j$-dependent factor of a certain term of $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{WW}}(x_j,x_k)|0\rangle$ is $\frac{x_{jl}^2}{(x_{jp}^2)^2}$ for some $l\neq p\in\{1,2,3\}$, then this term is uniquely completed[5] to a harmonic one in the argument $x_j$ by subtracting the same term multiplied by $\frac{x_{jk}^2x_{lp}^2}{x_{jl}^2x_{kp}^2}$;

All the terms, up to permutation of the indices $i$, $m$ and $n$, that we obtain in (5.28) are presented in figure 5.3 together with their complements to harmonic structures in the argument $x_4$. It is seen that structures 3) and 6) must be accompanied in the full 5-point function $\langle \mathcal{LLL}V_1^{\mathcal{WW}}\rangle$ by terms which we already eliminated. Then, we cannot obtain harmonic structures out of these graphs and they must be excluded. Because of the symmetry, it is easily seen that structures 1) and 4) are harmonic in the argument $x_5$, as well. The sum of both of them multiplied by twice the corresponding parameters gives $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{WW}}(x_4,x_5)|0\rangle$.

[5]Here we will use the conjecture that the leading terms of a harmonic rational function determine uniquely its part of lower singularities, as an established result. Note that in [3] a similar conjecture is proven for polynomials and using this result, together with Taylor expansions of degrees of arguments in the denominator, a lot is accomplished for the generalization [26].

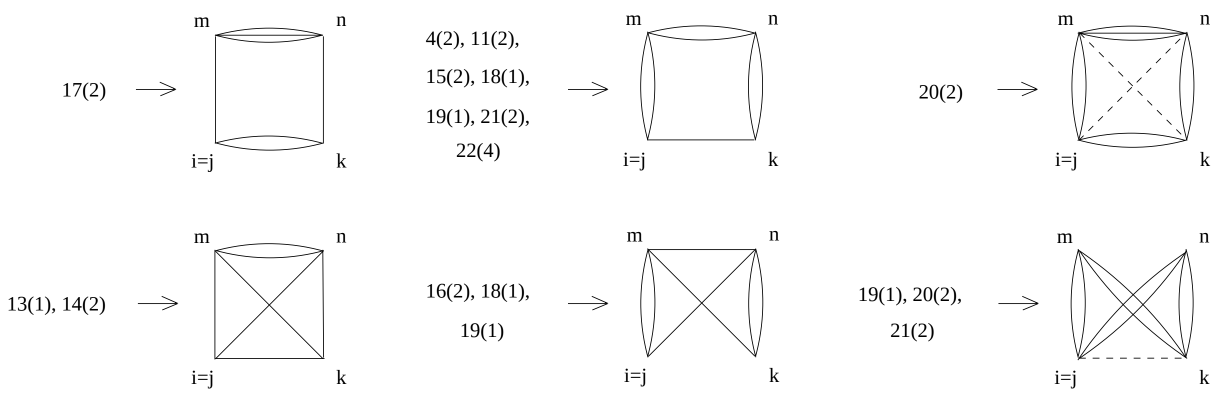


Figure 5.4: Performing the limit $\lim_{x_1\to x_5}(x_{15}^2)^2\langle\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)\rangle_T$

Now we are ready to perform the limit (5.33) and we come to the conclusion that *graphs 7), 8) and 10) are excluded and that the parameter of 11) is* $\alpha$.

4. *The lack of 3-dimensional scalar in the OPE of* $\mathcal{L}\cdot\mathcal{W}$ (section 5.2.5) leads to:

$$\lim_{x_1\to x_5}(x_{15}^2)^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T = 0 \tag{5.34}$$

The result from this limit is shown in figure 5.4. The numbers outside the brackets say which graph gives the corresponding structure. The numbers in the brackets stand for the number of permutations of this graph that produce this structure.

We observe that *graphs* 17) *and* 20) are the only ones that give rise in their limit to the first and the third structure. Then they *should vanish* as (5.34) dictates that the sum of all terms after the limit should be zero. For the same reason *the pairs* 13), 14) *and* 19), 21) *should have opposite coefficients, which we denote:* $2\delta$, $-\delta$ *and* $2\eta$, $-\eta$.

5. Last, we exploit the *absence of twist 5 structures in the OPE of* $\mathcal{L}\cdot\mathcal{W}$ (section 5.2.5). This condition annulates the following limit:

$$\begin{aligned}\lim_{x_{34}^2\to 0}x_{34}^2\big[&\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T - \\ &-(x_{34}^2)^{-2}\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)V_{\frac{3}{2}}^{\mathcal{L}\mathcal{W}}(x_3,x_4)\mathcal{W}(x_5)|0\rangle\big] = 0\end{aligned} \tag{5.35}$$

We find the function $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)V_{\frac{3}{2}}^{\mathcal{L}\mathcal{W}}(x_3,x_4)\mathcal{W}(x_5)|0\rangle$ in a similar way as we proceeded for $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)V_1^{\mathcal{W}\mathcal{W}}(x_4,x_5)|0\rangle$. Its leading terms, up to permutation of the indices $m$ and $n$, produced by the limit (5.35) are exposed on figure 5.5 together with their complements to a harmonic structure in the argument $x_4$; the numbers in the brackets say how many copies of the corresponding structure are given by this limit. Then, we show that in order (5.35) to hold, *the structures 12), 13) ,14), 16), 18), 19), 21) must be excluded. The couple 15)-22) has parameters respectively* $2\tau$ *and* $-\tau$.

*So, we are left with 5 structures: 1), 4), 11), 15) and 22), arising in the two combinations* $\alpha\big[\,1)-4)+11)\big]+\tau\big[\,2\cdot 15)-22)\big]$.

