Relations between 2D and 4D Conformal Quantum Field Theory

Diploma thesis

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1 Introduction

The study of conformal field theories (CFTs) has been a recurring theme in 20th century theoretical and mathematical physics. Conformal invariance is certainly not an exact symmetry of nature. The appearance of atomic structure and elementary particles with characteristic lengths and masses showed, that even the weaker dilatation symmetry is not directly realized.
We mention a few important results in the field: In 1909 Cunningham and Bateman showed conformal invariance of the Maxwell equations. Weyl discussed in his book [33] local scale transformations in general relativity in an attempt to unify gravitation and electromagnetism [33]. Although this approach failed, these ideas were very influential, because they marked the beginning of gauge theory.
Dirac proved in the 1930s, that the massless version of his famous equation in relativistic quantum mechanics is invariant under conformal transformations [4].
From the late 1960s on, results like the indication for a scaling law in the deep inelastic electron-proton scattering stimulated an active phase of research in the field (although this particular fact later turned out to be a manifestation of asymptotic freedom). Conformal field theories require either continuous or vanishing mass spectrum, so they were seen as an approximation in high energy physics, where the rest mass of the particles is neglectible, and also as a training ground for the study of field theories with a slightly bigger symmetry group than the Poincaré group, for which attempts to construct a non-pertubative scheme had failed to succeed. A review of and references to this early time can be found in [30].
The seminal papers [2] and [9] started an intense research in the field of twodimensional conformal field theories. Since any holomorphic function is angle preserving, after a Wick rotation from Minkowski to Euclidean space, methods from complex analysis could be applied and with the help of new results from mathematics (like the study of infinitedimensional Lie algebras and the calculation of the Kac determinant) one was able to identify certain models with models well known from statistical mechanics. Furthermore 2D CFT has found ample applications in string theory and made it possible to study several exact models based on affine Lie algebras. Textbooks and reviews of this subject are e.g. [11] or [3].
Interest in higherdimensional conformal field theory was revived in 1997, when the AdS/CFT correspondence was conjectured by Maldacena [20]. It states, that there is an equivalence between a string theory on some space, like e.g. a product of Anti-DeSitter space and a sphere, and a conformal field theory without gravity living on the boundary of that space.
A recent review of the history of research concerning conformal field theories can be found in [17], which also contains an extensive bibliography.
In 2000 Todorov and collaborators initiated a program, in which conformal invariance, Wightman axioms and insights from the twodimensional case were used to study higherdimensional, in particular $D = 4$, theories. Under the assumption of so-called global conformal invariance (GCI), fields fulfil the Huygens principle, which says, that the commutator of two fields is only supported on the light-cones. Together with energy positivity, this implies rationality of correlation functions with a very restricted pole structure. This can be used

to set up a higher dimensional analogon of twodimensional vertex operators [25]. It also makes possible the closer study of Wightman functions with more than three points [1]. For four-point functions, the partial wave expansion (PWE) has proven to be a useful tool for the study of positivity. Higher correlation functions in global conformally invariant QFT have the possibility to display a pole structure, which cannot arise from free field realizations ("double poles"). Therefore it would be desirable to find a method to determine, to which extent this property is compatible with positivity.

The organization of this work will be as followed: In section 2 we start with a summary of axiomatic or "general" quantum field theory, followed by a paragraph, in which we discuss the basics facts about the conformal group and its use in field theory.
Then we will discuss the properties of correlation functions in conformal field theory and describe the idea of partial wave expansions. We close the introductory part with a discussion of the mentioned program of studying global conformally invariant quantum field theory.

In two dimensions the action of the conformal group factorizes into an action on the chiral (light-cone) variables $x^0 \pm x^1$. We will later use this fact and restrict fourdimensional objects (namely the correlation functions) to two dimensions. The simplest conceivable restriction is the one to the submanifold $x^2 = x^3 = 0$. It is related to a specific embedding of the twodimensional conformal algebra into the fourdimensional one. There is however a second inequivalent embedding and we will investigate in section 3, whether it corresponds to another twodimensional submanifold, on which the twodimensional conformal group acts chirally.
In section 4 we relate the partial wave expansion of four-point functions in two and four dimensions by expanding the latter in a series of the former. This expansion is then compared to the group theoretic decomposition of a representation of the fourdimensional conformal group into representations of the twodimensional subgroup.
A six-point function was found, which cannot be realized by free fields in [26]. Unfortunately, general fourdimensional partial waves in this case are extremely difficult to obtain. One possibility to perform "a simpler version" of a PWE is to restrict the general form of a six-point function to two dimensions, obtain the PWE there, expand the "non-trivial" six-point function and see, what obstructions arise from demanding Wightman positivity in two dimensions. If a contradiction comes up, the double-pole structure can also be ruled out in four dimensions. This will be addressed in section 5.

[1]Note, that two and three point functions in conformal field theories are fixed up to normalization.

Notation:

We give a list of notations used in the main text.

$\mathcal{M}_D$	D-dimensional Minkowski space
D	Spacetime dimension
$x = x^\mu = (x^0, .., x^{D-1})$	Point in $\mathcal{M}_D$
$\vec{x} = (x^1, .., x^{D-1})$	Spatial components of a point x
$diag(..)$	Diagonal matrix
$\eta_{\mu\nu}$	Minkowski metric $diag(+, -, .., -)$ on $\mathbb{R}^D$
$x \cdot y = x_\mu y^\mu = \eta_{\mu\nu} x^\mu y^\nu$	Scalar product of $x, y \in \mathcal{M}_D$
x^2	$x \cdot x$
$\partial_\mu = \frac{\partial}{\partial x^\mu}$	Partial differentiation by x^μ
$\mathcal{H}$	Hilbert space of the QFT
$Aut(\mathcal{H})$	Automorphisms of $\mathcal{H}$
$U(g)$	Unitary operator associated with group element g
$\mathcal{L}_+^\uparrow$	Proper orthochronous Lorentz group
$\mathcal{P}_+^\uparrow$	Proper orthochronous Poincaré group
$\phi(x)$	Quantum field (operator valued distribution)
$\phi(f)$	Quantum field smeared with test function f
Ω	Vacuum vector in $\mathcal{H}$
W_n	Wightman distributions
$\overline{\mathcal{M}_D}$	D-dimensional compactified Minkowski space
$\overline{\mathcal{M}_{4,\mathbb{C}}}$	Complexification of $\overline{\mathcal{M}_D}$
$\mathcal{K}_D$	Dirac cone for $\overline{\mathcal{M}_D}$
η_{ab}	Metric $diag(+, -, .., -, +)$ on $\mathbb{R}^{D+2}$
ξ^a	Vector in the Dirac cone
$\mathcal{C}_D$	Conformal group in D dimensions
$SO(p, q)$	Pseudo-orthogonal group of signatur (p, q)
$SU(p, q)$	Pseudo-unitary group of signatur (p, q)
$C_D^{(n)}$	n-th order Casimir operator of $\mathcal{C}_D$
C_D^n	Value of n-th order Casimir operator of $\mathcal{C}_D$
$B_{nm}(u, v)$	Twodimensional partial wave
$G_n(u)$	Onedimensional partial wave
$F(a, b; c; x)$	Hypergeometric function
$(a)_n$	Pochhammer symbol

2 The setting

2.1 General quantum field theory

Since the study of conformal field theories mostly takes place in the axiomatic Wightman setting, we will give a short account of it in this section.

In the 1950s, quantum field theory (QFT) had turned into a great success story since its birth thirty years before. Using pertubation theory and so-called renormalization, quantum electrodynamics was able to explain experimental observations such as the anomalous magnetic moment of the electron and the Lamb shift with astonishing precision. The study of non-abelian gauge groups started with the works of Yang and Mills, which led to the electroweak theory and later to the standard model of elementary particle physics. In spite of these successes, there remained some open problems:

- A theory based entirely on pertubation theory seemed conceptually unsatisfactory.
- Haag's theorem showed, that mathematically the interaction picture used throughout pertubative QFT does not exist.
- It was unclear, why renormalizability should be a fundamental requirement for an admissible quantum field theory.
- The prescriptions for the correct calculations of higher order pertubations rapidly become very complicated and hard to communicate.
- There were situations like the strong interaction, where pertubation theory was not applicable at all.

Therefore Streater, Garding, Wightman and others attempted to give quantum field theory a mathematically solid foundation ([29],[15]), a program which was called axiomatic or general quantum field theory. They formulated a set of independent and compatible requirements or "axioms", that any reasonable relativistic quantum field theory on Minkowski space $\mathcal{M}_4$ should fulfil. The term "axiom" should not be taken too literally here, but more in the sense of "plausible working hypothesis", which might undergo some changes in the final formulation of the theory.
Within this setting, one was able to reproduce familiar results from QFT like the spin-statistics- and the CPT-theorem. Haag and Ruelle also succeeded in formulating a rigorous scattering theory.
One was however not yet able to construct field theories besides the (generalized) free ones, that fulfil the axioms.

Here is what Wightman et al. proposed:

- **Poincaré invariance:** States of the theory correspond to rays in a Hilbert space $\mathcal{H}$, which carries a unitary representation U of the proper, orthochronous Poincaré group $\mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{R}^4$:

$$\begin{aligned} U: \mathcal{P}_+^\uparrow &\rightarrow Aut(\mathcal{H}) \\ g &\mapsto U(g) \end{aligned} \tag{2.1}$$

  where g is of the form (Λ, a) with $\Lambda \in \mathcal{L}_+^\uparrow$ a proper, orthochronous Lorentz transformation and a a translational four vector. The Lorentz transformations are the homogeneous linear transformations Λ, such that $\Lambda x \cdot \Lambda y = x \cdot y$ for all $x, y \in \mathcal{M}_4$. They are called proper, if $\det(\Lambda) = +1$ and orthochronous, if $\Lambda^0_0 > 0$.
  The action of g on a spacetime point is then

$$x \mapsto \Lambda x + a. \tag{2.2}$$

  Demanding unitarity asserts the probabilistic interpretation of quantum theory.

- **Spectrum condition/Energy positivity:** The spectrum of the generator of translations P^μ in the unitary operator $U(1, a) = \exp(ia_\mu P^\mu)$ lies in the closure $\overline{V^+}$ of the forward light cone

$$V^+ = \{\, x = (x_0, \vec{x}) \in \mathbb{R}^4 \ , \ |x_0| > |\vec{x}| \,\} \tag{2.3}$$

  This reflects the idea, that the possible energy levels of a system should be bounded from below.

- **Fields:** The fields $\phi_i(x)$ ($i = 1..N$) of the theory are operator valued distributions

$$\phi_i(f) = \int dx\ \phi_i(x) f(x), \tag{2.4}$$

  which are in general unbounded [1] and together with their adjoints $\phi_i(f)^\dagger$ are defined on a dense, Poincaré invariant domain $\mathcal{D} \subset \mathcal{H}$. $\mathcal{D}$ contains the vacuum (see below) and is invariant under the action of the field algebra. The functions f are out of some appropiate test function space, e.g. the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ of smooth functions, which together with all their derivatives decrease faster at infinity than any polynomial.

- **Transformation of fields:** Under the action of the Poincaré group $\mathcal{P}_+^\uparrow$ the fields transform as

$$U(a, \Lambda)\phi_i(f)U(a, \Lambda)^{-1} = \sum_j S_{ij}(\Lambda^{-1})\phi_j(f_{a,\Lambda}), \tag{2.5}$$

  where $f_{a,\Lambda}(x) = f(\Lambda^{-1}(x - a))$ and S_{ij} is a suitable representation of $\mathcal{P}_+^\uparrow$.

[1] Therefore the integral sign should be understood symbolically.

- **Vacuum:** There exists a unique ray Ω in $\mathcal{D} \subset \mathcal{H}$ called the vacuum vector, which is invariant under the Poincaré action:

$$U(g)\Omega = \Omega \tag{2.6}$$

for all $g \in \mathcal{P}_+^\uparrow$. It is a cyclic vector for $\mathcal{H}$, i.e. the span of all terms of the form

$$\phi_{i_1}(f_1)...\phi_{i_k}(f_k)\Omega \ , \ \ k \in \mathbb{N} \ , \ \ f_i \in \mathcal{S}(\mathbb{R}^4) \ , \ \ i_j \in \{1, .., N\} \tag{2.7}$$

is dense in $\mathcal{H}$.

- **Locality:** If the supports of two test functions f and g are mutually spacelike separated, the commutator of any two field operators ϕ_i and ϕ_j smeared with them vanishes:

$$[\phi_i(f), \phi_j(g)] = 0 \tag{2.8}$$

The unsmeared version of this reads $[\phi_i(x), \phi_j(y)] = 0$, if $(x - y)^2 < 0$. For fermions appropiate incorporation of anticommutation relations can be done.
This requirement is also called Einstein causality and reflects the experimental fact, that no causal signal can travel faster than the speed of light.

- **Asymptotic completeness:** In scattering theory, the incoming and outgoing free particle states are labelled by their momenta and spin. These states span Hilbert spaces $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$. One requires, that these spaces coincide with the Hilbert space of the theory:

$$\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out}. \tag{2.9}$$

In this formulation of the axioms, the field algebra is the central object under consideration. One can also define the theory in terms of the Wightman distributions

$$\mathcal{W}_n(f_1, .., f_n) = \langle \Omega, \phi_1(f_1)..\phi_n(f_n)\Omega \rangle \tag{2.10}$$

with $f_i \in \mathcal{S}(\mathbb{R}^4)$ for all $i = 1, ..n$. They are also called correlation or n-point functions. $\mathcal{W}_n(f_1, .., f_n)$ is a tempered distribution in each f_i separately. Schwartz' nuclear theorem then implies, that there is a unique tempered distribution $\mathcal{W}_n(f)$ for $f \in \mathcal{S}(\mathbb{R}^{4n})$. We denote its unsmeared version by

$$W_n(x_1, .., x_n) = \langle \Omega, \phi_1(x_1)..\phi_n(x_n)\Omega \rangle. \tag{2.11}$$

The axioms can then be formulated as conditions on the $\mathcal{W}_n$ or the W_n, respectively. We will work only with the W_n in the following, bearing in mind, that for the involved objects to be well defined, in general one needs to smear them with appropiate test functions.
Both formulations are by construction equivalent and one has a scheme, which allows to reconstruct a unique QFT (up to unitary equivalence), once all the correlation functions are known. So finding a procedure, that facilitates their calculation within a given setting, is one possible approach to the formulation of a quantum field theory.

2.1.1 General conformal quantum field theory

It is clear, that if the symmetry group of spacetime is changed from the Poincaré group to the conformal group $\mathcal{C}_4 = SO_e(4,2)/\mathbb{Z}_2$, the Wightman axioms have to be changed in the appropiate places:

1. The Poincaré invariance is replaced by the invariance under $\mathcal{C}_4$ and the existence of a unitary representation of it on the Hilbert space $\mathcal{H}$.
2. Fields transform under induced representations of $\mathcal{C}_4$.
3. The vacuum is invariant under the whole action of $\mathcal{C}_4$.
4. The requirement of asymptotic completeness is dropped, because conformal quantum field theories are not particle theories (they have trivial S-matrix).

In this next section we consider in more detail, what the properties of a theory with those requirements are.

2.2 The conformal group and conformal field theory

In conformal field theory (CFT) the Poincaré group, the group of spacetime symmetries in ordinary QFT, is expanded to a bigger one, the conformal group. It is the maximal group preserving the causal structure of a spacetime (i.e. timelike/spacelike/null vectors in the tangent space are mapped to timelike/spacelike/null ones).
We will now introduce the group of conformal transformations and its properties, especially the ones relevant for field theory. We will stepwise reduce the generality, going from general manifolds to D-dimensional Minkowski space $\mathcal{M}_D$ and finally to the cases, which are of central interest in this work, namely $D = 2$ and $D = 4$.

2.2.1 Conformal transformations

Let $x \to x'(x)$ be a map from a (Semi-)Riemannian manifold N with metric $g_{\mu\nu}$ to itself. Then the mapping is called *conformal*, if it leaves $g_{\mu\nu}$ invariant up to a local scale factor $\Omega(x) > 0$:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}(x). \tag{2.12}$$

The name conformal stems from the fact, that these transformations preserve angles between two vectors, but they can change their respective lengths.
One possible way to introduce their precise form is to consider the conformal Killing equation (CKE)

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = f(x) g_{\mu\nu}(x). \tag{2.13}$$

Here $f(x)$ is a scalar function and ∇_ν is the covariant derivative associated with the Levi-Civita connection of $g_{\mu\nu}$.
Just as the ordinary Killing equation arises from demanding invariance of the metric under an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, the CKE arises from the requirement, that the associated change of the metric is proportional to the original metric (see e.g. [32]).

2.2.2 The conformal transformations of Minkowski space $\mathcal{M}_D$

We are in this work only interested in the flat Minkowski space $\mathcal{M}_D$, so

$$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, , .., -1) \tag{2.14}$$

and $\nabla_\mu = \partial_\mu$. We first take arbitrary spacetime dimension D and consider the solutions of the CKE.
For $f(x) = 0$ we have the ordinary Killing equation, which has two types of solutions, the translations

$$P^\alpha_\mu = -\delta^\alpha_\mu \tag{2.15}$$

and the Lorentz transformations

$$M^\alpha_{\mu\nu} = x_\mu \delta^\alpha_\nu - x_\nu \delta^\alpha_\mu. \tag{2.16}$$

They generate the isometries of Minkowski space (the Poincaré group). Furthermore, for $f(x) = -2$ there are the dilatations

$$D^\alpha = -x^\alpha \tag{2.17}$$

and for $f = 4x_\mu$ the special conformal transformations

$$K_\mu^\alpha = 2x_\mu x^\alpha - x^2 \delta_\mu^\alpha. \tag{2.18}$$

The α is a Killing vector index and the μ, ν are spacetime indices. The chosen nomenclature will become clear from the finite transformations belonging to these generators (see below). One can show, that any other solution of the CKE can be expanded in these

$$n = \frac{(D+1)(D+2)}{2} \tag{2.19}$$

solutions. This is also the maximum number of solutions a CKE in a D dimensional space V can have. The conformal group of V is then called maximal and this is the case, if and only if V is conformally flat [30], i.e. if its metric $g_{\mu\nu}$ is related to the Minkowskian as

$$g_{\mu\nu}(x) = \Lambda(x)\eta_{\mu\nu} \tag{2.20}$$

for some smooth, strictly positive function $\Lambda(x)$.

From the conformal Killing vectors ξ_i^α $(i = 1..n)$ we form the vector fields

$$X_i = \xi_i^\alpha \partial_\alpha \tag{2.21}$$

with $\{\partial_\alpha = \frac{\partial}{\partial x^\alpha}\}$ being the coordinate basis of $T\mathcal{M}_D$.
They generate the following finite transformations:

$$\text{Translations} \quad x^\mu \to x^\mu + a^\mu \tag{2.22}$$

$$\text{Lorentz transformations} \quad x^\mu \to \Lambda^\mu_\nu x^\nu \tag{2.23}$$

$$\text{Dilatations} \quad x^\mu \to \lambda x^\mu \tag{2.24}$$

$$\text{Special conformal transformations} \quad x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \tag{2.25}$$

The special conformal transformations K^μ can be understood as the composition of a Weyl reflection

$$I_r : x^\mu \to -\frac{x^\mu}{x^2} \tag{2.26}$$

and a translation P_μ, followed by another Weyl reflection:

$$K^\mu = I_r \circ P_\mu \circ I_r. \tag{2.27}$$

We note at this point, that the conformal analogue of Wightman positivity usually is to require, that the "conformal Hamiltonian" $H^0 = \frac{1}{2}(P^0 + K^0)$ has non-negative spectrum. Because of $K^0 = I_r \circ P_0 \circ I_r = I_r \circ P^0 \circ I_r$, this notion is equivalent to Poincaré positivity $P^0 \geq 0$.

2.2.3 The conformal algebra and conformal fields

The generators of the conformal transformations (the conformal Killing vector fields X_i) form a Lie algebra, which is called the conformal algebra. We focus now on the algebraic properties of this algebra. From the viewpoint of the Wightman axioms, we are interested in

the unitary implementations of the associated Lie group on the Hilbert space of the theory and the induced field representations.
We first have to clarify, what the properties of fields in conformally invariant theories are. We recall, that Wigner's analysis [34] of the positive energy representations of the Poincaré group led to a classification of covariant fields according to their mass m and spin s. In conformal field theory one does the same and labels fields by the quantum numbers of the unitary positive energy representations of the conformal group $\mathcal{C}_D$. We will discuss these representation later for the cases $D = 2$ and $D = 4$ in more detail.
A common feature for fields in all spacetime dimensions D is, that one of the quantum numbers, the so-called scaling dimension d, describes the behaviour of a field $\psi(x)$ under a conformal transformation $x \to g(x)$.
The transformation is implemented by a unitary operator $U(g)$ and has the form

$$\psi(x) \mapsto U(g)\psi(x)U(g)^\dagger = \left|\frac{\partial g(x)^\mu}{\partial x^\nu}\right|^{-d/D} \cdot \psi(g(x)), \tag{2.28}$$

where $\left|\frac{\partial g(x)^\mu}{\partial x^\nu}\right|$ is the Jacobi determinant of the transformation $g(x)$ and $U(g)^\dagger = U(g)^{-1}$ the adjoint of $U(g)$. Since the translations and Lorentz transformations are isometries, the prefactor is only non-trivial for the scale transformations and the special conformal transformations.
By Stone's theorem, unitary transformations are generated by hermitean generators Y_i as

$$U(g) = \exp(it^i Y_i) \tag{2.29}$$

with certain parameters t^i. For the conformal transformations we denote these generators in accordance with the previous names by P_μ, $M_{\mu\nu}$, D and K_μ ($\mu, \nu = 0, .., D-1$).
They fulfil the following list of commutation relations among each other:

$$[D, P_\mu] = -iP_\mu \tag{2.30}$$
$$[D, K_\mu] = iK_\mu \tag{2.31}$$
$$[D, M_{\mu\nu}] = 0 \tag{2.32}$$
$$[K_\mu, P_\nu] = -2i(\eta_{\mu\nu}D - M_{\mu\nu}) \tag{2.33}$$
$$[K_\rho, M_{\mu\nu}] = -i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \tag{2.34}$$
$$[P_\rho, M_{\mu\nu}] = -i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \tag{2.35}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma}) \tag{2.36}$$

Apparently the generators of the isometries form a subalgebra, the Poincaré algebra.
At this point, one can use the Baker-Campbell-Hausdorff formula (see appendix .1.2) to calculate, that

$$e^{i\alpha D}P^2 e^{-i\alpha D} = e^{2\alpha}P^2, \tag{2.37}$$

so the Lorentzian momentum square $P^2 = P_\mu P^\mu$, which in Poincaré QFT equals the mass square m^2, is not invariant under dilatations. Therefore conformal field theories have either vanishing or continous mass spectrum.

Setting

$$J_{\mu\nu} = M_{\mu\nu} \ , \ J_{\mu D} = \frac{1}{2}(P_\mu - K_\mu) \ , \ J_{\mu D+1} = \frac{1}{2}(P_\mu + K_\mu) \ , \ J_{DD+1} = D \tag{2.38}$$

and $J_{ba} = -J_{ab}$, it is easily checked, that the conformal algebra is isomorphic to the Lie algebra of the pseudoorthogonal group $SO(D,2)$, whose commutation relations can be written compactly as

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} - \eta_{ad}J_{bc} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad}), \tag{2.39}$$

with $a, b, ... = 0, 1, ..., D+1$ and the metric $\eta_{ab} = diag(+, -, ..., -, +)$.
$SO(D,2)$ is the matrix Lie group of the $(D+2) \times (D+2)$ matrices M satisfying

$$M^t \eta M = \eta, \tag{2.40}$$

where M^t is the transpose of M, and the corresponding matrix Lie algebra $so(D,2)$ are the matrices m satisfying

$$m^t \eta + \eta m = 0. \tag{2.41}$$

One can express an arbitrary generator m in terms of a basis of the n basis elements J_{ab} as

$$m = \sum_{a,b} m_{ab} J_{ab}. \tag{2.42}$$

The J_{ab} generate either rotational (sine/cosine type) or boost (sinh/cosh type) group elements in a $(D+2)$-dimensional space.

Dirac realization of conformally compactified Minkowski space

How the action on the $(D+2)$-dimensional space induces the conformal transformations in Minkowski space is reflected in a realization by Dirac, in which $SO(D,2)$ acts on a $(D+1)$-(pseudo)-cone. We consider $\mathbb{R}^{D+2}$ with metric $\eta_{ab} = \mathrm{diag}(+, -, .., -, +)$. Then the cone is defined as

$$\mathcal{K}_D = \{\xi \in \mathbb{R}^{D+2} \mid \xi \cdot \xi = 0 \ , \ \xi \neq 0\} \tag{2.43}$$

We identify ξ and $\lambda\xi$ ($\lambda \neq 0$), since they will correspond to the same point in Minkowski space (see below). In other words, we go over to the factor space

$$\overline{\mathcal{M}}_D = \mathcal{K}_D / \mathbb{R}^*, \tag{2.44}$$

where $\mathbb{R}^* = \{\lambda \in \mathbb{R} | \lambda \neq 0\}$. This space is the compactification $\overline{\mathcal{M}_D}$ of Minkowski space, which is

$$\overline{\mathcal{M}_D} = S^{D-1} \times S^1, \tag{2.45}$$

where S^n is the unit n-sphere. For $\xi^D + \xi^{D+1} \neq 0$ we define the Minkowski space coordinates as

$$x^\mu = \frac{\xi^\mu}{\xi^D + \xi^{D+1}} \tag{2.46}$$

for $\mu = 0, 1, .., D-1$. $\overline{\mathcal{M}_D}$ is a homogeneous space for the conformal group

$$\mathcal{C}_D = SO_e(D,2)/\mathbb{Z}_2, \tag{2.47}$$

where we divided out the center of $SO_e(D,2)$, which is isomorphic to the cylic group $\mathbb{Z}_2$ of two elements.
One can check now, that the $SO_e(D,2)/\mathbb{Z}_2$ action on an element ξ^a induces the correct conformal transformations on the Minkowski coordinates x^μ. We illustrate this for the dilatations, the other transformations can be calculated similarly. The corresponding transformation on the ξ-space is a pseudo-rotation in the $(D, D+1)$ plane:

$$\begin{pmatrix} \xi^0 \\ \vdots \\ \xi^{D-1} \\ \xi^D \\ \xi^{D+1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cosh(t) & \sinh(t) \\ 0 & 0 & 0 & \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \xi^0 \\ \vdots \\ \xi^{D-1} \\ \xi^D \\ \xi^{D+1} \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \vdots \\ \xi^{D-1} \\ \cosh(t)\xi^D + \sinh(t)\xi^{D+1} \\ \sinh(t)\xi^D + \cosh(t)\xi^{D+1} \end{pmatrix}$$

On the Minkowski space coordinates this has the effect

$$x^\mu \to x'^\mu = \frac{\xi^\mu}{\cosh(t)\xi^D + \sinh(t)\xi^{D+1} + \sinh(t)\xi^D + \cosh(t)\xi^{D+1}} = e^{-t}x^\mu \tag{2.48}$$

for $\mu = 0..D-1$, which is indeed a dilatation.

2.2.4 Conformal field theory on $\mathcal{M}_2$

The case $D = 2$ is special in the sense, that besides the usual global conformal transformations there is also an infinitedimensional local conformal algebra, which however does not leave the vacuum invariant.
We emphasize, that whenever we speak of the twodimensional conformal group or algebra in this work, we mean the six-dimensional global one, i.e. $SO(2,2)$ or $so(2,2)$.
By inversion of the relations (2.38), the basis $\{J_{ab}\}$ of $so(2,2)$ with $a, b = 0, .., 3$ can be replaced by the set of physical generators

$$\{P_0, P_1, K_0, K_1, D, M_{01}\}. \tag{2.49}$$

One find, that $so(2,2)$ splits into a direct sum of two identical algebras:

$$so(2,2) \simeq so(1,2) \oplus so(1,2). \tag{2.50}$$

To see this, we set

$$P_\pm = \frac{1}{2}(P_0 \pm P_1),\ K_\pm = \frac{1}{2}(K_0 \mp K_1),\ D_\pm = \frac{1}{2}(D \mp M_{01}) \tag{2.51}$$

and get two $sl(2,\mathbb{R})$ algebras commuting with each other:

$$[D_\pm, P_\pm] = -iP_\pm\ ,\ [D_\pm, K_\pm] = iK_\pm\ ,\ [P_\pm, K_\pm] = 2iD_\pm, \tag{2.52}$$

where $sl(2,\mathbb{R})$ is the Lie algebra of the group $SL(2,\mathbb{R})$ of real unimodular 2x2 matrices

$$SL(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1\ ,\ a, b, c, d \in \mathbb{R} \right\}. \tag{2.53}$$

Then we form the linear combinations

$$J_{01} = \frac{1}{2}(P_\pm - K_\pm)\ ,\ J_{02} = D_\pm\ ,\ J_{12} = \frac{1}{2}(P_\pm + K_\pm), \tag{2.54}$$

which fulfil the $so(1,2)$ commutation relations (cf. (2.39))

$$[J_{01}, J_{02}] = iJ_{12}\ ,\ [J_{01}, J_{12}] = iJ_{02}\ ,\ [J_{02}, J_{12}] = -iJ_{01}, \tag{2.55}$$

hence

$$sl(2,\mathbb{R}) \simeq so(1,2). \tag{2.56}$$

Let the coordinates of twodimensional Minkowski space be denoted by (x^0, x^1), then we define the so-called light-cone coordinates $x^+ = x^0 + x^1$ and $x^- = x^0 - x^1$. The splitting of the Lie algebra implies the factorization of $SO(2,2)$ into two $SL(2,\mathbb{R})$s on the group level. This leads to a chiral action of the two copies on the light-cones. This action has the form of fractional (Möbius) transformations

$$f(x^+) = \frac{ax^+ + b}{cx^+ + d} \tag{2.57}$$

(and correspondingly for x^-), where the parameters are precisely the components of a matrix $A \in SL(2,\mathbb{R})$.
For translations, dilatations and special conformal transformations, respectively, the explicit form of the matrices is

$$P: \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\ ,\ D: \begin{pmatrix} e^{b/2} & 0 \\ 0 & e^{-b/2} \end{pmatrix}\ ,\ K: \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}. \tag{2.58}$$

with $a, b, c \in \mathbb{R}$.
Since a simultaneous change of sign of all components of $A \in SL(2,\mathbb{R})$ leads to the same fractional transformation, the onedimensional conformal group is really isomorphic to the projective $SL(2,\mathbb{R})$,

$$\mathcal{C}_1 \simeq PSL(2,\mathbb{R}). \tag{2.59}$$

We saw, that we needed a unitary representation of the group of spacetime symmetries on the Hilbert space of the theory, so we now consider representations of the conformal group. Since there are no (non-trivial) finitedimensional unitary representations of non-compact groups, all relevant representations are necessarily infinitedimensional. Furthermore in accordance with the Wightman setting, we will demand the representations to be of positive energy.
We can get the representations of $SL(2,\mathbb{R})$ from those of its Lie algebra $sl(2,\mathbb{R})$. We saw, that

$$so(2,2) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}), \tag{2.60}$$

so the representations are tensor products of two $sl(2,\mathbb{R})$ representations. Those are in turn labelled by a positive number k [2], so a representation of the twodimensional conformal group can be labelled by a pair of quantum numbers (k, k').

