String-localized fields and point-localized currents in massless Wigner representations with infinite spin

Diplomarbeit

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List of symbols

$\mathcal{H}$	Hilbert space
$\langle\cdot\vert\cdot\rangle$	Hilbert space scalar product, linear in the right entry
$\mathbb{C}$	set of complex numbers
$\mathbb{P}(\mathcal{H})$	projective Hilbert space
$\hat{U}$	projective representation
U, D, S	Hilbert space representations
σ_i	i-th Pauli matrix
$\vec{\sigma}$	formal 3-vector whose components are the Pauli matrices
$\Re$, $\Im$	real and imaginary part
G^c	universal covering of the group G
$\mathbb{R}^n$	Euclidean n-dimensional space
$\mathbb{C}^n$	unitary n-dimensional space
$\vec{v}$	element of $\mathbb{R}^3$
$\times$	cross-product in $\mathbb{R}^3$
$\hbar$	Planck's constant
$\mathbb{M}$	four-dimensional Minkowski space
η	Minkowski metric
$\mathcal{P}$	Poincaré group
$\mathcal{P}_+^\uparrow$	proper orthochronous Poincaré group
$G_1 \ltimes G_2$	semidirect product (action of G_1 on G_2 stated separately)
$\mathcal{L}$	Lorentz group
$\mathcal{L}_+$	proper Lorentz group
$\mathcal{L}_+^\uparrow$	proper orthochronous Lorentz group
$\underset{\sim}{x}$	contravariant matrix representation of $x \in \mathbb{M}$
$\widetilde{x}$	covariant matrix representation of $x \in \mathbb{M}$
$\mathbf{1}_V$	unit operator on the vector space V (V usually omitted)
δ_{ij}	Kronecker delta
ϵ_{ijk}	three-dimensional Levi-Civita symbol
$\vert\vert\cdot\vert\vert$	Hilbert space norm
$B(\psi,\vec{n})$	Lorentz boost with rapidity ψ in the direction $\vec{n}$
$D(\varphi,\vec{n})$	spatial rotation about angle φ around the $\vec{n}$-axis

$B(\mathcal{H})$	bounded operators on $\mathcal{H}$
O_m	Lorentz-orbit with mass $m \geq 0$ and positive energy
H_m^+	upper mass shell with mass m ($= O_m$ for $m > 0$)
∂V^+	boundary of the forward lightcone V^+ ($= O_0$)
s	particle spin
κ	Pauli-Lubanski parameter
$\widetilde{\mathrm{d}p}$	$\mathcal{L}$-invariant measure
$B_p = \Lambda(R_p)$	Wigner boost
$\mathcal{H}_q$	little Hilbert space for reference momentum q
$\int_0^\oplus \widetilde{\mathrm{d}p}$	direct integral of Hilbert spaces
$L^2(O_m)$	Hilbert space of square-integrable functions on O_m
U_1	representation on the one-particle space
$\mathrm{stab}\, q$	stabilizer of momentum q under the action of $\mathcal{L}_+^\uparrow$
$\otimes_\mathrm{S}$	symmetrized tensor product
$\lvert\uparrow\rangle$, $\lvert\downarrow\rangle$	standard basis of $\mathbb{C}^2$
$\mathrm{Sym}(V^{\otimes n})$	symmetrized n-fold tensor power of the vector space V
$\mathcal{F}$	Fock space
$\lvert 0\rangle$	Fock vacuum state
$a^\dagger(p)$	creation operator with sharp momentum p
$a(p)$	annihilation operator with sharp momentum p
$[\cdot,\cdot]$	commutator
$\{\cdot,\cdot\}$	anticommutator
$[\cdot,\cdot]_s$	s-dependent commutator
$\bullet$	contraction over the basis of $\mathcal{H}_{(m,\vec{0})}$
$\circ$	contraction over the basis of $\mathcal{H}_{(1/2,0,0,1/2)}$
$\Phi(x,v)$	massive field with localization point x and spin vector v
$\Phi(x,e)$	massless string-localized field with endpoint x and string direction vector e
$B(x,\tilde{x})$	massless current field with localization points x and $\tilde{x}$
$W(x-x',\ldots)$, $M(p,\ldots)$	two-point function and its Fourier transform
$\mathcal{T} = \mathbb{M} - \mathrm{i}V^+$	tube in Minkowski space $\mathbb{M}$
W	standard wedge
W'	causal complement of the standard wedge
$\Lambda(t)$	one-parameter group of Lorentz transformations preserving the wedge W
j_0	reflection across the edge of the wedge W
H	Lorentz hyperboloid of spacelike direction vectors
SC, FC	simply and fully contracted terms in the Wick theorem

1. Introduction

Quantum field theory describes the physics of quantum particles consistently with Einsteins theory of Special Relativity, which means that the quantities involved are subject to formulas that determine how these quantities are related between two observers in Minkowski space, who are rotated or boosted w.r.t. each other, i.e. relativistic transformation laws. These are usually identified by stating a particle's mass and spin, for example m_e and $1/2$ for the electron, $m_{\mathrm{W/Z}}$ and 1 for the weak interaction particles etc., or it's helicity, if the particle is massless, for example $n = \pm 1$ for the photon. Conversely, the set of possible types of particles, which are subject to these transformations, is assumed to be determined by the irreducible representations of the underlying symmetries of Minkowski space, i.e. it's isometry group, the Poincaré group.

The positive energy representations of the Poincaré group, analyzed in the framework of Wigner's particle classification, fall into two classes, namely the massive and the massless ones. A vector in the representation Hilbert space can be considered as a multi-component wavefunction in momentum-space, which transforms under the so-called little group, introduced in [**?**, p. 193], dependening on the choice of an irreducible representation. One can thus reduce the construction of representations to this group.

For massive particles, the little group is the twofold covering group of the three-dimensional rotations whose irreducible representations are classified by a half-integral number which can be identified as the particle's spin which is the given by the value of the corresponding Casimir operator. Massless wavefunctions, on the other hand, transform under the covering group of rigid motions in the two-dimensional Euclidean plane, whose representation theory is a bit more delicate. In this case, there is a Casimir operator corresponding to the modulus of the momenta for shifts in the plane. The whole Wigner construction presented here is based on the script [Fre00, Chapter 3]. A more application-oriented introduction to the representation theory of the Lorentz group may be found in [Sre07, p. 205].

Usually, the aforementioned momenta in the Euclidean plane are taken to be vanishing, in which case the shift subgroup is represented trivially. These are the so-called helicity representations, which are distinguished by a photon helicity. The present work deals with the case where the little group is faithfully represented, hence incorporating the shifts in the Euclidean plane, which make it necessary to deal with the representation theory of a noncompact group. At various places, statements regarding the usual exclusion of these representations can be found, for example "These representations have so far found no application in physics" [**?**, p. 31], "Massless particles are not observed to have any continuous degree of freedom [...]" [**?**, p. 72] and "[...] we exclude zero mass infinite spin representations as not occuring in Nature." [SW64, p. 30]. Of course, a deeper understanding of the corresponding fields might lead to new ideas how these representations could occur in nature. It is also argued in [MSY05, p. 43], that the condition of positive energy puts them on a better theoretical footing than tachyons.

On the other hand, while in the massive case the resulting fields are pointlike localized, there is a structural theorem by Yngvason stating the impossibility of such a field in the massless case. [Yng69] Still, it is possible to weaken the localization properties, allowing for the construction of a field which is localized along a semiinfinite string extending from a given endpoint into a spacelike direction. This means that for two such strings, which are pointwise spacelike separated, the corresponding fields will commute. The construction is due to Mund, Schroer and Yngvason [MSY05]. There are functions entering the definition of the field operators which trade the Wigner transformation in the little group for a Lorentz transformation of the spacelike direction vector, which are called intertwiners. The causal commutativity is then shown using the analyticity properties of the intertwiners in [MSY04].

For massive fields, it is easily possible to construct a so-called current, i.e. an operator quadratic in the creation and annihilation operators which fulfils the usual spacelike commutation relations. A different approach is proposed in [MSY05] and [Sch08] for the massless stringlike fields. Instead of building the current in terms of an algebraic combination of fields, the intertwiner can be modified instead in order to obtain a new function which is suitable for a quadratic operator and additionally does not depend on spacelike directions anymore. The question discussed here is if these operators are in fact relatively local to the stringlike fields, i.e. whether a current commutes with a field if its localization point is spacelike to the string. The analogous problem for the commutator between two current fields is also discussed.

Operators which satisfy the locality requirement on a formal level, which is presented in this thesis, have in fact be found, but it is still unclear whether the analytic continuations can be done, which are necessary for the locality of the currents among each other as well as relatively to the string fields. Thus the question of existence of such a kind of current operator is answered up to the existence problem of certain analytic functions. For the latter case, it has already been pointed out in [Sch08] that the natural appearance of non-polynomial functions in the commutator can be considered as an obstacle to the construction of local currents. The warnings given here and in the later chapters, regarding the necessary analytic continuations, reflect this phenomenon.

The structure of the thesis is as follows: First of all the basic procedure of Wigner's particle classification theory are recaptured and the construction of a point-local covariant quantum field in a theory of massive particles with arbitrary spin is shown. The discussion will then continue to the massless case by introducing the mentioned string-localized fields. An analogous discussion is then given for the desired currents, which leads to a more detailed view on the aforementioned analytic questions. Also, some ideas are presented about how one could proceed to tame the behaviour of these operators.

2. Wigner analysis

> The simplicities of natural laws arise through the complexities of the language we use for their expression.
>
> E. P. Wigner

The formalism introduced in this chapter allows for a systematic characterization of the irreducible representations of the Poincaré group. In the first step, some motivational examples are given which illustrate the role group representations play when studying a quantum mechanical system, which features a certain symmetry. The discussion proceeds to a description of the special case of the Poincaré group, describing the most important symmetry of quantum field theory on Minkowski spacetime. Wigner's framework then leads to a very concrete description of the one-particle Hilbert space in terms of wavefunctions. The chapter is concluded with an outlook regarding the steps that follow this description once a choice of the representation has been made, which are put up to that point in order to achieve the desired generality.

2.1. Motivation

In this section the notion of a symmetry group in quantum mechanics is introduced, some examples for the possible obstacles to the construction of group representations on the Hilbert space of the system are given and it is indicated in how far these apply to the realm of relativistic quantum physics.

2.1.1. Symmetries of a system in quantum physics

A quantum physical system is described by a Hilbert space $\mathcal{H}$, whose inner product will be denoted by $\langle\cdot|\cdot\rangle$. Any state Ψ of the system is given by a ray

$$\Psi := \mathbb{C}\,|\psi\rangle \in \mathbb{P}(\mathcal{H}) \tag{2.1}$$

and hence determines the vector $|\psi\rangle$ up to a nonzero prefactor. This is sufficient to calculate the transition probability

$$P(\Psi \to \Phi) := \frac{\langle\psi|\phi\rangle\,\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle\,\langle\psi|\psi\rangle}$$

to another ray $\Phi = \mathbb{C}\,|\phi\rangle \in \mathbb{P}(\mathcal{H})$.[1] A group G is called a symmetry group of the system, if its action on $\mathbb{P}(\mathcal{H})$,

$$\begin{array}{rcl} \hat{U}\ :\ G & \to & \mathrm{Aut}(\mathbb{P}(\mathcal{H})) \\ g & \mapsto & \hat{U}(g)\ , \end{array}$$

fulfilling the ray representation property

$$\hat{U}(g)\hat{U}(h) = \hat{U}(gh)\ \forall\ g, h \in G\ ,$$

preserves the transition probabilities, that is

$$P(\hat{U}(g)\Psi \to \hat{U}(g)\Phi) = P(\Psi \to \Phi)\ \forall\ g \in G,\ \ \Psi, \Phi \in \mathbb{P}(\mathcal{H})\ .$$

Wigner's Theorem [**?**] states that in this case there is a unitary or antiunitary operator $U'(g)$ for each $g \in G$, such that

$$\hat{U}(g)\Psi = \hat{U}(g)\mathbb{C}\,|\psi\rangle = \mathbb{C}U'(g)\,|\psi\rangle\ \ \forall\ g \in G,\ \ |\psi\rangle \in \mathcal{H}$$

and that this operator is unique up to a phase factor. Because

$$\mathbb{C}U'(g)U'(h) = \hat{U}(g)\hat{U}(h)\mathbb{C} = \hat{U}(gh)\mathbb{C} = \mathbb{C}U'(gh)\ \forall\ g, h \in G$$

and both $U'(g)U'(h)$ and $U'(gh)$ are unitary or antiunitary, it follows that

$$U'(g)U'(h) = \mathrm{e}^{\mathrm{i}\omega(g,h)}U'(gh)\ \forall\ g, h \in G$$

with $\mathrm{e}^{\mathrm{i}\omega(g,h)} \in U(1)$ which in general prevents the map $g \mapsto U'(g)$ from having the representation property without an extra phase factor. In a finite-dimensional representation with $n := \mathrm{dim}\mathcal{H} < \infty$, it is possible to try the redefinition $U(g) := U'(g)/(\mathrm{det}U'(g))^{1/n}$ to obtain the ordinary representation property

$$U(g)U(h) = U(gh) \tag{2.2}$$

but in general, the n-th root on $U(1)$ is not uniquely defined. However, if G is simply connected, there is a unique definition of $g^{1/n}$ on the level of the group elements themselves and hence of $\mathrm{det}U(g)^{1/n}$. In a general (not necessarily finite-dimensional) representation, proceeding to the universal covering group of G^c is an option if Bargmann's lifting criterion[2] holds, which is the case for $\mathrm{SO}(3)^c = \mathrm{SU}(2)$ and ${\mathcal{L}_+^\uparrow}^c = \mathrm{SL}(2, \mathbb{C})$, the groups that are of special interest in relativistic quantum physics.[3]

[1] Note that the given expression is well-defined, that is, the result is independent of the choice of $|\phi\rangle\,, |\psi\rangle$ within each ray.