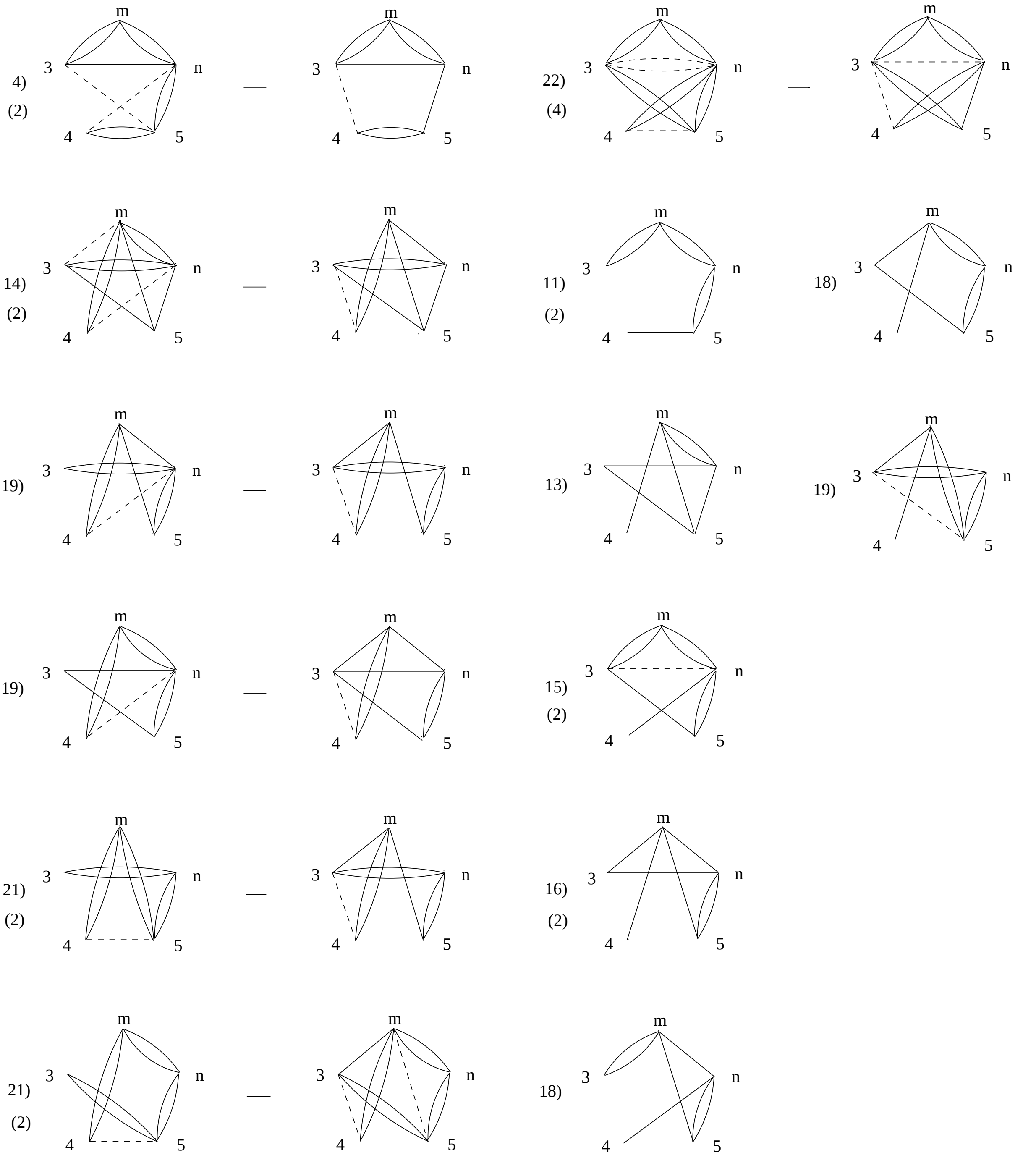


Figure 5.5: Terms of $\lim_{x_{34}^2\to 0}(x_{34}^2)^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(3,x_4)\mathcal{W}(x_5)|0\rangle_T$ and their complement to harmonic structures

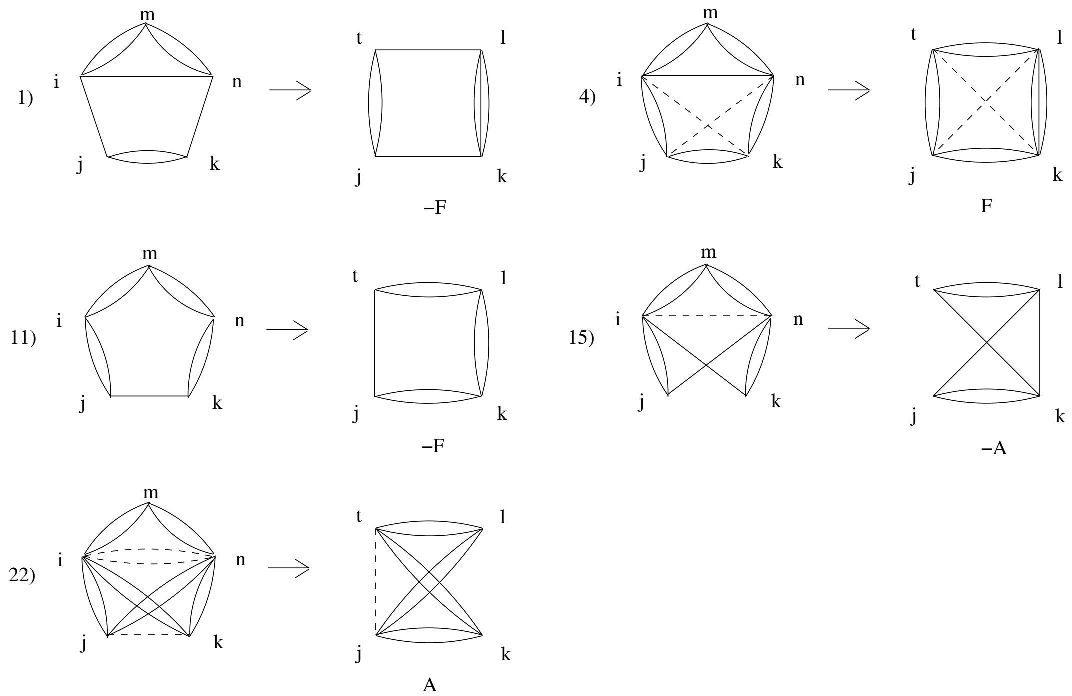


Figure 5.6: The transition $\langle \widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T \rightarrow \langle \mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$

## 5.5 The transition from the 5-point to the 4-point function

In the previous section we constructed $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$ under the conditions of our task. Now we will obtain from it $\langle 0|\widetilde{\mathcal{L}}(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\mathcal{W}(x_4)\mathcal{W}(x_5)|0\rangle_T$ by choosing consequently every of the vertices $i$, $m$ or $n$ (figure 5.1) to be the $\widetilde{\mathcal{L}}$-vertex. Then permutation symmetry will require invariance only under exchange between the two $\mathcal{L}$-vertices or between the two $\mathcal{W}$-vertices. Let us use the following notation: all those terms of $\langle \mathcal{L}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ which had a parameter $X$ will have a parameter $X_i$ in $\langle \widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ if the vertex $i$ has been distinguished to be the $\widetilde{\mathcal{L}}$-vertex.