[2]The onedimensional trivial representation corresponds to $k = 0$

We will often call $d = k + k'$ the scaling dimension and $s = k - k'$ the spin of the representation.
Much information about a representation ρ can be written compactly in the form of a character

$$\chi_\rho(\cdot) = \sum_\lambda \mathrm{mult}_\lambda e^\lambda(\cdot) \tag{2.61}$$

where the sum extends over all weights λ of ρ. The weights of the positive energy representation k of $sl(2,\mathbb{R})$ are of the form $k + n$ with $n \in \mathbb{N}_0$, so its character is

$$\chi_k(t) = \sum_{n=0}^{\infty} t^{k+n} = t^k \sum_{n=0}^{\infty} t^n = \frac{t^k}{1-t}. \tag{2.62}$$

The character of a representation of the twodimensional algebra is then the product $\chi_k \cdot \chi_{k'}$ of two characters of $sl(2,\mathbb{R})$ representations (c.f. appendix .1.4).
A Casimir element is an element in the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, that commutes with all generators of $\mathfrak{g}$ (see appendix .1.3). Therefore by Schur's lemma it has a constant value within a representation and since the number of independent Casimir operators for an algebra is equal to the number of quantum numbers necessary to label a representation of it, they can be used to characterize a representation.
In accordance with the decomposition of the twodimensional conformal algebra, also its Casimir operator is the sum of two onedimensional ones:

$$C^{(2)} = C_+^{(1)} + C_-^{(1)} \tag{2.63}$$

where

$$C_\pm^{(1)} = -D_\pm^2 + \frac{1}{2}(P_\pm K_\pm + K_\pm P_\pm). \tag{2.64}$$

The Casimir operators will become important in the context of the partial wave expansions. Their values in a $k_\pm$ representation are

$$C_\pm^1 = k_\pm(k_\pm - 1). \tag{2.65}$$

Coming to field representations, we note, that in this work we will only need those of the Lie algebra, which are obtained by inserting (2.29) into (2.28) and differentiating by the parameters t^i at $t^i = 0$. For a scalar field $\varphi(x)$ of scaling dimension d in one spacetime dimension we obtain

$$[P, \varphi(x)] = -i\partial_x \varphi \tag{2.66}$$

$$[K, \varphi(x)] = -i(x^2\partial_x + 2dx)\varphi \tag{2.67}$$

$$[D, \varphi(x)] = -i(d + x\partial_x)\varphi \tag{2.68}$$

Since in two dimensions the Hilbert space, that is obtained by applying the field operators on the vacuum, is a tensor product of two onedimensional ones, one has two of these lists for the fields, that live on the respective light cones.

2.2.5 Conformal field theory on $\mathcal{M}_4$

In $D \geq 3$ spacetime dimensions there exists only the group of global conformal transformation, which for $D = 4$ is a 15-dimensional group.
The physically interesting positive energy representations are lowest weight representations characterized by three numbers [19]

$$\lambda = (d, j_1, j_2). \tag{2.69}$$

They can be thought of as being induced from the maximal compact subgroup

$$K = U(1) \times SU(2) \times SU(2) \tag{2.70}$$

of the universal covering group $SU(2,2)$ of the conformal group. The d is the scaling dimension and corresponds to the $U(1)$ representation. The j_i are integer or half-integer and correspond to the two angular momentum algebras $SU(2)$. Unitarity sets certain bounds on the possible values of the scaling dimension:

- $d \geq j_1 + j_2 + 2$ for $j_1 j_2 \neq 0$
- $d \geq j_1 + j_2 + 1$ for $j_1 j_2 = 0$
- $d = j_1 = j_2 = 0$ (the trivial representation)

Some details on this are given in appendix .2.
The field representations for a field $\psi(x)$ of scaling dimension d in four dimensions read

$$[P_\mu, \psi(x)] = -i\partial_\mu \psi(x) \tag{2.71}$$
$$[M_{\mu\nu}, \psi(x)] = -[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}]\psi(x) \tag{2.72}$$
$$[D, \psi(x)] = -i(d + x^\mu \partial_\mu)\psi(x) \tag{2.73}$$
$$[K_\mu, \psi(x)] = [i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2d x_\mu) - 2x^\nu \Sigma_{\mu\nu}]\psi(x) \tag{2.74}$$

The matrices $\Sigma_{\mu\nu}$ describe the spin-tensor structure of the field, e.g. $\Sigma_{\mu\nu} = 0$ for a scalar field, $\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ for a spinor field (γ_μ are the Dirac γ matrices) and $\Sigma_{\mu\nu} = i(\eta_{\mu\tau}\delta^\rho_\nu - \eta_{\nu\tau}\delta^\rho_\mu)$ for a vector field.

Since $so(4,2)$ is a rank three algebra, there are three Casimir elements $C_2^{(4)} = \frac{1}{2}J_{ab}J^{ab}$, $C_3^{(4)}$ and $C_4^{(4)}$, which in terms of the physical generators read [8]

$$C_2^{(4)} = \frac{1}{2}M_{\mu\nu}M^{\mu\nu} - K_\mu P^\mu - 4iD - D^2 \tag{2.75}$$
$$C_3^{(4)} = -\frac{1}{4}(W_\mu K^\mu + K_\mu W^\mu) - \frac{1}{8}\epsilon_{\mu\nu\rho\tau}M^{\mu\nu}M^{\rho\tau} \tag{2.76}$$
$$\begin{aligned} C_4^{(4)} = &\frac{1}{4}\{K_\mu K^\mu P_\nu P^\nu - 4K_\mu M^{\mu\nu}M_{\nu\rho}P^\rho - 4K_\mu M^{\mu\nu}P_\nu(D + 6i) \\ &+ \frac{3}{4}(M_{\mu\nu}M^{\mu\nu})^2 + \frac{1}{16}(\epsilon_{\mu\nu\rho\tau}M^{\mu\nu}M^{\rho\tau})^2 + M_{\mu\nu}M^{\mu\nu}(D^2 + 8iD - C_2^{(4)} - 22) \\ &- D^4 - 16iD^3 + 80D^2 + 128iD + 36C_2^{(4)} - 16iC_2^{(4)}D - 2C_2^{(4)}D^2\} \end{aligned} \tag{2.77}$$

where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\tau}P^\nu M^{\rho\tau}$ is the Pauli-Lubanski vector. One can calculate their value for any field representation, but we will only need the so-called symmetric tensor representations.

We set the "angular momentum" quantum numbers $j_1 = j_2 = \frac{l}{2}$ and because of the unitary bounds and the fact, that the scaling dimension will be an integer in globally conformal invariant QFT, the scaling dimension is conveniently parametrized as

$$d = 2k + l, \tag{2.78}$$

where l is called the "spin" and $2k$ the "twist". Obviously k must be positive. In terms of these quantities, the Casimir values become

$$C_2^4 = (2k + l)(2k + l - 4) + l(l + 2) \tag{2.79}$$

$$C_3^4 = 0 \tag{2.80}$$

$$C_4^4 = -(2k + l - 2)^2 \left[-\frac{3}{4} - \frac{l(l + 2)}{2}\right] + \frac{1}{4} l^2 (l + 2)^2 \tag{2.81}$$

Character formulae for higherdimensional conformal groups were obtained e.g. by [1] and [7]. We will only need the $D = 4$ case. Low values of the scaling dimension, which are in principle allowed by unitarity, have to be treated separately [3]. These are called "short representations" in [1], the "normal" cases are called "long representations".

- Long representations with $j_1 \neq 0$, $j_2 \neq 0$, $d > j_1 + j_2 + 2$

$$A_{[d,j_1,j_2]}(s, x, y) = s^d \chi_{j_1}(x)\chi_{j_2}(y)P(s, x, y), \tag{2.82}$$

- Short representations with $j_1 \neq 0$, $j_2 \neq 0$, $d = j_1 + j_2 + 2$

$$D_{[j_1+j_2+2,j_1,j_2]}(s, x, y) = s^{j_1+j_2+2} \left(\chi_{j_1}(x)\chi_{j_2}(y) - s\chi_{j_1-\frac{1}{2}}(x)\chi_{j_2-\frac{1}{2}}(y)\right) P(s, x, y), \tag{2.83}$$

- Short representations with $j_1 \neq 0$, $j_2 = 0$, $d = j_1 + 1$

$$D_{[j_1+1,j_1]+}(s, x, y) = s^{j_1+1} \left(\chi_{j_1}(x) - s\chi_{j_1-\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y) + s^2\chi_{j_1-1}(x)\right) P(s, x, y), \tag{2.84}$$

- Short representations with $j_1 = 0$, $j_2 \neq 0$, $d = j_2 + 1$

$$D_{[j_2+1,j_2]-}(s, x, y) = s^{j_2+1} \left(\chi_{j_2}(y) - s\chi_{j_2-\frac{1}{2}}(y)\chi_{\frac{1}{2}}(x) + s^2\chi_{j_2-1}(y)\right) P(s, x, y) \tag{2.85}$$

with the $su(2)$ character

$$\chi_j(x) = \sum_{n=-j}^{j} e^{nx} = \frac{\sinh((j + 1/2)x)}{\sinh(x/2)} \tag{2.86}$$

and the function

$$P(s, x, y) = \left[(1 - \frac{s}{\sqrt{xy}})(1 - s\sqrt{xy})(1 - s\sqrt{\frac{x}{y}})(1 - s\sqrt{\frac{y}{x}})\right]^{-1}. \tag{2.87}$$

Since the lowest weight of symmetric tensor representations is of the form $(d = 2k+l, l/2, l/2)$, we are only interested in the long representations, equation (2.82), and the short representations of the first type, equation (2.83).

[3] This corresponds to certain conservation laws for twist 2 fields, see next section.

2.3 Correlations, operator product expansion and partial wave expansions

We mentioned in the first section, that the problem of constructing a non-trivial quantum field theory can be addressed by finding a recipe to calculate (at least in principle) all of its correlation functions

$$W_n(x_1, .., x_n) = \langle \Omega, \phi_1(x_1)..\phi_n(x_n)\Omega \rangle. \tag{2.88}$$

We give an overview about the strategies in this direction, that have been employed in conformal field theory.
The effect of requiring conformal invariance of a theory on its correlation functions is more restricting than in the case of a Poincaré invariant QFT. The two- and three-point functions are fixed up to normalization. For scalar fields ϕ_i with scaling dimensions d_i ($i = 1, 2, 3$) we have

$$\langle \Omega, \phi_1(x_1)\phi_2(x_2)\Omega \rangle = \begin{cases} \frac{C_{12}}{x_{12}^{2d}} & d_1 = d_2 \equiv d \\ 0 & d_1 \neq d_2 \end{cases} \tag{2.89}$$

and

$$\langle \Omega, \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\Omega \rangle = \frac{C_{123}}{x_{12}^{d_1+d_2-d_3} x_{13}^{d_1+d_3-d_2} x_{23}^{d_2+d_3-d_1}} \tag{2.90}$$

C_{12} and C_{123} are constants of normalization and

$$\frac{1}{x^n} = \left(\frac{-1}{(x^0 - i\epsilon)^2 - \vec{x}^2} \right)^{n/2}. \tag{2.91}$$

The argument, that (2.89) and (2.90) must have this form, uses invariance under all conformal transformations (see e.g. [3]).
The general n-point correlation function ($n \geq 4$) of scalar fields ϕ_i with scaling dimensions d_i is not that determined, but one can show, that it must have the form

$$\langle \Omega, \phi_1(x_1)..\phi_n(x_n)\Omega \rangle = \prod_{i,j} \rho_{ij}^{\mu_{ij}} f(s_1, .., s_{m_n}) \tag{2.92}$$

where $\rho_{ij} = (x_i - x_j)^2$ and f is an arbitrary function, that depends on the points only through the $m_n = n(n-3)/2$ conformally invariant cross ratios. A cross-ratio involves four-points x_i, x_j, x_k, x_l in the form

$$\frac{\rho_{ij}\rho_{kl}}{\rho_{ik}\rho_{jl}} \tag{2.93}$$

and one can check, that such a ratio indeed remains invariant under any conformal transformation. Therefore also f remains invariant under any conformal transformation. The number m_n emerges from the combinatorics of finding the number of independent ratios of this type. The exponents μ_{ij} have to fulfil the homogeneity relation $\sum_j (\mu_{ij} + \mu_{ji}) = -d_i$ for every fixed i. This expresses the fact, that under scaling, a field behaves as

$$\phi(\lambda x) \rightarrow \frac{1}{\lambda^d} \phi(x). \tag{2.94}$$

Since the distance squares transform as

$$\rho_{ij} \rightarrow (\lambda x_i - \lambda x_j)^2 = \lambda^2 \rho_{ij}, \tag{2.95}$$

the homogeneity relation ensures, that both sides of (2.92) transform in the same way.

A useful tool in the analysis of correlation functions in conformal field theory is the so-called partial wave expansion (PWE). This notion arose in non-relativistic scattering theory, where for central potentials one has common eigenstates of the Hamiltonian H and the angular momentum operators L_z and L^2. Then the wave functions are expanded in terms of them. Also in relativistic scattering PWEs were used, where it amounts to the tensor product expansion of two irreducible positive energy representations of the Poincaré group.
In conformal field theories PWEs of four-point-functions were employed for various spacetime dimensions as well as in Euclidean and Minkowskian setting. Here it amounts correspondingly to the tensor product expansion of two irreducible positive energy representations of the conformal group. For the Minkowskian case these representations were mentioned in section 2.2.

PWEs can be combined very well with the operator product expansion. This notion has been introduced by Wilson [35] and was employed in CFT as well as in ordinary QFT or even on curved spacetimes. One makes the assumption, that the product of two local fields $A(x)$ and $B(y)$ can for x inside a certain radius of convergence around y be written as

$$A(x)B(y) \sim \sum_i C_i^{AB}(x-y)\Phi_i(y), \tag{2.96}$$

for some local fields Φ_i. The $\sim$ should indicate, that the expression is meant to be exact inside a correlation function. One might hope, that if one finds a consistent way to carry this out, one can reduce any conformal n-point function successively down to the known three-point function. In practise however this turns out to be very cumbersome and has only been accomplished in CFT for the case, where just one of these steps is needed, i.e. for the four-point function of fields $\phi_i(x)$ (i=1,2,3,4).

Partial wave expansion and positivity in twodimensional CFT

We first consider the twodimensional case, which was investigated in [27] to study positivity for general scalar fields in 2D conformal field theory, trying to generalize previously obtained results for the energy-momentum field [9].
A problem with local conformal fields $\phi_{d\bar{d}}(x^+,x^-)$ (with $x^\pm = x^0 \pm x^1$ the light-cone coordinates and d and $\overline{d}$ the light-cone scaling dimensions) is, that they cannot transform irreducibly, if one requires Einstein causality. It was found however, that one can decompose the fields as

$$\phi_{d\bar{d}}(x^+,x^-) = \sum_{\xi\bar{\xi}} \phi_{d\bar{d}}^{\xi\bar{\xi}}(x^+,x^-) \tag{2.97}$$

into non-local fields $\phi_{d\bar{d}}^{\xi\bar{\xi}}(x^+,x^-)$ with ξ non-integer real numbers, which do so, e.g.

$$U(\lambda)\phi_{d\bar{d}}^{\xi\bar{\xi}}(x^+,x^-)U(\lambda)^{-1} = \lambda^d\phi_{d\bar{d}}^{\xi\bar{\xi}}(\lambda x^+,x^-), \tag{2.98}$$

where $U(\lambda)$ is the unitary operator representing a scale transformation.
The decomposition (2.97) takes place with respect to central elements Z and $\overline{Z}$ of the universal covering of the conformal group, such that

$$Z\phi_{d\bar{d}}^{\xi\bar{\xi}}(x^+,x^-)Z^{-1} = e^{-2\pi i(d-\xi)}\phi_{d\bar{d}}^{\xi\bar{\xi}}(x^+,x^-) \tag{2.99}$$

and

$$\overline{Z}\phi^{\xi\overline{\xi}}_{d\overline{d}}(x^+,x^-)\overline{Z}^{-1} = e^{2\pi i(\overline{d}-\overline{\xi})}\phi^{\xi\overline{\xi}}_{d\overline{d}}(x^+,x^-). \tag{2.100}$$

There are global operator product expansions for the non-local fields [28], which involve integration over the whole light cone and take the form

$$\phi^{\xi\overline{\xi}}_{d_1\overline{d_1}}(x_1^+,x_1^-)\phi_{d_2\overline{d_2}}(x_2^+,x_2^-)\Omega\rangle = \sum_{d_3,\overline{d_3}} 2e^{i\pi(d_3-\overline{d_3})}c_{123}\int d^2x_3\, K(d_i,\overline{d_i},x_i^+,x_i^-)\cdot\phi_{d_3\overline{d_3}}(x_3^+,x_3^-)\Omega\rangle \tag{2.101}$$

with $i = 1,2,3$. The expansion of local fields is then obtained from this by summation over ξ and $\overline{\xi}$ (cf. (2.97)). The integral kernel in the expression factorizes as

$$K(d_i,\overline{d_i},x_i^+,x_i^-) = K(d_i,x_i^+)\cdot\overline{K}(\overline{d_i},x_i^-) \tag{2.102}$$

with

$$K(d_i,x_i^+) = \frac{(2\pi)^{-1}\Gamma(2d_3)\Gamma(\lambda_3)\Gamma(1-\lambda_2)^{-1}}{(x_1^+-x_2^+-i\epsilon)^{\lambda_1}(x_1^+-x_3^++i\epsilon)^{\lambda_2-1+2d_3}(x_1^+-x_3^+-i\epsilon)^{1-2d_3}(x_2^+-x_3^++i\epsilon)^{\lambda_3}} \tag{2.103}$$

and the corresponding expression for $\overline{K}(\overline{d_i},x_i^-)$.
If one orthonormalizes the fields with respect to their two-point function, the c_{123} are equal to the constants C_{123} in the three-point function.

To obtain the partial wave expansion of the four-point function, we multiply the local counterpart of equation (2.101) from the left with a "bra"-vector of the form $\langle\phi_2(x_2)\phi_1(x_1)\Omega$. Then within the integral there appears the three-point-function, whose form up to normalisation is known. If one plugs in the expression, it turns out, that one can perform the integrations to yield the general form of the four-point function as

$$\langle\Omega,\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\Omega\rangle = g(x_i^+,d_i)\cdot g(x_i^-,\overline{d_i})\cdot\sum_{h_0,\overline{h_0}} e^{i\pi(h_0-\overline{h_0})}\, b^{\text{prim}}_{120}b^{\text{prim}}_{034}\,\mathcal{F}_{h_0}(u)\overline{\mathcal{F}}_{\overline{h_0}}(v) \tag{2.104}$$

with the function

$$g(x_i^+,d_i) = \frac{\left(\frac{x_2^+-x_4^+}{x_1^+-x_4^+}\right)^{d_1-d_2}\left(\frac{x_1^+-x_3^+}{x_1^+-x_4^+}\right)^{d_4-d_3}}{(x_1^+-x_2^+)^{d_1+d_2}(x_3^+-x_4^+)^{d_3+d_4}}, \tag{2.105}$$

the conformal blocks

$$\mathcal{F}_{h_0}(u) = u^{h_0}\sum_{n_0=0}^{\infty} N_{120}(n_1.n_2,n_0)N_{034}(n_0,n_3,n_4)\cdot(-u)^{n_0}\cdot F(d_2-d_1+d_0,d_3-d4+d_0;2d_0;u) \tag{2.106}$$

and the cross ratio

$$u = \frac{(x_1^+-x_2^+)(x_3^+-x_4^+)}{(x_1^+-x_3^+)(x_2^+-x_4^+)}. \tag{2.107}$$

For $\overline{\mathcal{F}}(v)$, $g(x_i^-,\overline{d_i})$ and v corresponding expressions hold. The $F(a,b;c;x)$ are the so-called hypergeometric functions, whose definition and basic properties are listed in appendix 3.

We note, that in the case of four fields with identical scaling dimension, the terms appearing in (2.104) have up to prefactors the form

$$B_{nm}(s,t) := G_n(u) \cdot G_m(v) := u^n F(n,n;2n;u) \cdot v^m F(m;m;2m;v). \tag{2.108}$$

In the expansion and the conformal blocks there appeared new coefficients N_{ijk} and b_{ijk}^{prim}. We clarify their meaning now. The chiral components $T(x^+)$ and $\overline{T}(x^-)$ of the (traceless, conserved) energy-momentum tensor act independently on the fields within a conformal family $\{\phi_{h+n}(x^+,x^-)\}_{n=0,1,2,\ldots}$. Hence the amplitudes factorize into a family dependent part and two parts depending on the chiral actions, which also include the distinction between fields of different level (the descendants) within a family:

$$c_{312} = b_{312}^{\text{prim}} \cdot N_{312}(n_1,n_2,n_3) \cdot \overline{N}_{312}(\overline{n}_1,\overline{n}_2,\overline{n}_3) \tag{2.109}$$

For scalar fields all three coefficient parts are real numbers. One further has the symmetry

$$N_{kji} = (-1)^{n_i+n_j+n_k} N_{ijk} \tag{2.110}$$

How is all this used now to study positivity? Let us assume, that a field in the expansion (2.101) creates a negative norm state, when applied to the vacuum. Then the sign of the three-point function changes, which is an undesirable feature. We could now introduce a minus sign in the expansion (2.101) for every field of this type to circumvent this, in other words a sign factor $s_{d_3\overline{d_3}} = \pm 1$. This sign factor can be interpreted as a metric in a certain $\mathbb{R}^p$, where p is the number of different fields in the expansion.
If one takes for example two of the fields to be the energy-momentum tensor, one can use this metric to define a (in general non-definite) "scalar product" on the space of the coefficients [27]. For positivity (i.e. no appearance of negative-norm states) it should be a true scalar product, i.e. positive definite.
If one studies the implications of this requirement, one can recover the Friedan-Qiu-Shenker quantization of scaling dimensions [9] for the minimal models with central charge $c < 1$.

Partial wave expansion in four dimensions

We now come to the fourdimensional case. In [5] the partial waves were obtained using the operator product expansion as well. The contribution of a spin l operator of scaling dimension $d = 2k + l$ to the operator product expansion of two fields ϕ_1 and ϕ_2 with scaling dimensions d_1 and d_2 can be written as

$$\phi_1(x_1)\phi_2(x_2) \sim B(\phi_1,\phi_2,O^{2k+l}) \frac{1}{\rho_{12}^{(d_1+d_2-d)/2}} K_k^{\mu_1\ldots\mu_l}(x_{12},\partial_{x_2}) O^{2k+l}_{\mu_1\ldots\mu_l} \tag{2.111}$$

The derivative operator $K_k^{\mu_1\ldots\mu_l}(x_{12},\partial_{x_2})$ is rather complicated and can be found e.g. in [23]. It is determined by the form of the associated two- and three-point-functions.
If the form (2.111) of the OPE is applied to the four-point function, one gets the contribution

$$\begin{aligned}\langle \Omega, \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\Omega\rangle \sim & \frac{1}{\rho_{12}^{(d_1+d_2)/2}\rho_{34}^{(d_3+d_4)/2}} \cdot \left(\frac{\rho_{24}}{\rho_{14}}\right)^{(d_1-d_2)/2} \cdot \left(\frac{\rho_{14}}{\rho_{13}}\right)^{(d_3-d_4)/2} \\ & \times B(\phi_1,\phi_2,O^{2k+l}) B(\phi_3,\phi_4,O^{2k+l}) s^k H^l(s,t)\end{aligned} \tag{2.112}$$

with the cross ratios

$$s = \frac{\rho_{12}\rho_{34}}{\rho_{13}\rho_{24}} \quad , \quad t = \frac{\rho_{14}\rho_{23}}{\rho_{13}\rho_{24}}. \tag{2.113}$$

The $H^l(s,t)$ depend on the scaling dimensions d_i and spins via the parameters $a = k - (d_1 - d_2)/2$, $b = k + (d_3 + d_4)/2$ and $c = 2k + l$. They are determined by an equation involving the differential operator $K_k^{\mu_1 \dots \mu_l}$. One can then derive a recurrence relation for the H^l, which can be explicitly solved in $D = 2$ and $D = 4$. In the former case one gets a symmetrized version of (2.108), the result of the latter is the same as in an alternative method, which we sketch now.

In [6] the partial wave expansion of four-point functions was approached by solving an eigenvalue equation for the Casimir operator of $O(D,2)$ with certain boundary conditions. This reproduced the results for the conformal partial wave representing the contribution of a field of scaling dimension d and spin l in $D = 2$ and $D = 4$, but also made it possible to get them for $D = 6$ (which will however be of no interest in this work).
We will sketch the procedure here, because this will be the method we use later to perform the partial wave expansion of six-point functions. Again we consider four scalar conformal fields $\phi_i(x)$ with scaling dimensions d_i. We choose a form for their four-point function, in which the prefactor matches the one of the contribution (2.111) from above:

$$\langle \Omega, \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\Omega \rangle = \frac{1}{\rho_{12}^{\frac{d_1+d_2}{2}} \rho_{34}^{\frac{d_3+d_4}{2}}} \cdot \left(\frac{\rho_{24}}{\rho_{14}}\right)^{\frac{d_1-d_2}{2}} \cdot \left(\frac{\rho_{14}}{\rho_{13}}\right)^{\frac{d_3-d_4}{2}} \cdot f(s,t) \tag{2.114}$$

The cross ratios s and t are the same as before.
We insert now a projector onto a symmetric tensor representation $\Pi_{d,l} \equiv \Pi_{(d=2k+l,l/2,l/2)}$ of the conformal group between the field ϕ_2 and ϕ_3 and define the function

$$\mathcal{G}_d^l = \langle \phi_1(x_1)\phi_2(x_2) \cdot \Pi_{d,l} \cdot \phi_3(x_3)\phi_4(x_4) \rangle \tag{2.115}$$

Then we put the quadratic Casimir operator $C_2^{(D)} = \frac{1}{2} J_{ab} J^{ab}$, which in terms of the physical generators was displayed in (2.75), just before this projector. On the one hand it can be applied on the projector to yield the Casimir value

$$C_2^D = d(d-D) + l(l+D-2), \tag{2.116}$$

where D is the space-time dimension (cf. the special case (2.79)). On the other hand one can commute the generators, that make up the Casimir operator, past the fields using the field representations and then express the emerging differential operator in terms of the cross-ratios only.
This results in a differential eigenvalue equation for the $\mathcal{G}_d^l$. Employing formally the factorization

$$s = uv \quad \text{and} \quad t = (1-u)(1-v) \tag{2.117}$$

known from two dimensions also for general D, this eigenvalue equation can be rewritten as follows:

$$\mathcal{D}_\epsilon \mathcal{G}_d^l = \frac{1}{2} C_2^D \mathcal{G}_d^l \tag{2.118}$$

with the differential operator

$$\mathcal{D}_\epsilon = u^2(1-u)\partial_u^2 + v^2(1-v)\partial_v^2 + c(u\partial_u + v\partial_v) - (a+b+1)(u^2\partial_u + v^2\partial_v) - ab(u+v) + \epsilon\frac{uv}{u-v}((1-u)\partial_u - (1-v)\partial_v). \tag{2.119}$$

The parameters take the values

$$a = -\frac{1}{2}(d_1 - d_2)\ ,\ b = \frac{1}{2}(d_3 - d_4)\ ,\ c = 0\ ,\ \epsilon = D - 2 \tag{2.120}$$

so $\epsilon = 0$ corresponds to $D = 2$ and $\epsilon = 2$ to $D = 4$. The solutions should be symmetric in u and v and behave at $u, v \to 0$ as

$$F_{nm}(u,v) \ \sim\ u^n v^m \tag{2.121}$$

for $u, v \to 0$, where $n - m \in \mathbb{N}_0$ and the limit with respect to v is taken first. This requirement can be explained, if one also considers the quartic Casimir operator $C_4^{(D)}$. Further consideration of the behaviour for $u \to 0$ (see [6]) enforces

$$n = \frac{1}{2}(d+l) = k + l \ \text{ and }\ m = \frac{1}{2}(d-l) = k, \tag{2.122}$$

where $2k$ is the twist and l the spin of the projector in (2.115) and $d = 2k + l$.