[2] This is a topological statement about G^c beyond the property of simple connectedness. In the case at hand it states that the second cohomology group $H^2(G^c, U(1))$ vanishes. [**?**]

[3] $\mathcal{L}_+^\uparrow$, the proper orthochronous Lorentz group, is the identity component of $\mathrm{SO}(1,3)$, which itself has two disconnected components.

2.1.2. An example from quantum mechanics

Now an illustration is given of how an additional phasefactor can arise if G is not simply connected. In nonrelativistic quantum mechanics, the space[4] $\mathcal{H} := L^2(S^2)$ is the representation space for SO(3), the Lie group of 3d-rotations. The action of $R(\varphi,\vec{n}) \in \mathrm{SO}(3)$[5] on an element $\psi \in \mathcal{H}$ is then given by a linear operator $U(R)$:

$$\langle\vec{x}|\, U(R)\, |\psi\rangle = \langle\vec{x}|\, \mathrm{e}^{\mathrm{i}\varphi\vec{L}\cdot\vec{n}/\hbar}\, |\psi\rangle = \langle R^{-1}\vec{x}|\psi\rangle$$

with the angular momentum operator[6] $\vec{L} = \frac{\hbar}{\mathrm{i}}\vec{X} \times \vec{\nabla}$. Because

$$R(\pi, -\vec{n}) = R(\pi, \vec{n})$$

the group SO(3) can be parametrized by the points inside a ball of radius π, where each extremal point has to be identified with its antipodal point. Connecting such a pair of points with a straight line gives an example of a closed path in the group which cannot continuously be deformed to a point. This means that SO(3) is not simply connected. Correspondingly, the action on an irreducible subrepresentation space

$$\begin{aligned} \mathcal{H} = \mathbb{C}^{2s+1} &= \mathrm{span}\{|m\rangle \mid m = -s, \ldots, s\} \\ U(\varphi, \vec{e}_3)\,|m\rangle &= \mathrm{e}^{\mathrm{i}m\varphi}\,|m\rangle \end{aligned} \tag{2.3}$$

where the given basis diagonalizes L_3, that is $L_3\,|m\rangle = m\hbar\,|m\rangle$, and $s \in \mathbb{N}/2$, does not in general fulfil the representation property. For example, the above closed path yields

$$U(R(\pi,\vec{e}_3))U(R(\pi,\vec{e}_3))\,|m\rangle \overset{(2.3)}{=} \mathrm{e}^{2\mathrm{i}m\pi}\,|m\rangle \overset{(2.3)}{=} \mathrm{e}^{2\mathrm{i}m\pi}U(R(0,\vec{e}_3))\,|m\rangle \ ,$$

while $R(\pi,\vec{e}_3)^2 = R(0,\vec{e}_3)$. Hence a representation for SO(3) is obtained only if $s \in \mathbb{N}$.
The condition $s \in \mathbb{N}/2$ is a consequence of the commutation relations $[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_k$ which only concern the corresponding Lie-Algebra $\mathfrak{so}(3)$. The problem of $U(R)$ not necessarily fulfilling the representation property appears in the form that the exponential map does not return the original Lie group SO(3).
Since SO(3) is a subgroup of SO(1,3), fixing this problem will be discussed explicitly for the covering group of the latter only. For SO(3) itself, proceeding to SU(2) will solve the problem of nontrivial closed paths due to the group not being simply connected because as a set[7]

$$\mathrm{SU}(2) = \{a\mathbf{1} + \mathrm{i}\vec{a}\cdot\vec{\sigma} \mid a^2 + \vec{a}^2 = 1\} \tag{2.4}$$

which, considering the condition for the coefficients $a, \vec{a}$, is topologically equivalent to S^3 or the manifold of unit quaternions, i.e. a simply connected space. It is also shown how the ambiguity of the antipodal points in SO(3) is resolved in SU(2).

[4] S^2 stands for the unit sphere embedded in $\mathbb{R}^3$, i.e. $S^2 = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x}^2 = 1\}$
[5] The element $R(\varphi,\vec{n})$ denotes a rotation about the angle φ around the $\vec{n}$-axis.
[6] The operator is given here in Cartesian coordinates on $\mathbb{R}^3$, but it can be restricted to an operator on functions on S^2 because since $\vec{X}$ is normal to the surface, no derivatives in this direction will appear.
[7] The coefficients $a, \vec{a}$ will be constructed explicitly later on.

2.1.3. Relativistic particles

Since quantum field theory is a method to formulate quantum physics on Minkowski spacetime $\mathbb{M} = (\mathbb{R}^4, \eta)$, the symmetry group G is taken to be the proper orthochronous Poincaré group

$$\mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{M}$$

where the semidirect multiplication law is given by the action of $\mathcal{L}_+^\uparrow$ on $\mathbb{M}$ by matrix multiplication

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$$

As indicated before, the relation of $\mathcal{P}$ to its universal cover $\mathcal{P}^c$ is now needed, before its irreducible representations can be constructed.[8]

2.2. Covering of the Poincaré group

Here, the relation between the Poincaré group, the isometry group of $\mathbb{M} = (\mathbb{R}^4, \eta)$, and its universal covering group as well as several formulas for the action on four-vectors and Minkowski products are provided. In the remainder of this section, the appearance of typical elements like Lorentz boosts and spatial rotations in the covering group is discussed.

2.2.1. Definition and properties of the covering map

Any four-vector $x = (x^0, \vec{x}) \in \mathbb{M}$ where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{2.5}$$

can be uniquely represented as a 2×2 complex matrix via one of the following maps

$$\begin{aligned} \underset{\sim}{\ } : \mathbb{M} &\to M_{2\times 2}(\mathbb{C}) \\ x &\mapsto \underset{\sim}{x} := x^0\mathbf{1} + \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} \\ \widetilde{\ } : \mathbb{M} &\to M_{2\times 2}(\mathbb{C}) \\ x &\mapsto \widetilde{x} := x^0\mathbf{1} - \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x^0 - x^3 & -x^1 + \mathrm{i}x^2 \\ -x^1 - \mathrm{i}x^2 & x^0 + x^3 \end{pmatrix} \end{aligned} \tag{2.6}$$

i.e. a linear combination of the unit matrix $\mathbf{1}$ and the Pauli matrices σ_i

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{2.7}$$

[8] The following construction is based on [Fre00, chapter 3].

fulfilling the useful property[9]

$$\sigma_i \sigma_j = \delta_{ij}\mathbf{1} + \mathrm{i}\epsilon_{ijk}\sigma_k \ , \tag{2.8}$$

implying that in products of linear combinations of $\mathbf{1}$ and $\vec{\sigma}$, all mixed terms are spanned by $\vec{\sigma}$ and all quadratic terms are multiples of $\mathbf{1}$. The Minkowski product η can be recovered from the matrix representations $\underset{\sim}{\cdot}$ and $\widetilde{\cdot}$, simply using $\mathrm{Tr}\mathbf{1} = 2$ and $\mathrm{Tr}\vec{\sigma} = \vec{0}$:

$$\begin{aligned} \underset{\sim}{x}\widetilde{y} &\overset{(2.6)}{=} (x^0\mathbf{1} + \vec{x}\cdot\vec{\sigma})(y^0\mathbf{1} - \vec{y}\cdot\vec{\sigma}) \\ &\overset{(2.8)}{=} (x^0y^0 - \vec{x}\cdot\vec{y})\mathbf{1} + (y^0\vec{x} - x^0\vec{y} - \mathrm{i}(\vec{x}\times\vec{y}))\cdot\vec{\sigma} \end{aligned} \tag{2.9}$$

$$\Rightarrow x^\mu y_\mu = \eta_{\mu\nu}x^\mu y^\nu \overset{(2.5)}{=} x^0y^0 - \vec{x}\cdot\vec{y} \overset{(2.9)}{=} \frac{1}{2}\mathrm{Tr}\underset{\sim}{x}\widetilde{y} \tag{2.10}$$

The covering homomorphism

$$\begin{aligned} \Lambda \ : \ \mathrm{SL}(2,\mathbb{C}) &\rightarrow \mathcal{L}_+^\uparrow \\ A &\mapsto \Lambda(A) \end{aligned}$$

is defined implicitly[10] by

$$\underset{\sim}{(\Lambda(A)x)} := A\underset{\sim}{x}A^\dagger \ . \tag{2.11}$$

Now check the homomorphism property:

$$\begin{aligned} \underset{\sim}{(\Lambda(A)\Lambda(B)x)} = A\underset{\sim}{(\Lambda(B)x)}A^\dagger = AB\underset{\sim}{x}B^\dagger A^\dagger = AB\underset{\sim}{x}(AB)^\dagger = \underset{\sim}{(\Lambda(AB)x)} \ \forall\, x \in \mathbb{M} \\ \Rightarrow \Lambda(A)\Lambda(B) = \Lambda(AB) \end{aligned} \tag{2.12}$$

The matrix representation $\underset{\sim}{x}$ can be used to compute the Minkowski square $x^2 := x^\mu x_\mu$ in the following way:

$$\det \underset{\sim}{x} \overset{(2.6)}{=} \det \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} = \eta_{\mu\nu}x^\mu x^\nu = x^2 \tag{2.13}$$

With this formula it can be seen that 2.11 indeed defines a Lorentz transformation:

$$(\Lambda(A)x)^2 \overset{(2.13)}{=} \det \underset{\sim}{(\Lambda(A)x)} \overset{(2.11)}{=} \det A\underset{\sim}{x}A^\dagger = \det \underset{\sim}{x} \overset{(2.13)}{=} x^2 \tag{2.14}$$

This proves at least $\Lambda(A) \in \mathcal{L}$.[11] The corresponding formula for an arbitrary Minkowski product can be obtained by a standard polarization argument:

$$(\Lambda(A)x)(\Lambda(A)y) = \frac{1}{4}[(\Lambda(A)(x+y))^2 - (\Lambda(A)(x-y))^2] \overset{(2.14)}{=} \frac{1}{4}[(x+y)^2 - (x-y)^2] = xy \tag{2.15}$$

[9] This property is shown by first checking that each σ_i squares to $\mathbf{1}$ and by applying this rule repeatedly to the equation $\sigma_1\sigma_2 = \mathrm{i}\sigma_3$ and its hermitean conjugate, $\sigma_2\sigma_1 = -\mathrm{i}\sigma_3$.

[10] The components of $\Lambda(A)$ can be recovered by the formula $\Lambda(A)_{\mu\nu} = e_\mu \cdot \Lambda(A)e_\nu \overset{(2.10)}{=} \frac{1}{2}\mathrm{Tr}\widetilde{e_\mu}\underset{\sim}{(\Lambda(A)e_\nu)} \overset{(2.11)}{=} \frac{1}{2}\mathrm{Tr}\widetilde{e_\mu}A\underset{\sim}{e_\nu}A^\dagger$, denoting by e_μ the μ-th basis vector of $\mathbb{M}$.

[11] It will become clear that in fact $\Lambda(A) \in \mathcal{L}_+^\uparrow$.

The formula which corresponds to 2.11, if a four-vector $y \in \mathbb{M}$ is to be transformed in the representation $\widetilde{y}$, can be derived using

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\underset{\sim}{x}\widetilde{y} &\overset{(2.10)}{=} xy \overset{(2.15)}{=} (\Lambda(A)x)(\Lambda(A)y) \overset{(2.10)}{=} \frac{1}{2}\mathrm{Tr}\underset{\sim}{(\Lambda(A)x)}\widetilde{(\Lambda(A)y)} \\ &\overset{(2.11)}{=} \frac{1}{2}\mathrm{Tr}A\underset{\sim}{x}A^\dagger\widetilde{(\Lambda(A)y)} = \frac{1}{2}\mathrm{Tr}\underset{\sim}{x}A^\dagger\widetilde{(\Lambda(A)y)}A\ .\end{aligned}$$

In the last step cyclicity of the trace has been used. Because $x \in \mathbb{M}$ is arbitrary,

$$\widetilde{y} = A^\dagger\widetilde{(\Lambda(A)y)}A \Rightarrow \widetilde{(\Lambda(A)y)} = (A^\dagger)^{-1}\widetilde{y}A^{-1}\ . \tag{2.16}$$

A similar technique can be applied to derive the transformation law for covariant components:

$$\frac{1}{2}\mathrm{Tr}(\widetilde{(p\Lambda(A))}\underset{\sim}{x}) \overset{(2.10)}{=} p\Lambda(A)x \overset{(2.10)}{=} \frac{1}{2}\mathrm{Tr}(\widetilde{p}\underset{\sim}{(\Lambda(A)x)}) \overset{(2.11)}{=} \frac{1}{2}\mathrm{Tr}(\widetilde{p}A\underset{\sim}{x}A^\dagger) = \frac{1}{2}\mathrm{Tr}(A^\dagger\widetilde{p}A\underset{\sim}{x})$$

Again, cyclicity of the trace has been used and arbitrariness of $x \in \mathbb{M}$ implies

$$\widetilde{(p\Lambda(A))} = A^\dagger\widetilde{p}A \tag{2.17}$$

If $\underset{\sim}{(p\Lambda(A))}\widetilde{x}$ is used instead of $\widetilde{(p\Lambda(A))}\underset{\sim}{x}$, 2.16 instead of 2.11 is inserted, the result is

$$\underset{\sim}{(p\Lambda(A))} = A^{-1}\underset{\sim}{p}(A^\dagger)^{-1}\ .$$

So far, it is still unclear that $\Lambda(A)$, as defined in 2.11, only produces Lorentz transformations which can be reached continuously from the identity transformation. This would mean $\Lambda(A) \in \mathcal{L}_+^\uparrow$. However, it is already known from 2.4 that $A \in \mathrm{SU}(2)$ can be continuously moved to the identity matrix $\mathbf{1} \in \mathrm{SL}(2,\mathbb{C})$. But the components of $\Lambda(A)$ depend continuously on the components of A, therefore a corresponding continuous path from $\Lambda(A)$ to $\mathbf{1} \in \mathcal{L}_+^\uparrow$ is obtained which proves the claim.
Still, this is not a very satisfactory description of the relation between $\mathrm{SL}(2,\mathbb{C})$ and $\mathcal{L}_+^\uparrow$. The form of the Lorentz transformation $\Lambda(A)$ corresponding to A can be split into a pure rotation and a pure Lorentz boost. This decomposition on the level of the covering group is discussed in the next section.