Now we will perform the transition from $\langle \widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ (which consists of the terms 1), 4), 11), 15) and 22) from figure 5.1, with parameters respectively $\alpha_x$, $-\alpha_x$, $\alpha_x$, $2\tau_x$ and $-\tau_x$, where $x$ takes the values $i$, $m$ and $n$), to $\langle \mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ as in (5.27). This transition will eliminate all the permutations which have less than two edges between the vertices 1 and 2. In figure 5.6 it is shown what structures produces each graph after the transition. We find the following correspondence between the parameters of $\langle \widetilde{\mathcal{L}}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ and $\langle \mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W}\rangle_T$ (5.2):

$$F = -\alpha_n - \alpha_i - 2\alpha_m, \quad A = -\tau_n - \tau_i - 2\tau_m \tag{5.36}$$

finding out that the transitions 1) $\rightarrow -F$, 4) $\rightarrow F$, 11) $\rightarrow -F$, 15) $\rightarrow -A$, 22) $\rightarrow A$ are consistent with each other.

Then, we conclude that the 5-point function analysis does not provide us with contradictory information about the parameters $A$ and $F$.

* * *

Applying partial wave and operator product content analysis and Ward identities to $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W} \rangle$, $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L} \rangle$ and $\langle \mathcal{L}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W} \rangle$ we did not find an inconsistency.

We cannot, therefore, exclude at this level of the analysis the existence of a 4-dimensional scalar coupled nontrivially to the Dirac square $\overline{\psi}\psi$.

# Chapter 6

# Model 3

## 6.1 The Setting

In this chapter we will give an example of a theory [26], where using Ward identities in combination with Wightman positivity and operator product content analysis and Cauchy–Schwarz inequalities proves to be really powerful and allows us to determine completely the model. We assume the existence of a dimension 2 scalar $\phi(x) := \frac{1}{2} : \varphi^2 : (x)$, where $\varphi(x)$ is a massless free scalar, and of a dimension 4 hermitian scalar $\mathcal{L}(x)$ in a theory which has a unique conserved symmetric rank 2 tensor (which implies that the 2-dimensional scalar is unique). In particular, this SET is the canonical SET of a massless free field and its central charge is $c = 1$. Under these assumptions we prove that $\mathcal{L} \sim: \varphi^4 :$

In the following section we will give a sketch of the argumentation, as we already demonstrated in details how the general techniques work in the previous two chapters.

## 6.2 Outline of the strategy

- *Constructing $\langle \mathcal{L}\mathcal{L}\phi\phi \rangle$, $\langle \mathcal{L}\mathcal{L}\phi\phi \rangle$ and $\langle \phi\phi\phi\phi \rangle$:*

  The central object in this analysis are the 4-point functions $\langle \mathcal{L}\mathcal{L}\phi\phi \rangle$ and $\langle \phi\mathcal{L}\mathcal{L}\phi \rangle$. By standard arguments for a GCI Wightman theory as in section 4.3 and using Proposition 5.2 from [20] we find the general expression for $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\phi(x_3)\phi(x_4)|0\rangle$ and it has the same structure as $\langle 0|\phi(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\phi(x_4)|0\rangle$ up to the appropriate rearrangement of variable indices. The disconnected part is of the form $\langle \mathcal{L}\mathcal{L} \rangle \langle \phi\phi \rangle$. Using (4.8) with normalization $C_{\mathcal{L}\mathcal{L}} = 1$ we show that:

$$\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)|0\rangle = \frac{1}{(x_{12}^2)^4} \tag{6.1}$$

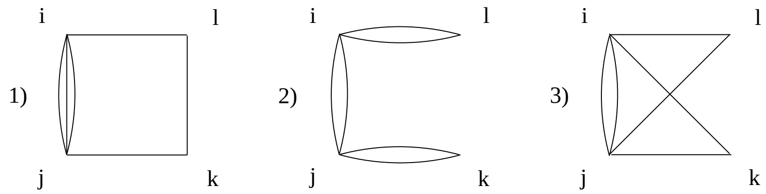


Figure 6.1: Terms of $\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\phi(x_3)\phi(x_4)|0\rangle_T$

We find the 2- and the 4-point functions of $\phi(x) = \frac{1}{2} : \varphi^2 : (x)$ using Wick's theorem:

$$\begin{aligned}
\langle 0|\phi(x_1)\phi(x_2)|0\rangle &= \frac{1}{2}\frac{1}{(x_{12}^2)^2}, &(6.2)\\
\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle &= \frac{1}{4}\left(\frac{1}{(x_{12}^2)^2(x_{34}^2)^2} + \frac{1}{(x_{13}^2)^2(x_{24}^2)^2} + \frac{1}{(x_{14}^2)^2(x_{23}^2)^2}\right) + \\
&+ \frac{1}{x_{12}^2x_{34}^2x_{13}^2x_{24}^2} + \frac{1}{x_{12}^2x_{34}^2x_{14}^2x_{23}^2} + \frac{1}{x_{13}^2x_{24}^2x_{14}^2x_{23}^2} = \\
&= \frac{1}{4}\frac{1}{(x_{12}^2)^2(x_{34}^2)^2}\left(1 + s^2 + \frac{s^2}{t^2} + 4\left(s + \frac{s}{t} + \frac{s^2}{t}\right)\right) &(6.3)
\end{aligned}$$

We use the graph representation for the disconnected terms of $\langle\mathcal{L}\mathcal{L}\phi\phi\rangle$ and $\langle\phi\mathcal{L}\mathcal{L}\phi\rangle$ as explained in section 4.5 and the result is displayed on figure 6.1. Vertices $k$ and $l$ refer to the arguments of $\phi(x)$ and vertices $i$ and $j$ — to the arguments of $\mathcal{L}(x)$. We will call the coefficients of the structures 1), 2) and 3) respectively $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$.