Solution in two dimensions:
(2.118) is solved by

$$u^{k+l}v^k F(k+l+a, k+l+b; 2k+2l+c; u)F(k+a, k+b; 2k+c; v) + (u \leftrightarrow v). \tag{2.123}$$

For four fields of equal scaling dimension ($a = b = 0$) the eigenvalue equation becomes

$$(u^2(1-u)\partial_u^2 + v^2(1-v)\partial_v^2) - u^2\partial_u - v^2\partial_v)\mathcal{G}_d^l = \frac{1}{2}(d(d-2) + l^2)\mathcal{G}_d^l \tag{2.124}$$

and the solution

$$u^{k+l}v^k F(k+l, k+l; 2k+2l; u)F(k,k;2k;v) + u^k v^{k+l} F(k,k;2k;u)F(k+l,k+l;2k+2l;v) = G_{k+l}(u)G_{k-1}(v) + G_{k+l}(v)G_{k-1}(u). \tag{2.125}$$

This is indeed the structure (2.108) in a symmetrized form.

Solution in four dimensions:
We can use the fact, that

$$\mathcal{D}_2 \frac{1}{u-v} = \frac{1}{u-v}\left(\mathcal{D}_0(a \to a-1, b \to b-1, c \to c-2) - c + 2\right), \tag{2.126}$$

where the arrows indicate, how the parameters in the operator $\mathcal{D}_0$ have to be changed. Because of this identity, we can put in the two-dimensional solution to get

$$\frac{1}{u-v}\left(u^{k+l+1}v^k F(k+l+a, k+l+b; 2k+2l; u)F(k+a-1, k+b-1; 2k-2; v) - (u \leftrightarrow v)\right) \tag{2.127}$$

These partial waves coincide with the ones, that were obtained by solving the recurrence relation for the H^l in the first method.
If the four fields are of equal scaling dimension ($a = b = 0$), we get

$$\begin{aligned}\beta_{kl}(s,t) &= \frac{uv}{u-v}\left(u^{k+l}v^{k-1}F(k+l,k+l;2k+2l;u)F(k-1,k-1;2k-2;v) - (u \leftrightarrow v)\right) \\ &= \frac{uv}{u-v}\left(G_{k+l}(u)G_{k-1}(v) - G_{k+l}(v)G_{k-1}(u)\right). \end{aligned} \tag{2.128}$$

Partial wave expansion for higher n-point function

Considering the proceeding one could wonder, why no projectors were inserted between the first and second as well as the third and forth field resulting in a system of three eigenvalue equations. The answer is quite simple: if we do this for the latter and apply the Casimir operator on the vector $\phi_4(x)\Omega$ it would just force the projector in front of ϕ_4 to take the values of the representation of ϕ_4. This trivial equation contains no relevant information and so the indices corresponding to it are dropped.
For higher n-point functions it is not sufficient to expand just in symmetric tensor fields. One has to insert all three Casimir operators and gets a system of $3(n-3)$ eigenvalue equations with a greater number of conformal cross ratios, which rapidly increases the complexity of the problem. In two dimensions, the situation simplifies a little, because the problem factorizes into two onedimensional ones and also the number of cross ratios is reduced from $\frac{n(n-3)}{2}$ to $n-3$. We will employ these facts in section 5.

2.4 Globally conformally invariant QFT

We had listed the conformal transformations of Minkowski space in (2.22)-(2.25). They all have an immediate physical interpretation, except for the special conformal transformations

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 + x^2}. \tag{2.129}$$

A very apparent problem of them is, that points on the surface $1-2b\cdot x+b^2x^2 = 0$ are mapped to infinity. Compactification of Minkowski space leads to the space with closed timelike curves and therefore no global causal ordering. It was suggested in [18] to consider invariance of the Euclidean correlation functions under the Euclidean conformal group $SO_e(5,1)$ [4], which however only implies invariance under infinitesimal conformal transformations back on the Minkowski side, a rather weak notion. Then by going over to the (infinite-sheeted) covering space $\widetilde{\mathcal{M}_4}$ one receives a space, that admits a global causal ordering. In this case one has to deal with projective representations of the conformal group.
In 2000, Todorov et al. started to investigate quantum field theories with so-called global conformal invariance (GCI) [21]. One works in the framework of axiomatic quantum field theory on the compactified complexified Minkowski space $\overline{\mathcal{M}_{4,\mathbb{C}}}$. This space is endowed with a true representation of $SO_e(4,2)/\mathbb{Z}_2$. The closed timelike curves (periodicity of time) are a feature of these theories, that has to be accepted.
The complex compactification of Minkowski space $\overline{\mathcal{M}_{4,\mathbb{C}}}$ yields the so-called z-picture. $\overline{\mathcal{M}_{4,\mathbb{C}}}$ can be parametrized the following way:

$$\overline{\mathcal{M}_{4,\mathbb{C}}} = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z = \frac{\overline{z}}{z^2}\} \tag{2.130}$$

and the (dense) embedding of $\mathcal{M}_4$ into $\overline{\mathcal{M}_{4,\mathbb{C}}}$ is the complex conformal transformation

$$z_i = \frac{x_i}{\omega(x)}\ ,\ z_4 = \frac{1-x^2}{2\omega(x)}\ ,\ \omega(x) = \frac{1+x^2+2ix^0}{2}\ ,\ x^2 = -(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2 \tag{2.131}$$

for $i = 1, 2, 3$. The correlation functions look essentially the same in this picture as they do in the x-picture on Minkowski space and the description is equivalent to the usual Wightmanian one.
We define now the precise notion of GCI used in this approach.

> **Definition 2.1:** A QFT (obeying Wightman axioms) satisfies global conformal invariance, if the n-point functions $W_n(z_1, .., z_n)$ remain invariant for any conformal transformation g, such that the image $(gz_1, .., gz_n)$ of the points z_i lies in Minkowski space.

[4] In this section, we will restrict to the case $D = 4$.

2.4.1 Properties of GCI QFT

The local commutativity condition in the Wightman axioms and GCI imply the Huygens principle:

Theorem 2.2 (based on Lemma 3.2 from [21]): Local GCI fields $\phi(z_1)$ and $\psi(z_2)$ commute, whenever the distance $x_1 - x_2$ is not lightlike:

$$[\phi(z_1), \psi(z_2)] = 0 \text{ for } (z_1 - z_2)^2 \neq 0 \tag{2.132}$$

This seems very much like a free theory, since one expects causal propagation of signals within the light cones. Indeed we noted, that a quantum field theory does not describe particles. Whenever we speak of a non-trivial theory in the following, we mean a theory, in which the correlation functions are not those of a sum of Wick products.
The previous theorem has its roots within the possibility to map any pair of points into any other pair by a conformal transformation. It is equivalent to strong locality,

$$((z_1 - z_2)^2)^N [\phi(z_1), \phi(z_2)] = 0 \tag{2.133}$$

for N sufficiently high, a property well known from the theory of twodimensional vertex algebras [16]. Together with energy positivity, this implies rationality of all Wightman functions

$$W_n(z_1, .., z_n) = \langle \Omega, \phi_1(z_1)..\phi_n(z_n)\Omega \rangle \tag{2.134}$$

of a GCI theory:

Theorem 2.3 (3.1 from [21]): The tempered distribution $W_n(z_1, .., z_n)$ satisfies GCI, locality, translation invariance and the spectral condition (positivity), if and only if it can be expressed in terms of a rational function of the following type:

$$W_n(z_1, .., z_n) = P_n(z_1, .., z_n) \prod_{1 \leq j < k \leq n} \rho_{jk}^{-\mu_{jk}^n}, \tag{2.135}$$

where $\rho_{jk} = z_{jk}^2 + i0z_{jk}^0$, μ_{jk}^n are positive integers and $P_n(z_1, .., z_n)$ is a polynomial with values in an n-fold product of a certain complex vector space.

The $i0z_{jk}^0$ prescription has been added to fulfil the spectrum condition and is only necessary, if the corresponding ρ_{jk} appears in the denominator (see [29]). The spin-tensor structure of the fields is taken into account by the vector-valuedness of the polynomial P_n. The scaling dimensions of GCI bose fields can only be integer.
Furthermore, Wightman positivity sets constraints on the possible pole degrees of the n-point functions.

Theorem 2.4 [21]: The poles of the rational Wightman functions $W_n(z_1, .., z_n)$ are uniformly bounded, which means the exponents μ_{jk}^n can be chosen indepedent of n. Let $\varphi(x)$ and $\psi(y)$ be two fields in a system of fields in $D = 4$, which transform under the elementary induced representation of the conformal group of weights (d, j_1, j_2) and (d', j'_1, j'_2) , respectively. Then the pole degree μ of $((x - y)^2 + i0(x^0 - y^0))^{-\mu}$ in any Wightman function $\langle ..\varphi(x)..\psi(y).. \rangle$ has the upper limit

$$\mu \leq [[\frac{d + j_1 + j_2 + d' + j'_1 + j'_2}{2} - \frac{1 - \delta_{j_1 j'_2}\delta_{j_2 j'_1}\delta_{dd'}}{2}]] \tag{2.136}$$

where $[[m]]$ stands for the integer part of m.

If in particular $\varphi = \psi^*$, then

$$\mu \leq d + j_1 + j_2 - 1 \tag{2.137}$$

This result enables one to label the correlators by a finite number of parameters, since only a finite number of pole structure is admissible for every correlation function.

2.4.2 Partial wave expansion and positivity in GCI models

In the following, models containing scalar fields of scaling dimensions $d \geq 2$ (the case $d = 1$ corresponds to a free massless scalar field) were considered. $d = 2$ also corresponds to the free case [22].

We discuss now, how the partial wave expansions from the previous section were used in GCI to study the implications of positivity for a hermitean scalar field φ of scaling dimension 4. Its general four point function is

$$\begin{aligned}\langle 1234\rangle &:= \langle \Omega, \varphi(z_1)\varphi(z_2)\varphi(z_3)\varphi(z_4)\Omega\rangle \\ &= \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle + \frac{B_\varphi^2 (z_{13}^2 z_{24}^2)^2}{(z_{12}^2 z_{23}^2 z_{34}^2 z_{14}^2)^3} P(s,t) \\ &= \langle 12\rangle\langle 34\rangle \left[1 + s^4 + s^4 t^{-4} + s t^{-3} P(s,t)\right]\end{aligned} \tag{2.138}$$

where $\langle ij\rangle = \langle \Omega, \varphi(z_i)\varphi(z_j)\Omega\rangle = B_\varphi (z_{ij}^2)^{-4}$ and s and t are the conformal cross ratios. In the second line we see, that the correlator is of the form "free Wick part" plus something else. Using the concrete form of the two-point functions and of the cross ratios this can be transformed into the expression in the third line. The crossing symmetric polynomial $P(s,t)$ is given by

$$P(s,t) = a_0 J_0(s,t) + a_1 J_1(s,t) + a_2 J_2(s,t) + st[bD(s,t) + b' Q(s,t)] \tag{2.139}$$

with the structure functions

$$J_0(s,t) = (t^2 + t^3) + s^2(1 + t^3) + s^3(1 + t^2) \tag{2.140}$$

$$\begin{aligned}J_1(s,t) =& (t + t^4) - (t^2 + t^3) + s(1 + t^4) - 2s(t + t^3) - \\ & - s^2(1 + t^3) - s^3(1 + t^2) - 2s^3 t + s^4(1 + t)\end{aligned} \tag{2.141}$$

$$\begin{aligned}J_2(s,t) =& (1 + t^5) - 2(t + t^4) + (t^2 + t^3) - 2s(1 + t^4) \\ & + s(t + t^3) + s^2(1 + t^3) + s^3(1 + t^2) + s^3 t - 2s^4(1 + 5) + s^5\end{aligned} \tag{2.142}$$

$$D(s,t) = 1 - 2t + t^2 - 2s(1 + t) \tag{2.143}$$

$$Q(s,t) = t + s(1 + t) \tag{2.144}$$

The J_ν, $\nu = 0, 1, 2$, emerge from symmetrized contributions of four point functions of twist 2 bifields (see below) to the total four point function. D and Q contribute to twist 4 partial-waves.

We do partial wave expansion in symmetric traceless tensor fields now by inserting projectors Π_{kl} onto the corresponding positive energy representation of the (universal covering of the) fourdimensional conformal group $SU(2,2)$ with highest weight $(2k+l, l/2, l/2)$ between the second and third field and then sum over all k and l:

$$\langle 1234\rangle = \sum_{k,l\geq 0} \langle 1234\rangle_{kl} \equiv \sum_{k,l\geq 0} \langle \Omega, \varphi(z_1)\varphi(z_2)\Pi_{kl}\varphi(z_3)\varphi(z_4)\Omega\rangle \tag{2.145}$$

A single partial wave then reads

$$\langle 1234\rangle_{kl} = \langle 12\rangle\langle 34\rangle B_{kl}\beta_{kl}(s,t) \tag{2.146}$$

where the β_{kl} are the universal partial waves

$$\beta_{\kappa l}(s,t) = \frac{uv}{u-v}\left(G_{k+l}(u)G_{k-1}(v) - G_{k+l}(v)G_{k-1}(u)\right), \tag{2.147}$$

obtained in the previous section. Since they are fixed by the representation theory, the only model specific information is contained in the coefficients B_{kl}.
We equate (2.138) and (2.145). Since the contribution β_{00} of the identity operator is 1, the leading term in (2.138) corresponds to $B_{00} = 1$ and we end up with

$$s^4 + s^4 t^{-4} + st^{-3}P(s,t) = \sum_{k,l>0} B_{kl}\beta_{kl}(s,t). \tag{2.148}$$

In [24] an expansion formula for monomials in terms of hypergeometric functions was used to obtain closed formulae for the B_{kl} in terms of the parameters a_ν, b and b'.
Wightman positivity forces all of the B_{kl} to be positive, which makes it possible to derive certain bounds for the parameters. Free field constructions are within these bounds, but there also room left for other, possibly nontrivial, fields.

2.4.3 Bifields and pole structures

We are dealing with a single hermitean, scalar field $\phi(z)$ of general scaling dimension d.
We consider the operator product expansion of ϕ with itself and it will turn out, that the singular part of the OPE of $\phi(z)$ with itself can be written as the two-point function plus d-1 conformal bifields $V_k(z_1,z_2)$ of dimension (k,k):

$$\phi(z_1)\phi(z_2) = B_\phi(z_{12}^2)^{-d} + B_\phi(z_{12}^2)^{-d}\sum_{k=1}^{d-1}(z_{12}^2)^k V_k(z_1,z_2) + :\phi(z_1)\phi(z_2):, \tag{2.149}$$

where $:\phi(z_1)\phi(z_2):$ is defined to be the regular part

$$B_\phi\sum_{k=0}^{\infty}(z_{12}^2)^k V_{k+d}(z_1,z_2) \tag{2.150}$$

of the expansion and B_ϕ is a normalization constant.

The fields $V_k(z_1,z_2)$ arise as follows: the bifield

$$U(z_1,z_2) = (\rho_{12})^{d-1}\left[\phi(z_1)\phi(z_2) - \langle\Omega,\phi(z_1)\phi(z_2)\Omega\rangle\right] \tag{2.151}$$

is Taylor expanded in the difference $z_{12} = z_1 - z_2$:

$$U(z_1,z_2) = \sum_{n=0}^{\infty}\sum_{\mu_1,..,\mu_n=0}^{\infty} z_{12}^{\mu_1}..z_{12}^{\mu_n} X^n_{\mu_1..\mu_n}(z_2) \tag{2.152}$$

Here the $X^n_{\mu_1..\mu_n}(z_2)$ are (Huygens) local fields of scaling dimension $n+2$, which are in general not quasi-primary (i.e. they do not transform irreducibly under conformal transformations).
One can subtract certain lower dimensional fields from the $X^n_{\mu_1..\mu_n}(z_2)$ to obtain quasi-primary fields $O^k_{\mu_1..\mu_l}(z_2)$. They are symmetric traceless tensor fields of scaling dimension d and tensor rank l. The difference $d-l$ is the twist $2k$ and the Taylor expansion can be rewritten as

$$U(x_1,x_2)=\sum_{k=1}^{\infty}(\rho_{12})^{k-1}V_k(z_1,z_2), \tag{2.153}$$

where the $V_k(z_1,z_2)$ are regular at $z_1=z_2$ and correspond to all twist $2k$ contributions:

$$V_k(z_1,z_2)=\sum_{l=0}^{\infty}K_k^{\mu_1..\mu_l}(z_{12},\partial_{z_2})O^{2k+l}_{\mu_1..\mu_l}(z_2). \tag{2.154}$$

The differential operator $K_k^{\mu_1..\mu_l}$ is the same here as in the contribution (2.111) of a $(2k+l,l/2,l/2)$ operator to the OPE. It should be noted, that it is not clear, whether these series converge and if they do, whether they are bilocal. We will see that one can find conditions for $k=1$, such that this is the case (see theorem 2.9).
We note in advance, that for $d=2$ these conditions are always fulfilled and the OPE reads

$$\phi(z_1)\phi(z_2)=B_\phi(z_{12}^2)^{-2}+B_\phi(z_{12}^2)^{-1}V_1(z_1,z_2)+:\phi_1(z_1)\phi(z_2):. \tag{2.155}$$

The higher twist contribution are all regular. With V_1 bilocal one can show, that it can be expressed as a sum of normal products of free massless fields and hence the correlators in such a theory always equal sums of Wick products. In fact, not only one, but any (countable) system of real GCI scalar fields of dimension 2 can equivalently be realized as a sum of normal products of free massless fields [26].
Coming back to arbitrary d, the twist 2 contribution is always biharmonic (i.e. it fulfils d'Alembert's equation in both arguments):

$$\Box_1 V_1(z_1,z_2)=\Box_2 V_1(z_1,z_2)=0, \tag{2.156}$$

where $\Box_i=\partial_{z^i}\partial_{z_i}$. This is equivalent to the conservedness of the twist 2 fields:

Theorem 2.5: Biharmonicity of the field V_1 is equivalent to the fact, that all the symmetric tensor fields in its expansion (2.154) are conserved:

$$\partial_{z_{\mu_1}}O^{l+2}_{\mu_1..\mu_l}(z)=0 \tag{2.157}$$

for $l\in\mathbb{N}$.

The proof involves the conformal invariance of the two-point-function, the Reeh-Schlieder theorem and the explicit knowledge of the $K_k^{\mu_1..\mu_l}$.
To exploit the biharmonicity, we introduce the so-called harmonic decomposition of a power series:

Lemma 2.6: Let $u(z)$ be a formal power series in $z\in\mathbb{C}^4$ with coefficients in a vector space V. Then there exist unique formal power series $v(z)$ and $\tilde{u}(z)$, such that

$$u(z)=v(z)+z^2\tilde{u}(z), \tag{2.158}$$

with $v(z)$ harmonic. This is called the harmonic decomposition of $u(z)$ and $v(z)$ is the harmonic part of $u(z)$.

Now we single out the twist 2 contribution in the expansion (2.153) by writing

$$U(z_1, z_2) = V_1(z_1, z_2) + \rho_{12}\tilde{U}(z_1, z_2) \tag{2.159}$$

and consider, what happens, if $U(z_1, z_2)$ appears in a correlation function:

$$\langle ...U(z_1, z_2)...\rangle = \langle ...V_1(z_1, z_2)...\rangle + \rho_{12}\langle ...\tilde{U}(z_1, z_2)...\rangle. \tag{2.160}$$

Then all these correlators are power series

$$F(z_1, z_2, ..) = H(z_1, z_2, ..) + \rho_{12}\tilde{F}(z_1, z_2, ..), \tag{2.161}$$

just like in the lemma, where the ... stand for the other points in the correlator. Because of the biharmonicity of V_1, $H(z_1, z_2, ..)$ can be interpreted as the harmonic part of $F(z_1, z_2, ..)$ both with respect to z_1 and to z_2. We suppress from now on the other arguments and write e.g. $F(z_1, z_2) \equiv F(z_1, z_2, ..)$.

The bilocality of $V_1(z_1, z_2)$ can be achieved by finding a criterion for the rationality of $H(z_1, z_2)$ (by theorem 2.3). In other words, we want to find a condition on the bilocal field $U(z_1, z_2)$, such that any correlation function involving it has a rational harmonic decomposition.
First we use the fact, that there are two harmonic decompositions (one with respect to z_1 and the other with respect to z_2), which should coincide. This can be used to derive a third order partial differential equation for the leading part of F. We write F as

$$F(z_1, z_2) = \sum_{p=0}^{M} (\rho_{12})^p F_p(z_1, z_2), \tag{2.162}$$

where the $F_p(z_1, z_2)$ depend on the points in the correlation function only through all the distance squares ρ_{ij} (except for ρ_{12}).
If H is the harmonic part of F, then the leading part F_0 is also the leading part of H (since everything else is of $\mathcal{O}(\rho_{12})$). Demanding coinciding harmonic decomposition has the following consequence:

> **Theorem 2.7:** [26] Let $F_0(z_1, z_2)$ be as above. Then $H(z_1, z_2)$ is its harmonic part both with respect to z_1 and to z_2, if and only if F_0 satisfies the differential equation
>
> $$(E_1 D_2 - E_2 D_1)F_0 = 0, \tag{2.163}$$
>
> where $E_1 = \sum_{i=3}^{n} \rho_{2i}\partial_{1i}$ and $D_1 = \sum_{3\leq j<k\leq n} \rho_{jk}\partial_{1j}\partial_{1k}$ and E_2 and D_2 correspondingly with $1 \leftrightarrow 2$ ($\partial_{jk} = \partial_{kj} = \frac{\partial}{\partial \rho_{jk}}$).

This differential equation implies two things:

- F_0 cannot have triple poles of the form

  $$(\rho_{1i})^{\mu_{1i}}(\rho_{1j})^{\mu_{1j}}(\rho_{1k})^{\mu_{1k}}$$

  with $\mu_{1i}, \mu_{1j}, \mu_{1k} < 0$ (and the same for z_2).

- If F_0 has a double pole

$$(\rho_{1i})^{\mu_{1i}}(\rho_{1j})^{\mu_{1j}}$$

with $\mu_{1i}, \mu_{1j} < 0$, then the coefficient of the corresponding term must be regular in distance squares ρ_{2k} for $k \neq i, j$ (and vice versa for $1 \leftrightarrow 2$).

Aiming towards the characterizing of bilocality of V_1, we introduce the so-called "single-pole property":

Definition 2.8: [26] Let $f(z_1, .., z_n)$ be a Laurent polynomial in the variables ρ_{ij}, i.e. regarded as a function of z_1 only, it is a finite linear combination of functions of the form

$$\prod_{j\geq 2} \rho_{1j}^{\mu_{1j}} = \prod_{j\geq 2}(z_1 - z_j)^{2\mu_{1j}} \tag{2.164}$$

Then f is said to satisfy the single pole property with respect to z_1, if it contains no terms for which there are $j \neq k$ $(j, k \geq 2)$, such that μ_{1j} and μ_{1k} are negative.

This is just a formal phrasing of what the name suggests, no two or more distance squares involving the same point appear in the denominator of the expression (2.164).
The goal of introducing this notion was to determine, whether harmonic bilocal fields exists, which do not stem from free field realizations like

$$: \phi(z_1)\phi(z_2) : , \tag{2.165}$$

$$: \overline{\psi}(z_1)\gamma_\mu(z_1 - z_2)^\mu \psi(z_2) : \tag{2.166}$$

$$\text{or} \quad (z_1 - z_2)^\mu (z_1 - z_2)^\nu : F_{\mu\sigma}(z_1) F^\sigma_\nu(z_2) : \tag{2.167}$$

where ϕ is a scalar field, ψ a Dirac field, $F_{\mu\nu}$ the Maxwell field and : .. : denotes normal ordering. These constructions only have single-poles and therefore are always bilocal (see theorem 2.9).
We now formulate the criterion for a correlator involving the twist 2 contribution $V_1(x_1, x_2)$ to be a rational function, which implies by theorem 2.3, that it is convergent to a bilocal field.

Theorem 2.9: The field $V_1(z_1, z_2)$ weakly converges on bounded energy states to a Huygens local field, which is conformal of weight $(1, 1)$ $\Longleftrightarrow$ the leading parts F_0 of the Laurent polynomial F satisfy the single pole property with respect to both z_1 and z_2. In this case, the formal series H converges to Laurent polynomials in $(z_i - z_j)^2$ subject to the same pole bounds as F.

The fourpoint function of the higher twist contributions are not rational [23], so they certainly do not converge to Huygens bilocal fields.
The leading contribution to a six-point function involving $d = 4$ hermitean scalar field violating the SPP was displayed in [26]. One needs two fields $\mathcal{L}_1(z_1)$ and $\mathcal{L}_2(z_2)$, such that the bilocal field $U(z_1, z_2)$ in its operator product expansion has a skew-symmetric part. Then the contribution to $\langle \Omega, U(z_1, z_2)\mathcal{L}(z_3)\mathcal{L}(z_4)U(z_5, z_6)\Omega \rangle$ with $\mathcal{L}$ any linear combination of $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$F_0(z_1, z_2) = A_{12}A_{56}\left[\frac{\rho_{15}\rho_{26}\rho_{34} - 2\rho_{15}\rho_{23}\rho_{46} - 2\rho_{15}\rho_{24}\rho_{36}}{\rho_{13}\rho_{14}\rho_{23}\rho_{24}\rho_{34}\rho_{35}\rho_{45}\rho_{36}\rho_{46}}\right], \tag{2.168}$$

where A_{ij} denotes antisymmetrization with respect to x_i and x_j. It is admissible, because it satisfies all pole bounds and fulfils the differential equation (2.163).
It would be desirable to be able to perform partial wave expansions of this correlator to see, whether the property of having double poles is compatible with Wightman positivity. Since this is a very difficult task in four dimensions, we will later try to obtain partial waves for the six-point function in two dimensions and restrict this correlation function to the twodimensional $z_2 = z_3 = 0$ plane, to see, if any obstructions for the fourdimensional case arise from this.

3 Twodimensional conformal orbits in $\mathcal{M}_4$

3.1 Chiral action on space time functions

We noted in the introduction, that in two dimensions the conformal group factorises into two copies of $SL(2,\mathbb{R})$, which act on the light-cone coordinates $x^{\pm} = x^0 \pm x^1$ of twodimensional Minkowski space $\mathcal{M}_2$ as fractional Möbius transformations. If we want to restrict general fourdimensional objects, like correlation functions, to a twodimensional submanifold in order to use this factorization property, we have to consider, in which ways this is possible.
This problem is related to the question, in which ways the twodimensional conformal algebra $so(2,2)$ can be embedded into the fourdimensional $so(4,2)$. This can be done in two inequivalent ways, namely the "block" type

$$\left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & so(2,2) & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

with metric $\eta'_{ab} = diag(+,+,-,-,-,-)$ and the direct "diagonal" type

$$\left(\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & so(1,2) & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & so(1,2) & \\ 0 & 0 & 0 & & & \end{array}\right)$$

with metric $\eta_{ab} = diag(+,-,-,-,-,+)$, which uses the splitting of $so(2,2)$.
For the block embedding, we chose a different order of the signs to emphasize, why this type is called block embedding. With the common η_{ab}, the algebra is embedded in such a way, that the associated twodimensional group (pseudo-)rotates the ξ^0, ξ^1, ξ^4 and ξ^5 components of a vector in the Dirac cone, while it leaves ξ^2 and ξ^3 fixed. Therefore in this case the restriction amounts to simply setting $x^2 = x^3 = 0$, which is obviously a twodimensional submanifold. Then one has the usual chiral action of the group $SO(2,2)$ on the light-cone coordinates x^+ and x^-.

We now want to determine, whether the diagonal embedding also corresponds to a certain submanifold, which can be parametrized by two spacetimes functions $u(x^\mu)$ and $v(x^\mu)$ and on which $SO(1,2) \times SO(1,2)$ acts chirally.
We begin by recalling the Dirac pseudo-cone for $D = 4$

$$\mathcal{K}_4/\mathbb{R}_* = \left\{ \xi \in \mathbb{R}^6 \mid (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - (\xi^4)^2 + (\xi^5)^2 = 0 \right\} /\mathbb{R}_*. \quad (3.1)$$

We will let the two $SO(1,2)$ blocks act on the six-dimensional ξ-space independently. The upper acts on ξ^0, ξ^1 and ξ^2 and the lower on ξ^3, ξ^4 and ξ^5.
From this we will calculate the effect, that these actions have on the associated Minkowski coordinates

$$x^\mu = \frac{\xi^\mu}{\xi^4 + \xi^5}. \tag{3.2}$$

The actions will be denoted by

$$x^\mu \to (A_i^t(x))^\mu \text{ and } x^\mu \to (B_i^t(x))^\mu \tag{3.3}$$

for the upper and lower part, respectively ($i = 1, 2, 3$). t here denotes parameter of the generated subgroup.
In two dimensions the conformal transformations act on just one light-cone coordinate, while leaving the other one invariant. We will now demand, that there exists a "chiral" space-time function $u(x)$, such that the effects of the upper $SO(1,2)$ on the x combine in such a way, that precisely this transformation behaviour arises:

$$u(A_1^t(x)) = e^{-t}u \tag{3.4}$$
$$u(A_2^t(x)) = u + t \tag{3.5}$$
$$u(A_3^t(x)) = \frac{u}{1 + tu} \tag{3.6}$$
$$u(B_1^t(x)) = u \tag{3.7}$$
$$u(B_2^t(x)) = u \tag{3.8}$$
$$u(B_3^t(x)) = u \tag{3.9}$$

Then we will demand the analogous relation for the action of the lower $SO(1,2)$ on another spacetime function $v(x)$. The two systems of equations will be solved by differentiating at $t = 0$, giving a system of partial differential equation for u and v, respectively.