2.2.2. Explicit form of typical elements of the covering group

For a given $A \in \mathrm{SL}(2,\mathbb{C})$, the so-called *polar decomposition* will be derived. Namely, there are matrices R (with $R^\dagger = R$ and $\det R = 1$) and U (with $U^\dagger = U^{-1}$ and $\det U = 1$) such that $A = RU$. It is then shown that the matrices which occur in this decomposition the covering matrices of Lorentz boosts $\Lambda(A)$ and spatial rotations $\Lambda(U)$, respectively. For an eigenvector $|\lambda\rangle$ of A (that is, $A\,|\lambda\rangle = \lambda\,|\lambda\rangle$), $|\lambda|^2 \in \mathbb{R}^+$ is the eigenvalue of $A^\dagger A$ since $A^\dagger A\,|\lambda\rangle = \lambda A^\dagger\,|\lambda\rangle = \overline{\lambda}\lambda\,|\lambda\rangle$. Thus the hermitean matrix $R := \sqrt{A^\dagger A}$ is defined because $\sqrt{\cdot}$

is defined on $\mathbb{R}^+$. Since all $\lambda \neq 0$ (otherwise $\det A = 0$ would follow) also A^{-1} and R^{-1} exist. Define $U := AR^{-1}$. This is unitary because $UU^\dagger = AR^{-1}(AR^{-1})^\dagger = A(R^{-1})^2 A^\dagger = A(A^\dagger A)^{-1} A^\dagger = AA^{-1}{A^\dagger}^{-1} A^\dagger = \mathbf{1}$. It follows: $UU^\dagger UU^\dagger = UU^\dagger \Rightarrow U^\dagger U = 1$.
Now the polar decomposition $A = RU$ has been established. Calculating the determinant of R in a basis where it is diagonal, it can be seen that $\det R = \sqrt{|\lambda|^2 \cdots} = \lambda \cdots = \det A = 1$. It follows that also $\det U = 1$. Because $|\det U|^2 = \overline{\det U} \det U = \det U^\dagger \det U = \det U^\dagger U = 1$ and $\det R \in \mathbb{R}^+$, it can be concluded that since $\det R \det U = \det RU = \det A = 1$, $\det R = \det U = 1$. Therefore, both the hermitean part R and the unitary part U can be written as exponential maps of linear combinations of the Pauli matrices.

- The 2×2-matrix R can be decomposed as[12]

$$R = a^0 \mathbf{1} + \vec{a} \cdot \vec{\sigma} \tag{2.18}$$

Since R is hermitean, as well as $\mathbf{1}$ and $\vec{\sigma}$, it follows that

$$\overline{a^0}\mathbf{1} + \overline{\vec{a}} \cdot \vec{\sigma} = R^\dagger = R = a^0 \mathbf{1} + \vec{a} \cdot \vec{\sigma},$$

hence, the coefficients $a^0, \vec{a}$ are real. Using 2.6 and 2.13, the condition for the determinant now reads $\det R = (a^0)^2 - \vec{a}^2 = 1$. R can be parametrized by $a^0 = \cosh\psi$ and $\vec{a} = \vec{n}\sinh\psi$. The constraints on the parameters are $\psi \in \mathbb{R}$ and $\vec{n} \in S^2$. The parametrization can be made unique by imposing the additional restrictions $\psi \geq 0$ and $\vec{n}$ arbitrary but fixed for $\psi = 0$. They account for the symmetry $R(-\psi, \vec{n}) = R(\psi, -\vec{n})$. The result is

$$R(\psi, \vec{n}) \overset{(2.18)}{=} \cosh\psi \mathbf{1} + \sinh\psi\, \vec{n} \cdot \vec{\sigma} = \mathrm{e}^{\psi \vec{n} \cdot \vec{\sigma}} \tag{2.19}$$

The last equality can be shown using $(\vec{n} \cdot \vec{\sigma})^2 = \mathbf{1}$. By 2.8, it implies the following formula for the product of two matrices given by the same $\vec{n}$:

$$R(\psi_1, \vec{n}) R(\psi_2, \vec{n}) \overset{(2.19)}{=} R(\psi_1 + \psi_2, \vec{n}) \tag{2.20}$$

- On the other hand, bringing U into an analogous form

$$U = \mathrm{e}^{\mathrm{i}\alpha}(b^0 \mathbf{1} + \vec{b} \cdot \vec{\sigma}) , \tag{2.21}$$

(with $b^0 \in \mathbb{R}$) it can be seen that because U is unitary, that is[13]

$$U^\dagger U = (b^0 \mathbf{1} + \overline{\vec{b}} \cdot \vec{\sigma})(b^0 \mathbf{1} + \vec{b} \cdot \vec{\sigma}) = (b^0)^2 \mathbf{1} + (b^0 \overline{\vec{b}} + b^0 \vec{b}) \cdot \vec{\sigma} + ||\vec{b}||^2 \mathbf{1} + \mathrm{i}(\overline{\vec{b}} \times \vec{b}) \cdot \vec{\sigma} \overset{!}{=} \mathbf{1}$$

the equations

$$\begin{aligned} (b^0)^2 + ||\vec{b}||^2 &= 1 \\ 2b^0 \Re(\vec{b}) + \mathrm{i}(\overline{\vec{b}} \times \vec{b}) &= \vec{0} \end{aligned}$$

[12] The Pauli matrices from 2.7 are used in the form $\vec{e}_i \cdot \vec{\sigma} = \sigma_i$.
[13] $|| \cdot ||^2$ denotes the standard norm in $\mathbb{C}^3$.

follow. Multiplying the second one by $\vec{b}$ as well as $\bar{\vec{b}}$ yields $\Re(\vec{b})\cdot\vec{b} = \Re(\vec{b})\cdot\bar{\vec{b}} = 0$ and thus $2\Re(\vec{b})^2 = \Re(\vec{b})\cdot(\vec{b}+\bar{\vec{b}}) = 0$. Therefore, $\vec{b}$ is purely imaginary. Thus the solutions of the first equation can be parametrized by $b^0 = \cos\varphi$, $\vec{b} = \mathrm{i}\vec{n}\sin\varphi$. The constraints on the parameters are $\varphi\in\mathbb{R}$ and $\vec{n}\in S^2$.[14] Using 2.6 and 2.13, the condition for the determinant now reads $1 = \det U = \mathrm{e}^{2\mathrm{i}\alpha}((b^0)^2 - \vec{b}^2) = \mathrm{e}^{2\mathrm{i}\alpha}$ which implies $\alpha\in\pi\mathbb{Z}$. Consequently,

$$U(\varphi,\vec{n}) \overset{(2.21)}{=} \cos\varphi\mathbf{1} + \sin\varphi(\mathrm{i}\vec{n}\cdot\vec{\sigma}) = \mathrm{e}^{\mathrm{i}\varphi\vec{n}\cdot\vec{\sigma}}\ . \tag{2.22}$$

The exponential form can be shown using $(\mathrm{i}\vec{n}\cdot\vec{\sigma})^2 = -\mathbf{1}$. It implies

$$U(\varphi_1,\vec{n})U(\varphi_2,\vec{n}) = U(\varphi_1+\varphi_2,\vec{n})\ , \tag{2.23}$$

analogously to 2.19

- Now the Lorentz transformations corresponding to the hermitan part $R(\psi,\vec{n})$ are derived. Using

$$\{\vec{m}\cdot\vec{\sigma},\vec{n}\cdot\vec{\sigma}\} \overset{(2.8)}{=} 2\vec{m}\cdot\vec{n}\mathbf{1} \tag{2.24}$$

and the identities

$$2\cosh\psi\sinh\psi = \sinh(2\psi) \tag{2.25}$$

$$2\sinh^2\psi = \cosh(2\psi) - 1 \tag{2.26}$$

for the hyperbolic functions, it follows:

$$\begin{aligned}
& (\Lambda(R(\psi,\vec{n}))x)_{\sim} \\
\overset{(2.11)}{=}\ & R(\psi,\vec{n})\underset{\sim}{x}R(\psi,\vec{n})^\dagger \overset{(2.6)}{=} R(\psi,\vec{n})(x^0\mathbf{1}+\vec{x}\cdot\vec{\sigma})R(\psi,\vec{n})^\dagger \\
\overset{(2.19)}{=}\ & R(\psi,\vec{n})(x^0\mathbf{1}+\vec{x}\cdot\vec{\sigma})(\cosh\psi\mathbf{1}+\sinh\psi\vec{n}\cdot\vec{\sigma}) \\
\overset{(2.24)}{=}\ & R(\psi,\vec{n})R(\psi,\vec{n})x^0\mathbf{1} + R(\psi,\vec{n})R(-\psi,\vec{n})\vec{x}\cdot\vec{\sigma} + 2R(\psi,\vec{n})\sinh\psi\vec{n}\cdot\vec{x}\mathbf{1} \\
\overset{(2.20)}{=}\ & R(2\psi,\vec{n})x^0\mathbf{1} + \vec{x}\cdot\vec{\sigma} + 2R(\psi,\vec{n})\sinh\psi\vec{n}\cdot\vec{x}\mathbf{1} \\
\overset{(2.25)(2.26)}{=}\ & R(2\psi)x^0\mathbf{1} + \vec{x}\cdot\vec{\sigma} + \sinh(2\psi)\vec{n}\cdot\vec{x}\mathbf{1} + (\cosh(2\psi)-1)(\vec{n}\cdot\vec{x})(\vec{n}\cdot\vec{\sigma}) \\
\overset{(2.19)}{=}\ & (\cosh(2\psi)x^0 + \sinh(2\psi)\vec{n}\cdot\vec{x})\mathbf{1} \\
& +(\cosh(2\psi)\vec{n}(\vec{n}\cdot\vec{x}) + \sinh(2\psi)\vec{n}x^0 + \vec{x} - \vec{n}(\vec{n}\cdot\vec{x}))\cdot\vec{\sigma} \\
\overset{(2.6)}{=}\ & \left[\begin{pmatrix} \cosh(2\psi) & \sinh(2\psi)\vec{n}\cdot \\ \sinh(2\psi)\vec{n} & \cosh(2\psi)\vec{n}\otimes\vec{n}+\mathbf{1}-\vec{n}\otimes\vec{n}\end{pmatrix}\begin{pmatrix} x^0 \\ \vec{x}\end{pmatrix}\right]_{\sim} \\
=:\ & (B(2\psi,\vec{n})x)_{\sim}
\end{aligned}$$

which is a Lorentz boost $B(2\psi,\vec{n})$ with rapidity 2ψ in the direction $\vec{n}$.

[14] The parametrization can be made unique by imposing the additional restrictions $\varphi\in[0,2\pi)$, $\vec{e}_z\cdot\vec{n}\geq 0$, $\vec{e}_y\cdot\vec{n}\geq 0$ for $\vec{e}_z\cdot\vec{n} = 0$ and $\vec{e}_x\cdot\vec{n} = 1 \geq 0$ if also $\vec{e}_y\cdot\vec{n} = 0$. They account for the symmetries $U(\varphi+2\pi,\vec{n}) = R(\varphi,\vec{n})$ and $R(-\varphi,\vec{n}) = R(\varphi,-\vec{n})$.