The corresponding expressions are:

$$\begin{aligned}
\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\phi(x_3)\phi(x_4)|0\rangle &= \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)|0\rangle\langle 0|\phi(x_3)\phi(x_4)|0\rangle\left(1 + 2\widetilde{a}\left(s + \frac{s}{t}\right) + \right.\\
&\left. + 2\widetilde{b}\left(s^2 + \frac{s^2}{t^2}\right) + 2\widetilde{c}\frac{s^2}{t}\right) &(6.4)\\
\langle 0|\phi(x_1)\mathcal{L}(x_2)\mathcal{L}(x_3)\phi(x_4)|0\rangle &= \frac{1}{(x_{12}^2)^2(x_{23}^2)^2(x_{34}^2)^2}\left(\frac{1}{2}\frac{s^2}{t^2} + \widetilde{a}\left(\frac{s}{t} + \frac{s^2}{t}\right) + \right.\\
&\left. + \widetilde{b}\left(1 + s^2\right) + \widetilde{c}s\right) &(6.5)
\end{aligned}$$

- *Ward identities allow us to fix the parameter $\widetilde{a} = 4$:*

  Let us work in a theory with a unique SET. As we discussed in section 3.4, if we apply the differential operator $\mathcal{D}_{34}(v)$ from (3.24) to the 4-point function $\langle 0|V_1^{\mathcal{A}\mathcal{A}}(x_1, x_2)V_1^{\mathcal{B}\mathcal{B}}(x_3, x_4)|0\rangle$, where $\mathcal{A}(x)$ and $\mathcal{B}(x)$ are two hermitian scalar fields, the result will be proportional to a universal function $\omega_3^{(1)}(x_1; x_2; x_3, v)$ as in (3.26). The coefficient of proportionality is $\sigma_{\mathcal{A}\mathcal{B}} = \frac{d_{\mathcal{A}}}{\gamma^{\mathcal{B}}}$, where $d_{\mathcal{A}}$ is the scaling dimension of $\mathcal{A}(x)$ and for our purposes here it will be sufficient to recall that $\gamma^{\mathcal{B}} = \frac{c}{d_{\mathcal{B}}}$ (3.30). Here $c$ is the central charge from the 2-point function of SET (3.28) and it is shown in [20] that if the SET is given by a free massless scalar, then $c = 1$. Hence, in our case we find that $\sigma_{\mathcal{L}\phi} = \frac{d_{\mathcal{L}}d_{\phi}}{c} = 8$.

Then, we need to find:

$$\langle 0|V_1^{\mathcal{LL}}(x_1,x_2)V_1^{\phi\phi}(x_3,x_4)|0\rangle = (x_{12}^2)^3 x_{34}^2 \sum_L \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\Pi_{1L}\phi(x_3)\phi(x_4)|0\rangle \tag{6.6}$$

where $\Pi_{1L}$ is the projection on the twist 2 contributions in the OPE of $\phi \cdot \phi$ and the terms from the sum from the right hand side are the twist 2 partial waves from the partial wave expansion as defined in section 3.3. Similarly to how we argued before (section 4.4), we show that the leading twist 2 contributions in the PWE will be given by and only by the part of (6.4) which contains terms with $s$ to the power 1, i.e. by the terms with coefficient $\widetilde{a}$. Inserting them into (6.6) we obtain a structure which is already harmonic in $x_3$ and $x_4$, hence (see section 5.4.2) this is the full 4-point function $\langle V_1^{\mathcal{LL}} V_1^{\phi\phi}\rangle$:

$$\langle 0|V_1^{\mathcal{LL}}(x_1,x_2)V_1^{\phi\phi}(x_3,x_4)|0\rangle = (x_{12}^2)^3 x_{34}^2 \langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)|0\rangle\langle 0|\phi(x_3)\phi(x_4)|0\rangle 2\widetilde{a}\left(s+\frac{s}{t}\right) \tag{6.7}$$

Now we are ready to apply the Ward identities and the final result is that one of the parameters of (6.4) is fixed: $\widetilde{a} = 4$.

- *Wightman positivity analysis of $\langle\phi\mathcal{LL}\phi\rangle$ allows us to fix a second of the parameters* $\widetilde{b} = 3$.

  We showed in section 3.3 that Wightman positivity of a function of the type $\langle\phi\mathcal{LL}\phi\rangle$ is equivalent to the positivity of all the coefficients in its partial wave expansion and these coefficients are linear functions of the parameters $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$. This expansion is performed according to the described in section 3.3 procedure and we obtain the following relations, coming from $B_{\kappa L} \geq 0$ for $\kappa$ and $L$ respectively:

$$\kappa = 2,\ L = 0 \quad \text{and} \quad \kappa = 2,\ L = 1 \quad \rightarrow \quad -\widetilde{a} \leq \widetilde{c} \leq 3\widetilde{a} \ \rightarrow\ -4 \leq \widetilde{c} \leq 12 \tag{6.8}$$

$$\kappa = 3,\ L = 1 \quad \text{and} \quad \kappa = 4,\ L = 0 \quad \rightarrow \quad 3 \leq \widetilde{b} \leq 3 \ \rightarrow\ \widetilde{b} = 3 \tag{6.9}$$

- *Hilbert space positivity yields a Cauchy–Schwarz inequality, which allows us to fix the third remaining parameter* $\widetilde{c} = 12$*:*

  Let us denote with $\mathfrak{L}(x)$ the dimension 4 scalar in the OPE of $\mathcal{L}\cdot\phi$. Then we form the product:

$$\langle 0|\left(\mathcal{L}(x_1) - \lambda\mathfrak{L}(x_1)\right)\left(\mathcal{L}(x_2) - \lambda\mathfrak{L}(x_2)\right)|0\rangle \tag{6.10}$$

  which must be positive due to Hilbert space positivity. It contains the functions $\langle\mathcal{LL}\rangle$, $\langle\mathcal{L}\mathfrak{L}\rangle$, $\langle\mathfrak{LL}\rangle$ and we will now find them.

  Note that:

$$\langle 0|\mathcal{L}(x_1)\mathfrak{L}(x_2)|0\rangle = \lim_{x_3\to x_2} x_{23}^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\phi(x_3)|0\rangle \tag{6.11}$$

  Then we apply the Wick's theorem to $\phi\cdot\phi$ and show that the 2-dimensional scalar contained in it is $2\phi(x)$. This implies the following relation:

$$\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\phi(x_3)|0\rangle = \frac{1}{2}\lim_{x_4\to x_3} x_{34}^2\langle 0|\mathcal{L}(x_1)\mathcal{L}(x_2)\phi(x_3)\phi(x_4)|0\rangle_T \tag{6.12}$$

and together with (6.11) and (6.4) gives:

$$\langle 0|\mathcal{L}(x_1)\mathfrak{L}(x_2)|0\rangle = \widetilde{a}\frac{1}{(x_{12}^2)^4} \tag{6.13}$$