3.2 Action on the ξ^a and x^μ

The upper $SO(1,2)$ with metric $(+,-,-)$ acts on the triple (ξ^0, ξ^1, ξ^2), while leaving (ξ^3, ξ^4, ξ^5) invariant:

$$\left(\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & SO(1,2) & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

On the Lie algebra side there would be zeros on the lower half of the diagonal. We leave out the trivial part of the matrix and just consider the upper left 3x3 block.
A simple basis of the matrix realization of $so(1,2)$ is given by

$$m_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} m_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} m_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.10}$$

We switch to the linear combinations $a_1 = m_1$, $a_2 = m_2 + m_3$ and $a_3 = m_2 - m_3$, which fulfil the $sl(2,\mathbb{R})$ commutation relations

$$[a_1, a_3] = a_3 \ , \ [a_1, a_2] = -a_2 \ , \ [a_3, a_2] = 2a_1 \tag{3.11}$$

We exponentiate them to the finite transformations (the one-parameter subgroups generatored by the a_i):

$$A_1 = \exp(ta_1) = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.12}$$

$$A_2 = \exp(ta_2) = \begin{pmatrix} 1+\frac{t^2}{2} & \frac{t^2}{2} & 0 \\ -\frac{t^2}{2} & 1-\frac{t^2}{2} & -t \\ t & t & 1 \end{pmatrix} \tag{3.13}$$

$$A_3 = \exp(ta_3) = \begin{pmatrix} 1+\frac{t^2}{2} & -\frac{t^2}{2} & t \\ \frac{t^2}{2} & 1-\frac{t^2}{2} & t \\ t & -t & 1 \end{pmatrix} \tag{3.14}$$

Here the exponential of a square matrix M is defined the usual way as

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}. \tag{3.15}$$

The following effect on the x^μ results:

$$(A_1^t(x))^0 = \cosh(t)x^0 + \sinh(t)x^1 \tag{3.16}$$
$$(A_1^t(x))^1 = \sinh(t)x^0 + \cosh(t)x^1 \tag{3.17}$$
$$(A_1^t(x))^2 = x^2 \tag{3.18}$$
$$(A_1^t(x))^3 = x^3 \tag{3.19}$$
$$(A_2^t(x))^0 = (1+\frac{t^2}{2}) \cdot x^0 + \frac{t^2}{2} \cdot x^1 + t \cdot x^2 \tag{3.20}$$
$$(A_2^t(x))^1 = -\frac{t^2}{2} \cdot x^0 + (1-\frac{t^2}{2}) \cdot x^1 - t \cdot x^2 \tag{3.21}$$
$$(A_2^t(x))^2 = x^2 + t(x^0 + x^1) \tag{3.22}$$
$$(A_2^t(x))^3 = x^3 \tag{3.23}$$
$$(A_3^t(x))^0 = (1+\frac{t^2}{2}) \cdot x^0 - \frac{t^2}{2} \cdot x^1 + t \cdot x^2 \tag{3.24}$$
$$(A_3^t(x))^1 = +\frac{t^2}{2} \cdot x^0 + (1-\frac{t^2}{2}) \cdot x^1 + t \cdot x^2 \tag{3.25}$$
$$(A_3^t(x))^2 = x^2 + t(x^0 - x^1) \tag{3.26}$$
$$(A_3^t(x))^3 = x^3 \tag{3.27}$$

For the lower part acting on (ξ^3, ξ^4, ξ^5) we have to take into account, that the metric has the opposite order $(-,-,+)$. We have the corresponding generators

$$n_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} n_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} n_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.28}$$

we analogously set $b_1 = n_3$, $b_2 = n_1 + n_2$ and $b_3 = n_2 - n_1$ and the actions of $B_1 = \exp(tb_1)$, $B_2 = \exp(tb_2)$ and $B_3 = \exp(tb_1)$ on the x^μ in this case are

$$(B_1^t(x))^0 = e^{-t}x^0 \tag{3.29}$$
$$(B_1^t(x))^1 = e^{-t}x^1 \tag{3.30}$$
$$(B_1^t(x))^2 = e^{-t}x^2 \tag{3.31}$$
$$(B_1^t(x))^3 = e^{-t}x^3 \tag{3.32}$$
$$(B_2^t(x))^0 = x^0 \tag{3.33}$$
$$(B_2^t(x))^1 = x^1 \tag{3.34}$$
$$(B_2^t(x))^2 = x^2 \tag{3.35}$$
$$(B_2^t(x))^3 = x^3 + t \tag{3.36}$$
$$(B_3^t(x))^0 = \frac{x^0}{1 + 2tx^3 - t^2 x_\mu x^\mu} \tag{3.37}$$
$$(B_3^t(x))^1 = \frac{x^1}{1 + 2tx^3 - t^2 x_\mu x^\mu} \tag{3.38}$$
$$(B_3^t(x))^2 = \frac{x^2}{1 + 2tx^3 - t^2 x_\mu x^\mu} \tag{3.39}$$
$$(B_3^t(x))^3 = \frac{x^3 - tx_\mu x^\mu}{1 + 2tx^3 - t^2 x_\mu x^\mu} \tag{3.40}$$

3.3 Calculation of $u(x^\mu)$

Now that we have the action on the Minkowski coordinates, we can derive the partial differential equation system by differentiating the expressions (3.4)-(3.9) at $t = 0$.
The result is

$$x^1 \cdot \partial_0 u + x^0 \cdot \partial_1 u = -u \tag{3.41}$$
$$x^2 \cdot \partial_0 u - x^2 \cdot \partial_1 u + (x^0 + x^1) \cdot \partial_2 u = 1 \tag{3.42}$$
$$x^2 \cdot \partial_0 u + x^2 \cdot \partial_1 u + (x^0 - x^1) \cdot \partial_2 u = -u^2 \tag{3.43}$$
$$-x^0 \partial_0 u - x^1 \cdot \partial_1 u - x^2 \cdot \partial_2 u - x^3 \cdot \partial_3 u = 0 \tag{3.44}$$
$$\partial_3 u = 0 \tag{3.45}$$
$$-2x^0x^3 \cdot \partial_0 u - 2x^1x^3 \cdot \partial_1 u - 2x^2x^3 \cdot \partial_2 u - (x_\mu x^\mu + 2x^3x^3) \cdot \partial_3 u = 0 \tag{3.46}$$

From (3.45) it is clear, that there is no x^3 dependence, which implies the equivalence of the (3.44) and (3.46). We add and subtract (3.42) and (3.43) from one another

$$2\partial_0 u \cdot x^2 + 2\partial_2 u \cdot x^0 = 1 - u^2 \tag{3.47}$$
$$2\partial_1 u \cdot x^2 - 2\partial_2 u \cdot x^1 = -u^2 - 1 \tag{3.48}$$

We can eliminate the derivatives by taking $2x^2 \cdot (3.41) - x^1 \cdot (3.47) - x^0 \cdot (3.48)$:

$$0 = -2x^2u - x^1 + x^1u^2 + x^0u^2 + x^0 \iff u^2 - 2\frac{x^2}{x^0 + x^1}u + \frac{x^0 - x^1}{x^0 + x^1} = 0$$
$$\implies u_\pm = u_\pm(x^\mu) = \frac{1}{x^0 + x^1}\left[x^2 \pm \sqrt{-(x^0)^2 + (x^1)^2 + (x^2)^2}\right]. \tag{3.49}$$

This is compatible with (3.44) as well.

3.4 Calculation of $v(x^\mu)$

Now we turn to the lower part. The PDEs are similar, only the first three and the last three right-hand sides are exchanged.

$$x^1 \cdot \partial_0 v + x^0 \cdot \partial_1 v = 0 \tag{3.50}$$
$$x^2 \cdot \partial_0 v - x^2 \cdot \partial_1 v + (x^0 + x^1) \cdot \partial_2 v = 0 \tag{3.51}$$
$$x^2 \cdot \partial_0 v + x^2 \cdot \partial_1 v + (x^0 - x^1) \cdot \partial_2 v = 0 \tag{3.52}$$
$$-x^0 \cdot \partial_0 v - x^1 \cdot \partial_1 v - x^2 \cdot \partial_2 v - x^3 \cdot \partial_3 v = -v \tag{3.53}$$
$$\partial_3 v = 1 \tag{3.54}$$
$$-2x^0x^3 \cdot \partial_0 v - 2x^1x^3 \cdot \partial_1 v - 2x^2x^3 \cdot \partial_2 v - (x_\mu x^\mu + 2x^3x^3) \cdot \partial_3 v = -v^2 \tag{3.55}$$

Addition and subtraction of (3.51) and (3.52) and insertion of (3.54) into (3.53) and (3.55) yields

$$x^2 \cdot \partial_0 v + x^0 \cdot \partial_2 v = 0 \tag{3.56}$$
$$x^2 \cdot \partial_1 v - x^1 \cdot \partial_2 v = 0 \tag{3.57}$$
$$x^0 \partial_0 v + x^1 \cdot \partial_1 v + x^2 \cdot \partial_2 v + x^3 = v \tag{3.58}$$
$$-2x^0x^3 \cdot \partial_0 v - 2x^1x^3 \cdot \partial_1 v - 2x^2x^3 \cdot \partial_2 v - (x_\mu x^\mu + 2x^3x^3) = -v^2 \tag{3.59}$$

We multiply (3.58) by $2x^3$ and plug it into (3.59) to get

$$-2vx^3 - ((x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2) = -v^2 \tag{3.60}$$

which has the solution

$$v_\pm(x^\mu) = x^3 \pm \sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2} \tag{3.61}$$

These functions also fulfil the equations (3.50), (3.56) and (3.57) not used in the derivation. It can be checked that these u and v indeed transform covariantly under the induced transformations.

3.5 The hypersurface parametrized by u and v

We want to find a twodimensional hypersurface, defined by two equations

$$f_1(x^\mu) = 0 \tag{3.62}$$
$$f_2(x^\mu) = 0, \tag{3.63}$$

that is left invariant under the transformations A_i^t and B_i^t ($i = 1, 2, 3$) induced by the $SO(1,2) \times SO(1,2)$ action.

It is sufficient to do this for the infinitesimal transformations $\zeta^\mu_{A_i} = \epsilon \cdot \partial_t (A^t_i)^\mu{}_{t=0}$ and $\zeta^\mu_{B_i} = \epsilon \cdot \partial_t (B^t_i)^\mu{}_{t=0}$, i.e. we require

$$f_1(x^\mu + \zeta^\mu_{A_i}) = \zeta^\nu_{A_i}\partial_\nu f_1(x^\mu) = 0 \tag{3.64}$$
$$f_2(x^\mu + \zeta^\mu_{A_i}) = \zeta^\nu_{A_i}\partial_\nu f_2(x^\mu) = 0 \tag{3.65}$$

and the analogous equations for $\zeta^\mu_{B_i}$.
The chiral variables have to be real, which is only fulfilled on the threedimensional submanifold defined by

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = 0. \tag{3.66}$$

Therefore this has to be one of the equations, say $f_1(x^\mu) = 0$. For this equation we can even for finite transformations check easily, that they preserve the submanifold:

$$\begin{aligned} A^t_1 \ : \ f_1 \mapsto &[\cosh(t)x^0 + \sinh(t)x^1]^2 - [\sinh(t)x^0 + \cosh(t)x^1]^2 - [x^2]^2 \\ &= (x^0)^2 - (x^1)^2 - (x^2)^2 = f_1 = 0 \end{aligned} \tag{3.67}$$

$$\begin{aligned} A^t_2 \ : \ f_1 \mapsto &[(1+\frac{t^2}{2})x^0 + \frac{t^2}{2}x^1 + tx^2]^2 - [-\frac{t^2}{2}x^0 + (1-\frac{t^2}{2})x^1 - tx^2]^2 - [x^2 + t(x^0+x^1)]^2 \\ &= (x^0)^2 - (x^1)^2 - (x^2)^2 = 0 \end{aligned} \tag{3.68}$$

$$\begin{aligned} A^t_3 \ : \ f_1 \mapsto &[(1+\frac{t^2}{2})x^0 - \frac{t^2}{2}x^1 + tx^2]^2 - [\frac{t^2}{2}x^0 + (1-\frac{t^2}{2})x^1 + tx^2]^2 - [x^2 + t(x^0-x^1)]^2 \\ &= (x^0)^2 - (x^1)^2 - (x^2)^2 = 0 \end{aligned} \tag{3.69}$$

$$B^t_1 \ : \ f_1 \mapsto [e^{-t}x^0]^2 - [e^{-t}x^1]^2 - [e^{-t}x^2]^2 = e^{-2t}f_1 = 0 \tag{3.70}$$

$$B^t_2 \ : \ f_1 \mapsto [x^0]^2 - [x^1]^2 - [x^2]^2 = f_1 = 0 \tag{3.71}$$

$$B^t_3 \ : \ f_1 \mapsto \frac{1}{(1+2tx^2 - t^2x^\mu x_\mu)^2}\left([x^0]^2 - [x^1]^2 - [x^2]^2\right) = \frac{1}{(1+2tx^2 - t^2x^\mu x_\mu)^2} f_1 = 0 \tag{3.72}$$

For $f_2(x^\mu) = 0$ we determine the consequences of requiring conservation under the infinitesimal transformation $\zeta^\mu_{A_i}$ and $\zeta^\mu_{B_i}$. How a general spacetime dependent function behaves under such a transformation, was already reflected in the systems of partial differential equations. The case at hand just differs in that all the equations have zero on the right hand side:

$$x^1 \cdot \partial_0 f_2 + x^0 \cdot \partial_1 f_2 = 0 \tag{3.73}$$
$$x^2 \cdot \partial_0 f_2 - x^2 \cdot \partial_1 f_2 + (x^0 + x^1) \cdot \partial_2 f_2 = 0 \tag{3.74}$$
$$x^2 \cdot \partial_0 f_2 + x^2 \cdot \partial_1 f_2 + (x^0 - x^1) \cdot \partial_2 f_2 = 0 \tag{3.75}$$
$$-x^0 \cdot \partial_0 f_2 - x^1 \cdot \partial_1 f_2 - x^2 \cdot \partial_2 f_2 - x^3 \cdot \partial_3 f_2 = 0 \tag{3.76}$$
$$\partial_3 f_2 = 0 \tag{3.77}$$
$$-2x^0x^3 \cdot \partial_0 f_2 - 2x^1x^3 \cdot \partial_1 f_2 - 2x^2x^3 \cdot \partial_2 f_2 - (x_\mu x^\mu + 2x^3x^3) \cdot \partial_3 f_2 = 0 \tag{3.78}$$

We add and subtract (3.74) and (3.75) from one another and put (3.77) into (3.76) and (3.78) (which are therefore equivalent) to get the system

$$x^1 \cdot \partial_0 f_2 + x^0 \cdot \partial_1 f_2 = 0 \tag{3.79}$$
$$x^2 \cdot \partial_0 f_2 + x^0 \cdot \partial_2 f_2 = 0 \tag{3.80}$$
$$x^2 \cdot \partial_1 f_2 - x^1 \cdot \partial_2 f_2 = 0 \tag{3.81}$$
$$x^0 \cdot \partial_0 f_2 + x^1 \cdot \partial_1 f_2 + x^2 \cdot \partial_2 f_2 = 0 \tag{3.82}$$

Using (3.81) to eliminate $\partial_2 f_2$, (3.79) and (3.80) become equivalent. We multiply (3.79) by x^0 to get

$$x^0 x^1 \cdot \partial_0 f_2 + (x^0)^2 \cdot \partial_1 f_2 = 0 \tag{3.83}$$

$$x^0 x^1 \cdot \partial_0 f_2 + (x^1)^2 \cdot \partial_1 f_2 + (x^2)^2 \cdot \partial_1 f_2 = 0. \tag{3.84}$$

This system of linear equations in two variables has rank one (if $f_1 = 0$ holds), so the solution space is onedimensional. We take the special solution $\partial_0 f_2 = x^0$ and $\partial_1 f_2 = -x^2$ as a basis vector, then any other solution has the form $(\partial_0 f_2, \partial_1 f_2) = (\gamma x^0, -\gamma x^1)$ with $\gamma \in \mathbb{R}$. By (3.81), we have $\partial_2 f_2 = -\gamma x^2$ and with (3.77) we find, that any solution of the whole system has the form

$$\partial_\mu f_2 = \gamma(x^0, -x^1, -x^2, 0), \tag{3.85}$$

that is to say, it is proportional to the gradient $\partial_\mu f_1$. Hence f_2 is a function of f_1,

$$f_2 = f_2(f_1), \tag{3.86}$$

so there is no independent second equation and no twodimensional submanifold, that is parametrized by u and v and that is invariant under the diagonally embedded conformal transformations.

4 Partial wave expansions in D=2 and D=4

4.1 Decomposition of the partial waves

We considered conformal partial wave expansions of four-point functions of hermitean scalar fields in two spacetime dimensions in the introduction. We saw, that the partial waves factorize as

$$B_{mn}(u,v) = G_m(u)G_n(v) \tag{4.1}$$

with

$$G_m(u) = u^m \cdot F(m,m;2m;u). \tag{4.2}$$

They involve the chiral cross ratios u and v, the hypergeometric function $F(a,b;c;x)$ and they are labelled by two numbers (m,n), which correspond to a representation of $SO(2,2)$.
If one performs the PWE in four dimensions for a correlation function with four fields of equal scaling dimensions, the different contributions

$$\beta_{kl}(u,v) = \frac{uv}{u-v}\left[G_{k+l}(u)G_{k-1}(v) - G_{k+l}(v)G_{k-1}(u)\right]. \tag{4.3}$$

are also labelled by two quantum numbers, the twist $2k$ and the spin l. Here they correspond to (unitary, positive energy) symmetric tensor representations of the universal covering group $SU(2,2)$ of the fourdimensional conformal group with lowest weight

$$\lambda = (2k+l, l/2, l/2). \tag{4.4}$$

The chiral variables u and v are introduced here implicitly by

$$s = uv \ , \ \ t = (1-u)(1-v) \tag{4.5}$$

in formal analogy to the two-dimensional case (s and t are the ordinary conformally invariant cross-ratios). If one restricts the correlator and therefore s and t to the $x^2 = x^3 = 0$ plane, u and v coincide with the two-dimensional chiral cross-ratios.
To achieve a better understanding of the "almost-factorization" property of (4.3), it is of interest to investigate the following problem

> How can a given 4D partial wave be expanded in terms of the $B_{mn}(u,v)$ (with u and v being just formal variables) and which representations of the 2D conformal group appear in this expansion, if one interprets the $B_{mn}(u,v)$ by restriction to the $x^2 = x^3 = 0$ plane as two-dimensional partial waves?

We derive the expansion in this section and compare it in the next one to the branching of the representations of $so(4,2)$ into representations of its subalgebra $so(2,2)$.

Generically the stated problem could be posed as finding the coefficients C_{nm} in the expansion

$$\beta_{kl}(u,v) = \sum_{n,m} C_{nm}(k,l) G_n(u) G_m(v). \tag{4.6}$$

We choose a way to obtain this decomposition, that relies on the use of identities of hypergeometric functions, beginning with the simplest case $l = 0$. The results obtained there will be used, when we go over to the general case. The $l = 0$ partial waves are

$$\beta_k(u,v) \equiv \beta_{k0}(u,v) = \frac{uv}{u-v}\left(G_k(u)G_{k-1}(v) - G_k(v)G_{k-1}(u)\right), \tag{4.7}$$

Their decomposition can be obtained from the following

Lemma 4.1: The partial waves β_k can be written as

$$\beta_k(u,v) = G_k(u)G_k(v) + c_1(k)\beta_{k+1}(u,v) \tag{4.8}$$

with the coefficient

$$c_n(k) = \frac{1}{4^n}\Pi_{i=1}^n \frac{(k+i-1)^2}{(2k+2i-3)(2k+2i-1)} = \frac{(k)_n^2}{16^n (k-\frac{1}{2})_n (k+\frac{1}{2})_n}. \tag{4.9}$$

We write $c(k) \equiv c_1(k)$.

Proof: We introduce the shortcut $F_k(x) := F(k,k;2k;x)$ and state the identity

$$F_{k-1}(x) + c(k)x^2 F_{k+1}(x) - (1-\frac{x}{2})F_k(x) = 0. \tag{4.10}$$

which can be checked by comparing the coefficients of the power series:

$$\begin{aligned} x^n: \frac{(k-1)_n^2}{(2k-2)_n n!} + \frac{k^2}{4(4k^2-1)}\frac{(k+1)_{n-2}^2}{(2k+2)_{n-2}(n-2)!} - \frac{(k)_n^2}{(2k)_n n!} + \frac{1}{2}\frac{(k)_{n-1}^2}{(2k)_{n-1}(n-1)!} = 0 \\ \Longleftrightarrow (k-1)(2k+n-1)(2k+n-2) + kn(n-1) \\ -2(k+n-1)^2(2k-1) - (2k+n-1)n(2k-1) = 0 \end{aligned} \tag{4.11}$$

which is fulfilled. Next we multiply (4.10) by x^k to get

$$\mathcal{J}_k(x) := x\left(G_{k-1}(x) + c(k)G_{k+1}(x)\right) - (1-\frac{x}{2})G_k(x) = 0. \tag{4.12}$$

and consider

$$\begin{aligned} 0 =& vG_k(v)\mathcal{J}_k(u) - uG_k(u)\mathcal{J}_k(v) = (u-v)G_k(u)G_k(v) \\ &+ uv\left[G_{k-1}(u)G_k(v) - G_k(u)G_{k-1}(v) + c(k)(G_{k+1}(u)G_k(v) - G_k(u)G_{k+1}(v))\right] \end{aligned} \tag{4.13}$$

This can be rearranged as

$$\begin{aligned} &\frac{uv}{u-v}\left[G_k(u)G_{k-1}(v) - G_{k-1}(u)G_k(v)\right] \\ =& G_k(u)G_k(v) + c_1(k)\frac{uv}{u-v}\left[G_{k+1}(u)G_k(v) - G_k(u)G_{k+1}(v)\right], \end{aligned} \tag{4.14}$$

which is equation (4.8). $\square$

We note, that shifting the product index in the coefficient $c_n(k+1)$ and multiplying by $c_1(k)$ gives the identity

$$c_{n+1}(k) = c_1(k)c_n(k+1). \tag{4.15}$$

If we plug in the higher twist partial waves recursively, we can use (4.15) to rewrite (4.8) as

$$\begin{aligned}\beta_k(u,v) &= G_k(u)G_k(v) + c_1(k)[G_{k+1}(u)G_{k+1}(v) + c_1(k+1)\beta_{k+2}(u,v)] \\ &= G_k(u)G_k(v) + c_1(k)G_{k+1}(u)G_{k+1}(v) + c_2(k)\beta_{k+2}(u,v) \\ &= .. = \sum_{n=0}^{\infty} c_n(k)G_{k+n}(u)G_{k+n}(v) \end{aligned} \tag{4.16}$$

So in this case the appearing 2D partial waves correspond to representations of the two-dimensional conformal group with lowest weight

$$(k+n, k+n) \tag{4.17}$$

$n \in \mathbb{N}$. In terms of the alternative quantum numbers scaling dimension and spin the representations labelled by

$$(2k+2n, 0) \tag{4.18}$$

appear.

Closer investigation of low l cases motivates the following ansatz for general l:

$$\beta_{kl}(u,v) - \underbrace{\sum_{n+m=l} G_{k+n}(u)G_{k+m}(v)}_{=:S(k,l)} = c(k+l)\beta_{k+1,l}(u,v) + X(k,l). \tag{4.19}$$

The $X(k,l)$ stands for possible extra 4D partial waves. They do not appear for $l=0$ and $l=1$, but we will see, that such terms occur for higher l. We rewrite (4.19) as

$$\begin{aligned} (u-v)S(k,l) =& (u-v)\left[\beta_{kl}(u,v) - c(k+l)\beta_{k+1,l}(u,v) - X(k,l)\right] \\ \Longrightarrow (u-v)\sum_{n+m=l} G_{k+n}(u)G_{k+m}(v) =& uv\left[G_{k+l}(u)G_{k-1}(v) - G_{k+l}(v)G_{k-1}(u)\right] - (u-v)X(k,l) \\ & - c(k+l)uv\left[G_{k+l+1}(u)G_k(v) - G_k(u)G_{k+l+1}(v)\right] \end{aligned} \tag{4.20}$$

The $c(k+l)$ can be eliminated using the identity (4.12):

$$\begin{aligned} (u-v)S(k,l) =& uv\left[G_{k+l}(u)G_{k-1}(v) - G_{k+l}(v)G_{k-1}(u)\right] - vG_k(v)[(1-\frac{u}{2})G_{k+l}(u) \\ & - uG_{k+l-1}(u)] + uG_k(u)[(1-\frac{v}{2})G_{k+l}(v) - vG_{k+l-1}(v)] - (u-v)X(k,l) \end{aligned} \tag{4.21}$$

We subtract now $S(k+1, l-2)$ from $S(k,l)$ to cancel out the middle of the summation:

$$(u-v)[S(k,l) - S(k+1,l-2)] = (u-v)[G_{k+l}(u)G_k(v) + G_k(u)G_{k+l}(v)]. \tag{4.22}$$

If we also perform this with the corresponding expressions (4.21) for $S(k,l)$ and (4.20) for $S(k+1,l-2)$, we get

$$\begin{aligned}(u-v)[S(k,l)-S(k+1,l-2)] =&uv(G_{k+l}(u)G_{k-1}(v)-G_{k+l}(v)G_{k-1}(u))-\\&-vG_k(v)[(1-\frac{u}{2})G_{k+l}(u)-uG_{k+l-1}(u)]\\&+uG_k(u)[(1-\frac{v}{2})G_{k+l}(v)-vG_{k+l-1}(v)]-(u-v)X(k,l)\\&-uv(G_{k+l-1}(u)G_k(v)-G_{k+l-1}(v)G_k(u))\\&+vG_{k+1}(v)c(k+l-1)uG_{k+l}(u)\\&-uG_{k+1}(u)c(k+l-1)vG_{k+l}(v)+(u-v)X(k+1,l-2)\end{aligned} \tag{4.23}$$

The $G_k(u/v)G_{k+l-1}(v/u)$ terms cancel out here.
We equate the two expressions (4.22) and (4.23) and collect the terms involving $G_k(u)G_{k+l}(v)$. Their coefficients can be summed up as $u-\frac{uv}{2}-u+v=v(1-\frac{u}{2})$ and analogously for $(u\leftrightarrow v)$, so we have

$$\begin{aligned}0 =&uv(G_{k+l}(u)G_{k-1}(v)-G_{k+l}(v)G_{k-1}(u))-u(1-\frac{v}{2})G_k(v)G_{k+l}(u)\\&+v(1-\frac{u}{2})G_k(u)G_{k+l}(v)-(u-v)X(k,l)+vG_{k+1}(v)c(k+l-1)uG_{k+l}(u)\\&-uG_{k+1}(u)c(k+l-1)vG_{k+l}(v)+(u-v)X(k+1,l-2)\end{aligned} \tag{4.24}$$

We can now summarize terms by using the identity (4.12) again:

$$\begin{aligned}0=-c(k)uv(G_{k+l}(u)G_{k+1}(v)-G_{k+1}(u)G_{k+l}(v))-(u-v)X(k,l)\\+c(k+l-1)uvG_{k+1}(v)G_{k+l}(u)-c(k+l-1)uvG_{k+1}(u)G_{k+l}(v)+(u-v)X(k+1,l-2)\end{aligned} \tag{4.25}$$

Dividing by $(u-v)$ finally gives a recurrence relation for the $X(k,l)$

$$X(k,l)-X(k+1,l-2)=[c(k+l-1)-c(k)]\beta_{k+2,l-2}(u,v), \tag{4.26}$$

which is solved by

$$X(k,l)=\sum_{i=0}^{2i<l}[c(k+l-i-1)-c(k+i)]\beta_{k+2+i,l-2i-2}(u,v). \tag{4.27}$$

Plugging this into the original ansatz (4.19) results in

$$\begin{aligned}\beta_{kl}(u,v) =&\sum_{n=0}^{l}G_{k+n}(u)G_{k+l-n}(v)+c(k+l)\beta_{k+1,l}(u,v)\\&+\sum_{i=0}^{2i<l}[c(k+l-i-1)-c(k+i)]\beta_{k+2+i,l-2i-2}(u,v).\end{aligned} \tag{4.28}$$

Inserting the terms $\beta_{k+1,l}$, $\beta_{k+2,l}$, .. in the first line, we can use the coefficent identity (4.15) to rewrite this as

$$\beta_{kl}(u,v) = \sum_{m=0}^{\infty} c_m(k+l)[\sum_{n=0}^{l} G_{k+m+n}(u)G_{k+m+l-n}(v) \\ + \sum_{i=0}^{2i<l}(c(k+m+l-i-1) - c(k+m+i))\beta_{k+m+2+i,l-2-2i}(u,v)] \tag{4.29}$$

where the prefactor c_m is the one from Lemma 4.1.
In this formula there are only terms, why can be interpreted as 2D partial waves, and terms with 4D partial waves of lower spins. This means, that by inserting the $l-2, l-4, ..$ partial waves in the second line of this formula we can express any given $\beta_{kl}(u,v)$ entirely in terms of 2D partial waves.
We formulate the result of doing so:

Theorem 4.2: A 4D partial wave $\beta_{kl}(u,v)$ corresponding to a $(2k+l, l/2, l/2)$ symmetric tensor representation of $SU(2,2)$ can be decomposed as a series in 2D partial waves $B_{mn}(u,v) = G_m(u)G_n(v)$, which correspond to the following representations of $SO(2,2)$:

- $(k+m+n, k+m+l-n)$ for $m \in \mathbb{N}$ and $n = 0, .., l$,
- $(k+m+n, k+m+l-2-n)$ for $m \in \mathbb{N}$ and $n = 0, 1, .., l-2$
- $\vdots$
- $(k+m, k+m)$ for l even - or - $(k+m+n, k+m+1-n)$ with $n = 0, 1$ for l odd (in both cases, $m \in \mathbb{N}$).