- The Lorentz transformations for the unitary part U can be calculated using

$$[\vec{m}\cdot\vec{\sigma},\vec{n}\cdot\vec{\sigma}] = 2\mathrm{i}(\vec{m}\times\vec{n})\cdot\vec{\sigma} \tag{2.27}$$

  and the trigonometric identities

$$\begin{aligned} 2\cos\varphi\sin\varphi &= \sin(2\varphi) && (2.28)\\ 2\sin^2\varphi &= 1-\cos(2\varphi)\ . && (2.29)\end{aligned}$$

  The result is:

$$\begin{aligned}
& (\Lambda(U(\varphi,\vec{n})))_{\sim} \\
\overset{(2.11)}{=}\ & U(\varphi,\vec{n})\underset{\sim}{x}U(\varphi,\vec{n})^{\dagger} \overset{(2.6)}{=} U(\varphi,\vec{n})(x^0\mathbf{1}+\vec{x}\cdot\vec{\sigma})U(\varphi,\vec{n})^{\dagger} \\
\overset{(2.22)}{=}\ & U(\varphi)(x^0\mathbf{1}+\vec{x}\cdot\vec{\sigma})(\cos\varphi\mathbf{1}-\sin\varphi\mathrm{i}\vec{n}\cdot\vec{\sigma}) \\
\overset{(2.27)}{=}\ & U(\varphi)U(-\varphi)x^0\mathbf{1}+U(\varphi)U(-\varphi)\vec{x}\cdot\vec{\sigma}-2U(\varphi)\sin\varphi(\vec{n}\times\vec{x})\cdot\vec{\sigma} \\
\overset{(2.23)}{=}\ & x^0\mathbf{1}+\vec{x}\cdot\vec{\sigma}-2U(\varphi,\vec{n})\sin\varphi(\vec{n}\times\vec{x})\cdot\vec{\sigma} \\
\overset{(2.28)(2.29)}{=}\ & x^0\mathbf{1}+\vec{x}\cdot\vec{\sigma}-\sin(2\varphi)(\vec{n}\times\vec{x})\cdot\vec{\sigma}+(1-\cos(2\varphi))(\vec{n}\times(\vec{n}\times\vec{x}))\cdot\vec{\sigma} \\
=\ & x^0\mathbf{1}+(\vec{n}(\vec{n}\cdot\vec{x})+\cos(2\varphi)(\vec{x}-\vec{n}(\vec{n}\cdot\vec{x}))-\sin(2\varphi)(\vec{n}\times\vec{x}))\cdot\vec{\sigma} \\
\overset{(2.6)}{=}\ & \left[\begin{pmatrix} 1 & 0 \\ 0 & \cos(2\varphi)(\mathbf{1}-\vec{n}\otimes\vec{n})-\sin(2\varphi)\vec{n}\times+\vec{n}\otimes\vec{n}\end{pmatrix}\begin{pmatrix} x^0 \\ \vec{x}\end{pmatrix}\right]_{\sim} \\
=:\ & (D(-2\varphi,\vec{n})x)_{\sim}
\end{aligned}$$

  which is a rotation $D(-2\varphi,\vec{n})$ by -2φ around the axis given by the direction $\vec{n}$.

- Since both $\Lambda(A(\psi,\vec{n}))$ and $\Lambda(U(\varphi,\vec{n}))$ are in $\mathcal{L}_+^{\uparrow}$ it is clear that Λ is a covering of the identity component.

- The arbitrariness of $\alpha\in\pi\mathbb{Z}$ has reappeared in the form $D(\vec{n},\pi)=D(\vec{n},2\pi)=\mathbf{1}_{\mathbb{R}^3}$, hence Λ is a twofold covering of the (a priori) not simply connected identity component $\mathcal{L}_+^{\uparrow}$, namely SO(3). Its cover SU(2)

- This property can be shown using the fact that any two matrices in the group $\mathrm{SL}(2,\mathbb{C})$ can be mapped to each other by another such matrix due to the group structure. This matrix can in turn be decomposed into an hermitean part and a unitary part given by two spatial directions $\vec{n}$, a rapidity ψ and an angle φ. Starting from $\psi=\varphi=0$ a continuous path from one matrix to the other can be constructed.

In summary, a twofold covering homomorphism Λ from the simply connected group $\mathrm{SL}(2,\mathbb{C})$ onto $\mathcal{L}_+^{\uparrow}$ has been constructed. In fact it is the so-called universal cover. While the ordinary

Poincaré group is the semidirect product $\mathcal{P} = \mathcal{L} \ltimes \mathbb{M}$ (with the usual action of $\mathcal{L}$ on $\mathbb{M}$), the covering of the orthochronous Poincaré group $\mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{M}$ is defined by

$$\begin{aligned} \mathcal{P}^c &= \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{M} \\ (A_1, a_1)(A_2, a_2) &= (A_1 A_2, \Lambda(A_1) a_2 + a_1) \end{aligned} \tag{2.30}$$

This means that the covering homomorphism Λ is used in the semidirect product to provide an action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{M}$.

2.3. Relativistic transformation law of the state vector

Starting from the Hilbert space of a relativistic particle, the irreducible representations of the Poincaré group on this space are classified. Their action is translated into the transformation rule for a relativistic wavefunction taking values in the so-called little Hilbert space, which depends on the chosen representation.

2.3.1. Irreducible representation spaces

Let $\mathcal{H}$ be the Hilbert space describing the state of a single particle as in 2.1. It has already been discussed there that $U : \mathcal{P}^c \to B(\mathcal{H})$ can be chosen as a unitary representation on $\mathcal{H}$. U is also assumed to be irreducible. [SW64, S. 23] relates this assumption to the notion of an elementary system, for example a stable elementary particle. This is justified because the irreducibility condition of the representation U on $\mathcal{H}$ means that there is no nontrivial subspace $\mathcal{H}_{\mathrm{red}} \subset \mathcal{H}$ (nontrivial means that $\{0\} \neq \mathcal{H}_{\mathrm{red}} \neq \mathcal{H}$) which is stable under the representation U, i.e. $U\mathcal{H}_{\mathrm{red}} \subseteq \mathcal{H}_{\mathrm{red}}$. In this sense, $\mathcal{H}_{\mathrm{red}}$ would describe a “more elementary” system.

The irreducibility condition implies that nontrivial observables are excluded which have a trivial transformation behaviour. Trivial observables are those which are multiples of the identity, i.e. observables of the form $\lambda\mathbf{1}$ with $\lambda \in \mathbb{C}$. If for an observable A, there holds the relation $UAU^\dagger = A$ for all unitary representation elements U, then the transformation behaviour of A is said to be trivial.

The reasoning behind this relation uses Schur's Lemma: The equation $UAU^\dagger = A$ implies $[U, A] = 0$ which means that $U\ker A \subseteq \ker A$ because $|\psi\rangle \in \ker A \Leftrightarrow A|\psi\rangle = 0 \Rightarrow AU|\psi\rangle = UA|\psi\rangle = 0 \Rightarrow U|\psi\rangle \in \ker A$. The same holds true if A is replaced by $A - \lambda 1$. Now $\mathcal{H}_{\mathrm{red}} := \ker(A - \lambda 1) \neq \{0\}$ for some $\lambda \in \mathbb{C}$ because there is at least one (possibly complex) eigenvalue for A. Since U was assumed to be irreducible, the only possibility is $\ker(A - \lambda\mathbf{1}) = \mathcal{H}_{\mathrm{red}} = \mathcal{H}$, which means $(A - \lambda\mathbf{1})|\psi\rangle = 0 \ \forall |\psi\rangle \in \mathcal{H}$ and therefore $A = \lambda 1$ on $\mathcal{H}$. Therefore, if A has a trivial transformation behaviour under an irreducible representation, it must in fact be a trivial observable.

To classify the possible irreducible representations, the value of suitable trivial[15] observables will be used, the so-called *quadratic Casimir elements*. For notational convenience, the definition $U(a) := U((\mathbf{1}, a))$ is used for the pure translations by a four-vector $a \in \mathbb{M}$, which can be written

$$U(a) = \mathrm{e}^{\mathrm{i}Pa} \tag{2.31}$$

with the generators P_μ of the translation subgroup. Similarly, $U(A) := U((A, 0))$ denotes the purely homogeneous elements. Because[16]

$$\begin{aligned} U(A)\mathrm{e}^{\mathrm{i}Pa} &\overset{(2.31)}{=} U(A)U(a) \overset{(2.2)}{=} U((A,0)(0,a)) \overset{(2.30)}{=} U((A,\Lambda(A)a)) \\ &\overset{(2.2)}{=} U(\Lambda(A)a)U(A) \overset{(2.31)}{=} \mathrm{e}^{\mathrm{i}P\Lambda(A)a}U(A) \end{aligned} \tag{2.32}$$

the transformation law of the generators P_μ can be deduced by taking partial derivatives w.r.t. the components a_μ and evaluating the result at $a = 0$:

$$U(A)P = -\mathrm{i}\frac{\partial}{\partial a} U(A)\mathrm{e}^{\mathrm{i}Pa}\big|_{a=0} \overset{(2.32)}{=} -\mathrm{i}\frac{\partial}{\partial a} \mathrm{e}^{\mathrm{i}P\Lambda(A)a}U(A)\Big|_{a=0} = P\Lambda(A)U(A) \tag{2.33}$$

i.e. the generators P_μ transform as four-vectors under the representation U of $\mathrm{SL}(2,\mathbb{C})$. It follows that

- The joint spectrum[17] $\operatorname{sp} P$ of P is invariant under $\Lambda(A)$. This can be seen by considering a state $|\psi\rangle$ of sharp momentum $p \in \mathbb{M}$, i.e.

  $$P\,|\psi\rangle = p\,|\psi\rangle \tag{2.34}$$

  This eigenvalue equation implies[18]

  $$P[U(A)\,|\psi\rangle] \overset{(2.33)}{=} U(A)P\Lambda(A)\,|\psi\rangle \overset{(2.34)}{=} U(A)p\Lambda(A)\,|\psi\rangle = p\Lambda(A)[U(A)\,|\psi\rangle]$$

  Thus, for every state $|\psi\rangle$ with sharp momentum p there is the state $U(A)\,|\psi\rangle$ with sharp momentum $p\Lambda(A)$. If p belongs to the spectrum of P, then the whole Lorentz orbit $p\Lambda(\mathrm{SL}(2,\mathbb{C}))$ does.

- The operator $P^2 := P_\mu P^\mu$ (quadratic Casimir element) transforms as a Lorentz scalar because

  $$\begin{aligned} U(A)P^2 &= U(A)P_\mu P^\mu \overset{(2.33)}{=} P_\nu \Lambda(A)_\mu{}^\nu U(A)P^\mu \overset{(2.33)}{=} P_\nu \underbrace{\Lambda(A)_\mu{}^\nu \Lambda(A)^\mu{}_\kappa}_{=\delta^\nu{}_\kappa} P^\kappa U(A) \\ &= P^2U(A) \end{aligned}$$

[15] In general, these observables will only be trivial on the single-particle space, not on direct sums, which coud allow superpositions of states with different values for λ. In other words, such states might not transform unter irreducible representations.

[16] The representation equation 2.2 with U' replaced by U has been used.

[17] The joint spectrum is defined as the set of those vectors p in Minkowski space $\mathbb{M}$, whose components are in the spectrum of the corresponding component of the operator P, i.e. $\operatorname{sp} P := \{p \in \mathbb{M} \mid p_\mu \in \operatorname{sp} P_\mu,\ \mu = 0,\dots,3\}$.

[18] Notice that once the eigenvalue equation 2.34 has been inserted, $p\Lambda(A)$ and $U(A)$ can be exchanged.

and hence

$$[U(A), P^2] = 0$$

Thus, in an irreducible representation, Schur's Lemma implies $P^2 = m^2\mathbf{1}$, where $m \in \mathbb{R}_0^+$ is interpreted as the mass of the corresponding particle. However, the representations under consideration are restricted to those where the spectrum of P^0 is positive, i.e. the positive energy representations.[19]

In summary, the spectrum of P in an irreducible representation U is precisely the $\mathcal{L}_+^\uparrow$-orbit

$$O_m := \{p \in \mathbb{M} \mid p^2 = m^2, p^0 \geq 0\} \tag{2.35}$$

For $m > 0$ this set is $H_m^+ := O_m$, the upper mass shell, while for $m = 0$ it is $\partial V^+ := O_0$, the boundary of the forward light cone. Let $\mathcal{H}$ is the representation space for one of these cases.

2.3.2. Subspaces of sharp momentum

For a purely spatial translation along the four-vector $(0, \vec{a})$ the translation operator is defined by

$$U(\vec{a}) := U((0, \vec{a})) \overset{(2.31)}{=} U((\mathbf{1}, (0, \vec{a}))) \ .$$

Considering two vectors $|\phi\rangle, |\psi\rangle$ in the subspace $\mathcal{D}$ of $\mathcal{H}$ where the matrix elements of the translations $\langle\phi|U(\vec{x})|\psi\rangle$ are rapidly decreasing as $|\vec{x}| \to \infty$, the Fourier transform[20] can be defined by

$$\langle\phi|\psi\rangle_{\vec{p}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} \langle\phi|U(\vec{x})|\psi\rangle \tag{2.36}$$

which is positive semidefinite for each $\vec{p}$. W.l.o.g., a representation is considered where the operators P_μ, which enter the matrix element because of definition 2.31, are diagonal. Only the eigenvectors of $\vec{P}$[21] to the eigenvalue $\vec{p}$ give the stationary contributions to the integral. Denoting the corresponding norm on $\mathcal{D}$ by $||\psi||_{\vec{p}} := \sqrt{\langle\psi|\psi\rangle_{\vec{p}}}$ the null-space

$$\mathcal{N}_{\vec{p}} := \{|\psi\rangle \in \mathcal{D}| \, ||\psi||_{\vec{p}} = 0\}$$

is divided out to obtain the Hilbert space

$$\mathcal{H}_{\vec{p}} := \overline{\mathcal{D}/\mathcal{N}_{\vec{p}}}^{||\cdot||_{\vec{p}}}$$

[19] This assumption does not violate the $\mathcal{L}_+^\uparrow$-invariance of the spectrum of P because $\operatorname{sgn} p^0$ is invariant.