Then we observe that:

$$\langle 0|\mathfrak{L}(y)\mathfrak{L}(z)|0\rangle \;=\; x_{12}^2 x_{34}^2 \sum_L \langle 0|\phi(x_1)\mathcal{L}(x_2)\Pi_{2L}\mathcal{L}(x_3)\phi(x_4)|0\rangle|_{\substack{x_1=x_2=y\\ x_3=x_4=z}} \tag{6.14}$$

Using (3.12) and (3.13) we show that every partial wave from (6.14) has the following expression:

$$\langle 0|\phi(x_1)\mathcal{L}(x_2)\Pi_{2L}\mathcal{L}(x_3)\phi(x_4)|0\rangle \;=\; \frac{1}{(x_{12}^2)^2(x_{23}^2)^2(x_{34}^2)^2} B_{2L}\beta_{2L}^2(s,t) \tag{6.15}$$

$$\beta_{2L}^2(s,t) \;=\; \frac{uv}{u-v}\left(G_{L+1}^2(u) - G_{L+1}^2(v)\right) \tag{6.16}$$

As we argued before (section 4.4), we prove that such structure as $\beta_{2L}^2(s,t)$ is produced by and only by the following part of (6.5):

$$\frac{1}{(x_{12}^2)^2(x_{23}^2)^2(x_{34}^2)^2}\left(\widetilde{a}\frac{s}{t} + \widetilde{c}s\right) \tag{6.17}$$

and that these terms produce only the relevant partial waves. Then we find:

$$\langle 0|\mathfrak{L}(x_1)\mathfrak{L}(x_2)|0\rangle = (\widetilde{a}+\widetilde{c})\frac{1}{(x_{12}^2)^4} \tag{6.18}$$

Inserting (6.1), (6.11) and (6.18) in (6.10) we obtain the following inequality:

$$1 - 2\lambda\widetilde{a} + \lambda^2(\widetilde{a}+\widetilde{c}) \geq 0 \tag{6.19}$$

and as it should be valid for all values of $\lambda$:

$$\widetilde{a}^2 \leq \widetilde{a} + \widetilde{c} \quad \rightarrow \quad \widetilde{c} \geq 12 \tag{6.20}$$

Then, having in mind the Wightman positivity bound $\widetilde{c} \leq 12$, we conclude that $\widetilde{c} = 12$.

- *By Reeh–Schlieder theorem we obtain $\mathcal{L} \sim : \phi^2 :$.*

  We want to show that for some $\lambda$ the 2-point function

$$\langle 0|(:\phi^2:(x_1) - \lambda\mathcal{L}(x_1))(:\phi^2:(x_2) - \lambda\mathcal{L}(x_2)|0\rangle \tag{6.21}$$

vanishes. It involves the functions $\langle\mathcal{L}\mathcal{L}\rangle$, $\langle\mathcal{L}:\phi^2:\rangle$ and $\langle :\phi^2::\phi^2:\rangle$.

$\langle\mathcal{L}:\phi^2:\rangle$ is related to $\langle\mathcal{L}\phi\phi\rangle$, which is fixed only by its conformal invariance property to the form (see [7]):

$$\langle 0|\mathcal{L}(x_1)\phi(x_2)\phi(x_3)|0\rangle = \frac{\mathtt{x}}{(x_{12}^2)^2(x_{13}^2)^2} \tag{6.22}$$

The connection between the two functions is:

$$\begin{aligned}\langle 0|\mathcal{L}(x_1):\phi^2:(x_2)|0\rangle &= \lim_{x_3\to x_2}\left(\langle 0|\mathcal{L}(x_1)\phi(x_2)\phi(x_3)|0\rangle - (\text{singular terms in } x_{23}^2)\right)\\ &= \frac{\mathtt{x}}{(x_{12}^2)^4}\end{aligned} \tag{6.23}$$

We will show that the uniqueness of the 2-dimensional scalar field in the theory allows us to fix the parameter $\mathtt{x}$. Let us denote the 2-dimensional scalar field in the OPE of $\mathcal{L}\cdot\phi$ by $\widetilde{\phi}(x)$ and then $\widetilde{\phi}(x)\sim\phi(x)$. We will find that the coefficient of proportionality is $\chi = 2\mathtt{x}$. Indeed:

$$\langle 0|\widetilde{\phi}(x_1)\phi(x_3)|0\rangle = \lim_{x_2\to x_1}(x_{12}^2)^2\langle 0|\mathcal{L}(x_1)\phi(x_2)\phi(x_3)|0\rangle = \mathtt{x}\frac{1}{(x_{13}^2)^2} \tag{6.24}$$

$$\langle 0|\widetilde{\phi}(x_1)\phi(x_2)|0\rangle = \chi\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{2}\chi\frac{1}{(x_{12}^2)^2} \tag{6.25}$$

Here we have used (6.2) for $\langle\phi\phi\rangle$.

On the other hand:

$$\langle 0|\widetilde{\phi}(x_1)\widetilde{\phi}(x_2)|0\rangle = \chi^2\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \frac{\chi^2}{2}\frac{1}{(x_{12}^2)^2} \tag{6.26}$$

$$\langle 0|\widetilde{\phi}(y)\widetilde{\phi}(z)|0\rangle = (x_{12}^2)^2(x_{34}^2)^2\langle 0|\phi(x_1)\mathcal{L}(x_2)\Pi_{10}\mathcal{L}(x_3)\phi(x_4)|0\rangle|_{\substack{x_1=x_2=y\\ x_3=x_4=z}} \tag{6.27}$$

The partial wave in (6.27) has the form:

$$\langle 0|\phi(x_1)\mathcal{L}(x_2)\Pi_{10}\mathcal{L}(x_3)\phi(x_4)|0\rangle = \frac{1}{(x_{12}^2)^2(x_{23}^2)^2(x_{34}^2)^2}B_{10}\beta_{10}^2(s,t) \tag{6.28}$$

We know from [20] that $\beta_{\kappa 0}^{+2\kappa} = 1$, so we conclude that the part of (6.5) that contributes to this partial wave is $\frac{\widetilde{b}}{(x_{12}^2)^2(x_{23}^2)^2(x_{34}^2)^2}$. Inserting it in (6.27) we find:

$$\langle 0|\widetilde{\phi}(x_1)\widetilde{\phi}(x_2)|0\rangle = \frac{\widetilde{b}}{(x_{12}^2)^2} \tag{6.29}$$

We remember that $\widetilde{b} = 3$, then (6.26) and (6.29) give $\chi^2 = 6$ and hence $\mathtt{x}^2 = \frac{3}{2}$.