In terms of the quantum numbers scaling dimension and spin, the list reads

- $(2k+2m+l, l-2n)$, $n = 0, 1, .., l$
- $(2k+2m+l-2, (l-2)-2n)$, $n = 0, 1, .., l-2$
- $\vdots$
- $(2k+2m+2, 2-2n)$, $n = 0, 1, 2$
- $(2k+2m, 0)$

for l even and

- $(2k+2m+l, l-2n)$, $n = 0, 1, .., l$
- $(2k+2m+l-2, (l-2)-2n)$, $n = 0, 1, .., l-2$
- $\vdots$
- $(2k+2m+3, 3-2n)$, $n = 0, 1, 2, 3$
- $(2k+2m+1, 1-2n)$, $n = 0, 1$

for l odd.

No statements are possible about multiplicities, which would be interesting in the light of the next section, where we will compare this decomposition to the decomposition of a unitary positive energy representation of $so(4,2)$ into representations of $so(2,2)$.

4.2 Character decomposition

In this section we want to see, into which representations of the twodimensional conformal algebra a representation of the fourdimensional one decomposes. Then we will compare this to the decomposition of the partial waves in the previous section. Since no multiplicity statements could be made there, we will only compare, which representations appear in both cases.
For the classical Lie algebras the problem of decomposing a representation into representation of a subalgebra is addressed using the so-called branching rules. There is a well developed theory for them and there exist tables and efficient computer programs, that facilitate their calculation. In the present case the matter is more complicated due to the non-compactness of both of the involved groups.
In any case, it is important to specify the way, in which the subalgebra is embedded. We had seen before, that the twodimensional conformal algebra can be embedded in two inequivalent ways into the fourdimensional algebra. We had called them the diagonal and the block embedding.
Our analysis of the branching of representations will make use of character formulae. They summarize in a compact way informations about the weight system of a representation. With suitable restrictions, one can also count the weights of a subrepresentation. For the unitary irreducible positive energy representations of $so(4,2)$ the characters were listed in section 2.2.
We start with the block type embedding. Since the Cartan elements used in the paper [7] are certain linear combinations of the ones, that one receives in the splting of $so(4)$ into $so(3) \oplus so(3)$, to achieve a decomposition of this character in terms of $so(1,2)$ characters

$$\tilde{\chi}_n(x) = \frac{x^n}{1-x} \tag{4.30}$$

we have to set $x = y$ in (2.82) and (2.83).
For a long symmetric tensor representations with highest weight $(d = 2k + l, l/2, l/2)$ the formula for the long representations becomes

$$A_{[2k+l,l/2,l/2]}(s,x,x) = s^{2k+l} \chi_{l/2}(x)^2 \frac{1}{(1-\frac{s}{x})(1-sx)(1-s)^2}. \tag{4.31}$$

We redefine the formal variables as

$$sx = p \text{ and } \frac{s}{x} = q, \tag{4.32}$$

and get

$$\begin{aligned}(1-p)(1-q) \cdot A_{[2k+l,l/2,l/2]}(s,x,x) &= \sqrt{pq}^{2k+l} \chi_{l/2}(\sqrt{\frac{p}{q}})^2 \frac{1}{(1-\sqrt{pq})^2} \\ &= (pq)^{k+l/2} \left[(\frac{p}{q})^{-l/4} + (\frac{p}{q})^{-l/4+1/2} + .. + (\frac{p}{q})^{l/4} \right]^2 \sum_{n=0}^{\infty} (n+1)(pq)^{n/2} \\ &= (pq)^k \left[p^{l/2} + p^{l/2-1/2} q^{1/2} + .. + q^{l/2} \right]^2 \sum_{n=0}^{\infty} (n+1)(pq)^{n/2}. \end{aligned} \tag{4.33}$$

Then we can write the character as a sum of products of characters of $so(1,2)$:

$$A_{[2k+l,l/2,l/2]}(s,x,x) = \sum_{n=0}^{\infty}\sum_{i,j=0}^{l}(n+1)\frac{p^{k+n/2+l-(i+j)/2}}{1-p}\frac{q^{k+n/2+(i+j)/2}}{1-q}. \qquad (4.34)$$

We can now list the appearing representations.

Theorem 4.3: The long symmetric tensor representation of $so(4,2)$ with lowest weight $(2k+l, l/2, l/2)$ decomposes for the block embedding into representations of $so(2,2)$ of scaling dimensions $2k+n+l$ and spins $l' = l-i-j$ for $n \in \mathbb{N}$ and $0 \leq i, j \leq l$. For fixed n every spin l′ appears $(n+1)\cdot(l+1-|l'|)$ times.

For the first short (or twist 2) representation we have the character

$$D_{[l+2,l/2,l/2]}(s,x,y) = s^{l+2}\left(\chi_{l/2}(x)\chi_{l/2}(y) - s\chi_{l/2-\frac{1}{2}}(x)\chi_{l/2-\frac{1}{2}}(y)\right)P(s,x,y), \qquad (4.35)$$

which for $x = y$ using our previous result is equal to

$$\sum_{n=0}^{\infty}(n+1)\left[\sum_{i,j=0}^{l}p^{1+n/2+l-(i+j)/2}q^{1+n/2+(i+j)/2} - \sum_{i,j=0}^{l-1}p^{2+n/2+(l-1)-(i+j)/2}q^{2+n/2+(i+j)/2}\right] \qquad (4.36)$$

We see, that we have the subrepresentations with scaling dimensions $(2+n+l)$ and spins $(l-i-j)$ $(i,j=0,..,l)$ as before, but certain representations are subtracted. Careful counting yields the surviving representations.

Theorem 4.4: The short symmetric tensor representation of $so(4,2)$ with lowest weight $(2+l, l/2, l/2)$ decomposes for the block embedding into the following representations of $so(2,2)$:

- the "wingers" $(2+n+l, \pm l)$ with multiplicity $(n+1)$ (since the subtraction does not reach them)
- the "insiders" $(2+n+l, l' = l-i-j)$ $(i,j = 0..l-1, n \geq 1)$ with multiplicities $n+l+1-|l'|$ (since less representations are subtracted then there were before)
- the $(2+l, l' = l-i-j)$ $(i,j = 0..l)$ from the $n = 0$-term with multiplicities $l+1-|l'|$ (since they are not reached by the subtraction either).

We compare our results to the analytic decomposition formula in the last section now. Starting with the long representations, we see, that for the scaling dimension as well as for the spin, in the analytic formula the index labelling the representations increases in steps of two, while in the character decomposition it increases in steps of one. If one interprets the terms in (4.29) by restriction as twodimensional partial waves, half of the representations do not contribute. Apparently the fields, that carry the corresponding quantum numbers of the partial waves, do not couple to two scalar fields of equal dimension (i.e. their three point function with them vanishes).
Also for the short representation this pattern of a double step width appears.

We note, that our restriction of the fourdimensional correlation functions of fields of equal scaling dimension could be interpreted as taking only the leading term ϕ_{00} of the fields in a

Taylor expansion of $\phi(x^\mu)$ with scaling dimension d in some region around the $x^2 = x^3 = 0$-plane:

$$\phi(x^\mu) = \sum_{n,m=0}^{\infty} \frac{(x^2)^n}{n!} \frac{(x^3)^m}{m!} \phi_{mn}(x^0, x^1). \tag{4.37}$$

A priori, the twodimensional fields

$$\phi_{mn}(x^0, x^1) = \frac{\partial^2 \phi}{\partial x^2 \partial x^3}_{(x^2=x^3=0)}, \tag{4.38}$$

that appear as transverse derivatives, need not be conformally covariantly transforming fields anymore. But we can check that a field differentiated in the x^2 or x^3 direction under commutation with the twodimensional generators $P_0, P_1, K_0, K_1, D, M_{01}$ does in fact behave covariantly for $x^2 = x^3 = 0$. This implies the covariance of higher and mixed derivatives. The action of the generators on fields was stated in section 2.2.
For the transformation under the $P_{0/1}$ this is obvious, since partial derivatives commute. The other cases need a little computation:

$$[M_{01}, \partial_{2|3}\phi] = \partial_{2|3}[(x_0\partial_1 - x_1\partial_0)\phi] = (x_0\partial_1 - x_1\partial_0)\partial_{2|3}\phi \tag{4.39}$$

$$[D, \partial_{2|3}\phi] = d\partial_{2|3}\phi + x^\nu\partial_\nu\partial_{2|3}\phi + \delta^\nu_{2|3}\partial_\nu\phi = (d+1)\partial_{2|3}\phi + x^\nu\partial_\nu\partial_{2|3}\phi \tag{4.40}$$

$$\begin{aligned}
[K_{0/1}, \partial_{2/3}\phi] &= \partial_{2/3}[(x^2\partial_{0/1} - 2x_{0/1}x^\nu\partial_\nu - 2x_{0/1}d)\phi] \\
&= (x^2\partial_{0/1} - 2x_{0/1}x^\nu\partial_\nu - 2x_{0/1}d)\partial_{2/3}\phi + (2x_{x/3}\partial_{0/1}\delta^\nu_{2/3}\partial_\nu)\phi \\
&= (x^2\partial_{0/1} - 2x_{0/1}x^\nu\partial_\nu - 2x_{0/1}(d+1))\phi
\end{aligned} \tag{4.41}$$

This also shows, that differentiating x^2 or x^3 direction raises the scaling dimension by one, so $\phi_{mn}(t,x)$ has scaling dimension $d + m + n$.
A next step would be to generalize our analytic decomposition formula to the case of four fields of different scaling dimensions. This would open the possibility to take also higher Taylor terms into account, to which the fields corresponding to the missing partial waves are expected to couple.
If all fields in a four-point function are Taylor expanded we then get

$$\begin{aligned}
&\langle \Omega, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Omega \rangle \\
&= \sum_{n_i,m_i} \frac{(x_1^2)^{n_1}(x_1^3)^{m_1}}{n_1!m_1!} \dots \frac{(x_4^2)^{n_4}(x_4^3)^{m_4}}{n_4!m_4!} \langle \phi_{m_1,n_1}(x_1^0, x_1^1)\phi_{m_2,n_2}(x_2^0, x_2^1)\phi_{m_3,n_3}(x_3^0, x_3^1)\phi_{m_4,n_4}(x_4^0, x_4^1)\rangle.
\end{aligned} \tag{4.42}$$

and expect, that if we first partial wave expand the fourdimensional four-point function and then decompose the partial waves, the result should be the same, as if we first Taylor expand and then partial wave expand the twodimensional four-point functions.
At last, we comment briefly on the diagonal embedding. In this case it turns out, that one has to set the formal variable s in the character formulae equal to one. Then we get for the long representations

$$A_{[2k+l,l/2,l/2]}(x,y) = \chi_{l/2}(x)\chi_{l/2}(y) \left[(1 - \frac{1}{\sqrt{xy}})(1 - \sqrt{xy})(1 - \sqrt{\frac{x}{y}})(1 - \sqrt{\frac{y}{x}})\right]^{-1} \tag{4.43}$$

We see, that we cannot transform this into geometric series as before, because this would require $|\sqrt{xy}| < 1$ and $|\frac{1}{\sqrt{xy}}| < 1$. This is also the case for the short representation. We will take a different approach to investigate this embedding in the next section.

4.3 A closer look at the diagonal embedding

In the previous section the decomposition of the representation could not be obtained, if the subalgebra is embedded diagonally. Here we will use a different approach to check for the scalar representation of $so(4,2)$ with lowest weight

$$(h,0,0),$$

what type of spectrum of representations of $so(2,2)$ it contains.
We had noticed, that the Casimir operator of a Lie algebra can be used to characterize its representations. We will use this fact and express the Casimir operators of the two diagonally embedded onedimensional conformal algebras $so(1,2)$ through ladder operators and Cartan elements of the fourdimensional one. Then we try to find eigenvectors of those operators within the representation $(h,0,0)$. The corresponding eigenvalues λ are then for positive energy representations related to scaling dimensions a of representations of the subalgebra via the expression $\lambda = a(a-1)$ (cf. 2.65).
First we recall, that the $so(4,2)$ commutation relations can be summarized as

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc}) \tag{4.44}$$

with $J_{ab} = -J_{ba}$, $\eta_{ab} = diag(+,-,-,-,-,+)$ and $a,b,.. = 0,1,..,5$. $so(4,2)$ is a rank three Lie algebra, so we can pick three independent commuting operators and we choose

$$h_1 = J_{12}\ ,\ h_2 = J_{34}\ ,\ h_3 = J_{05}. \tag{4.45}$$

and list the root system with respect to this Cartan subalgebra, i.e. twelve linear combinations e^α of generators, such that $[h^i, e^\alpha] = \alpha_i e^\alpha$.

Linear combination	$[h_1,.]$	$[h_2,.]$	$[h_3,.]$
$A_l^+ = \frac{1}{2}(J_{01} + iJ_{02} - iJ_{15} + J_{25})$	$-A_l^+$	0	A_l^+
$A_r^+ = \frac{1}{2}(J_{01} - iJ_{02} - iJ_{15} - J_{25})$	A_r^+	0	A_r^+
$A_d^+ = \frac{1}{2}(J_{03} + iJ_{04} - iJ_{35} + J_{45})$	0	$-A_d^+$	A_d^+
$A_u^+ = \frac{1}{2}(J_{03} - iJ_{04} - iJ_{35} - J_{45})$	0	A_u^+	A_u^+
$A_l^- = \frac{1}{2}(J_{01} + iJ_{02} + iJ_{15} - J_{25})$	$-A_l^-$	0	$-A_l^-$
$A_r^- = \frac{1}{2}(J_{01} - iJ_{02} + iJ_{15} + J_{25})$	A_r^-	0	$-A_r^-$
$A_d^- = \frac{1}{2}(J_{03} + iJ_{04} + iJ_{35} - J_{45})$	0	$-A_d^-$	$-A_d^-$
$A_u^- = \frac{1}{2}(J_{03} - iJ_{04} + iJ_{35} + J_{45})$	0	A_u^-	$-A_u^-$
$A_{ld} = [A_d^+, A_l^-] = \frac{i}{2}(-J_{13} - iJ_{14} - iJ_{23} + J_{24})$	$-A_{ld}$	$-A_{ld}$	0
$A_{rd} = [A_d^+, A_r^-] = \frac{i}{2}(-J_{13} - iJ_{14} + iJ_{23} - J_{24})$	A_{rd}	$-A_{rd}$	0
$A_{lu} = [A_u^+, A_l^-] = \frac{i}{2}(-J_{13} + iJ_{14} - iJ_{23} - J_{24})$	$-A_{lu}$	A_{lu}	0
$A_{ru} = [A_u^+, A_r^-] = \frac{i}{2}(-J_{13} + iJ_{14} + iJ_{23} + J_{24})$	A_{ru}	A_{ru}	0

The indices emphasize, which weight the operator raises or lowers, namely +,− for the h_3 weight, l(eft),r(ight) for the h_1 weight and u(p),d(own) for the h_3 weight.
We recall, that the $so(1,2)$ algebra with metric $(+,-,-)$ consists of three operators $\{L_{01}, L_{02}, L_{12}\}$ with commutation relations

$$[L_{01}, L_{02}] = iL_{12}\ ,\ [L_{01}, L_{12}] = iL_{02}\ ,\ [L_{02}, L_{12}] = -iL_{01} \tag{4.46}$$

We introduce the ladder operators

$$L_{\pm} = L_{01} \pm iL_{02}, \tag{4.47}$$

which leads to the commutation relations

$$[L_+, L_0] = L_+ \ , \ [L_-, L_0] = -L_- \ , \ [L_+, L_-] = 2L_0 \tag{4.48}$$

where we set $L_0 := L_{12}$.
The diagonal embedding $so(1,2) \oplus so(1,2) \hookrightarrow so(2,4)$ had the form

$$\left(\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & so(1,2) & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & so(1,2) & \\ 0 & 0 & 0 & & & \end{array}\right), \tag{4.49}$$

from which we can see, that the sets $\{J_{01}, J_{02}, J_{12}\}$ and $\{J_{34}, J_{35}, J_{45}\}$ form two $so(1,2)$ subalgebras. For the lower block one has to keep in mind, that the metric there is $(-,-,+)$ instead of $(+,-,-)$. The Cartan operators J_{12} and J_{34} are obviously equal to h_1 and h_2, respectively, and with the help of the table, we express the ladder operators through ladder operators of the ambient algebra:

$$J_{01} - iJ_{02} = (A_r^+ + A_r^-) =: A_r \tag{4.50}$$

$$J_{01} + iJ_{02} = (A_l^+ + A_l^-) =: A_l \tag{4.51}$$

$$J_{35} - iJ_{45} = \frac{1}{i}(A_u^- - A_u^+) =: B_u \tag{4.52}$$

$$J_{35} + iJ_{45} = \frac{1}{i}(A_d^- - A_d^+) =: B_d. \tag{4.53}$$

Inverting these relations, we obtain the Casimir operators in the form we wanted

$$\begin{aligned} C_a^{(1)} &= -J_{01}^2 - J_{02}^2 + J_{12}^2 = J_{12}^2 - \frac{1}{4}\left[(A_l + A_r)^2 - (A_l - A_r)^2\right] = J_{12}^2 - \frac{1}{2}\left[A_l A_r + A_r A_l\right] \\ &= h_1(h_1+1) - A_l A_r = h_1(h_1 - 1) - A_r A_l \end{aligned} \tag{4.54}$$

$$C_b^{(1)} = -J_{35}^2 - J_{45}^2 + J_{34}^2 = h_2(h_2+1) - B_d B_u = h_2(h_2-1) - B_u B_d \tag{4.55}$$

Because of the symmetry of the situation, we just consider, what the spectrum of $C_a^{(1)}$ is. The one of $C_b^{(1)}$ will be the same and the total spectrum of twodimensional representations will be the one of a tensor product.

4.3.1 Generating the representation

First we want to convince ourselves, that the operators with a plus sign span the whole representation.
We consider the vectors obtained by multiple application of the "plus" operators (which raise the scaling dimension by one and change one of the other weights) on the lowest weight vector

$$|h\rangle \equiv |h, 0, 0\rangle \tag{4.56}$$

and introduce the shortcut

$$|pqrs\rangle := (A_r^+)^p(A_l^+)^q(A_u^+)^r(A_d^+)^s|h\rangle. \tag{4.57}$$

Then the claim is

Lemma 4.6: The $|pqrs\rangle$ span the whole scalar representation.

First, we list the eigenvalues of a vector $|pqrs\rangle$ under application of the Cartan elements.

$$h_1(A_r^+)^p(A_l^+)^q(A_u^+)^r(A_d^+)^s|h\rangle = (p-q)|pqrs\rangle \tag{4.58}$$

$$h_2(A_r^+)^p(A_l^+)^q(A_u^+)^r(A_d^+)^s|h\rangle = (r-s)|pqrs\rangle \tag{4.59}$$

$$h_3(A_r^+)^p(A_l^+)^q(A_u^+)^r(A_d^+)^s|h\rangle = (h+p+q+r+s)|pqrs\rangle \tag{4.60}$$

Also the eigenvalues of the operators $E_\pm = h_3 \pm h_1$ will be useful.

$$E_+|pqrs\rangle = (h+p+q+r+s+p-q)|pqrs\rangle = (h+2p+r+s)|pqrs\rangle \tag{4.61}$$

$$E_-|pqrs\rangle = (h+p+q+r+s-p+q)|pqrs\rangle = (h+2q+r+s)|pqrs\rangle \tag{4.62}$$

Both are calculated by commuting the operator to the lowest weight vector $|h\rangle$, on which their value is known. In our convention of positive and negative roots, operators with a minus sign as well as A_{ld}, A_{rd}, A_{lu} and A_{ru} annihilate $|h\rangle$, so if they are involved, our strategy will be to commute them to the end as well.
We will prove the lemma by checking, that the application of any ladder operator on $|pqrs\rangle$ is again a linear combination of vectors of this type (with other pqrs).
For A_i^+ (i=r,l,u,d) this is obvious, since the operators with a $+$ commute with each other[1], so just one of the numbers is raised by one.
So at first we consider the lowering operators with respect to the scaling dimensions, e.g. A_r^-:

$$\begin{aligned}
A_r^-|pqrs\rangle =&(A_r^+)^pA_r^-(A_l^+)^q(A_u^+)^r(A_d^+)^s|h\rangle \\
=&(A_r^+)^p(A_l^+A_r^- + E_-)(A_l^+)^{q-1}(A_u^+)^r(A_d^+)^s|h\rangle \\
=&(A_r^+)^p(A_l^+A_r^- + (h+2(q-1)+r+s)(A_l^+)^{q-1}(A_u^+)^r(A_d^+)^s|h\rangle \\
=&(A_r^+)^p((A_l^+)^2A_r^-(A_l^+)^{q-2} + (h+2(q-2)+r+s)(A_l^+)^{q-1})(A_u^+)^r(A_d^+)^s|h\rangle \\
&+ (h+2(q-1)+r+s)|p,q-1,rs\rangle \\
=& .. = (A_r^+)^p(A_l^+)^qA_r^-(A_u^+)^r(A_d^+)^s|h\rangle + \sum_{i=1}^{q}(h+2(q-i)+r+s)|p,q-1,rs\rangle
\end{aligned} \tag{4.63}$$

We used the commutation relation $[A_r^-, A_l^+] = E_-$ and the eigenvalue (4.61) of E_-. The sum can be performed easily and we continue to commute using $[A_r^-, A_u^+] = -A_{ru}$:

$$\begin{aligned}
A_r^-|pqrs\rangle =&(A_r^+)^p(A_l^+)^q(A_u^+A_r^- - A_{ru})(A_u^+)^{r-1}(A_d^+)^s|h\rangle + q(h+q+r+s-1)|p,q-1,rs\rangle \\
=& .. = \underbrace{(A_r^+)^p(A_l^+)^q(A_u^+)^rA_r^-(A_d^+)^s|h\rangle}_{(1)} - \underbrace{r(A_r^+)^p(A_l^+)^q(A_u^+)^{r-1}A_{ru}(A_d^+)^s|h\rangle}_{(2)} \\
&+ q(h+q+r+s-1)|p,q-1,rs\rangle
\end{aligned} \tag{4.64}$$

[1] In fact any two operators with one coinciding index commute.

For the first term, commuting A_r^- past the A_d^+ produces A_{rd}:

$$(1) = (A_r^+)^p (A_l^+)^q (A_u^+)^r (A_d^+ A_r^- - A_{rd})(A_d^+)^{s-1}|h\rangle, \tag{4.65}$$

until at last A_r^- hits the lowest weight vector $|h\rangle$. But since $[A_{rd}, A_d^+] = 0$, also the second term is annihilated, so $(1) = 0$.
Now we consider term (2) with $[A_{ru}, A_d^+] = A_r^+$:

$$\begin{aligned}(2) &= r(A_r^+)^p (A_l^+)^q (A_u^+)^{r-1} (A_d^+ A_{ru} + A_r^+)(A_d^+)^{s-1}|h\rangle \\ &= r(A_r^+)^p (A_l^+)^q (A_u^+)^{r-1} (A_d^+ A_{ru} + A_r^+)(A_d^+)^{s-1}|h\rangle\end{aligned} \tag{4.66}$$

Further commutation produces s identical terms with only "+"-operators. The first term takes the form $...A_{ru}|h\rangle = 0$ and it remains, that

$$(2) = rs(A_r^+)^p (A_l^+)^q (A_u^+)^{r-1} A_r^+ (A_d^+)^{s-1}|h\rangle = rs|p+1, q, r-1, s-1\rangle \tag{4.67}$$

Therefore the result of the application of the A_r^- operator is a vector, which is a linear combination of vectors solely obtained by application of ladder operators with a plus sign:

$$A_r^-|pqrs\rangle = -rs|p+1, q, r-1, s-1\rangle + q(h+q+r+s-1)|p, q-1, rs\rangle \tag{4.68}$$

Analogously the application of the other "minus" operators yields

$$A_l^-|pqrs\rangle = -rs|p, q+1, r-1, s-1\rangle + p(h+p+r+s-1)|p-1, q, rs\rangle \tag{4.69}$$

$$A_d^-|pqrs\rangle = -pq|p-1, q-1, r, s+1\rangle + r(h+r+p+q-1)|pq, r-1, s\rangle \tag{4.70}$$

$$A_u^-|pqrs\rangle = -pq|p-1, q-1, r+1, s\rangle + s(h+s+p+q-1)|pq, r, s-1\rangle \tag{4.71}$$

Now we check, that the operators with two lower indices produce a linear combination of $|pqrs\rangle$-type vectors, where we exemplify the procedure with A_{ld}. We use the commutators $[A_{ld}, A_r^+] = -A_d^+$ and $[A_{ld}, A_u^+] = -A_l^+$ to get

$$\begin{aligned}A_{ld}|pqrs\rangle &= (A_l^+)^q (A_d^+)^s A_{ld} (A_r^+)^p (A_u^+)^r |h\rangle \\ &= (A_l^+)^q (A_d^+)^s (A_r^+ A_{ld} - A_d^+)(A_r^+)^{p-1} (A_u^+)^r |h\rangle \\ &= .. = (A_l^+)^q (A_d^+)^s ((A_r^+)^p A_{ld} - pA_d^+ (A_r^+)^{p-1})(A_u^+)^r |h\rangle \\ &= -p|p-1, q, r, s+1\rangle + (A_r^+)^p (A_l^+)^q (A_d^+)^s A_{ld} (A_u^+)^r |h\rangle \\ &= -p|p-1, q, r, s+1\rangle + (A_r^+)^p (A_l^+)^q (A_d^+)^s (A_u^+ A_{ld} - A_l^+)(A_u^+)^{r-1} |h\rangle \\ &= .. = -p|p-1, q, r, s+1\rangle - r|p, q+1, r-1, s\rangle\end{aligned} \tag{4.72}$$

This proves, that the $|pqrs\rangle$ indeed span the whole representation. □

Finally, we address the question of eigenvalue degeneracies. Suppose, that we are given a Casimir eigenvalue triple

$$(p-q, r-s, h+p+q+r+s) =: (k_1, k_2, h+2q+2s+k_1+k_2) =: (k_1, k_2, h+k_3), \tag{4.73}$$

where the first equality defines k_1 and k_2 and the second k_3. To determine its degeneracies, we count the numbers of different ways to to build the three given numbers k_1, k_2, k_3 out of p, q, r, s. For fixed k_1, k_2, the different combinations of q and s, that yield a given k_3, is

$$\#(k_1, k_2, h+k_3) = \frac{k_3 - k_2 - k_1}{2} + 1 = q + s + 1, \tag{4.74}$$

which corresponds to the numbers of combinations (q, s), whose sum $2q + 2s$ yields a fixed given number.