[20] No information about the momenta p involved is lost in this definition because the condition $p \in O_m$ uniquely determines p^0 depending on $\vec{p}$ by the formula $p_0^2 - \vec{p}^2 = p^2 = m^2$, $p_0 \geq 0$ from 2.35.

[21] In this notation, momentum operators $P = (P^0, \vec{P})$ have been grouped into a temporal part P^0 and a spatial part $\vec{P}$.

of sharp (spatial) momentum $\vec{p}$. To be precise, $\mathcal{Q} := \mathcal{D}/\mathcal{N}_{\vec{p}}$ is the quotient pre-Hilbertspace of equivalence classes in $\mathcal{H}$ where $|\phi\rangle$ and $|\psi\rangle$ are called equivalent if $||\phi - \psi||_{\vec{p}} = 0$ and $\overline{\mathcal{Q}}^{||\cdot||_{\vec{p}}}$ is the norm-closure of $\mathcal{Q}$ with respect to $||\cdot||_{\vec{p}}$.

Since m has been specified in 2.3.1, the spectrum of P is known and can thus (for fixed p) be used to recover p^0. To make the notation simpler, also the definition $\mathcal{H}_p := \mathcal{H}_{\vec{p}}$ can be used if the constraint that p must fulfil $p^2 = m^2$ and $p^0 \geq 0$ is taken care of. $|\psi\rangle(p) := |\psi\rangle / \mathcal{N}_{\vec{p}}$ denotes the equivalence class in $\mathcal{H}_p$ corresponding to the state vector $|\psi\rangle \in \mathcal{D}$. This definition suggests to consider $\mathcal{H}$ as the space of sections $p \mapsto (p, \mathcal{H}_p)$ in a vector bundle. The dependence $p(\vec{p})$ can be used to reconstruct the matrix element of a general (i.e. not necessarily purely spatial) translation:

$$\begin{aligned}\langle\phi|U(a)|\psi\rangle &\overset{(2.36)}{=} \int \mathrm{d}^3p\, \mathrm{e}^{\mathrm{i}p_0(\vec{p})a^0 - \vec{p}\cdot\vec{a}} \langle\phi|\psi\rangle_{\vec{p}} = \int \underbrace{\mathrm{d}^4p\, \delta(p^2 - m^2)\Theta(p^0)}_{=:\widetilde{\mathrm{d}p}}\, \mathrm{e}^{\mathrm{i}pa}\, \underbrace{2p^0 \langle\phi|\psi\rangle_{\vec{p}}}_{=:\langle\phi|\psi\rangle_p} \\ &= \int \frac{\mathrm{d}^3p}{2p^0}\, \mathrm{e}^{\mathrm{i}pa} \langle\phi|\psi\rangle_p \ , \end{aligned} \tag{2.37}$$

where in the second step, $\mathrm{d}^4p := \mathrm{d}p_0 \mathrm{d}^3p$ has been defined. Performing the integration over p_0 reproduces the correct dependence $p_0(\vec{p})$ from 2.35. In the last expression the dependence $p^0(\vec{p})$ has been hidden in the notation. The measure $\widetilde{\mathrm{d}p}$ and the phase $\mathrm{e}^{\mathrm{i}pa}$ transform as Lorentz scalars, hence $\langle\cdot|\cdot\rangle_p$ yields for $a = 0$ the Lorentz-covariant decomposition of the scalar product:

$$\langle\phi|\psi\rangle \overset{(2.37)}{=} \int \widetilde{\mathrm{d}p}\, \langle\phi|\psi\rangle_p \tag{2.38}$$

2.3.3. Identification of the subspaces

To change between the Hilbert spaces $\mathcal{H}_p$ of sharp momentum p the action of $U(A)$ on the state $|\psi\rangle \in \mathcal{H}$ and the projection on the corresponding space $\mathcal{H}_{p\Lambda(A)^{-1}}$ are used. This defines a new map $\underline{U}(A)$ via[22]

$$\underline{U}(A)\,|\psi\rangle(p) = (U(A)\,|\psi\rangle)(p\Lambda(A)^{-1}) \tag{2.39}$$

which preserves the scalar product between the spaces $\mathcal{H}_p$ and $\mathcal{H}_{p\Lambda(A)^{-1}}$. To verify this property, the scalar product in $\mathcal{H}_p$ is denoted by[23]

$$(|\phi\rangle(p), |\psi\rangle(p)) := \langle\phi|\psi\rangle_p \tag{2.40}$$

[22]While the left side only involves $|\psi\rangle(p) \in \mathcal{H}/\mathcal{N}_p$, the right side depends a priori on $|\psi\rangle \in \mathcal{H}$. However, this mapping is well defined because $U(A)\mathcal{N}_p = \mathcal{N}_{p\Lambda(A)^{-1}}$, since $\int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} ||U(A)\psi||^2_{p\Lambda(A)^{-1}} = \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}p\Lambda(A)a} ||U(A)\psi||^2_p = \langle U(A)\psi|\, U(\Lambda(A)a)U(A)\,|\psi\rangle = \langle\psi|\, U(a)\,|\psi\rangle = \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} ||\psi||^2_p$.

[23]At this point, this is clearly an abuse of notation because the spaces $\mathcal{H}_p$ belonging to different p have not yet been identified. However, the scalar product $(\cdot,\cdot)$ is well-defined, analogously to the argument as for 2.39, this time using the Cauchy-Schwarz inequality.

Then it is sufficient to consider the Fourier transforms:

$$
\begin{aligned}
&\int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} (\underline{U}(A) |\phi\rangle (p), \underline{U}(A) |\psi\rangle (p)) \\
\overset{(2.39)}{=} \;& \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} ((U(A) |\phi\rangle)(p\Lambda(A)^{-1}), (U(A) |\psi\rangle)(p\Lambda(A)^{-1})) \\
\overset{(2.40)}{=} \;& \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} \langle U(A)\phi | U(A)\psi\rangle_{p\Lambda(A)^{-1}} = \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}p\Lambda(A)a} \langle U(A)\phi | U(A)\psi\rangle_p \\
\overset{(2.38)}{=} \;& \langle U(A)\phi |\, U(\Lambda(A)a)\, | U(A)\psi\rangle \overset{(2.32)}{=} \langle \phi |\, U(a)\, |\psi\rangle \overset{(2.38)}{=} \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} \langle \phi | \psi\rangle_p \\
\overset{(2.40)}{=} \;& \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} (|\phi\rangle (p), |\psi\rangle (p)) \\
\Rightarrow \;& (\underline{U}(A) |\phi\rangle (p), \underline{U}(A) |\psi\rangle (p)) = (|\phi\rangle (p), |\psi\rangle (p))
\end{aligned}
\tag{2.41}
$$

$\underline{U}$ also inherits the representation property from U:

$$
\begin{aligned}
&\underline{U}(A)\underline{U}(B) |\psi\rangle (p) \overset{(2.39)}{=} \underline{U}(A)(U(B) |\psi\rangle)(p\Lambda(B)^{-1}) \\
\overset{(2.39)}{=} \;& (U(A)U(B) |\psi\rangle)(p\Lambda(B)^{-1}\Lambda(A)^{-1}) \overset{(2.2)}{=} (U(AB) |\psi\rangle)(p\Lambda(AB)^{-1}) \\
\overset{(2.39)}{=} \;& \underline{U}(AB) |\psi\rangle (p)
\end{aligned}
\tag{2.42}
$$

2.3.4. The little group construction

If a certain reference momentum q in the spectrum of P is fixed, an element $R_p \in \mathrm{SL}(2,\mathbb{C})$ can be constructed for each $p \in O_m$, yielding the so-called *Wigner boost* (which actually is not a pure boost in general)

$$
B_p := \Lambda(R_p) \text{ with the property } qB_p = p\,. \tag{2.43}
$$

Using the representation $\underline{U}(R_p)$ of the Wigner-boost, the sections $|\psi\rangle$ on $p \mapsto \mathcal{H}_p$ can therefore be transformed in a way that keeps the momentum in the projection $|\psi\rangle \mapsto |\psi\rangle (p)$ constant and equal to q. This is achieved by the map

$$
\begin{aligned}
V \;:\; \mathcal{H} \;&\to\; \mathcal{K} := L^2(\mathrm{sp}P) \otimes \mathcal{H}_q \\
|\psi\rangle \;&\mapsto\; [\psi \;:\; p \mapsto \psi(p) := \underline{U}(R_p) |\psi\rangle (p) = (U(R_p) |\psi\rangle)(q) \in \mathcal{H}_q]
\end{aligned}
\tag{2.44}
$$

under which the original Hilbert space $\mathcal{H}$ becomes an L^2-space on the spectrum of P whose elements ψ take values in $\mathcal{H}_q$. V can be used to transform the representation U on $\mathcal{H}$ into

$$
U_1 := VUV^{-1} \tag{2.45}
$$

on $\mathcal{K}$. Here, for the sake of clarity, the inverse of V is stated explicitly:[24]

$$
\begin{aligned}
V^{-1} :\; \mathcal{K} \;&\to\; \mathcal{H} \\
\psi \;&\mapsto\; |\psi\rangle := \int_{O_m}^{\oplus} \widetilde{\mathrm{d}p}\, \underline{U}(R_p)^{-1}\psi(p)
\end{aligned}
\tag{2.46}
$$

[24] While V involved the projection $|\psi\rangle \mapsto |\psi\rangle (p) = |\psi\rangle / \mathcal{N}_p$ onto the equivalence classes modulo $\mathcal{N}_p$, the direct integral recombines these projections to the original vector $|\psi\rangle \in \mathcal{H}$, such that $\underline{U}(R_p)$ and its inverse cancel and $\int_{O_m}^{\oplus} \widetilde{\mathrm{d}p}\, |\psi\rangle (p) = |\psi\rangle$. Hence V is in fact invertible.

Naturally, the conjugation 2.45 preserves the representation property:

$$\begin{aligned} U_1(A)U_1(B) &\overset{(2.45)}{=} VU(A)V^{-1}VU(B)V^{-1} = VU(A)U(B)V^{-1} \overset{(2.2)}{=} VU(AB)V^{-1} \\ &\overset{(2.45)}{=} U_1(AB) \end{aligned} \tag{2.47}$$

The momentum decomposition of the scalar product $\langle\cdot|\cdot\rangle$ in $\mathcal{H}$, while $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}_q$, now looks as follows:

$$\begin{aligned} \langle\phi|\psi\rangle &\overset{(2.38)}{=} \int \widetilde{\mathrm{d}p}\, \langle\phi|\psi\rangle_p \overset{(2.40)}{=} \int \widetilde{\mathrm{d}p}\, (|\phi\rangle(p), |\psi\rangle(p)) \\ &\overset{(2.41)}{=} \int \widetilde{\mathrm{d}p}\, (\underline{U}(R_p)\,|\phi\rangle(p), \underline{U}(R_p)\,|\psi\rangle(p)) \overset{(2.44)}{=} \int \widetilde{\mathrm{d}p}\, ([V\,|\phi\rangle](p), [V\,|\psi\rangle](p)) \end{aligned}$$

which is precisely the scalar product in $\mathcal{K}$, hence V is also a unitary map. Evaluating the action of $\mathcal{P}^c$ on $\mathcal{K}$ gives

$$\begin{aligned} [U_1(a)\psi](p) &\overset{(2.45)}{=} [VU(a)V^{-1}\psi](p) \overset{(2.44)}{=} \underline{U}(R_p)[U(a)V^{-1}\psi](p) \\ &\overset{(2.31)}{=} \underline{U}(R_p)e^{\mathrm{i}pa}[V^{-1}\psi](p) \overset{(2.46)}{=} \mathrm{e}^{\mathrm{i}pa}\psi(p) \end{aligned} \tag{2.48}$$

for a translation and

$$\begin{aligned} [U_1(A)\psi](p) &\overset{(2.45)}{=} VU(A)V^{-1}\psi(p) \overset{(2.44)}{=} \underline{U}(R_p)[U(A)V^{-1}\psi](p) \qquad (2.49) \\ &\overset{(2.39)}{=} \underline{U}(R_p)\underline{U}(A)[V^{-1}\psi](p\Lambda(A)) \\ &\overset{(2.46)}{=} \underline{U}(R_p)\underline{U}(A)\underline{U}(R_{p\Lambda(A)})^{-1}\psi(p\Lambda(A)) \overset{(2.42)}{=} \underline{U}(R_pAR^{-1}_{p\Lambda(A)})\psi(p\Lambda(A)) \end{aligned}$$

for a Lorentz transformation. This is consistent because $\Lambda(R_pAR^{-1}_{p\Lambda(A)}) \in \operatorname{stab} q \subset \mathcal{L}^\uparrow_+$, i.e. leaves the reference momentum q invariant. For the case where $\underline{U} : \mathcal{H}_q \to \mathcal{H}_q$, it remains to construct a representation (now called D) with

$$D(A)D(B) \overset{(2.42)}{=} D(AB) \tag{2.50}$$

of the so-called *little group* $\Lambda^{-1}(\operatorname{stab} q)$ on the space $\mathcal{H}_q$, which is accordingly called the *little Hilbert space*. Defining the *Wigner rotation*,

$$R(A,p) := R_pAR^{-1}_{p\Lambda(A)}\ , \tag{2.51}$$

the representation of the Lorentz transformation in 2.49 now reads

$$[U_1(A)\psi](p) \overset{(2.51)}{=} D(R(A,p))\psi(p\Lambda(A)) \tag{2.52}$$

Thus, for an arbitrary element $(A,a) \in \mathcal{P}^c$,

$$\begin{aligned} [U_1((A,a))\psi](p) &\overset{(2.30)}{=} [U_1((\mathbf{1},a)(A,0))\psi](p) \overset{(2.47)}{=} [U_1(a)U_1(A)\psi](p) \\ &\overset{(2.48)}{=} \mathrm{e}^{\mathrm{i}pa}[U_1(A)\psi](p) \overset{(2.52)}{=} \mathrm{e}^{\mathrm{i}pa}D(R(A,p))\psi(p\Lambda(A)) \end{aligned} \tag{2.53}$$