We find the function $\langle :\phi^2::\phi^2:\rangle$:

$$\begin{aligned}\langle 0|:\phi^2:(y):\phi^2:(z)|0\rangle &= \big(\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle -\\ &\quad -(\text{singular terms in } x_{12}^2 \text{ and } x_{34}^2)\big)\,|_{\substack{x_1=x_2=y\\ x_3=x_4=z}}\end{aligned} \tag{6.30}$$

Using (6.3) we arrive at:

$$\langle 0|:\phi^2:(x_1):\phi^2:(x_2)|0\rangle = \frac{3}{2}\frac{1}{(x_{12}^2)^4} \tag{6.31}$$

Let us now choose $\lambda = \mathtt{x}$. Then, substituting (6.1), (6.23), and (6.31) in (6.21) we end up with:

$$\langle 0|(:\phi^2:(x_1) - \mathtt{x}\mathcal{L}(x_1))(:\phi^2:(x_2) - \mathtt{x}\mathcal{L}(x_2))|0\rangle = 0 \tag{6.32}$$

and applying Reeh–Schlieder theorem we obtain:

$$:\phi^2:(x_2) - \mathtt{x}\mathcal{L}(x_2) = 0 \quad\rightarrow\quad :\phi^2:(x_2) = \mathtt{x}\mathcal{L}(x_2) \tag{6.33}$$

Hence, we reached the desired result that $\mathcal{L}\sim:\phi^2:$.

# Chapter 7

# Concluding remarks

In this thesis we illustrated that Wightman field theories, which are globally conformal invariant in the sense that their Wightman distributions are invariant under a conformal transformation in the domain of its definition, allow deep-reaching analysis. Their correlation functions, which in the Wightman approach contain all the information about the theory, are rational functions in the arguments differences with uniform bounds on the singularities. As we convinced ourselves in this work, this fact, together with the Huygens principle, which holds in every GCI theory, often makes the construction of 4- and of 5-point functions with finite number of parameters very easy. This is already a big profit, because 4-point functions can be explored to a great extent, as there is a technique of partial wave expansion applicable to them and allows exploiting of Wightman positivity. Furthermore, we know that if the connected part of the 4-point function of some field vanishes, then this field is a generalized free field. If the field in our theory with the quantum numbers of the stress–energy tensor is unique, then we obtain Ward identities for 4-point functions of the type $\langle \mathcal{A}\mathcal{A}\mathcal{B}\mathcal{B}\rangle$. Then studying only of 4- and of the related to them 5-point functions often will be enough for our purposes, for example in two of the three cases under consideration in this thesis we could obtain by such analysis full information about the models and we could gain some insight for the third model.

In model 1 we proved that the 3-dimensional scalar field $W(x)$, generating a theory without a stress–energy tensor, is a generalized free field. We achieved this, adding the 5-point functions consideration to the standard methods of Wightman positivity, operator product content analysis and Cauchy–Schwarz inequalities applied to 4-point functions composed of $W(x)$ and of the 4-dimensional scalar $\mathcal{L}(x)$ in its OPE. $\langle \mathcal{L}WWWW\rangle$ is related to $\langle \mathcal{L}WW\mathcal{L}\rangle$ via a projection on $\mathcal{L}(x)$ in the OPE of two copies of $W(x)$ in the 5-point function and its exploration gives a crucial new information.

In model 3 the important tool in our analysis were Ward identities. In combination with partial wave analysis and Cauchy–Schwarz inequalities and working exclusively on the 4-point level, they allowed us to prove the triviality of the model of the 4-dimensional scalar $\mathcal{L}(x)$ when it is coupled to a massless free scalar in a theory with a unique stress–energy tensor with a central charge $c = 1$.

Model 2 appeared to be not so susceptible to our analysis. There we explore a theory, containing a free massless Dirac field $\psi(x)$ and a 4-dimensional scalar field $\mathcal{L}(x)$. As there is no such a field in the Dirac theory, and we exclude the trivial Yukawa solution by the assumption of absence of 1-dimensional scalar in the OPE of $\mathcal{L}(x)$ with $\mathcal{W} :=: \overline{\psi}\psi : (x)$, we expect to find a contradiction.

We assume also the uniqueness of a stress–energy tensor and of a dimension 2 scalar. However, even Ward identities and 5-point functions analysis did not help us to find an inconsistency. But even then we still end up with a lot of structural information, eliminating 10 of the parameters of $\langle \mathcal{W}\mathcal{W}\mathcal{L}\mathcal{L} \rangle$ and $\langle \mathcal{W}\mathcal{L}\mathcal{L}\mathcal{W} \rangle$ by partial wave analysis, one of the remaining two of which is fixed by Ward identities and Wightman positivity leaves a narrow range for the other. The 5-point functions $\langle \mathcal{L}\mathcal{L}\mathcal{L}\mathcal{W}\mathcal{W} \rangle$ were also left with two parameters, related to those of the mixed 4-point functions.

Although the techniques for analysis, used in this thesis, are mainly good for obtaining of "negative results", i.e. for proving the triviality of models, they are of constructive interest because they help to distinguish the trivial cases and because they still provide us with some insight about the models even if they cannot be excluded by these methods. Such information will be useful in the efforts for constructing non-trivial models with more advanced techniques, such as the Vertex algebra apparatus.

# Appendix A

# Partial Wave Coefficients

## A.1 PWE coefficients of $\langle WWWW \rangle$ and Wightman positivity

Following the prescription of section 3.3, we perform the partial wave expansion of $\langle WWWW \rangle$. The coefficients $B_{\kappa L}$ are listed below. We remind that when $A(x) = B(x)$ in (3.10), then all the coefficients with odd $L$ vanish.