4.3.2 Casimir spectrum

The action of the operators $C_a^{(1)}$ and $C_b^{(1)}$ on a state $|pqrs\rangle$ yields after diagonalization the spectrum of the Casimir operator of the twodimensional conformal algebra within a representation of the fourdimensional one.
To find their eigenvectors, we first have to determine the result of their application on $|pqrs\rangle$.
The result of the "Cartan element part" of the Casimir $C_a^{(1)}$ is

$$h_1(h_1+1)|pqrs\rangle = (p-q)(p-q+1)|pqrs\rangle \tag{4.75}$$

We calculate now the application of the second part.

$$\begin{aligned} A_l A_r|pqrs\rangle = [&\underbrace{A_l^+ A_r^+ (A_r^+)^p (A_l^+)^q (A_u^+)^r (A_d^+)^s |h\rangle}_{(a)} + \underbrace{A_l^- A_r^+ (A_r^+)^p (A_l^+)^q (A_u^+)^r (A_d^+)^s |h\rangle}_{(b)} \\ + &\underbrace{A_l^+ A_r^- (A_r^+)^p (A_l^+)^q (A_u^+)^r (A_d^+)^s |h\rangle}_{(c)} + \underbrace{A_l^- A_r^- (A_r^+)^p (A_l^+)^q (A_u^+)^r (A_d^+)^s |h\rangle}_{(d)}] \end{aligned} \tag{4.76}$$

(a) equals $|p+1, q+1, rs\rangle$, for the others we can use the equations (4.68)-(4.71). Then the second term is

$$(b) = -rs|p+1, q+1, r-1, s-1\rangle + (p+1)(h+p+1+r+s-1)|pqrs\rangle, \tag{4.77}$$

the third

$$(c) = -rs|p+1, q+1, r-1, s-1\rangle + q(h+q+r+s-1)|pqrs\rangle \tag{4.78}$$

and the fourth

$$\begin{aligned} (d) =& A_l^- [-rs|p+1, q, r-1, s-1\rangle + q(h+q+r+s-1)|p, q-1, rs\rangle] \\ =& -rs\,[-(r-1)(s-1)|p+1, q+1, r-2, r-2\rangle + (p+1)(h+p+q+r+s-2)|pq, r-1, s-1\rangle] \\ &+ q(h+q+r+s-1)\,[-rs|pq, r-1, s-1\rangle + p(h+p+r+s-1)|p-1, q-1, rs\rangle]\,. \end{aligned} \tag{4.79}$$

Altogether this becomes

$$\begin{aligned} A_l A_r|pqrs\rangle =& |p+1, q+1, r, s\rangle + (q(h+q+r+s-1) + (p+1)(h+p+r+s))|pqrs\rangle \\ -2rs|p+1, q+1, r-1, s-1\rangle &- rs(q(h+q+r+s-1) + (p+1)(h+p+r+s-2))|p, q, r-1, s-1\rangle \\ +p(h+p-1+r+s)q(h+q+r+s-1)&|p-1, q-1, r, s\rangle + rs(r-1)(s-1)|p+1, q+1, r-2, s-2\rangle. \end{aligned} \tag{4.80}$$

The application of $A_r A_l$ is obtained by exchanging $p \leftrightarrow q$ and as a check, we calculate the application of the commutator:

$$\begin{aligned} [A_l A_r - A_r A_l]|pqrs\rangle =& [q(h+q+r+s-1) + (p+1)(h+p+r+s) - p(h+p+r+s-1) \\ &-(q+1)(h+q+r+s)]|pqrs\rangle - rs[q(h+q+r+s-1) + (p+1)(h+p+r+s-2) \\ &-p(h+p+r+s-1) - (q+1)(h+q+r+s-2)]|p, q, r-1, s-1\rangle \\ =& [-q + (h+p+r+s) + p - (h+q+r+s)]|pqrs\rangle - rs[(h+p+r+s-1) - (p+1) \\ &-(h+q+r+s-1) + (q+1)]|p, q, r-1, s-1\rangle = (2p-2q)|pqrs\rangle = 2h_1|pqrs\rangle \end{aligned} \tag{4.81}$$

The application of the full Casimir operator then yields

$$
\begin{aligned}
C_a^{(1)}|pqrs\rangle = & -|p+1,q+1,r,s\rangle + [(p-q)(p-q+1) - q(h+q+r+s-1) \\
& - (p+1)(h+p+r+s)]|pqrs\rangle + 2rs|p+1,q+1,r-1,s-1\rangle \\
& + rs[q(h+q+r+s-1) + (p+1)(h+p+r+s-2)]|p,q,r-1,s-1\rangle \\
& - p(h+p-1+r+s)q(h+q+r+s-1)|p-1,q-1,r,s\rangle \\
& - rs(r-1)(s-1)|p+1,q+1,r-2,s-2\rangle
\end{aligned}
\tag{4.82}
$$

4.3.3 Spectrum of representations on a subspace

Since the general diagonalization problem of finding linear combinations of states $|pqrs\rangle$, which are eigenvectors under the action (4.82) of the Casimir operators, is very complicated, we consider this problem on an invariant subspace of the whole space.
Let $|h\rangle$ again be a lowest weight vector of a scalar representation with scaling dimension h, then application of $A_l^+ A_r^+ = A_r^+ A_l^+$ leads to a state with a h_3 weight $h+2$ and the h_1 and h_2 weights are still zero. We set

$$
|n\rangle = (A_l^+ A_r^+)^n |h\rangle = (A_r^+ A_l^+)^n |h\rangle \tag{4.83}
$$

as the n-fold repetition of this operation. Then the h_3 weight of $|n\rangle$ is $h+2n$ for $n \in \mathbb{N}$.
We consider the expression (4.82) with $p = q = n$ and $r = s = 0$:

$$
\begin{aligned}
C_a^{(1)}|n\rangle &= -|n+1\rangle - [n(h+n-1) + (n+1)(h+n)]|n\rangle - n^2(h+n-1)^2|n-1\rangle \\
&= -|n+1\rangle - (2n^2 + 2nh + h)|n\rangle - n^2(h+n-1)^2|n-1\rangle
\end{aligned}
\tag{4.84}
$$

We rewrite this equation in terms of normalized vectors $\psi_n = \frac{1}{\||n\rangle\|}|n\rangle$. The norm of a vector $|n\rangle$ is

$$
\||n\rangle\| = \frac{n!(n+h-1)!}{(h-1)!} \tag{4.85}
$$

Dividing by $\||n\rangle\|$ we get the previous equation in terms of the normalized vectors

$$
C_a^{(1)}\psi_n = -(n+1)(n+h)\psi_{n+1} - (2n^2 + 2nh + h)\psi_n - n(n+h-1)\psi_{n-1}. \tag{4.86}
$$

Let us now suppose, that

$$
\psi = \sum_{n=0}^{\infty} f_n \psi_n \tag{4.87}
$$

is an eigenvector of $C_a^{(1)}$ with eigenvalue $-\lambda$. We map the eigenvector to a function

$$
f(x) = \sum_{n=0}^{\infty} f_n x^n \tag{4.88}
$$

with the same coefficients f_n, which we require to be square summable:

$$
(f_n)_{n\in\mathbb{N}} \in l^2 = \{(a_n)_{n\in\mathbb{N}} \mid \sum_n |a_n|^2 < \infty\}. \tag{4.89}
$$

The eigenvalue equation for ψ is then equivalent to the differential equation

$$\left[x(1+x)^2\partial_x^2 + \left(2x(1+x)+h(1+x)^2\right)\partial_x + h(1+x)\right] f(x) = \lambda f(x) \tag{4.90}$$

for $f(x)$, which can be seen by applying (-1) times the differential operator on a monomial x^n, which produces equation (4.86) with x^n replacing ψ_n.
For computational convenience we set

$$z = -x, \tag{4.91}$$

and consider the subspace of the solution space spanned by

$$f(z) = (1-z)^{-a} \cdot F(h-a, 1-a; h; z) \tag{4.92}$$

with $a = \frac{1}{2} + \frac{1}{2}\sqrt{1-4\lambda}$. We put in series representations of both factors. Then the Cauchy product formula yields

$$\begin{aligned} f(z) &= \left(\sum_{n=0}^{\infty} \frac{(1-a)_n \cdot (h-a)_n}{(h)_n \cdot n!} \cdot z^n\right) \cdot \left(\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k\right) \\ &= \sum_{n=0}^{\infty} z^n \cdot \sum_{k=0}^{n} \frac{(1-a)_k \cdot (h-a)_k \cdot (a)_{n-k}}{(h)_k \cdot k! \cdot (n-k)!} \\ &= \sum_{n=0}^{\infty} z^n \cdot \frac{(a)_n \cdot}{n!} F(-n, 1-a, h-a; h, 1-a-n; 1), \end{aligned} \tag{4.93}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ is the Pochhammer symbol. $F(a,b,c;d,e;x)$ is a generalized hypergeometric function of type $(3,2)$ and for n integer it fulfils the identity

$$F(-n,b,c;d,e;1) = \frac{(d-b)_n}{(d)_n} F(-n,b,e-c;e,b-d-n+1;1), \tag{4.94}$$

where in our case $b = 1-a$, $c = a$, $d = 1$ and $e = h$. With this relation the solution, again as a function of x, takes the form

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot F(-n, 1-a, a; 1, h; 1) \cdot x^n =: \sum_{n=0}^{\infty} f_n \cdot x^n \tag{4.95}$$

We note, that for the second solution of the differential equation, at this point we would have

$$g(x) = x^{1-h} \sum_{n=0}^{\infty} (-1)^n \cdot F(-n, 1-a, a; 1, 2-h; 1) \cdot x^n. \tag{4.96}$$

Since there seem to be no true eigenvectors $f(x)$ with square summable coefficients, we try to determine the spectrum of the Casimir operator $C_a^{(1)}$ using the concept of generalized eigenvalues [12]. We consider the partial sum

$$f_N = \sum_{n=0}^{N} f_n \psi_n \tag{4.97}$$

with the f_n we just found. The norm of f_N is defined as

$$\|f_N\| = \sum_{n=0}^{N} |f_n|^2. \tag{4.98}$$

Then λ is said to be a generalized eigenvalue of $C_a^{(1)}$, if the quantity

$$\frac{\left\|(C_a^{(1)} - \lambda) f_N\right\|}{\|f_N\|} \tag{4.99}$$

goes to zero for $N \to \infty$.
We use the action (4.86) of the Casimir operator on the normalized vectors ψ_n to get

$$\begin{aligned}\frac{\|(C_a^{(1)} - \lambda) f_N\|}{\|f_N\|} =& \frac{\|(C_a^{(1)} - \lambda)\sum_{n=0}^{N} f_n \psi_n\|}{\|\sum_{n=0}^{N} f_n \psi_n\|} \\ =& \frac{1}{\|\sum_{n=0}^{N} f_n \psi_n\|} \cdot \|\sum_{n=0}^{N} f_n[-(n+1)(n+h)\psi_{n+1} - (2n^2 + 2nh + h + \lambda)\psi_n \\ & - n(n+h-1)\psi_{n-1}]\|\end{aligned} \tag{4.100}$$

We shift now the index of summation, so that every sum includes ψ_n and we can use the formula for the norm. The zeroth and the Nth term are treated separately and there appears a $(N+1)$th term, so we have to study the convergence behaviour for $N \to \infty$ of the expression

$$\begin{aligned}\frac{\|(C_a^{(1)} - \lambda) f_N\|}{\|f_N\|} =& \left[\sum_{n=0}^{N} |f_n|^2\right]^{-1} \{|(\lambda + h) + \frac{(\lambda - h}{h} \cdot 2(h+1)|^2 \\ & + [\sum_{n=1}^{N-1} |f_{n-1} \cdot n(n+h-1) + f_n \cdot (\lambda + 2n^2 + 2nh + h) \\ & + f_{n+1} \cdot (n+1)(n+h)|^2] + |f_{N-1} \cdot N(N+h-1) \\ & + f_N \cdot (\lambda + 2N^2 + 2Nh + h)|^2 + |f_{N+1}(N+2)(N+h+1)|^2\}\end{aligned} \tag{4.101}$$

From the introductory considerations about positive energy representations of $so(1,2)$, we expect the eigenvalues λ_i[2] be related to lowest weights k_i of representations via

$$\lambda_i = k_i(1 - k_i) \tag{4.102}$$

with $k_i > 0$. It is also conceivable, that there appear "negative energy representations" with highest weight $k_i < 0$ and weights of the form $k_i - n$ for $n \in \mathbb{N}$. We see from (4.102), that for those cases $\lambda \leq \frac{1}{4}$.
The trivial representation given by $k_0 = 0$ never appears, since this would require $\lambda = 0$ to be a generalized eigenvalue, which would imply $f_n = (-1)^n$, for which (4.99) does not go to zero.
A numerical treatment shows, that for $\lambda \leq \frac{1}{4}$ the expression does not converge either, hence there are no positive (or negative) energy representations within our subspace of the scalar

[2]With i out of some appropiate index set $\mathcal{I}$, e.g. $\mathcal{I} \subseteq \mathbb{N}$ (discrete spectrum) or $\mathcal{I} \subseteq \mathbb{R}$ (continuous spectrum).

representation of the fourdimensional algebra.
For $\lambda > \frac{1}{4}$ there are indications, that there might be continous spectrum of generalized eigenvalues, which however would make k_i in (4.102) complex. These values correspond to representations of $so(1,2)$ with a spectrum, which is neither bounded from below nor from above. They are of no interest in Wightman QFT, where energy boundedness in one direction is a fundamental requirement.

5 Six-point restricted partial wave expansion

5.1 Strategy and conventions

Since partial wave expansions were such a useful tool to study implications of Wightman positivity for four-point functions of GCI scalar fields in $D = 4$ dimensional Minkowski space, it is desirable to be able to perform them also for higher n-point functions.
Unfortunately the procedure quickly gets very involved. We could write the four-point function as a sum of contributions of symmetric tensor field, because we only needed one OPE, which only involves fields of this type for scalar fields. Therefore it sufficed to insert just one of the three Casimir operators of the fourdimensional conformal group. If we need to do more OPEs, this nice property does not hold anymore and we have to insert all three Casimirs. Considering their concrete form (2.75)-(2.77) makes it not difficult to see, that this procedure will be very complicated. For the six-point function the non-trivial part (i.e. the one not just fixing the quantum numbers of the corresponding projector to certain values) consists of inserting $C_2^{(4)}$, $C_3^{(4)}$ and $C_4^{(4)}$ into three spots, which would yield system of nine partial differential equations in nine variables (the cross ratios).

We will take a simpler step in this section. We take the general form of a conformal six-point function, restrict it to the two dimensional surface $x^2 = x^3 = 0$ and then use the fact, that there the cross ratios factorize and the twodimensional Casimir operator $C^{(2)}$ is a sum of two onedimensional chiral Casimir operators $C_\pm^{(1)}$. Then we can treat the problem as two separate, identical ones. After we have obtained the partial waves, we want to study the implications, that Wightman positivity has for a correlator violating the single-pole property (see section 2.4.3).
As mentioned in the introduction, every n-point function in a conformally invariant QFT can be written in the form

$$\langle \Omega, \phi_1(x_1)..\phi_n(x_n)\Omega\rangle = \prod_{i,j} \rho_{ij}^{\mu_{ij}} f(s_1, .., s_{m_n}) \tag{5.1}$$

where $\rho_{ij} = x_{ij}^2 = (x_i - x_j)^2$ and f is an arbitrary function of the cross ratios $s_1, .., s_{m_n}$.
The following arguments shows, that the number of independent cross ratios, that can be formed out n points, in two dimensions is reduced to $n - 3$. First we note, that the Lorentz squares factorize into

$$\begin{aligned}\rho_{ij} &= (x_i - x_j)^2 = (x_i^0 - x_j^0)^2 - (x_i^1 - x_j^1)^2 \\ &= ((x_i^0 + x_i^1) - (x_j^0 + x_j^1)) \cdot ((x_i^0 - x_i^1) - (x_j^0 - x_j^1)) =: (x_i^+ - x_j^+) \cdot (x_i^- - x_j^-)\end{aligned} \tag{5.2}$$

and that this immediately implies that so do the cross ratios

$$s = \frac{(x_i - x_j)^2 \cdot (x_k - x_l)^2}{(x_i - x_k)^2 \cdot (x_j - x_l)^2} = \frac{x^+_{ij} x^+_{kl}}{x^+_{ik} x^+_{jl}} \cdot \frac{x^-_{ij} x^-_{kl}}{x^-_{ik} x^-_{jl}} =: s^+ \cdot s^-. \tag{5.3}$$

with $x^\pm_{ij} = x^\pm_i - x^\pm_j$. In higher dimensions one has two independent cross-ratios involving the same four points x_i, x_j, x_k, x_l, which for example can be taken as

$$s_{ijkl} = \frac{\rho_{ij}\rho_{kl}}{\rho_{ik}\rho_{jl}} \text{ and } t_{ijkl} = \frac{\rho_{il}\rho_{jk}}{\rho_{ik}\rho_{jl}} \tag{5.4}$$

We saw, that in two dimensions any cross ratio is a product of onedimensional ones. In one dimensions in turn we can see, that there is just one independent chiral cross-ratio for four given points, because one of the two can be expressed through the other:

$$t^\pm_{ijkl} = \frac{x^\pm_{il} x^\pm_{jk}}{x^\pm_{ik} x^\pm_{jl}} = \frac{(x^\pm_{ik} x^\pm_{jl} - x^\pm_{ij} x^\pm_{kl})}{x^\pm_{ik} x^\pm_{jl}} = 1 - s^\pm_{ijkl} \tag{5.5}$$

To see, that the total number of independent chiral cross-ratios for an n-point function is $n-3$, we consider now the cross ratios (leaving out the $\pm$)

$$s_n = \frac{x_{12} x_{3n}}{x_{13} x_{2n}}. \tag{5.6}$$

They are obviously independent of each other, because they contain different points. The three points x_1, x_2, x_3 take a somewhat distinguished role among the points, which however poses no loss of generality. We have to show now, that any cross-ratio with only two, one and zero of these points can be reduced to a function of certain s_n. By the considerations above, we are done, if we have expressed them through any cross ratios involving x_1, x_2 and x_3, i.e. they need not have the precise form (5.6). We consider

$$\begin{aligned} t_{pq} = \frac{x_{12} x_{pq}}{x_{1p} x_{2q}} &= 1 - \frac{x_{1q} x_{2p}}{x_{1p} x_{2q}} \\ &= 1 - \frac{x_{1q} x_{23}}{x_{13} x_{2q}} \cdot \frac{x_{13} x_{2p}}{x_{1p} x_{23}}, \end{aligned} \tag{5.7}$$

next

$$t_{pqr} = \frac{x_{1p} x_{qr}}{x_{1q} x_{pr}} = \frac{1 - \frac{x_{1q} x_{2r}}{x_{1r} x_{2q}}}{\frac{x_{1q} x_{2p}}{x_{1p} x_{2q}} - \frac{x_{1q} x_{2r}}{x_{1r} x_{2q}}} \tag{5.8}$$

and finally

$$t_{pqrs} = \frac{x_{pq} x_{rs}}{x_{pr} x_{qs}} = \frac{x_{1r} x_{pq}}{x_{1q} x_{pr}} \cdot \frac{x_{1q} x_{rs}}{x_{1r} x_{qs}}. \tag{5.9}$$

This proofs the proposition.

For the six point function the number of independent cross ratios after the restriction is reduced three and we choose

$$s = \frac{x_{14} x_{56}}{x_{15} x_{46}}\ ,\ t = \frac{x_{24} x_{56}}{x_{25} x_{46}}\ ,\ u = \frac{x_{34} x_{56}}{x_{35} x_{46}}. \tag{5.10}$$

For the prefactor of distance squares, the simplest and most symmetric choice in the six-point case seems to be the product of two three-point-function type (cf. (2.90)) structures:

$$\begin{aligned}&\langle\Omega,\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\phi_5(x_5)\phi_6(x_6)\Omega\rangle\\ =&\frac{1}{\rho_{12}^{l_1+l_2-l_3}\rho_{13}^{l_1+l_3-l_2}\rho_{23}^{l_2+l_3-l_1}\rho_{45}^{l_4+l_5-l_6}\rho_{46}^{l_4+l_6-l_5}\rho_{56}^{l_5+l_6-l_4}}f(s,t,u).\end{aligned} \tag{5.11}$$

The l_i are the scaling dimensions of the fields ϕ_i ($i=1,..,6$). The prefactor obviously fulfils the homogeneity requirement.
We perform now the partial wave expansion by inserting projectors $\Pi_{n,m}$ on representations with lowest weight (n,m) of the twodimensional conformal group $SO(2,2)$ between all fields.
If we insert the quadratic Casimir operator $C_2^{(2)}$ in front of a projector and commute it to the vacuum Ω, we obtain a differential operator $\mathcal{D}_2^{(2)}$, that is the sum of two identical differential operators, that act on the light-cone coordinates $x_i^\pm$ and involve the chiral scaling dimensions $l_i^\pm$:

$$\mathcal{D}_2^{(2)}=\mathcal{D}_+^{(1)}+\mathcal{D}_-^{(1)}, \tag{5.12}$$

because

$$C_2^{(2)}=C_+^{(1)}+C_-^{(1)}. \tag{5.13}$$

Therefore we can make a product ansatz

$$f(s,t,u)=f^+(s^+,t^+,u^+)f^-(s^-,t^-,u^-). \tag{5.14}$$

Then the problem splits into two simpler (and identical) onedimensional problems, so it suffices to do one of them and take the product of the partial solutions in the end. To relax notation, we leave out the $+/-$ in the following. We have

$$\begin{aligned}&\langle\Omega,\phi_1(x_1)\Pi_a\phi_2(x_2)\Pi_b\phi_3(x_3)\Pi_c\phi_4(x_4)\Pi_d\phi_5(x_5)\Pi_e\phi_6(x_6)\Omega\rangle\\ =&\frac{1}{\rho_{12}^{l_1+l_2-l_3}\rho_{13}^{l_1+l_3-l_2}\rho_{23}^{l_2+l_3-l_1}\rho_{45}^{l_4+l_5-l_6}\rho_{46}^{l_4+l_6-l_5}\rho_{56}^{l_5+l_6-l_4}}f_{abcde}(s,t,u)=:A(1,..,6)f_{abcde}(s,t,u)\end{aligned} \tag{5.15}$$

and now we insert into every spot a Casimir operator $C^{(1)}$. On the one hand, we apply it on the projector Π_α to yield the Casimir value $\alpha(\alpha-1)$ of the corresponding representation, on the other hand we commute the Casimir past the fields, until it hits the vacuum, using the following list of commutators. It takes into account, how the generators have to be commuted past the non-conformally covariant fields, which appear in the process.

$$[P,\varphi(x)]=-i\partial_x\varphi \tag{5.16}$$

$$[K,\varphi(x)]=-i(x^2\partial_x+2lx)\varphi \tag{5.17}$$

$$[D,\varphi(x)]=-i(l+x\partial_x)\varphi \tag{5.18}$$

$$[P,x^2\partial_x\varphi(x)]=-ix^2\partial_x^2\varphi \tag{5.19}$$

$$[P,x\varphi(x)]=-ix\partial_x\varphi \tag{5.20}$$

$$[D,x\partial_x\varphi(x)]=-i(x(l+1)\partial_x+x^2\partial_x^2)\varphi \tag{5.21}$$

5.2 Calculation

Commuting past one scalar field $\phi(x)$ with scaling dimension l applied on the vacuum yields not a differential operator, but just the value $l(l-1)$, so the differential equation for the first and the last insertion are rather trivial, namely

$$a(a-1) = l_1(l_1-1) \text{ and } e(e-1) = l_6(l_6-1). \tag{5.22}$$

We drop these indices in the remainder ($f_{bcd} \equiv f_{abcde}$). For more fields, we have the following

Theorem 5.1: The differential operator, that one obtains by commuting the Casimir operator $C^{(1)}$ past a string of n fields $\phi_1(x_1)..\phi_n(x_n)$ with scaling dimensions l_i, that is applied on the vacuum Ω, is

$$\begin{aligned}\mathcal{D}_n =& \sum_{i=1}^{n} l_i(l_i-1) + \sum_{i<j}\left[2l_il_j - (x_i-x_j)^2\partial_i\partial_j\right] \\ &+ 2\left(\sum_{i=1}^{n} l_i\right)\left(\sum_{j=1}^{n} x_j\partial_j\right) - 2\left(\sum_{i=1}^{n} l_ix_i\right)\left(\sum_{j=1}^{n}\partial_j\right)\end{aligned} \tag{5.23}$$

Proof: (by induction) For $n=1$ the expression is $l_1(l_1-1)$, so we consider

$$\begin{aligned} C\phi_1..\phi_{n+1}\Omega =& \phi_1 C\phi_2..\phi_{n+1}\Omega - \partial_1\sum_{i=2}^{n+1}(x_i^2\partial_i + 2l_ix_i) - (x_1^2\partial_1 + 2l_1x_1)\sum_{i=2}^{n+1}\partial_i \\ &-(x_1^2\partial_1^2 + 2l_1x_1\partial_1) - (l_1 + x_1\partial_1) + 2(l_1+x_1\partial_1)\sum_{i=2}^{n+1}(l_i + x_i\partial_i) + l_1(l_1+x_1\partial_1) \\ &+(l_1+1)x_1\partial_1 + x_1^2\partial_1^2\end{aligned}$$

We insert the hypothesis of induction and group the terms together to form the corresponding $n+1$ form of (5.23)

$$\begin{aligned} ... =& \underbrace{\sum_{i=2}^{n+1} l_i(l_i-1)}_{(I)} + \sum_{\substack{2\le i,j\le n+1\\ i<j}}\left[2l_il_j - (x_i-x_j)^2\partial_i\partial_j\right] + 2\left(\sum_{i=2}^{n+1} l_i\right)\left(\sum_{j=2}^{n+1} x_j\partial_j\right) \\ &- 2\left(\sum_{i=2}^{n+1} l_ix_i\right)\left(\sum_{j=2}^{n+1}\partial_j\right) - \partial_1\sum_{i=2}^{n+1}\left(x_i^2 + 2l_ix_i\right) - \left(x_1^2\partial_1 + 2l_1x_1\right)\sum_{j=2}^{n+1}\partial_i \\ &- \left(x_1^2\partial_1^2 + 2l_1x_1\partial_1\right) - (l_1+x_1\partial_1) + 2\left(l_1+x_1\partial_1\right)\sum_{i=2}^{n+1}\left(l_i + x_i\partial_i\right) \\ &+ l_1(l_1+x_1\partial_1) + (l_1+1)x_1\partial_1 + x_1^2\partial_1^2 \\ =& \sum_{i=1}^{n+1} l_i(l_i-1) + \sum_{\substack{1\le i,j\le n+1\\ i<j}}\left[2l_il_j - (x_i-x_j)^2\partial_i\partial_j\right] \\ &+ 2\left(\sum_{i=1}^{n+1} l_i\right)\left(\sum_{j=1}^{n+1} x_j\partial_j\right) - 2\left(\sum_{i=1}^{n+1} l_ix_i\right)\left(\sum_{j=1}^{n+1}\partial_j\right). \end{aligned}$$

□

One can actually derive an expression for the application of the quadratic Casimir operator $C_2^{(D)}$ on n conformal fields in D dimensions (cf. [31]).
We only need the operators, which appear after commuting the Casimir past two and three fields respectively. We choose to commute to the beginning for

$$\langle\Omega,\phi_1\phi_2\cdot C^{(1)}\cdot\Pi_b\phi_3\Pi_c\phi_4\Pi_d\phi_5\phi_6\Omega\rangle = b(b-1)A(1,..,6)f_{bcd}(s,t,u). \tag{5.24}$$

and to the end for

$$\langle\Omega,\phi_1\phi_2\Pi_b\phi_3\cdot C^{(1)}\cdot\Pi_c\phi_4\Pi_d\phi_5\phi_6\Omega\rangle = c(c-1)A(1,..,6)f_{bcd}(s,t,u) \tag{5.25}$$

and

$$\langle\Omega,\phi_1\phi_2\Pi_b\phi_3\Pi_c\phi_4\cdot C^{(1)}\cdot\Pi_d\phi_5\phi_6\Omega\rangle = d(d-1)A(1,..,6)f_{bcd}(s,t,u). \tag{5.26}$$

We apply the corresponding differential operators on the right hand side of equation (5.15). This yields three new different equations of the type

$$\tilde{\mathcal{D}}_\alpha f_{bcd} = \alpha(\alpha-1)A(1,..,6)f_{bcd}(s,t,u) \tag{5.27}$$

We multiply by the homogeneity prefactor from the right side and remain with a differential equation for f_{bcd}, which has to be put together to be purely in terms of the cross ratios s, t and u. Some details on this lengthy procedure are given in appendix .4.
The result is the following system of partial differential equations. For insertion between the second and the third field we get

$$\begin{aligned} l_3(l_3-1)f_{bcd} + (s-t)^2\partial_s\partial_t f_{bcd} - (l_1+l_3-l_2)\frac{(s-t)(t-u)}{s-u}\partial_t f_{bcd} \\ -(l_1-l_2-l_3)\frac{(s-t)(u-s)}{u-t}\partial_s f_{bcd} = b(b-1)f_{bcd}, \end{aligned} \tag{5.28}$$

between the third and the fourth field

$$\begin{aligned} -(l_4+l_5{+}_6)(l_4+l_5+l_6-1)f_{bcd} + (s-t)^2\partial_s\partial_t f_{bcd} + (t-u)^2\partial_t\partial_u f_{bcd} \\ +(s-u)^2\partial_s\partial_u f_{bcd} = c(c-1)f_{bcd}, \end{aligned} \tag{5.29}$$

and between the fourth and fifth field

$$\begin{aligned} l_4(l_4-1)f_{bcd} + (1-s)s^2\partial_s^2 f_{bcd} + (1-t)t^2\partial_t^2 f_{bcd} + (1-u)u^2\partial_u^2 f_{bcd} + (2-s-t)st\partial_s\partial_t f_{bcd} \\ +(2-t-u)tu\partial_t\partial_u f_{bcd} + (2-s-u)su\partial_s\partial_u f_{bcd} + (2l_4-(l_4-l_5+l_6+1)s)s\partial_s f_{bcd} \\ +(2l_4-(l_4-l_5+l_6+1)t)t\partial_t f_{bcd} + (2l_4-(l_4-l_5+l_6+1)u)u\partial_u f_{bcd} = d(d-1)f_{bcd}. \end{aligned} \tag{5.30}$$

Unfortunately even under these extremely simplifying assumptions, we still get a very complicated system of equations, for which it seems very difficult to find a solution.