2.4. Further steps

There are some steps in the construction that have not been carried out explicitly up to this point because they crucially depend on the mass m:

- choice of reference momentum q, such that $q^0 > 0$ and $q^2 = m^2$
- construction of the covering preimage $R_p \in \mathrm{SL}(2,\mathbb{C})$ in such a way that $q\Lambda(R_p) = p$
- identification of the little group $\Lambda^{-1}(\operatorname{stab} q) = \{A \in \mathrm{SL}(2,\mathbb{C}) | q\Lambda(A) = q\}$
- construction and classification of the irreducible representatios D of $\operatorname{stab} q$ on the little Hilbert space $\mathcal{H}_q$
- construction the corresponding representation U_1 of the full group $\mathcal{P}^c$ on the one-particle space $\mathcal{H}$ from the above discussion by inserting D into 2.52

The discussion in the next two chapters is dedicated to performing these steps for the cases $m > 0$ and $m = 0$, respectively. Following the above discussion then directly gives the $\mathcal{P}^c$-covariantly transforming $\mathcal{H}_q$-valued wavefunctions on O_m. The description of D is then chosen in a particular way, such that the complicated Wigner-boost $R(\Lambda, p) \in \operatorname{stab} q$ can be handled. Proceeding to the corresponding field operators that are to be investigated is then a matter of second quantization.

3. Massive representations

> My soul, seek not the life of immortals;
> but enjoy to the full the resources that
> are within your reach.
>
> PINDAROS

In this chapter, it is shown how the Wigner framework developed before can be applied in the case of positive mass. The single particle states transforming under the massive representations are constructed and then the corresponding fields obtained by second quantization are studied.

3.1. Construction of single particle states

The transformation law of the single particle states is fixed by the representation of the little group, which is determined first. It turns out to be convenient to introduce so-called *intertwiner* functions, which render the effect of a little group transformation significantly simpler. This technique is introduced in the present chapter in the context of a well-known class of representations because it is essential to the representations with infinite spin.

3.1.1. Little group

Calculation of the little group

Since the case of massive irreducible representations is investigated, $m > 0$. According to 2.35, $\operatorname{sp} P = H_m^+$, the upper mass shell is the set of possible momenta p. The recipe in 2.4 from the previous chapter indicates that a reference momentum $q \in O_m = H_m^+$ has to be chosen. It is convenient to pick[1]

$$q := (m, 0, 0, 0) \Rightarrow \widetilde{q} = m\mathbf{1} \ . \tag{3.1}$$

[1] The possibility to chose this simple form of q reflects the fact that for a massive particle, there is a distinguished Lorentz frame, the rest frame.

The Wigner Boost B_p from 2.43 is then given by the element[2]

$$R_p := \sqrt{\widetilde{p}/m} = \frac{1}{\sqrt{m}}\sqrt{(p_0\mathbf{1} - \vec{\sigma}\cdot\vec{p})} \tag{3.2}$$

$$= \frac{1}{2\sqrt{m}}\left[\left(\sqrt{p_0-|\vec{p}|}+\sqrt{p_0+|\vec{p}|}\right)\mathbf{1} + \left(\sqrt{p_0-|\vec{p}|}-\sqrt{p_0+|\vec{p}|}\right)\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|}\right] \tag{3.3}$$

because

$$(qB_p)\widetilde{\ }\overset{(2.43)}{=}(q\Lambda(R_p))\widetilde{\ }\overset{(2.17)}{=}R_p^\dagger\widetilde{q}R_p\overset{(3.1)}{=}mR_p^\dagger R_p\overset{(3.2)}{=}m\sqrt{\widetilde{p}/m}\sqrt{\widetilde{p}/m}\overset{(3.3)}{=}\widetilde{p}\,. \tag{3.4}$$

From the choice made in eq. 3.1 the covering of the little group[3] can be derived as

$$G_q = \{R \in \mathrm{SL}(2,\mathbb{C}) | q\Lambda(R) = q \overset{(2.17)}{\Leftrightarrow} R^\dagger R = \mathbf{1}\} = \mathrm{SU}(2)\ .$$

Irreducible representations of the little group

It is well known that the irreducible representations D of G_q can be classified by the *spin* $s \in \mathbb{N}_0/2$. The wavefunction ψ takes values in $\mathcal{H}_q = \mathbb{C}^{2s+1} \overset{\sim}{=} \mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$.

In fact, $\dim \mathbb{C}^{2s+1} = \dim \mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$: Denoting the basis elements of $\mathbb{C}^2$ by $|\uparrow\rangle$ and $|\downarrow\rangle$, the basis elements of $\mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$ are uniquely specified by the number of $|\uparrow\rangle$'s in the (symmetrized) tensor product, for which there are $2s+1$ possibilities, ranging from $\underbrace{|\downarrow\rangle \otimes_S |\downarrow\rangle \otimes_S \cdots \otimes_S |\downarrow\rangle}_{2s \text{ factors}}$ over $|\uparrow\rangle \otimes_S |\downarrow\rangle \otimes_S \cdots \otimes_S |\downarrow\rangle$ up to $|\uparrow\rangle \otimes_S |\uparrow\rangle \otimes_S \cdots \otimes_S |\uparrow\rangle$. The last isomorphism is called the Majorana description.[4]

The action of the representation D on $\mathcal{H}_q$ is given by[5]

$$D(R)(v_1 \otimes_S v_2 \otimes_S \cdots \otimes_S v_{2s}) := (Rv_1) \otimes_S (Rv_2) \otimes_S \cdots \otimes_S (Rv_{2s}) \tag{3.5}$$

[2] In the following, the stated explicit form of $\sqrt{\widetilde{p}}$ is not needed. Its existence can be shown alternatively by examining the (due to $\widetilde{p}^\dagger = \widetilde{p}$ real) eigenvalues λ of $\widetilde{p}$: Using $0 < m^2 = p^2 = (p_0, \vec{p})^2 = p_0^2 - \vec{p}^2$, the equation for the roots of the characteristic polynomial $0 \overset{!}{=} \det(\widetilde{p} - \lambda 1) = (p - (\lambda, \vec{0}))^2 = (p_0 - \lambda)^2 - \vec{p}^2$ implies $\lambda = p_0 \pm |\vec{p}| > 0$. Hence $\widetilde{p}$ is hermitean and positive definite and therefore $\sqrt{\widetilde{p}}$ is defined.

[3] We emphasize the fact that $G_q = SU(2)$ is (referring to 2.4) a compact group because the absence of this property will lead to considerable difficulties then studying the case $m = 0$ later.

[4] This type of representation of the spin degrees of freedom is explained in e.g. [Pen04, p. 560]

[5] No linear extension to sums of elementary tensors is necessary in this definition: For an arbitrary element $v \in \mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$ and $\zeta := (1, z)^{\otimes 2s}$ consider the polynomial $v \cdot \zeta =: v_0 + v_1 z + \ldots + v_{2s} z^{2s}$. Now v_i is the $\underbrace{|\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle}_{2s-i \text{ factors}} \otimes \underbrace{|\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle}_{i \text{ factors}}$-component of v which can be recovered from the polynomial by linear independence of the monomials of different degree. According to the fundamental theorem of Algebra, the polynomial admits the factorization $v \cdot \zeta = w_0(z - w_1) \cdots (z - w_{2s})$ with $w_0, \ldots, w_{2s} \in \mathbb{C}$. Hence $v \cdot \zeta = \underbrace{(w_0(-w_1, 1) \otimes \cdots \otimes (-w_{2s}, 1))}_{=:w} \cdot \zeta$. If v is recovered from this polynomial, this implies $v = w$, an elementary tensor. This remark is based on exercise [22.30] from [Pen04, p. 560].

D fulfils the representation property

$$\begin{aligned} & D(R_1)D(R_2)(v_1 \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} v_{2s}) = D(R_1)((R_2 v_1) \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} (R_2 v_{2s})) \\ = \quad & ((R_1 R_2 v_1) \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} (R_1 R_2 v_{2s})) = D(R_1 R_2)(v_1 \otimes_{\mathrm{S}} \cdots \otimes_{\mathrm{S}} v_{2s}) \; . \end{aligned} \tag{3.6}$$

This property immediately generalizes from $D: \mathrm{SU}(2) \to \mathrm{Aut}(\mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s}))$ to a new representation $\widetilde{D}$, whose domain of definition is the whole group $\mathrm{SL}(2,\mathbb{C})$ instead. In this case, a factorization of the representation of the Wigner rotation $R(\Lambda, p)$, namely

$$D(R(A,p)) \overset{(2.51)}{=} D(R_p A R^{-1}_{p\Lambda(A)}) = \widetilde{D}(R_p A R^{-1}_{p\Lambda(A)}) \overset{(3.6)}{=} \widetilde{D}(R_p)\widetilde{D}(A)\widetilde{D}(R^{-1}_{p\Lambda(A)}) \tag{3.7}$$

is possible. In the next section, this possibility is exploited to simplify the transformation behaviuor of the wavefunctions $\psi \in \mathcal{K}$.

3.1.2. Intertwiners for the fundamental representation

It is already known from Sec. 2.4 what the action of $\mathcal{P}^c$ on $L^2(H_m^+) \otimes \mathcal{H}_q$ looks like, especially how it involves the complicated Wigner rotation $R(A,p)$ defined in 2.51. The purpose of the following intertwiner construction is essentially to get rid of this expression.

Identity component $\mathcal{L}_+^\uparrow$

The application of the Lorentz transformation $\Lambda(A)$ in the representation U_1 reads

$$U_1(A)\psi(p) \overset{(2.52)}{=} D(R(A,p))\psi(p\Lambda(A)) \overset{(2.51)}{=} D(R_p A R^{-1}_{p\Lambda(A)})\psi(p\Lambda(A)) \; .$$

To simplify the transformation behaviour, another intertwiner[6]

$$\begin{aligned} u \;:\; H_m^+ \times \mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s}) \;&\to\; \mathcal{H}_q \\ (p, v) \;&\mapsto\; \widetilde{D}(R_p)v \end{aligned} \tag{3.8}$$

is introduced, which makes it possible to trade the Wigner rotation for a matrix multiplication of v if the *intertwiner equation*

$$D(R(A,p))u(p\Lambda(A), v) \overset{(3.7)}{=} u(p, \widetilde{D}(A)v) \tag{3.9}$$

is used.

[6] As shown before, $\mathcal{H}_q$ and $\mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$ are the same Hilbert-space. The difference is that the elements of $\mathcal{H}_q$ are symmetrized tensor products of vectors transforming under SU(2), while for $\mathcal{H}_q$ and $\mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$ the same holds true with SU(2) replaced by $\mathrm{SL}(2,\mathbb{C})$.