1. *for* $\kappa = 1$*,* $L$ *even:*

$$B_{1L} = 2\frac{(L!)^2}{(2L)!}\Big\{\big[1 + L(L+1)\big]a + b\Big\} \tag{A.1}$$

2. *for* $\kappa \geq 2$*,* $L$ *even:*

$$\begin{aligned} B_{kL} \;\; = \;\; & 2\frac{((\kappa+L-1)!)^2}{(2\kappa+2L-2)!}\frac{((\kappa-2)!)^2}{(2\kappa-4)!}\bigg\{\frac{1}{4}(\kappa+L-1)(\kappa+L)(\kappa-2)(\kappa-1)\times \\ & \times(2\kappa+L-2)(L+1) + a(-1)^\kappa(\kappa+L-1)(\kappa+L)(\kappa-2)(\kappa-1) + \\ & +(a+b)\Big(2(\kappa+L-1)(\kappa+L) - 2(\kappa-2)(\kappa-1) - \\ & -(-1)^\kappa\big[(\kappa-2)(\kappa-1) - (\kappa+L-1)(\kappa+L)\big]\Big) + c(-1)^\kappa\bigg\} \end{aligned} \tag{A.2}$$

According to section 3.3, all these coefficients must be non-negative in order Wightman positivity to be satisfied. In the 3-dimensional space, defined by the three parameters $a$, $b$ and $c$, the region in which these conditions are fulfilled is restricted by the following inequalities:

1. $a \geq 0$, (coming from $B_{\kappa L} \geq 0$ for $\kappa = 1$, $L = \infty$);

2. $a + b \geq 0$, (coming from $B_{\kappa L} \geq 0$ for $\kappa = 1$, $L = 0$);

3. $c \geq -(Xa + Yb + Z)$ where $X$, $Y$, $Z$ are positive monotonously growing polynomial functions in $\kappa$, calculated straightforward from (A.2) under the condition $B_{\kappa L} \geq 0$ for even $\kappa$ greater than 1 and $L = 0$;

4. $c \leq -Ua + Vb + W$ where $U$, $V$, $W$ are positive monotonously growing polynomial functions in $\kappa$, calculated straightforward from (A.2) under the condition $B_{\kappa L} \geq 0$ for odd $\kappa$ greater than 1 and $L = 0$.

## A.2 PWE coefficients of $\langle W\mathcal{L}\mathcal{L}W\rangle$ under the condition of absence of SET (Model 1)

In this section we perform partial wave analysis of $\langle W\mathcal{L}\mathcal{L}W\rangle$ in the conditions of Model 1. The partial wave coefficients, in combination with the Wightman positivity analysis, are as follows:

1.
$$B_{\frac{3}{2}L} = \frac{L!(L+1)!}{(2L+1)!}\Big\{(-1)^L\big[A(L+1)^2+B\big]+D(L+1)\Big\} \tag{A.3}$$

We observe that at the limit $L\to\infty$ the dominating term $(-1)^L A(L+1)^2\frac{L!(L+1)!}{(2L+1)!}$ has an alternating sign. Then Wightman positivity will be violated either for even or for odd values of $L$ unless the parameter $A=0$;

2.
$$B_{\frac{5}{2}L} = \frac{(L+1)!(L+2)!}{(2L+3)!}\Big\{B(-1)^L(L+1)(L+3)+C\big[(-1)^L+L+2\big]\Big\} \tag{A.4}$$

with the same argumentation as in the previous case we show that $B=0$;

3.
$$\begin{aligned} B_{\frac{7}{2}L} \quad = \quad & \frac{(L+2)!(L+3)!}{(2L+5)!}\Bigg\{\frac{c+2}{12}(L+1)(L+2)(L+3)(L+4)(L+5)- \\ & -C\left[\frac{1}{3}(L+3)+\frac{2}{3}(-1)^L\right]+D\Big[(-1)^L\big((L+2)(L+4)-1\big)+L+3\Big]\Bigg\} \end{aligned} \tag{A.5}$$

4. For $\kappa = m+\frac{1}{2} > \frac{7}{2}$ we obtain:

$$\begin{aligned} B_{m+\frac{1}{2},L} &= \frac{(m+L-1)!(m+L)!}{(2m+2L-1)!}\frac{(m-2)!(m-1)!}{(2m-3)!}\times \\ \times\Bigg\{\frac{c+2}{24}&(m+L-1)(m+L)(m+L+1)(m-2)(m-1)m(2m+L-1)(L+1)+ \\ &+C\Big[(-1)^{m+L}(m-1)+(-1)^m(m+L)\Big] \\ &+D\Big((-1)^L\big[(m+L-1)(m+L+1)-(m-2)m\big]- \\ &\Big[(-1)^{m+L}(m+L-1)(m+L+1)(m-1)+(-1)^m(m-2)m(m+L)\Big]\Big)\Bigg\} \end{aligned} \tag{A.6}$$

Further analysis would show that Wightman condition restricts the coefficients $C$ and $D$ to span a finite region in the 2-dimensional space defined by $C$ and $D$. This region is bounded by $C\geq 0$, $\frac{C}{6}-(c+2)\leq D$ and $6(c+2)-\frac{C}{15}\geq D$. These conditions come from the non-negativity of the coefficients $B_{\kappa L}$ respectively for $\kappa=\frac{5}{2},\ L=0$; $\kappa=\frac{7}{2},\ L=0$ and $\kappa=\frac{7}{2},\ L=1$. These are the strongest restrictions that $B_{\kappa L}\geq 0$ implies.

## A.3 Partial wave analysis of $\langle W\mathcal{L}\mathcal{L}W\rangle$ in the setting of Model 2

In this section we list the results from the partial wave analysis of $\langle W\mathcal{L}\mathcal{L}W\rangle$ in the setting of Model 2. The coefficients in the partial wave expansion are as follows:

1.
$$B_{\frac{3}{2}L} = \frac{L!(L+1)!}{(2L+1)!}\Big\{(-1)^L\big[A(L+1)^2+B\big]+\frac{1}{2}FL(L+1)(L+2)\Big\} \tag{A.7}$$

2.
$$B_{\frac{5}{2}L} \;=\; \frac{(L+1)!(L+2)!}{(2L+3)!}(A+B)(-1)^L(L+1)(L+3) \tag{A.8}$$

3.
$$\begin{aligned} B_{\frac{7}{2}L} \;=\; & \frac{(L+2)!(L+3!)}{(2L+5)!}\left\{\frac{1}{12}(L+1)(L+2)(L+3)(L+4)(L+5)-\right. \\ & +(A+B)(-1)^L\left[\frac{1}{3}(L+2)(L+4)-1\right]+ \\ & \left.+F(L+2)(L+4)\left[(-1)^{L+1}-\frac{1}{2}(L+3)\right]\right\} \end{aligned} \tag{A.9}$$