6 Summary and Outlook

We have investigated several relations between twodimensional and fourdimensional conformal field theory. We considered the two different ways, in which the twodimensional conformal algebra can be embedded into the fourdimensional one. One of our main results is, that the diagonal embedding has not proven to be of much interest. It does not lead to a chiral action on a twodimensional submanifold like the block embedding. Furthermore, representations of the fourdimensional conformal algebra do not decompose into positive energy representations of the diagonally embedded twodimensional one.
These results were important for our next step: when we interpreted terms in the decomposition of a 4D partial wave as 2D partial waves and performed the restricted partial wave expansion, we only needed to do one restriction, namely to the submanifold corresponding to the block embedding.
Our results can be seen as some first steps towards an understanding of the relation between partial wave expansions in 2D and 4D. We conjectured the reason for the discrepancies between the analytic and the character decomposition and indicated, how our procedure has to be expanded to resolve them.
In investigating the relation between analytic and group theoretic decompositions, more sophisticated application of the connections between Lie theory and the theory of special functions might be of help.
Our attempt to simplify the six-point partial wave expansion by restriction to two dimensions has unfortunately not led to an easily solvable system of differential equations. If their solutions (the partial waves) are found, the six-point function will have to be expanded in terms of them, which will presumably again be a tremendous task. This seems to show, that this "brute force" approach is not very promising, also having in mind, that the presented six-point function is just an example and that the final goal should be to facilitate a systematic study of all n-point functions. Therefore, other methods need to be found to study the double-pole property and its compatibility with positivity in particular and GCI QFT in general.
It might be hoped, that certain indications for infinitedimensional (Lie and associative) algebra structures lead the way.

7 Appendices

.1 Representation theory of Lie algebras

Here we give a short account of the representation theory of finitedimensional Lie algebras to the extent, in which it is used in the text. Proofs will generally be omitted, we refer for them to [10], [13], [14] and [25]. We assume familiarity with linear algebra and basic manifold theory.
We list some of the notations used in this appendix.

$\mathfrak{g}$	Lie algebra
V	vector space
$GL(V)$	general linear group of V
$gl(V)$	the Lie algebra of $GL(V)$
(ρ, V)	representation ρ of a Lie algebra on V
$\mathfrak{h}$	Cartan subalgebra of $\mathfrak{g}$
$\mathfrak{g}^*/\mathfrak{h}^*$	dual space of $\mathfrak{g}/\mathfrak{h}$
$\mathfrak{i}$	ideal of $\mathfrak{g}$
$ad_h(.) = [h, .]$	adjoint map of h
f_{ijk}	structure constants of $\mathfrak{g}$
μ	weight
$x, y, x_i, ..$	elements of $\mathfrak{g}$
$\alpha, \beta, ..$	roots
$h, h_i, ..$	elements of $\mathfrak{h}$
h^α	element of $\mathfrak{h}$ associated with a root α
G	Lie group
$\tilde{G}$	universal covering group of G
$C^{(n)}$	n-th order Casimir operator
$U(\mathfrak{g})$	universal enveloping algebra of $\mathfrak{g}$
χ_ρ	character of a representation ρ
$sl(r)$	special linear algebra of rank r
$so(r)$	special orthogonal algebra of rank r
$sp(r)$	sympletic algebra of rank r
$su(r)$	special unitary algebra of rank r
θ	Cartan involution on $\mathfrak{g}$
Θ	Cartan involution on G
$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$	Cartan decomposition of $\mathfrak{g}$
$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^\pm$	Iwasawa decomposition of $\mathfrak{g}$
$G = KAN^\pm$	Iwasawa decomposition of G

.1.1 Lie algebras and representations

Lie algebras are ubiquitous in physics. Generators of symmetry transformations in most cases fulfil Lie algebra type commutation relations. In high energy physics different species of particles can be put into multiplets of certain algebras. Furthermore they are used in the calculation of atomic, molecular and nuclear spectra and in gauge theories. Many other algebraic structures in modern mathematics and physics possess close connections to Lie algebras.
We first define, with what we are dealing:

> **Definition:** A *Lie algebra* $\mathfrak{g}$ is a vector space endowed with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (also called the commutator or Lie bracket), that fulfils
>
> - Antisymmetry: $[x, y] = -[y, x]$
> - Jacobi-identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
>
> for all $x, y, z \in \mathfrak{g}$.

Familiar examples in physics are the Poisson bracket in classical Hamiltonian mechanics or the commutator in quantum mechanics.
The dimension n of a Lie algebra is equal to its dimension as a vector space and we assume $n < \infty$. We mostly consider complex $\mathfrak{g}$, since the theory is best developed for this case. Some results are not available (or at least need more effort) for real Lie algebras, which makes their study considerably harder.
We note, that one can complexify a real Lie algebra $\mathfrak{g}$ by taking complex linear combinations of their basis elements, which can be written as $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$.
The commutator of two elements of $\mathfrak{g}$ must be a linear combination of other elements of $\mathfrak{g}$, which can be expressed as

$$[x_i, x_j] = \sum_{k=1}^{n} f_{ijk} x_k. \qquad (.1)$$

The f_{ijk} are called the *structure-constants* of $\mathfrak{g}$ and antisymmetry and the Jacobi-identity impose certain restrictions on them.
For two subspaces $A, B \subseteq \mathfrak{g}$ let $[A, B]$ denote the set $\{[a, b] \mid a \in A, b \in B\}$. A *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace, such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, an *ideal* $\mathfrak{i}$ of $\mathfrak{g}$ a subspace, such that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$.
One distinguishes between different types of Lie algebras:

- *Abelian*: $[x, y] = 0$ for all $x, y \in \mathfrak{g}$
- *Nilpotent*: the derived series $\mathfrak{g}^{\{i\}} = [\mathfrak{g}^{\{i-1\}}, \mathfrak{g}^{\{i-1\}}]$ ($\mathfrak{g}^{\{0\}} = \mathfrak{g}$) of $\mathfrak{g}$ ends up with $\{0\}$ for some $i \in \mathbb{N}$.
- *Solvable*: the lower central series $\mathfrak{g}_{\{i\}} = [\mathfrak{g}, \mathfrak{g}_{\{i-1\}}]$ ($g_{\{0\}} = \mathfrak{g}$) of $\mathfrak{g}$ ends up with $\{0\}$ for some $i \in \mathbb{N}$.
- *Simple*: $\mathfrak{g}$ has no proper ideals and is not abelian.
- *Semi-simple*: $\mathfrak{g}$ is a direct sum of simple Lie algebras.

The solvable and nilpotent cases are not of too much interest from a physical point of view, but we will need them in the section on Lie algebras of non-compact type.

For simple Lie algebras there is a complete classification available. There are four infinite series $A_r \simeq sl(r+1)$ $(r \geq 1)$, $B_r \simeq so(2r+1)$ $(r \geq 3)$, $C_r \simeq sp(r)$ $(r \geq 2)$ and $D_r \simeq so(2r)$ $(r \geq 4)$ as well as five isolated cases, which were named E_6, E_7, E_8, G_2 and F_4. The index indicates the rank of the Lie algebra (see below).
From now on we will only consider semisimple $\mathfrak{g}$ (which of course includes the simple case). An important example for a simple Lie algebra is $A_1 \simeq sl(2)$, which is spanned by three generators $\{h, x, y\}$ with commutation relations

$$[x, y] = h \ , \ [h, x] = 2x \ , \ [h, y] = -2y. \tag{.2}$$

We recognize the angular-momentum algebra $su(2)$ in the J_3, J^+, J^- form (up to renormalization of the generators).
A system in quantum mechanics is described by a maximal set of commuting (and therefore simultaneously diagonalizable) observables. With this in mind we introduce the following notion:

Definition: A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace, such that

- $[h_1, h_2] = 0$ for all $h_1, h_2 \in \mathfrak{h}$.
- If for some $x \in \mathfrak{g}$, $[h, x] = 0$ for all $h \in \mathfrak{h}$, then $x \in \mathfrak{h}$.
- For all $h \in \mathfrak{h}$, $ad_h(.) \equiv [h, .]$ is diagonalizable (i.e. there's a basis $\{x_i\}$ of $\mathfrak{g}$, such that $ad_h(x_i) \sim x_i$ for all i).

The first property states, that $\mathfrak{h}$ is a commuting subspace, the second that it is maximal and the third, that it consists entirely of so-called semisimple elements. There can be several choices of $\mathfrak{h}$ for a given $\mathfrak{g}$, but they are all related by automorphisms of $\mathfrak{g}$, so taking any convenient choice does not lead to an arbitrariness in the description of the Lie algebra. This also shows, that the dimension r of a Cartan subalgebra $\mathfrak{h}$ is a property of $\mathfrak{g}$ itself, which is called the *rank* of $\mathfrak{g}$.
The fact, that all ad_h are simultaneously diagonalizable, motivates the introduction of an own name for a simultaneous eigenvector:

Definition: A *root* of $\mathfrak{g}$ is a nonzero map $\alpha : \mathfrak{h} \to \mathbb{C}$ (in other words $\alpha \in \mathfrak{h}^*$), such that there is an $0 \neq x \in \mathfrak{g}$ fulfilling

$$ad_h(x) \equiv [h, x] = \alpha(h)x \tag{.3}$$

for all $h \in \mathfrak{h}$.

Such an x is called a *root vector* and all x fulfilling (.3) for a fixed root α form the *root space* $\mathfrak{g}_\alpha$.
The Lie algebra $\mathfrak{g}$ is spanned by such x and so it decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} g_\alpha, \tag{.4}$$

which is called the root space decomposition. The root spaces have the following properties

- For all $\alpha, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
- If $\alpha \in \mathfrak{h}^*$, also $-\alpha \in \mathfrak{h}^*$ and these are the only multiples of α, which are roots.

- All $\mathfrak{g}_\alpha$ are onedimensional.
- For all α, there are $X \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}$, that form a $sl(2)$ subalgebra.
- There is a basis $\mathcal{B} = \{h^i\} \cup \{e^\alpha\}$ of $\mathfrak{g}$, where $\{h^i\}$ is a basis of $\mathfrak{h}$ and the e^α satisfy

$$[h^i, e^\alpha] = \alpha(h^i)e^\alpha =: \alpha^i e^\alpha. \tag{.5}$$

The vector $(\alpha^i)_{i=1,..,r}$ in the last statement is also called a root, because for the given basis the α^i can be interpreted as the components of a vector in $\mathfrak{h}^*$. The e^α are called ladder operators. We will in the following need an inner product on the space of roots. Every inner product induces an isomorphism between a vector space and its dual, so here between $\mathfrak{h}^*$ and $\mathfrak{h}^{**} = \mathfrak{h}$.

> **Definition:** The *Cartan-Killing form* is the bilinear and symmetric map $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by
>
> $$\kappa(x, y) = tr(ad_x \circ ad_y)^1. \tag{.6}$$
>
> It fulfils the invariance property $\kappa([x, y], z) = \kappa(x, [y, z])$.

The Cartan-Killing form can be used to define semisimplicity of a Lie algebra $\mathfrak{g}$, which is equivalent to κ being non-degenerate on $\mathfrak{g}$ (i.e. if $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$).
One can show, that κ is a proper inner product on $\mathfrak{h}$ as well. Therefore we associate in the spirit of the Riesz representation theorem with any root $\alpha \in \mathfrak{h}^*$ up to a normalization constant c_α an element $h^\alpha \in \mathfrak{h}$, such that $\alpha(h) = c_\alpha \kappa(h^\alpha, h)$ for all $h \in \mathfrak{h}$. The inner product of two roots α and β is then defined as

$$(\alpha, \beta) := c_\alpha c_\beta \kappa(h^\alpha, h^\beta) \tag{.7}$$

and therefore by bilinearity on the space spanned by the roots.
The inner product on the roots provides a geometrical picture of them. One divides the root space into half-spaces $V^\pm$ by a hyperplane, which does not contain any of the roots. Then one says quite arbitrarily, that α is a *positive root*, if it lies in V^+ and a *negative root*, if it lies in V^-. We also write $\alpha > 0$ and $\alpha < 0$ respectively.
A convenient basis for the set of positive roots are given by the following notion.

> **Definition:** A *simple root* is a positive root $\alpha^{(i)}$, that cannot be obtained as a linear combination of other positive roots with positive coefficients.

There are r simple roots and any other positive root can be written as a linear combination of simple roots with integral coefficients.
Also with every simple root one can associate an element in $h^{\alpha^{(i)}} \in \mathfrak{h}$, such that $e^{\alpha^{(i)}}$, $e^{-\alpha^{(i)}}$ and $h^{\alpha^{(i)}}$ fulfil the $sl(2)$ commutation relations, where $e^{\alpha^{(i)}} \in \mathfrak{g}_{\alpha^{(i)}}$ and $e^{-\alpha^{(i)}} \in \mathfrak{g}_{-\alpha^{(i)}}$ are the elements in the fourth property of the root spaces. With a proper choice of the normalization constant c_α, namely $\frac{1}{2}(\alpha, \alpha)$, $h^{\alpha^{(i)}}$ coincides with the original Cartan generator h^i. All this will become important, when tracing the general finitedimensional representations of semisimple Lie algebras back to the $sl(2)$ case.

[1]The trace is well-defined since we assumed finite $dim(\mathfrak{g})$.

We now define, what a representation is:

> **Definition:** A *representation* ρ of a Lie algebra $\mathfrak{g}$ on a vector space V is a homomorphism
> $$\rho : \mathfrak{g} \to gl(V), \tag{.8}$$
> such that $\rho([x,y]) = [\rho(x), \rho(y)]$. We will also denote it by (ρ, V).

Representations establish the connection between the abstract algebras and concrete physical systems, which can often be described as vector spaces, e.g. the $\mathbb{R}^n$ or a Hilbert space $\mathcal{H}$. A representation is called *faithful*, if ρ is an injective map. We will in this section only be interested in the case, that V is finitedimensional.
For a representation ρ on V, a subspace W of V is called *invariant*, if $\rho(x)w \in W$ for all $x \in \mathfrak{g}$ and $w \in W$. A representation is called *irreducible*, if it has no non-trivial invariant subspaces. One usually only classifies the irreducible representations, since e.g. any finitedimensional representation of a semisimple Lie algebra is isomorphic to a direct sum of such representations.
We have

> **Schur's Lemma:** If (ρ, V) is an irreducible representation and an element $\rho(x_0) \in gl(V)$ commutes with all other elements $\rho(x)$, then it must be a multiple of the identity operator and therefore has constant value within the representation.

For every vector space V, $\rho(x) = 1 \in gl(V)$ for all $x \in \mathfrak{g}$ defines a representation of $\mathfrak{g}$, which is called the trivial representation.
We considered the map
$$ad \ : \ \mathfrak{g} \to gl(\mathfrak{g}) \tag{.9}$$
defined by
$$ad_x(y) = [x,y], \tag{.10}$$
before. It yields a representation, called the *adjoint representation* of $\mathfrak{g}$, where the vector space V the Lie algebra $\mathfrak{g}$ itself.
The following definition generalizes the notion of roots from the adjoint representation to any representation ρ:

> **Definition:** An element $\mu \in \mathfrak{h}$ is called a *weight* for (ρ, V), if there is a vector $0 \neq v \in V$, such that
> $$\rho(h)v = (\mu, h)v \tag{.11}$$
> for all $h \in \mathfrak{h}$. Such a v is called a *weight vector* for the weight μ, all vectors fulfilling this equation form the *weight space* with weight μ and the dimension of this weight space is called the *multiplicity* of the weight μ.

Comparing this defintion with the previously introduced roots, we see, that they are the weights of the adjoint representation. We adopted the scalar product notation for the roots also for the weights, since we saw, that any element in $\mathfrak{h}$ can be identified with one in $\mathfrak{h}^*$.
We can now also explain, why the e^α are called ladder operators: if their representatives $\rho(e^\alpha)$ are applied to a weight vector with weight μ, one receives other weight vectors with weights $\mu + \alpha$.
Taking tensor products of vector spaces is a well-known construction, which is often used in physics. Therefore we will also define the tensor product representations here.

Definition: Let $\mathfrak{g}$ be a Lie algebra and (ρ_1, V_1) and (ρ_2, V_2) representations of g, then the tensor product of ρ_1 and ρ_2 acting on $V_1 \otimes V_2$ is defined as $(\rho_1 \otimes \rho_2)(x) = \rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)$.

The probably most familiar example is the tensor product of two $sl(2)$ representations, which e.g. is used in the description of spin-orbit-coupling.
Let us therefore consider the finitedimensional irreducible representations of $sl(2)$ as an example. From the commutation relations (.2), we see, that $x(y)$ increases (decreases) the weight of h by 2. Since the representation space V is finitedimensional, there must be a highest weight Λ with weight vector v_Λ. All other weights then have the form $\Lambda - 2n$. For the same reason, there must be a n_0, such that the weight spaces of the weights $\Lambda - 2n$ for $n > n_0$ are zerodimensional. It turns out, that this can only be the case, if Λ is a non-negative integer. This in fact exhausts all finitedimensional irreducible representations of $sl(2)$. The weights of a representation with highest weight Λ are then of the form $\Lambda, \Lambda - 2, .. - \Lambda + 2, -\Lambda$ and each weight space is onedimensional.
If ρ_{Λ_1} and ρ_{Λ_2} are two representations of $sl(2)$, then by the Clebsch-Gordan decomposition their tensor product is isomorphic to the following direct sum of irreducible representations:

$$\rho_{\Lambda_1} \otimes \rho_{\Lambda_2} = \rho_{|\Lambda_1 - \Lambda_2|} \oplus \rho_{|\Lambda_1 - \Lambda_2|+2} \oplus .. \oplus \rho_{\Lambda_1 + \Lambda_2}. \tag{.12}$$

We will see, that the theory of finding the finitedimensional representations of semisimple Lie algebras is based on the idea to reduce the problem to the well established $sl(2)$ case. Here the $sl(2)$ subalgebra associated with the simple roots will play a mayor role.
This idea will lead to the most important theorem in the theory of irreducible finitedimensional representations (ρ, V) of semisimple Lie algebras $\mathfrak{g}$, the theorem of the highest weight. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. We saw, that with every Cartan element $h^i \in \mathfrak{h}$ there is associated a $sl(2)$ subalgebra involving the ladder operators corresponding to a simple root $\alpha^{(i)}$. From the $sl(2)$ representation theory it follows, that there is a basis of V on which the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts diagonally. Then there is a decomposition of V into weight spaces V_λ, such that

$$\rho(h^i) v_\lambda = \lambda^i v_\lambda \tag{.13}$$

for all $v_\lambda \in V_\lambda$ and $i = 1, .., r$. If a representation of the Lie algebra should be finitedimensional, then it must be finitedimensional with respect to the $sl(2)$ subalgebras as well.
We had seen, that $h^{\alpha^{(i)}} = h^i$ then has only integer eigenvalues, so all

$$\lambda^i = \frac{2(\alpha^{(i)}, \lambda)}{(\alpha^{(i)}, \alpha^{(i)})} \tag{.14}$$

are integer. λ is hence a weight of a finitedimensional representation, if and only if it is a integral linear combination

$$\lambda = \sum_{i=1}^{r} \lambda^i \Lambda_{(i)} \tag{.15}$$

of the fundamental weights $\Lambda_{(i)}$ (defined by $2 \cdot (\Lambda_{(i)}, \alpha^{(j)}) = \delta_i^j \cdot (\alpha^{(j)}, \alpha^{(j)})$) and the λ^i are then called *integral weights*.
Since ρ should be finitedimensional, there must exists a maximal weight Λ, such that for a weight vector $v_\Lambda \in V_\Lambda$

$$\rho(e^\alpha) v_\Lambda = 0 \tag{.16}$$

for all $\alpha > 0$. If this Λ is unique, it is called the highest weight of the representation. All numbers Λ^i in $\Lambda = \sum_i \Lambda^i \Lambda_{(i)}$ are non-negative and such Λ^i are called *dominant integral weights*.
With these remarks we can state the central theorem, which we subdivide into three parts.

Theorem of the highest weight:

- Every irreducible representation of a semisimple Lie algebra has a highest weight and two representations with the same highest weight are equivalent.
- The highest weight of a irreducible representation is a dominant integral element.
- Every dominant integral element occurs as the highest weight of an irreducible representation.

Sometimes in physics one works with lowest instead of highest weight, especially when some form of energy is involved, which is usually assumed to be bound from below. Then one argues the same way, that their must be a minimal weight, whose weight vector is annihilated by all negative roots.
The last part of the theorem is the hardest to prove. Given a dominant integral element Λ, one needs to find a way to construct a representation, which has Λ as a highest weight. We note, that this leads to the notion of Verma modules and refer to the literature for details.

.1.2 Lie algebras and Lie groups

Lie groups appear in physics as groups of symmetry transformations of a spacetime. Examples are the group in rotations in classical physics, the Lorentz group in special relativity and the conformal group. To every Lie group there is an associated Lie algebra, which as discussed in the previous section are linear spaces. This fact makes is easier to study properties of Lie groups on the Lie algebra side. Most informations about a Lie group are already encoded in its Lie algebra and they can be transferred back to the group via the exponential mapping, which in a way allows to reconstruct a Lie group from a Lie algebra. In this section, we will first make these statements more precise and then clarify the connection between representations of a Lie algebra and representations of associated Lie groups.
We first state, how a Lie group is defined:

Definition: A *Lie group* G is a differentiable manifold, which has a group structure, such that the group operations

$$\cdot : \quad G \times G \to G \quad , \ (g_1, g_2) \mapsto g_1 \cdot g_2 \quad \text{and} \tag{.17}$$

$$(.)^{-1} : \quad G \to G \quad , \ g \mapsto g^{-1} \tag{.18}$$

are differentiable.

Many of the Lie groups used in physics are matrix Lie groups, i.e. closed subgroups of the general linear groups $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.
Given a Lie group G, one can define its Lie algebra $\mathfrak{g}$ as the set of left invariant vector fields with the bracket being the commutator of two vector fields. We define the left translation $L_a : G \to G$ of an element g by a as

$$L_a g = ag \tag{.19}$$

If x is a vector field on G, then x is said to be *left-invariant*, if

$$L_{a*}x(g) = x(ag), \tag{.20}$$

where $(L_{a*}x)[f] = x[f \circ L_a]$ is the differential map induced by L_a ($f \in F(M) = \{h : G \to \mathbb{R}\}$ is any scalar function).
Consider now an element v of the tangent space T_eG at the identity element $e \in G$. Then for all $g \in G$,

$$x_v(g) = L_{g*}v \tag{.21}$$

is a left invariant vector. Conversely, a left invariant vector field x defines a unique vector $v = x(e) \in T_eG$. Hence T_eG and the set of left invariant vector fields, which we denote by $\mathfrak{g}$, are isomorphic as vector spaces. For general vector fields X, Y one defines the *Lie bracket* $[.,.]$ as

$$[X,Y]f = X[Y[f]] - Y[X[f]] \tag{.22}$$

for $f \in F(M)$. Then the set $\mathfrak{g}$ of left invariant vector fields endowed with the Lie bracket fulfils all the defining properties of a Lie algebra and is called the *Lie algebra of the Lie group* G[2].
Conversely, one can recover a Lie group (at least in the connected component of the identity element) from a Lie algebra via the so-called exponential mapping. We begin with the following definition

> **Definition:** A one-parameter subgroup of a Lie group G is a curve $\gamma : \mathbb{R} \to G$, that fulfils $\gamma(t)\gamma(s) = \gamma(t+s)$.

With every element $x \in T_eG$ there is associated a geodesic $s(t)$ (a straightest possible line in G), such that

$$s(0) = e \;\text{ and }\; \frac{ds}{dt}(t=0) = x. \tag{.23}$$

Then one defines the *exponential map* $\exp : T_eG \to G$ by

$$X \mapsto \exp(x) := s(1). \tag{.24}$$

With this map one can relate vectors $x \in T_eG \simeq \mathfrak{g}$ with (local) one-parameter subgroups of G via

$$t \in [-\epsilon, \epsilon] \mapsto g_t^x = \exp(tx). \tag{.25}$$

The exponential fulfils basically the usual properties of exponentials, except $\exp(x)\exp(y) = \exp(x+y)$, which is only true, if $[x,y] = 0$. The generalization of this relation is called the *Baker-Campbell-Hausdorff formula* and in lowest orders it reads

$$\exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[[x,y],y] + ..) \tag{.26}$$

If G is a matrix Lie group, the exponential mapping is the usual power series

$$\exp(tx) = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n \tag{.27}$$

[2] The construction could of course also have been done with right translations and with T_eG replaced by any other tangent space T_gG

for $x \in G$. In a neighborhood U of the unit element, every element $g \in U$ can be written as

$$g = \exp(x) \tag{.28}$$

for some suitable $x \in \mathfrak{g}$.
We will discuss compact and non-compact Lie groups later, but we note here the following facts: If G is connected and compact, then $U = G$. If it is compact, but not connected, then $U = G_0$, where G_0 is the connected component of the unit element $e \in G$. For non-compact G it is not necessarily true anymore, that exponentiation yields the full group.

The exponential mapping is crucial in relating the representations of a matrix Lie group with the representations of its Lie algebra. We define

> **Definition:** A *Lie group homomorphism* Φ between two Lie groups G and H is a continous group homomorphism. A *Lie algebra homomorphism* between two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a vector space homomorphism $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$, that fulfils $\pi([x_1, x_2]) = [\pi(x_1), \pi(x_2)]$ for all $x_1, x_2 \in \mathfrak{g}_1$.

Now we can say, what a representation of a Lie group is.

> **Definition:** A representation ρ of a Lie group G on a (real or complex) vector space V is a Lie group homomorphism
>
> $$\rho : G \to GL(V). \tag{.29}$$

Let now $\Phi : G \to GL(V)$ be such a representation and let $\mathfrak{g}$ and $gl(V)$ denote the respectively Lie algebras of G and $GL(V)$. Then we have the following special case of a theorem, that is valid for all homomorphisms between two Lie groups:

> **Theorem:** [13] There exists a unique real linear map $\phi : \mathfrak{g} \to gl(V)$, such that
>
> $$\Phi(\exp(x)) = \exp(\phi(x)) \tag{.30}$$
>
> for all $x \in \mathfrak{g}$. This ϕ is a Lie algebra homomorphism.

If in turn a Lie algebra homomorphism $\phi : \mathfrak{g} \to gl(V)$ is given and G is simply connected, also the converse holds. Thus we have the

> **Theorem:** Representations of a simply connected Lie group G are in one-to-one correspondence with the representations of its Lie algebra $\mathfrak{g}$.

For a connected Lie group G, a simply connected group $\tilde{G}$ is called its *universal covering group*, if there is surjective map $p : \tilde{G} \to G$ and if for each $g \in G$ there is a connected open set $U \subset G$ containing g, such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{G}$, each of which is mapped homeomorphically onto U by p. G has then the same Lie algebra as $\tilde{G}$.
We note, that the last theorem therefore implies, that for a general Lie group G with Lie algebra $\mathfrak{g}$, the representations of $\mathfrak{g}$ are one-to-one to the representations of the universal covering group $\tilde{G}$ of G.

.1.3 Casimir operators

We construct an associative algebra $\mathcal{T}\mathfrak{g}$ associated with a Lie algebra $\mathfrak{g}$, which contains $\mathfrak{g}$ as a subalgebra and whose commutator reduces to the Lie bracket on $\mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, we can take its n-fold tensor product

$$\mathfrak{g}^n = \bigotimes_{j=1}^{n} \mathfrak{g} \tag{.31}$$

and form the direct sum of all $\mathfrak{g}^n$:

$$\mathcal{T}\mathfrak{g} = \bigoplus_{n=0}^{\infty} \mathfrak{g}^n. \tag{.32}$$

This so-called *tensor algebra* of $\mathfrak{g}$ is endowed with the tensor product $\otimes : (x_1, x_2) \mapsto x_1 \otimes x_2$. Therefore the $\mathfrak{g}^n$ are graded:

$$\mathfrak{g}^n \otimes \mathfrak{g}^m \subseteq \mathfrak{g}^{n+m}. \tag{.33}$$

In the light of the underlying Lie algebra $\mathfrak{g}$ the two elements $x_1 \otimes x_2 - x_2 \otimes x_1 \in \mathfrak{g}^2$ and $[x_1, x_2] \in \mathfrak{g}^1$ should be "the same". If one identifies them and also does the corresponding identification for higher tensor products, one ends up with the so-called *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$. If $\mathcal{I}$ is the smallest ideal, that contains all elements

$$x_1 \otimes x_2 - x_2 \otimes x_1 - [x_1, x_2] \in \mathcal{T}\mathfrak{g} \tag{.34}$$

with $x_1, x_2 \in \mathfrak{g}$, one can describe this construction as taking the quotient

$$U(\mathfrak{g}) = \mathcal{T}\mathfrak{g}/\mathcal{I}. \tag{.35}$$

Let now $\{T_a\}$ be a basis of generators of $\mathfrak{g}$. We define the *quadratic Casimir operator*

$$C^{(2)} = \sum_{a,b=1}^{n} \kappa_{ab} T^a T^b, \tag{.36}$$

where

$$\kappa_{ab} = \frac{1}{I_{ad}} \kappa(T^a, T^b) \tag{.37}$$

and I_{ad} is a normalization constant [3]. Then $C^{(2)}$ commutes with all elements $x \in U(\mathfrak{g})$, so of course also with all $x \in \mathfrak{g}$. By Schur's Lemma it must therefore have a constant value within a representation.