Extension to the proper orthochronous Lorentz group $\mathcal{L}_+$

It is possible to extend the transformation law to $\mathcal{L}_+$ since the little group SU(2) is adjoined with the group $\mathbb{Z}/4\mathbb{Z}$ whose generating element is denoted by j.[7] It should satisfy $Rj = jR\ \forall A \in \mathrm{SU}(2)$.[8] The group which results from this procedure is the so-called *group extension* G. Formally, there is a short exact sequence of groups[9]

$$\{j^0\} \xrightarrow{f_0:\ j^0 \mapsto j^0} \mathbb{Z}/4\mathbb{Z} \xrightarrow{f_1:\ j^i \mapsto j^i \mathbf{1}} G \xrightarrow{f_2:\ j^i R \mapsto R} \mathrm{SU}(2) \xrightarrow{f_3:\ U \mapsto \mathbf{1}} \{\mathbf{1}\}$$

The representation $\widetilde{D}$ can be extended accordingly with the definition

$$\widetilde{D}(j)v = \widetilde{D}(\zeta)\overline{v} \tag{3.10}$$

where[10]

$$\zeta = \mathrm{i}\sigma_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} . \tag{3.11}$$

It has to be checked first that $\widetilde{D}$ preserves the representation property. This is the case because

$$\widetilde{D}(j)\widetilde{D}(j)v \overset{(3.10)}{=} \widetilde{D}(j)\widetilde{D}(\zeta)\overline{v} \overset{(3.10)}{=} \widetilde{D}(\zeta)\overline{\widetilde{D}(\zeta)}v \overset{(3.6)}{=} \widetilde{D}(-\mathbf{1})v \tag{3.12}$$

since $\zeta\overline{\zeta} = (i\sigma_2)^2 = -\mathbf{1}$ and therefore

$$\widetilde{D}(j)^4 v = (\widetilde{D}(j)\widetilde{D}(j))^2 v \overset{(3.12)}{=} \widetilde{D}(-\mathbf{1})^2 v \overset{(3.6)}{=} \widetilde{D}(\mathbf{1})v = \widetilde{D}(j^4)v\ .$$

For all $A \in \mathrm{SL}(2,\mathbb{C})$ the identity (using $\det A = A_{11}A_{22} - A_{12}A_{21} = 1$)

$$A^T\zeta A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{= \begin{pmatrix} A_{21} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix}} = (A_{11}A_{22} - A_{12}A_{21}) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \zeta \tag{3.13}$$

holds, which implies[11] $\zeta\overline{R} = R\zeta$ if $R \in \mathrm{SU}(2)$. With the help of this identity, it can be shown that[12] $\mathbb{Z}/4\mathbb{Z} \subset G'$ is compatible with the definition of $\widetilde{D}$:

$$D(R)\widetilde{D}(j)v \overset{(3.10)}{=} \widetilde{D}(R\zeta)\overline{v} = \widetilde{D}(\zeta\overline{R})\overline{v} \overset{(3.10)}{=} \widetilde{D}(j)D(R)v\ \forall R \in \mathrm{SU}(2) \tag{3.14}$$

[7] This means that $\mathbb{Z}/4\mathbb{Z}$ consists of elements of the form j^i, where $j^0 = j^4 = \mathbf{1}$, the neutral element. In other words $j^i j^{i'} = j^{(i+i')\bmod 4}$.

[8] The result will again be a group because all elements of the form j^iU have $j^{-i}U^{-1}$ as their inverse, while associativity is preserved because all j's can be collected bracketwise due to commutativity.

[9] This means that the group homomorphisms f_i satisfy $\ker f_1 = j^0 = \operatorname{ran} f_0$, $\ker f_2 = j^i\mathbf{1} = \operatorname{ran} f_2$ and $\ker f_3 = \mathrm{SU}(2) = \operatorname{ran} f_2$.

[10] σ_2 is defined in 2.7.

[11] Proof: $R \in \mathrm{SU}(2) \Rightarrow \overline{R}^T = R^\dagger = R^{-1}$ and the identity 3.13 implies $\zeta\overline{R} = (\overline{R}^T)^{-1}\zeta$. In conclusion, $\zeta\overline{R} = R\zeta$.

[12] G' denotes the center of G.

j turns out to implement the PCT-conjugation, which includes the spacetime reflection PT, extending the representation from $\mathcal{L}_+^\uparrow$ to $\mathcal{L}_+$.

3.1.3. Conjugate intertwiner

In order to define a correspondingly transforming quantum field with local (anti)commutation behaviour later, a so-called conjugate intertwiner u_c is needed as well, with the same properties as u, but antilinear in v. u_c is defined by the equation

$$u_c(p,v) = \widetilde{D}(j)u(p,v) \tag{3.15}$$

In the following, it is shown that u_c has indeed the same transformation behaviour as u. The appearance of ζ, for which the formula

$$\underset{\sim}{p}\zeta\overline{A} \overset{(3.13)}{=} \underset{\sim}{p}(A^\dagger)^{-1}\zeta \overset{(2.17)}{=} A(p\Lambda(A))\underset{\sim}{\zeta} \tag{3.16}$$

holds, in the conjugation $\widetilde{D}(j)$ allows for a concrete description of the conjugate intertwiner

$$\begin{aligned} u_c(p,v) &\overset{(3.15)}{=} \widetilde{D}(j)u(p,v) \overset{(3.8)}{=} \widetilde{D}(j)\widetilde{D}(R_p)v \overset{(3.10)}{=} \widetilde{D}(\zeta)\overline{\widetilde{D}(R_p)v} \overset{(3.6)}{=} \widetilde{D}(\zeta\overline{R_p})\overline{v} \\ &\overset{(3.1)}{=} \widetilde{D}(\underset{\sim}{q}\zeta\overline{R_p})\overline{v} \overset{(3.16)}{=} \widetilde{D}(R_p(q\Lambda(R_p))\underset{\sim}{\zeta})\overline{v} \overset{(3.4)}{=} \widetilde{D}(R_p\underset{\sim}{p\zeta})\overline{v} \overset{(3.6)}{=} \widetilde{D}(R_p)\widetilde{D}(\underset{\sim}{p}\zeta)\overline{v} \\ &\overset{(3.8)}{=} u(p,\widetilde{D}(\underset{\sim}{p}\zeta)\overline{v}) \; . \end{aligned} \tag{3.17}$$

For the transformation behaviour

$$\begin{aligned} D(R(A,p))u_c(p\Lambda(A),v) &\overset{(3.15)}{=} D(R(A,p))\widetilde{D}(j)u(p\Lambda(A),v) \\ &\overset{(3.14)}{=} \widetilde{D}(j)D(R(A,p))u(p\Lambda(A),v) \\ &\overset{(3.9)}{=} \widetilde{D}(j)u(p,\widetilde{D}(A)v) \overset{(3.15)}{=} u_c(p,\widetilde{D}(A)v) \end{aligned}$$

the same intertwiner equation is valid. Applying $D(j)$ again, $u(p,e)$ is recovered up to a possible change of sign:

$$\widetilde{D}(j)u_c(p,v) \overset{(3.15)}{=} \widetilde{D}(j)^2u(p,v) \overset{(3.12)}{=} \widetilde{D}(-\mathbf{1})u(p,v) \overset{(3.5)}{=} (-1)^{2s}u(p,v)$$

In conclusion, $\widetilde{D}(j)$ is an involution in a representation with integer spin $s \in \mathbb{N}$, while there is a change of sign if it is applied twice in a representation with half-integer spin $s \in (\mathbb{N}/2)\backslash\mathbb{N}$. A similar behaviour can be achieved in 2.3 for $\varphi = \pi$.

3.1.4. One particle states

The intertwiner u and its conjugate u_c can now be used to define wavefunctions $\psi(f,v)$ and $\psi_c(f,v)$ of the form

$$\begin{aligned} \psi(f,v)(p) &:= u(p,v)\widetilde{f}(p) \\ \psi_c(f,v)(p) &:= u_c(p,v)\widetilde{f}(p) \end{aligned} \tag{3.18}$$

where $\widetilde{f}$ is the restriction of the Fourier transform of $f : \mathbb{M} \to \mathbb{C}$ to the mass shell H_m^+:

$$\widetilde{f}(p) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\, f(x)$$

The basic transformation properties of the Fourier transform are:

$$\begin{aligned} ((\Lambda,a)_* f)\widetilde{\ }(p) &= \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\, ((\Lambda,a)_* f)(x) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} f(\Lambda^{-1}(x-a)) \\ &= \mathrm{e}^{\mathrm{i}pa} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\Lambda x} f(x) = \mathrm{e}^{\mathrm{i}pa}\widetilde{f}(p\Lambda) \end{aligned} \tag{3.19}$$

$$\begin{aligned} ((-\mathbf{1})_* f)\widetilde{\ }(p) &= \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}((-\mathbf{1})_* f)(x) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} f(-x) = \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px} f(x) \\ &= \overline{\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\overline{f(x)}} = \overline{\widetilde{\overline{f}}(p)} \end{aligned} \tag{3.20}$$

The vector $v \in \mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$ now transforms directly under $\mathrm{SL}(2,\mathbb{C})$ because

$$\begin{aligned} & U_1(A,a)\psi_{(c)}(f,v)(p) \overset{(2.53)}{=} \mathrm{e}^{\mathrm{i}pa} D(R(A,p))\psi_{(c)}(f,v)(p\Lambda(A)) \\ \overset{(3.18)}{=} \; & \mathrm{e}^{\mathrm{i}pa} D(R(p,a))u_{(c)}(p\Lambda(A),v)\widetilde{f}(p\Lambda(A)) \overset{(3.9)}{=} u_{(c)}(p,\widetilde{D}(A)v)\mathrm{e}^{\mathrm{i}pa}\widetilde{f}(p\Lambda(A)) \\ \overset{(3.19)}{=} \; & u_{(c)}(p,\widetilde{D}(A)v)((\Lambda(A),a)_* f)\widetilde{\ }(p) \overset{(3.18)}{=} \psi_{(c)}((\Lambda(A),a)_* f, \widetilde{D}(A)v) \; . \end{aligned} \tag{3.21}$$

The action of j is defined[13] by the antiunitary operator $U_1(j)$

$$\begin{aligned} & U_1(j)\psi(f,v)(p) := \widetilde{D}(j)\psi(f,v)(p) \overset{(3.18)}{=} \widetilde{D}(j)u(p,v)\widetilde{f}(p) \overset{(3.10)}{=} \overline{[\widetilde{D}(j)u(p,v)]\widetilde{\overline{f}}(p)} \\ \overset{(3.15)}{=} \; & u_c(p,v)\overline{\widetilde{\overline{f}}(p)} \overset{(3.20)}{=} u_c(p,v)((-\mathbf{1})_*\overline{f})(p) \overset{(3.18)}{=} \psi_c((-\mathbf{1})_*\overline{f},v)(p) \end{aligned} \tag{3.22}$$

while repeated application of $U_1(j)$ yields

$$U_1(j)\psi_c((-\mathbf{1}_*\overline{f},v)(p) \overset{(3.22)}{=} \widetilde{D}(j)^2\psi(f,v)(p) \overset{(3.12)}{=} (-1)^{2s}\psi(f,v)(p) \; . \tag{3.23}$$

3.2. Second quantization

In order to translate the transformation law of a wavefunction into the transformation law for the creation and annihilation operators, it is first shown how the momentum-space basis of the one-particle space transforms. These are then used to define the corresponding quantum field.

3.2.1. Basis kets

The one particle states are decomposed in the standard basis of $L^2(H_m^+) \otimes \mathcal{H}_q$, where $\bullet$ stands for summation over the basis of $\mathcal{H}_q$:[14]

$$|\psi_{(c)}(f,v)\rangle := \int_{H_m} \widetilde{\mathrm{d}p}\, \psi_{(c)}(f,v)(p) \bullet |p\rangle \overset{(3.18)}{=} \int_{H_m} \widetilde{\mathrm{d}p}\, \widetilde{f}(p)u_{(c)}(p,v) \bullet |p\rangle \tag{3.24}$$

[13]The first equality here is not a tautological definition because U_1 acts on $\psi(f,v)$, an element in $\mathcal{H}$, while D acts on $\psi(f,h)(p)$, an element in $\mathcal{H}_q$.

[14]q does not appear explicitly in the following formulas because it is fixed for the whole construction.

Evaluation of the action of U_1 on the one particle state

$$U_1(A,a)\,|\psi_{(c)}(f,v)\rangle \overset{(3.24)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\, U_1(A,a)\psi_{(c)}(f,v)(p) \bullet |p\rangle \tag{3.25}$$
$$\overset{(3.21)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} \widetilde{f}(p\Lambda(A)) u_{(c)}(p, \widetilde{D}(A)v) \bullet |p\rangle$$
$$= \int_{H_m^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a} \widetilde{f}(p) u_{(c)}(p\Lambda(A)^{-1}, \widetilde{D}(A)v) \bullet |p\Lambda(A)^{-1}\rangle$$

$$U_1(j)\,|\psi(f,v)\rangle \overset{(3.24)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\, U_1(j)\psi(f,v)(p) \bullet |p\rangle \overset{(3.22)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\overline{\widetilde{f}(p)} u_c(p,v) \bullet |p\rangle \tag{3.26}$$

$$U_1(j)\,|\psi_c(f,v)\rangle \overset{(3.24)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\, U_1(j)\psi_c(f,v)(p) \bullet |p\rangle \overset{(3.23)}{=} (-1)^{2s} \int_{H_m^+} \widetilde{\mathrm{d}p}\overline{\widetilde{f}(p)} u(p,v) \bullet |p\rangle \tag{3.27}$$

gives the matrix elements by comparison with 3.24

$$U_1(A,a)u(p,v) \bullet |p\rangle \overset{(3.25)}{=} \mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a} u(p\Lambda(A)^{-1}, \widetilde{D}(A)v) \bullet |p\Lambda(A)^{-1}\rangle \tag{3.28}$$
$$U_1(j)u(p,v) \bullet |p\rangle \overset{(3.26)}{=} u_c(p,v) \bullet |p\rangle \tag{3.29}$$
$$U_1(j)u_c(p,v) \bullet |p\rangle \overset{(3.27)}{=} (-1)^{2s} u(p,v) \bullet |p\rangle\ . \tag{3.30}$$