4. For $\kappa = m+\frac{1}{2} > \frac{7}{2}$ we obtain:

$$\begin{aligned} B_{m+\frac{1}{2},L} = & \frac{(m+L-1)!(m+L)!}{(2m+2L-1)!}\frac{(m-2)!(m-1)!}{(2m-3)!}\times \\ \times\Bigg\{ & \frac{1}{24}(m+L-1)(m+L)(m+L+1)(m-2)(m-1)m(2m+L-1)(L+1)+ \\ & +(A+B)(-1)^L\big[(m+L-1)(m+L+1)-(m-2)m\big]+ \\ & +\frac{1}{2}F(m+L-1)(m+L+1)(m-2)m\Big[(-1)^{m+L}(m-1)+(-1)^m(m+L)\Big]\Bigg\} \end{aligned} \tag{A.10}$$

Wightman positivity requires that $B=-A$, otherwise the condition $B_{\frac{5}{2}L}\geq 0$ will be violated because of the alternating sign. The strongest restrictions that we obtain for $F$ and $A$ are $0\leq F\leq \frac{1}{2}$ and $-F\leq A\leq \frac{3}{2}F$, coming from $B_{\kappa L}\geq 0$ for $\kappa=\frac{3}{2},\ L\to\infty$; $\kappa=\frac{7}{2},\ L=0$; $\kappa=\frac{3}{2},\ L=1$ and $\kappa=\frac{3}{2},\ L=2$.

# Appendix B

# Harmonicity of $V_{\frac{3}{2}}^{\mathcal{L}W}(x_1, x_2)$

In section 3.1 we discussed that $V_1^{\phi\phi}(x_1, x_2)$, where $\phi(x)$ is a hermitian scalar field of any dimension, is harmonic in both arguments. Now we will show that $V_{\frac{3}{2}}^{\mathcal{L}W}(x_1, x_2)$, where $W(x)$ and $\mathcal{L}(x)$ are hermitian scalars of dimensions 3 and 4, is harmonic in its second argument. Our analysis is based on the assumption that $V_{\frac{3}{2}}^{\mathcal{L}W}(x_1, x_2)$ is bilocal.

We will prove this statement relating it to the harmonicity of the 4-point function of $V_{\frac{3}{2}}^{\mathcal{L}W}(x_1, x_2)$. We will consider the following steps:

1. We find that $\langle 0|V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2)V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle = \frac{1}{x_{13}^2 x_{23}^2 x_{24}^2}\frac{1}{u-v}\left(f(u) - f(v)\right) = \frac{1}{x_{13}^2 x_{23}^2 x_{24}^2}F(s,t)$

   - because of the orthogonality property (3.7) (plus (3.12)) we know that:

$$\begin{aligned}
\langle 0|V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2)V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle &= (x_{12}^2)^2(x_{34}^2)^2\sum_L \langle 0|W(x_1)\mathcal{L}(x_2)\Pi_{\frac{3}{2}L}\mathcal{L}(x_3)W(x_4)|0\rangle \\
&= \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2}\sum_L B_{\frac{3}{2}L}\cdot\beta^1_{\frac{3}{2}L}(s,t) \qquad \text{(B.1)}
\end{aligned}$$

   - (3.13) and (3.18) say that:

$$\beta^1_{\frac{3}{2}L} = \frac{uv}{u-v}\left(G^1_{1+L}(u) - G^1_{1+L}(v)\right) \qquad \text{(B.2)}$$

   and that such partial waves are produced by and only by the part of $(x_{12}^2)^2(x_{34}^2)^2\langle W\mathcal{L}\mathcal{L}W\rangle$ of the form:

$$\frac{1}{x_{13}^2 x_{23}^2 x_{24}^2}\frac{1}{u-v}\left(f(u) - f(v)\right) = \frac{1}{x_{13}^2 x_{23}^2 x_{24}^2}F(s,t) \qquad \text{(B.3)}$$

   This part gives only partial waves twist 3 and hence these are all the terms of $\langle 0|V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2)V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle$.

2. We prove that $\partial_{x_1}^2\langle 0|V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2)V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle = 0$, i.e. that it is harmonic in its first argument.

- Following [20] and denoting $\rho_{ij} := x_{ij}^2,\ \rho = \{\rho_{ij},\ 1 \leq i < j \leq 4\}$, we prove the equivalence:

$$\partial_{x_1}^2 f(\rho) = -\mathcal{D}_1 f(\rho) = 0, \quad \mathcal{D}_1 := 4 \sum_{1<i<j\leq 4}, \quad \rho_{ij} \frac{\partial^2}{\partial \rho_{1i} \partial \rho_{1j}} \tag{B.4}$$

- Using (B.3) and (B.4) we show that:

$$\partial_{x_1}^2 \langle 0| V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2) V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle = 4 \frac{x_{23}^2 x_{34}^2}{(x_{13}^2)^3 x_{24}^2} \mathcal{D}_{st} F(s,t) \tag{B.5}$$

$$\mathcal{D}_{st} := \left( s\frac{\partial^2}{\partial s^2} + t\frac{\partial^2}{\partial t^2} + (s+t-1)\frac{\partial^2}{\partial s \partial t} + 2\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial t} \right) \tag{B.6}$$

- Then we remember that $F(s,t) = \frac{1}{u-v}(f(u) - f(v))$ and transform the differential operator $\mathcal{D}_{st}$ into another one in the variables $u$ and $v$ using (3.15):

$$\mathcal{D}_{st} F(s,t) = \frac{1}{u-v} \frac{\partial^2}{\partial u \partial v}(f(u) - f(v)) \tag{B.7}$$

  Obviously, the mixed differential operator will eliminate the function in the brackets, which consists only of mono-variable terms. This means that (B.5) vanishes, which proves our statement that the 4-point function of $V_{\frac{3}{2}}^{\mathcal{L}W}$ is harmonic in its first argument.

3. By similar argumentation we show that $\langle 0| V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2) V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle$ is harmonic in its argument $x_4$. This implies that:

$$\langle 0| \partial_{x_1}^2 V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2) \partial_{x_4}^2 V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle = 0 \tag{B.8}$$

4. (B.8) together with $V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2) = \left(V_{\frac{3}{2}}^{\mathcal{L}W}(x_2, x_1)\right)^*$ (provided that $V_{\frac{3}{2}}^{\mathcal{L}W}(x_1, x_2)$ is bilocal) leads to:

$$\partial_{x_4}^2 V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)|0\rangle = 0 \tag{B.9}$$

5. Then, we use Reeh–Schlieder theorem to conclude that $V_{\frac{3}{2}}^{\mathcal{L}W}(x_3, x_4)$ is harmonic in its argument $x_4$ and respectively $V_{\frac{3}{2}}^{W\mathcal{L}}(x_1, x_2)$ is harmonic in $x_1$. This was what we wanted to prove.

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