In the case of $sl(2)$ the quadratic casimir operator is

$$C^{(2)} = \frac{1}{4} h^2 + \frac{1}{2}(xy + yx) \tag{.38}$$

and its value within the Λ highest weight representation if $\frac{1}{4}\Lambda(\Lambda+2)$. In terms of the angular momentum generators $\{J_1, J_2, J_3\}$ and the spin quantum number $j = \Lambda/2$ it reads

$$C^{(2)} = \vec{J}^2 = J_1^2 + J_2^2 + J_3^2 \tag{.39}$$

[3] The index ad stands for the adjoint representation, on which we defined the Cartan-Killing form.

and has the value $j(j+1)$ within a j multiplet.

There are also higher order Casimir operators of the form

$$C^{(n)} = \sum_{a_1,..,a_n} d_{a_1,..,a_n} T^{a_1}..T^{a_n}, \tag{.40}$$

which also take constant values within a representation. $d_{a_1,..,a_n}$ are suitable "invariant tensors" of the adjoint representation. There are $r = rank(\mathfrak{g})$ Casimir operators for a semisimple Lie algebra $\mathfrak{g}$ and they form a basis of the center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra.
Since a representation is also determined by r quantum numbers (the components of the highest weight), a representation can equivalently be characterized by all the values of the Casimir operators in it.

.1.4 Characters

Characters are a compact way of encoding the weight system of a representation (ρ, V). The first step in to introduce a generating function. Normally, if one wants to generate a certain series f_n, one introduces a Laurent series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n x^n, \tag{.41}$$

which contains the same informations as the coefficients f_n, which can be obtained as Taylor coefficients of $f(x)$.
For our purposes, we have to modify this slightly. We do not sum over the integer numbers, but over the weights λ of a representation. We introduce a formal exponential e^λ, that fulfils $e^\lambda e^{-\lambda} = e^{-\lambda}e^\lambda = 1$ and $e^{\lambda+\mu} = e^\lambda e^\mu$ for all weights λ and μ. Then we consider the generating function

$$\chi_\rho(\mu) = \sum_\lambda \mathrm{mult}_\rho(\lambda) \exp[(\lambda, \mu)]. \tag{.42}$$

The argument μ is an r component vector of formal variables. The weights λ are linear functions on the Cartan subalgebra $\mathfrak{h}$, therefore so are e^λ and χ_ρ. As a function on $\mathfrak{h}$, χ_ρ is called the *character* of the representation (ρ, V).
For our standard example, the Λ highest weight representation of $sl(2)$, the weights are the numbers $\lambda = \Lambda, \Lambda-2, .., -\Lambda+2, -\Lambda$ and the character is

$$\chi_\Lambda(\mu) = \sum_\lambda \exp[(\lambda, \mu)] = \frac{\sinh(\mu(\Lambda+1)/2)}{\sinh(\mu/2)}. \tag{.43}$$

We mention two immediate properties

- For a representation (ρ, V), which is a direct sum $V = \bigoplus_i V_i$ of $\mathfrak{g}$-representations (ρ_i, V_i),

$$\chi_\rho = \sum_i \chi_{\rho_i} \tag{.44}$$

- For the tensor product $\rho_1 \otimes \rho_2$ of two $\mathfrak{g}$-representations (ρ_1, V_1) and (ρ_2, V_2),

$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_1}. \tag{.45}$$

.1.5 Non-compact Lie algebras

Not all results from the representation theory of Lie algebras of $\mathfrak{g}$ carry over to the case, when the associated Lie group G is non-compact. This is quite unfortunate, since several groups of physical interest, like the Poincaré group or the conformal groups are of this type.

We start with the observation, that the Killing form κ of $\mathfrak{g}$ can be used to characterize compact Lie groups:

> **Weyl's Theorem:** Let G be a Lie group with semisimple Lie algebra $\mathfrak{g}$. Then G is compact, if and only if the Killing form on $\mathfrak{g}$ is negative definite.

$\mathfrak{g}$ is then said to be of *compact type*, otherwise of *non-compact type*. It should be noted as quite remarkable, that the algebraic statement of definiteness of a bilinear form is equivalent to the topological property of compactness.
Every complex semisimple Lie algebra has a so-called *compact real form*. The real forms of a Lie algebra $\mathfrak{g}$ are the real Lie algebras, which have $\mathfrak{g}$ as their complexification. Among these there is a unique one, on which the Killing form is negative definite, therefore it is called the compact real form.
We aim now towards the Cartan decomposition, which is the decomposition of a Lie algebra or Lie group into a compact and a non-compact part. For a matrix Lie algebra $\mathfrak{g}$, we define the map $\theta(x) = -x^\dagger$for $x \in \mathfrak{g}$[4] . If $\mathfrak{g}$ is real and semisimple and if the symmetric bilinear form $\kappa_\theta(x, y) = -\kappa(x, \theta y)$ is positive definite, then θ is called a *Cartan involution*. Every such $\mathfrak{g}$ has essentially a unique θ (up to inner automorphisms), which yields a decomposition of $\mathfrak{g}$ into eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ corresponding to eigenvalues $+1$ and -1, the *Cartan decomposition*:

> **Theorem:** $\mathfrak{g}$ is a direct sum $\mathfrak{k} \oplus \mathfrak{p}$, where
>
> - The restriction of the Killing form κ of $\mathfrak{g}$ to $\mathfrak{k}$ ($\mathfrak{p}$) is negative (positive) definite.
> - $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$
>
> Any other subalgebra $\mathfrak{k}'$ with $\kappa(\mathfrak{k}', \mathfrak{k}')$ negative definite is related to a subalgebra of $\mathfrak{k}$ by an inner automorphism.

This theorem has an analogue on the group side:

> **Lemma:** Let G be a connected Lie group with a finite center. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of its Lie algebra $\mathfrak{g}$, then
>
> $$G = PK, \qquad (.46)$$
>
> where P is the image of $\mathfrak{p}$ under the exponential mapping and K the compact subgroup of G with Lie algebra $\mathfrak{k}$.

So every element $g \in G$ can be written as a unique product of certain $p \in P$ and $k \in K$. We note, that the finiteness of the center is needed to ensure the compactness of K. For semisimple Lie groups G it suffices to have discrete center.
We illustrate the Cartan decomposition in both forms with a simple matrix example: Let $G = SL(n, \mathbb{R})$ be the group of invertible unimodular real $n \times n$ matrices and $K = SO(n, \mathbb{R})$ be the

[4]The † stands for the adjoint, i.e. the complex conjugated transposed matrix.

orthogonal $n \times n$ matrices with determinant $+1$. The Lie algebra $\mathfrak{g}$ of G are the traceless $n \times n$ matrices. Because the groups are real, the Cartan involution becomes taking the negative transpose, $\theta(x) = -x^t$. Then for $x \in \mathfrak{k}$, we get $-x^t = x$, i.e. x is skewsymmetric, and for $x \in \mathfrak{p}$, we get $-x^t = -x$, i.e. x is symmetric. Therefore we can conclude, that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the separation into symmetric and antisymmetric matrices.
On the group level, one has a similar involution Θ and it is in this case $\Theta(g) = (g^{-1})^t$. We have $(g^{-1})^t = g$ for $g \in K$, i.e. g is an orthogonal matrix. For $g \in P$, because of $g^t = g$, g is symmetric and since $det(g) = +1$, it can be joined to the identity (which is positive definite) by a curve consisting of symmetric matrices, so g must be positive definite. Then $G = PK$ is the polar decomposition of a matrix as the product of a symmetric positive definite and an orthogonal one.
Based on the Cartan decomposition, there is another type of decomposition of semisimple Lie groups G of noncompact type, the so-called *Iwasawa decomposition*. It also needs a G with a finite center, but in contrast to the Cartan decomposition (cf. section .1.2), here the factors are closed subgroups (where P in general was not).
Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G.
We consider a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. The root space Δ with respect to $\mathfrak{a}$ can be splitted into the set of positive roots Δ^+ and negative roots Δ^-. This can e.g. be done by taking a fixed regular element $A_0 \in \mathfrak{a}$ [5] and saying, that all roots λ with $\lambda(a_0) > 0$ are positive.
We denote by $\mathfrak{n}^+$ and $\mathfrak{n}^-$ the subspaces of $\mathfrak{g}$ spanned the root vectors belonging to the positive and negative roots, respectively.
Then the following three statements involving $\mathfrak{n}^+$ (and of course similar versions with $\mathfrak{n}^-$) hold:

- $\mathfrak{n}^+$ is nilpotent.
- $\mathfrak{a} + \mathfrak{n}^+$ is solvable.
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+$ (unique).

The group version of the last fact is the Iwasawa decomposition:

> **Theorem:** Let K, A, N^+ denote the connected subgroups of G belonging to the subalgebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}^+$. Then
>
> $$G = KAN^+, \qquad (.47)$$
>
> i.e. every element $g \in G$ is $g = kan^+$ with $k \in K$, $a \in A$ and $n^+ \in N^+$.

Again we consider the example $G = SL(n, \mathbb{R})$ and $K = SO(n, \mathbb{R})$. $\mathfrak{a}$ can be taken as the Lie algebra of trace zero diagonal matrices. Since $\mathfrak{a}$ is abelian, all ad_a, $a = (a_{ij}) \in \mathfrak{a}$ can be simultaneously diagonalized:

$$ad_a(x) = [a, x] = \lambda(a)x, \qquad (.48)$$

where $x = (x_{ij}) \in \mathfrak{g}$. We consider a with unequal diagonal elements[6], which are ordered, such that $a_{ii} > a_{jj}$ for $i > j$. Writing the component version of (.48) and using the diagonality of a, we get the condition

$$a_{ii}x_{ij} - x_{ij}a_{jj} = \lambda x_{ij} \Longleftrightarrow x_{ij}(\lambda - a_{ii} + a_{jj}) = 0 \qquad (.49)$$

[5] A_0 regular means, that the centralizer $C(a_0) = \{B \in \mathfrak{p} \mid [B, A_0] = 0\}$ of A_0 in $\mathfrak{p}$ is a Cartan subalgebra.
[6] Then a is a regular element.

We now additionally demand, that no two pairs of diagonal elements differ by the same amount:

$$a_{ii} - a_{jj} \neq a_{kk} - a_{ll} \tag{.50}$$

for $(i, j) \neq (k, l)$. Then x_{ij} can be non-zero only for one pair (i, j) and because of the ordering of the diagonal elements and the fact, that $\lambda > 0$, we have $i > j$. So the space $\mathfrak{n}^+$ spanned by these matrices are of the strictly lower triangular matrices. It is then easily seen, that AN^+ is the group of lower triangular matrices.
Therefore in this case the Iwasawa decomposition $G = KAN^+$ is the classical statement (sometimes called QL decomposition), that any non-singular matrix can be written as a unique product of an orthogonal and a lower triangular matrix.
The advantage of such a decomposition into group factors is, that it makes it easier to use the theory of induced representations.
We comment briefly on this topic. The notion describes the induction of a representation of a whole Lie group G by a subgroup H. We state a fact, which is known as Weyl's unitary trick in the form, that will be relevant for us:

> **Theorem:** Every semisimple Lie group G has a complexification $G^{\mathbb{C}}$, which is a complex Lie group and has a maximal subgroup K. Then one can obtain the finitedimensional representations of G from those of K.

A slight modification of this will be used in the next section, where G is an infinite-sheeted covering of the conformal group $\mathcal{C}_4 = SO_e(4, 2)/\mathbb{Z}_2$ and $K = \mathbb{R} \times SU(2) \times SU(2)$.

.2 Unitary positive energy representations of $SU(2,2)$

The unitary positive energy representations of the universal covering group $\tilde{G}$ of the matrix group $G = SU(2,2)/\mathbb{Z}_4$ have been obtained by Mack [19]. G is locally isomorphic to the conformal group, i.e. they have the same Lie algebra $\mathfrak{g}$. The analysis relies on the theory sketched in the previous appendix.
The group $G = SU(2,2)$ is defined similarly to the pseudoorthogonal group. It is the following set of complex 4x4 matrices

$$G = \{m \in Mat_4(\mathbb{C}) \mid m^{-1}\beta = \beta m^* \ , \ \det(m) = 1\} \tag{.51}$$

for

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{.52}$$

Its Lie algebra $\mathfrak{g}$ is the set

$$\mathfrak{g} = \{m \in Mat_4(\mathbb{C}) \mid tr(m) = 0 \ , \ -m\beta = \beta m^*\}. \tag{.53}$$

and it is real.

The Lie algebra

We consider the real Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of all diagonal matrices and the following basis of $i\mathfrak{h}$

$$h_0 = \frac{1}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \ , \ h_1 = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & 0 \end{pmatrix} \ , \ h_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix} \tag{.54}$$

where $\mathbb{I}$ is the 2x2 unit matrix and $\sigma^3 = diag(+1,-1)$ the third Pauli matrix.
We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The explicit form of the matrices in $X \in \mathfrak{p}$ follows from the requirement, that $\theta(X) = -X$. With the definition on the Cartan involution and (.53) we conclude, that $X\beta = -\beta X$ and hence

$$X = \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \tag{.55}$$

with a complex 2x2 matrix z. Analogously, the matrices $X \in \mathfrak{k}$ fulfil $= X\beta = \beta X$. The associated maximal compact subgroup of G is $K = S(U(2) \times U(2))$ and consists of the pairs of two $U(2)$ matrices k_1 and k_2, such that $\det(k_1)\det(k_2) = 1$.
Let now $\mathfrak{g}_c$ and $\mathfrak{h}_c$ denote the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$.
If $\mathfrak{n}^{\pm}$ denotes the upper/lower triangular matrices in $\mathfrak{g}_c$, we can then obviously write

$$\mathfrak{g}_c = \mathfrak{h}_c + \mathfrak{n}^+ + \mathfrak{n}^-, \tag{.56}$$

because every matrix $x \in \mathfrak{g}_c$ can be decomposed into a sum of three matrices of the respective types.
The Cartan decomposition in turn yields a splitting into a compact and non-compact part

$$\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{p}_c, \tag{.57}$$

on which the Killing form is negative and positive definite respectively.
Forming the intersection of (.56) and (.57), we get

$$\mathfrak{g}_c = \mathfrak{h}_c + \mathfrak{n}^+ \cap \mathfrak{k}_c + \mathfrak{n}^- \cap \mathfrak{k}_c + \mathfrak{p}_c \cap \mathfrak{n}^+ + \mathfrak{p}_c \cap \mathfrak{n}^- \tag{.58}$$

We sketch now, how a basis of $\mathfrak{g}_c$, that suits this decomposition, can be found.
One quickly calculates, that the possible eigenvalues of ad_{h_1} and ad_{h_2} in $\mathfrak{p}_c \cap n^\pm$ are $\pm 1/2$, which we use to introduce a basis X^+_{jk} with $j,k = \pm 1/2$ of $\mathfrak{p}_c \cap n^+$, and out of this a basis $X^-_{jk} = (X^+_{-j-k})^*$ of $\mathfrak{p}_c \cap n^-$. If further we pick the basis X^0_{jk} of $(\mathfrak{n}^+ + \mathfrak{n}^-) \cap \mathfrak{k}_c$, that fulfils

$$ad_{h_0}(X^0_{jk}) = 0 \ , \ ad_{h_1}(X^0_{jk}) = jX^0_{jk} \ , \ ad_{h_2}(X^0_{jk}) = kX^0_{jk}, \tag{.59}$$

where either $(j,k) = (0, \pm 1)$ or $(j,k) = (\pm 1, 0)$, we end up with a basis of $\mathfrak{g}_c$, that has commutation relations in the Cartan normal form (with respect to the compact Cartan subalgebra $\mathfrak{h}$).

The Lie group

The universal covering group $\tilde{G}$ of G is an infinite sheeted covering. It consists of homotopic equivalence classes of directed paths in G starting at the identity. Because of the chain of isomorphisms

$$\mathcal{C}_4 \simeq SO_e(4,2)/\mathbb{Z}_2 \simeq G/\mathbb{Z}_4 \simeq \tilde{G}/\mathbb{Z}_2 \times \mathbb{Z}, \tag{.60}$$

the center of $\tilde{G}$ is not finite, but we will assume that this poses no problem and the results from the first appendix still hold.
The Iwasawa decomposition of G is then

$$G = K A_\mathfrak{p} N_\mathfrak{p}, \tag{.61}$$

where K is the maximal compact subgroup mentioned before.
Furthermore, $A_\mathfrak{p} = A_\mathfrak{m} A$ and $N_\mathfrak{p} = N_\mathfrak{m} N$, with $A_\mathfrak{m}$ and $N_\mathfrak{m}$ come from the Iwasawa decomposition

$$\mathcal{L} = U A_\mathfrak{m} N_\mathfrak{m} \tag{.62}$$

of the Lorentz group.
Explicitly, A and N are the dilatations and the special conformal transformations in an approppiate 4x4 matrix form, $A_\mathfrak{m}$ are the Lorentz boosts in z-direction and $N_\mathfrak{m}$ is a twodimensional abelian group of a lightlike vector pointing in z-direction. $U = SU(2)$ is the maximal compact subgroup of the Lorentz group.
Because $A_\mathfrak{p} N_\mathfrak{p}$ is simply connected, any two paths with the same end points are homotopic to each other and we can write $\tilde{G}$ as

$$\tilde{G} = \tilde{K} A_\mathfrak{p} N_\mathfrak{p}, \tag{.63}$$

where $\tilde{K}$ is the universal covering of K (which is not compact anymore and which we again asuume to pose no problems.). Explicitly

$$\tilde{K} = \mathbb{R} \times SU(2) \times SU(2). \tag{.64}$$

Since the center of $\tilde{G}$ is contained in $\tilde{K}$ it suffices to consider representations of $\tilde{K}$, because it induces representations of $\tilde{G}$. We have hence, that the representations of $\tilde{G}$ can be labelled by three numbers

$$\lambda = (d, j_1, j_2). \tag{.65}$$

Energy positivity

If $\mathcal{H}$ is a Hilbert space with a unitary representation U^7 of $\tilde{G}$, the condition of energy positivity is imposed by requiring

$$\langle \psi, U(H^0)\psi \rangle \geq 0 \tag{.66}$$

for all vectors $\psi \in \mathcal{D} \subset \mathcal{H}$, where $\mathcal{D}$ is the $\tilde{G}$ invariant domain of $U(H^0)$ (cf. section 2.1). H^0 is the conformal Hamiltonian $\frac{1}{2}(P^0 + K^0)$. Since $K^0 = I_r \circ P_0 \circ I_r = I_r \circ P^0 \circ I_r$, one can equivalently demand

$$\langle \psi, U(P^0)\psi \rangle \geq 0. \tag{.67}$$

By the spectral theorem, the spectrum of $U(H^0)$ is discrete and by (.67), it is bounded from below, let us say by d. The eigenvalues then take values $d + m$, $m \in \mathbb{N}_0$.

Lowest weight vectors

Let T be an irreducible representation of $\mathfrak{g}_c$ on a vector space V. We call $\Omega \in V$ a *lowest weight vector* with lowest weight λ, if it is annihilated by all operators $T(X)$ for $X \in \mathfrak{n}^-$ a lower triangular matrix:

$$T(X)\Omega = 0 \tag{.68}$$

and

$$T(h)\Omega = \lambda(h)\Omega \tag{.69}$$

for all Cartan elements $h \in \mathfrak{h}_c$.
If V is finitedimensional, one can use the classical theorems, which state, that T is a lowest weight representation, which restricts to a representation of $\mathfrak{k}_c$. The lowest weight is then of the form

$$(d, -j_1, -j_2) \tag{.70}$$

and group and algebra representations are equivalent.
For V infinitedimensional, T need not have a lowest weight, but for unitary positive energy representations U (assuming they exist) this will turn out to be the case.
Because of the discrete spectrum of $U(H^0)$, the Hilbert space decomposes into Hilbert spaces V^μ, which decompose into copies of one unitary irreducible representation of $\tilde{K}$ with lowest weights μ of the form

$$(d + N, -J_1, -J_2). \tag{.71}$$

Another result in representation theory says, that for finite center the V^μ are finitedimensional, when one decomposes with respect to the maximal compact subgroup. Here neither the center of $\tilde{G}$ is finite, nor is $\tilde{K}$ compact, but we will again assume that the result remains valid. Then the algebraic sum V of all the V^μ is a vector space and a common dense domain for all $X \in \mathfrak{g}$. So there is an irreducible representations of the Lie algebra by operators $U(X)$ on V associated to the unitary irreducible representation (UIR) U of the group on $\mathcal{H}$. Conversely, any representation of $\mathfrak{g}$ by skew-hermitean operators on V can be exponentiated to a UIR of the group. This is the classical result of equivalence of representations of a simply connected Lie group and its Lie algebra. If d is the lowest eigenvalue of $U(H^0)$, there must be a weight $\lambda = (d, -j_1, -j_2)$ with $2j_1$,$2j_2$ non-negative integer and an associated weight

[7] We denote a general representation of $\tilde{G}$ by T and a unitary one by U.

vector, which can be checked to be annihilated by all $X \in \mathfrak{n}^-$ [8], so it is a lowest weight vector and we have the following result:

> **Theorem:** A unitary irreducible representation of $\tilde{G}$ with positive energy has a unique lowest weight. Any two representations with same lowest weight are unitarily equivalent.

Here saying, that a representation of $\tilde{G}$ has a lowest weight, means, that the associated representation of $\mathfrak{g}_c$ does.

Unitarity of the representation

Demanding unitarity of the representation imposes restrictions on the possible lowest weights, more precisely on the scaling dimension. It is imposed via an algorithm for computing norms of the V-spanning vectors

$$\psi_{\{n\}} = T(X_1)^{n_1} ... T(X_6)^{n_6} \Omega \qquad (.72)$$

where the X_i are the basis vectors of $\mathfrak{n}^+$ from above and $n_i \in \mathbb{N}$ ($i = 1, .., 6$). The scalar product of two vectors $\psi_{\{n\}}$ and $\psi_{\{n'\}}$ is computed by commuting operators to the vacuum vector using the commutation relations mentioned above. This scalar product should induce a positive definite norm on these vectors.
There appear five different cases depending on the quantum numbers, for which this requirement has to be imposed separately. The resulting restriction on the possible d and the Poincaré content (mass and spin/helicity) of the representation are listed in the following table:

Spin quantum numbers	Scaling dimensions	Poincaré content $[m, s]$
$j_1 j_2 = 0$	$d = j_1 + j_2 + 1$	$[0, j_1 - j_2]$
$j_1 j_2 = 0$	$d > j_1 + j_2 + 1$	$[m, j_1 + j_2]$
$j_1 \neq 0, j_2 \neq 0$	$d = j_1 + j_2 + 2$	$[m, j_1 + j_2]$
$j_1 \neq 0, j_2 \neq 0$	$d > j_1 + j_2 + 2$	$[m, \lvert j_1 - j_2 \rvert, .., j_1 + j_2]$

In addition, there is a onedimensional trivial representation $d = j_1 = j_2 = 0$.

[8] Here one uses the commutation relations of the chosen basis and the fact, that weights j in a $SU(2)$ representation j_1 have to fulfil the equation $|j| \leq j_1$.

.3 The hypergeometric functions ${}_pF_q$

We summarize the most important properties of the (generalized) hypergeometric function. It is defined as

$$ {}_pF_q(a_1,..,a_p;b_1,..,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot .. \cdot (a_p)_n}{(b_1)_n \cdot .. \cdot (b_q)_n} \frac{x^n}{n!}, \tag{.73} $$

where

$$ (a)_n = a(a+1)..(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{.74} $$

is the Pochhammer symbol. The definition of the $(a)_n$ makes clear, that if one of the a_i is a negative integer $-m$, ${}_pF_q$ has just finitely many terms:

$$ {}_pF_q(a_1,..,-m,..,a_p;b_1,..,b_q;x) = \sum_{n=0}^{m-1} \frac{(a_1)_n \cdot .. \cdot (-m)_n \cdot .. \cdot (a_p)_n}{(b_1)_n \cdot .. \cdot (b_q)_n} \frac{x^n}{n!} \tag{.75} $$

The most important of the ${}_pF_q$ is Gauss' hypergeometric function $F(a,b;c;x) \equiv {}_2F_1(a,b;c;x)$. It generalizes various special functions and solves the hypergeometric differential equation

$$ x(1-x)\frac{d^2f}{dx^2} + [c-(a+b+1)x]\frac{df}{dx} - abf = 0. \tag{.76} $$

The other solution to this second order equation is

$$ x^{\lambda} \cdot F(a+\lambda, b+\lambda; c+\lambda; x) \tag{.77} $$

with $\lambda = 1-c$, which is essentially a hypergeometric series as well.
Beside other series representations than (.73), there are also integral representations of the hypergeometric function like

$$ F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}. \tag{.78} $$

Tables list relations between hypergeometric functions with different values a_i, b_i and also with derivatives of the function. They are proven using the different representations.
The identity

$$ F(k-1,k-1;2k-2;x) + \frac{k^2}{4(2k-1)(2k+1)} \cdot x^2 \cdot F(k+1,k+1;2k+2;x) - (1-\frac{x}{2}) \cdot F(k,k;2k;x) = 0 $$

was proven in section 4.1.
There are also identities, which just hold for certain parameter or function values like

$$ F(-n,b,c;d,e;1) = \frac{(d-b)_n}{(d)_n} F(-n,b,e-c;e,b-d-n+1;1) \tag{.79} $$

for $n \in \mathbb{N}$, which was used in 4.3.3.

.4 Details of the restricted six-point function PWE

Here we display some details of the derivation of the system of three partial differential equations, that determine the partial waves of the conformal six-point function (5.11) restricted to two dimensions.
When the differential operators $\mathcal{D}_2$ and $\mathcal{D}_3$ hit the expression (5.15), both the distance squares in front and the cross ratios within the function $f_{bcd} \equiv f_{abcde}$ have to be differentiated using the product and chain rule. Therefore the innocuously looking expression (5.27) is in fact very long for the three cases. The calculation was done computer aided.
The first important step in putting the expression back together in terms of the cross ratios was to note, that if

$$s_{ijkl} = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}} \tag{.80}$$

is differentiated e.g. by x_i, then

$$\partial_i s_{ijkl} = \frac{x_{kl}}{x_{ik}x_{jl}} - \frac{x_{ij}x_{kl}}{x_{ik}^2 x_{jl}} = \frac{1}{x_{ij}} s_{ijkl} - \frac{1}{x_{ik}} s_{ijkl}. \tag{.81}$$

Appropiate adjustments of signs have to be made, depending on where the points appear in a cross ratio, since obviously

$$\partial_i x_{ij} = -\partial_i x_{ij}. \tag{.82}$$

For second derivatives we get terms like

$$\begin{aligned} \partial_l \partial_i s_{ijkl} &= \partial_l \left[\frac{x_{kl}}{x_{ik}x_{jl}} - \frac{x_{ij}x_{kl}}{x_{ik}^2 x_{jl}} \right] \\ &= -\frac{1}{x_{ik}x_{jl}} + \frac{x_{kl}}{x_{ik}x_{jl}^2} + \frac{x_{ij}}{x_{ik}^2 x_{jl}} - \frac{x_{ij}x_{kl}}{x_{ik}^2 x_{jl}^2} \\ &= -\frac{1}{x_{ij}x_{kl}} s_{ijkl} + \frac{1}{x_{ij}x_{jl}} s_{ijkl} + \frac{1}{x_{kl}x_{ik}} s_{ijkl} - \frac{1}{x_{ik}x_{jl}} s_{ijkl} \end{aligned} \tag{.83}$$

The factors in front of the s_{ijkl} must then be combined with other terms to expressions, that only contain the cross ratios s, t, u.
If one does this correctly, the following three partial differential equations arise. For the Casimir insertion between the second and the third field

$$\begin{aligned} l_3(l_3-1)f_{bcd} + (s-t)^2 \partial_s \partial_t f_{bcd} - (l_1 + l_3 - l_2) \frac{(s-t)(t-u)}{s-u} \partial_t f_{bcd} \\ -(l_1 - l_2 - l_3) \frac{(s-t)(u-s)}{u-t} \partial_s f_{bcd} = b(b-1) f_{bcd}, \end{aligned} \tag{.84}$$

between the third and the fourth field

$$\begin{aligned} -(l_4 + l_5 +_6)(l_4 + l_5 + l_6 - 1) f_{bcd} + (s-t)^2 \partial_s \partial_t f_{bcd} + (t-u)^2 \partial_t \partial_u f_{bcd} \\ +(s-u)^2 \partial_s \partial_u f_{bcd} = c(c-1) f_{bcd}, \end{aligned} \tag{.85}$$

and between the fourth and fifth field

$$\begin{aligned} l_4(l_4-1)f_{bcd} + (1-s)s^2 \partial_s^2 f_{bcd} + (1-t)t^2 \partial_t^2 f_{bcd} + (1-u)u^2 \partial_u^2 f_{bcd} + (2-s-t)st \partial_s \partial_t f_{bcd} \\ +(2-t-u)tu \partial_t \partial_u f_{bcd} + (2-s-u)su \partial_s \partial_u f_{bcd} + (2l_4 - (l_4 - l_5 + l_6 + 1)s)s \partial_s f_{bcd} \\ +(2l_4 - (l_4 - l_5 + l_6 + 1)t)t \partial_t f_{bcd} + (2l_4 - (l_4 - l_5 + l_6 + 1)u)u \partial_u f_{bcd} = d(d-1) f_{bcd}. \end{aligned} \tag{.86}$$

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