Via second quantization, $|p\rangle = a^\dagger(p)\,|0\rangle$, the $\bullet$-products of $u_{(c)}$ and creation operators $a^\dagger(p)$ transform correspondingly:

$$U(A,a)u_{(c)}(p,v) \bullet a^\dagger(p,k)U^\dagger(A,a) \overset{(3.28)}{=} \mathrm{e}^{\mathrm{i}p\Lambda^{-1}a} u_{(c)}(p\Lambda(A)^{-1}, \widetilde{D}(A)v) \bullet a^\dagger(p\Lambda(A)^{-1}) \tag{3.31}$$
$$U(j)u(p,v) \bullet a^\dagger(p)U^\dagger(j) \overset{(3.29)}{=} u_c(p,v) \bullet a^\dagger(p) \tag{3.32}$$
$$U(j)u_c(p,v) \bullet a^\dagger(p)U^\dagger(j) \overset{(3.30)}{=} (-1)^{2s} u(p,v) \bullet a^\dagger(p) \tag{3.33}$$

A complex conjugate intertwiner $\overline{u_{(c)}}$ transforms analogously if the $\bullet$-product with the annihilation operators $a(p)$ is taken:

$$U(A,a)\overline{u_{(c)}(p,v)} \bullet a(p)U^\dagger(A,a) = (U(A,a)u_{(c)}(p,v) \bullet a^\dagger(p)U^\dagger(p))^\dagger \tag{3.34}$$
$$\overset{(3.31)}{=} (\mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a} u_{(c)}(p\Lambda(A)^{-1}, \widetilde{D}(A)v) \bullet a^\dagger(p\Lambda(A)^{-1}))^\dagger$$
$$= \mathrm{e}^{-\mathrm{i}p\Lambda(A)^{-1}a} \overline{u_{(c)}(p\Lambda(A)^{-1}, \widetilde{D}(A)v)} \bullet a(p\Lambda(A)^{-1})$$
$$U(j)\overline{u(p,v)} \bullet a(p)U^\dagger(j) = (U(j)u(p,v) \bullet a^\dagger(p)U^\dagger(j))^\dagger \tag{3.35}$$
$$\overset{(3.32)}{=} (u_c(p,v) \bullet a^\dagger(p))^\dagger = \overline{u_c(p,v)} \bullet a(p)$$
$$U(j)\overline{u_c(p,v)} \bullet a(p)U^\dagger(j) = (U(j)u_c(p,v) \bullet a^\dagger(p)U^\dagger(j))^\dagger \tag{3.36}$$
$$\overset{(3.33)}{=} (-1)^{2s}(u(p,v) \bullet a^\dagger(p))^\dagger = (-1)^{2s}\overline{u(p,v)} \bullet a(p)$$

The relations which have been derived in this section will be used to compute the transformation law of the corresponding quantum field.

3.2.2. Fields

The field operator $\Phi(x,v)$ for the measurement of Φ localized at $x \in \mathbb{M}$ in the spin state v is defined by

$$\Phi(x,v) := \int_{H_m^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}px} u(p,v) \bullet a^\dagger(p) + \mathrm{e}^{-\mathrm{i}px} \overline{u_c(p,v)} \bullet a(p) \tag{3.37}$$

It should be noted for later reference that hermitean conjugation of $\Phi(x,v)$ exchanges the normal and conjugate intertwiners $u_{(c)}$:

$$\Phi(x,v)^\dagger \overset{(3.37)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}px} u_c(p,v) \bullet a^\dagger(p) + \mathrm{e}^{-\mathrm{i}px} \overline{u(p,v)} \bullet a(p) \tag{3.38}$$

The canonical (anti)commutation relations are imposed on the creation and annihilation operators depending on the chosen spin s in the representation D of SU(2) by requiring that the s-dependent commutator $[A,B]_s := AB - (-1)^{2s} BA$, i.e.

$$[A,B]_s = \begin{cases} [A,B] = AB - BA & s \in \mathbb{N} \\ \{A,B\} = AB + BA & s \in (\mathbb{N}/2)\backslash\mathbb{N} \end{cases} \tag{3.39}$$

is δ-peaked in $L^2(H_m^+)$ for identical $\mathcal{H}_q$ indices for one annihilation and one creation operator and vanishing for all other possible combinations:

$$[a(p), a^\dagger(p')]_s = 2p_0 \delta(\vec{p} - \vec{p}')\mathbf{1}_{\mathcal{F}}\mathbf{1}_{\mathcal{H}_q} \text{ and } [a^\dagger(p), a^\dagger(p')]_s = [a(p), a(p')]_s = 0$$

In other words, fields with integer spin give rise to bosonic states while fields with half-integer spin give rise to fermionic states. This choice shall be justified by the spin-statistics theorem which arises from the discussion of the locality properties of the field $\Phi(x,v)$.

3.3. Properties of the fields

It is shown how the fields behave under a proper Poincaré transformation. The resulting formulas are used to prove corresponding identities for the two-point function. Together with the analyticity properties of this function, it can be shown that the fields are compatible with Einstein causality, i.e. they commute if their localization points are spacelike separated.

3.3.1. Transformation laws

The covariant transformation law as well as the form of the PCT symmetry can be derived with a straightforward calculation because all the necessary ingredients have already been collected.

Covariance

The action of U on the field is

$$
\begin{aligned}
&U(A,a)\Phi(x,v)U^\dagger(A,a)\\
\overset{(3.37)}{=}\;&\int_{H_m^+}\widetilde{\mathrm{d}p}\,[\mathrm{e}^{\mathrm{i}px}U(A,a)u(p,v)\bullet a^\dagger(p)U^\dagger(A,a)\\
&\qquad\qquad +\mathrm{e}^{-\mathrm{i}px}U(A,a)\overline{u_c(p,v)}\bullet a(p)U^\dagger(A,a)]\\
\overset{(3.31)(3.34)}{=}\;&\int_{H_m^+}\widetilde{\mathrm{d}p}\,[\mathrm{e}^{\mathrm{i}p(x+\Lambda(A)^{-1}a)}u(p\Lambda(A)^{-1},\widetilde{D}(A)v)\bullet a^\dagger(p\Lambda(A)^{-1})\\
&\qquad\qquad +\mathrm{e}^{-\mathrm{i}p(x+\Lambda(A)^{-1}a)}\overline{u_c(p\Lambda(A)^{-1},\widetilde{D}(A)v)}\bullet a(p\Lambda(A)^{-1})]\\
=\;&\int_{H_m^+}\widetilde{\mathrm{d}p}\,\mathrm{e}^{\mathrm{i}p(\Lambda(A)x+a)}u(p,\widetilde{D}(A)v)\bullet a^\dagger(p)+\mathrm{e}^{-\mathrm{i}p(\Lambda(A)x+a)}\overline{u_c(p,\widetilde{D}(A)v)}\bullet a(p)\\
\overset{(3.37)}{=}\;&\Phi(\Lambda(A)x+a,\widetilde{D}(A)v)
\end{aligned}
\tag{3.40}
$$

which shows that the localization point x transforms as a four-vector, while the spin-state v is sensitive to A itself without the covering map.

PCT symmetry

If $U(j)$ acts on the field operator, the result is

$$
\begin{aligned}
&U(j)\Phi(x,v)U^\dagger(j)\\
\overset{(3.37)}{=}\;&\int_{H_m^+}\widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}px}U(j)u(p,v)\bullet a^\dagger(p)U^\dagger(j)+\mathrm{e}^{\mathrm{i}px}U(j)\overline{u_c(p,v)}\bullet a(p)U^\dagger(j)\\
\overset{(3.32)(3.36)}{=}\;&\int_{H_m^+}\widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}px}u_c(p,v)\bullet a^\dagger(p)+\mathrm{e}^{\mathrm{i}px}(-1)^{2s}\overline{u(p,v)}\bullet a(p)\\
\overset{(3.38)}{=}\;&F\Phi(-x,v)^\dagger
\end{aligned}
\tag{3.41}
$$

The definition is analogous to 3.37, where F is used as a reminder that there is a prefactor $(-1)^{2s}$ in front of the annihilation part.

3.3.2. Locality

Since the fields are linear in the creation/annihilation operators, their (anti)commutator will be a multiple of identity. Therefore it is sufficient to consider the two-point function

$$\langle 0|\,\Phi(x,v)\Phi(x',v')\,|0\rangle=\int_{H_m^+}\widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p(x-x')}\underbrace{\overline{u_c(p,v)}\bullet u(p,v')}_{=:M(p,v,v')}=:\mathcal{W}(x-x',v,v')\tag{3.42}$$

Using the identities

$$\zeta^\dagger\overset{(3.11)}{=}-\zeta\text{ and }\zeta A\overset{(3.13)}{=}(A^T)^{-1}\zeta\tag{3.43}$$

the function

$$M(p,v,v') \stackrel{(3.8)(3.17)}{=} \overline{\widetilde{D}(\underset{\sim}{p}\zeta R_p)v} \bullet \widetilde{D}(R_p)v' = v \bullet \widetilde{D}((R_p\underset{\sim}{p}\zeta)^\dagger R_p)v' = v \bullet \widetilde{D}(\zeta^\dagger \underset{\sim}{p} R_p^\dagger R_p)v'$$
$$\stackrel{(3.43)(3.2)}{=} v \bullet \widetilde{D}(-\zeta \underset{\sim}{p}\widetilde{p})v' \stackrel{(2.35)}{=} v \bullet \widetilde{D}(-m^2\zeta)v' \tag{3.44}$$

can be evaluated. This form of $M(p,v,v')$ immediately implies the covariance property $M(p\Lambda,v,v) = M(p,v,v')$. It follows that

$$\begin{aligned} \langle 0|\,\Phi(x,)\Phi(x',v')\,|0\rangle &\stackrel{(3.42)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p(x-x')} M(p,v,v') \\ &= \int_{H_m^+} \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p\Lambda(A)(x-x')} \underbrace{M(p\Lambda(A),v,v')}_{=M(p,v,v')} \\ &\stackrel{(3.42)}{=} \langle 0|\,\Phi(\Lambda(A)x+a,v)\Phi(\Lambda(A)x'+a,v')\,|0\rangle \ . \end{aligned}$$

The covariance property for 3.44 can alternatively be stated in the form

$$\begin{aligned} M(p,\widetilde{D}(A)v,\widetilde{D}(A)v') &\stackrel{(3.44)}{=} \widetilde{D}(A)v \bullet \widetilde{D}(-m^2\zeta)\widetilde{D}(A)v' \stackrel{(3.6)}{=} v \bullet \widetilde{D}(-m^2 \underbrace{A^T\zeta A}_{\stackrel{(3.43)}{=}\zeta})v' \\ &\stackrel{(3.44)}{=} M(p,v,v') \end{aligned} \tag{3.45}$$

which directly implies $\mathcal{W}(x-x',\widetilde{D}(A)v,\widetilde{D}(A)v') = \mathcal{W}(x-x',v,v')$. Using $U(A,a)\,|0\rangle = |0\rangle$ this formula follows more systematically

$$\begin{aligned} \langle 0|\,\Phi(x,v)\Phi(x',v')\,|0\rangle &= \langle 0|\,U(A,a)\Phi(x,v)U(A,a)^\dagger U(A,a)\Phi(x',v')U(A,a)^\dagger\,|0\rangle \\ &\stackrel{(3.40)}{=} \langle 0|\,\Phi(\Lambda(A)x+a,\widetilde{D}(A)v)\Phi(\Lambda(A)x'+a,\widetilde{D}(A)v')\,|0\rangle \\ &\stackrel{(3.45)}{=} \langle 0|\,\Phi(\Lambda(A)x+a,v)\Phi(\Lambda(A)x'+a,v')\,|0\rangle \end{aligned}$$

This yields for the two-point function:

$$\mathcal{W}(x-x',v,v') = \mathcal{W}(\Lambda(x-x'),v,v') \tag{3.46}$$

Also the PCT symmetry can be translated into a property of the two-point function. From

$$\begin{aligned} M(p,v,v') &\stackrel{(3.44)}{=} v \bullet \widetilde{D}(-m^2\zeta)v' \stackrel{(3.5)}{=} (-1)^{2s} v \bullet \widetilde{D}(m^2\zeta)v' \\ &\stackrel{(3.43)}{=} (-1)^{2s} v \bullet \widetilde{D}(-m^2\zeta^\dagger)v' \stackrel{(3.5)}{=} (-1)^{2s} v \bullet \widetilde{D}(-m^2\zeta)^\dagger v' \\ &\stackrel{(3.5)}{=} (-1)^{2s} v' \bullet \widetilde{D}(-m^2\zeta)v \stackrel{(3.44)}{=} (-1)^{2s} M(p,v',v) \end{aligned} \tag{3.47}$$

it follows that

$$\begin{aligned} &\langle 0|\,\Phi(x,v)\Phi(x',v')\,|0\rangle \stackrel{(3.42)}{=} \int_{H_m^+} \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p(x-x')} M(p,v,v') \\ \stackrel{(3.47)}{=}\ &(-1)^{2s} \int_{H_m^+} \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p(-x'-(-x))} M(p,v',v) \stackrel{(3.42)}{=} (-1)^{2s} \langle 0|\,\Phi(-x',v')\Phi(-x,v)\,|0\rangle \ . \end{aligned}$$

Again, using $U(j)\left|0\right\rangle = \left|0\right\rangle$ as the reflection symmetry of the vacuum state, this formula can be derived systematically as well:[15]

$$\begin{aligned} \langle 0|\,\Phi(x,v)\Phi(x',v')\,|0\rangle &= \overline{\langle 0|\,U(j)\Phi(x,v)U(j)^\dagger U(j)\Phi(x',v')U(j)^\dagger\,|0\rangle} \\ &\overset{(3.41)}{=} \overline{\langle 0|\,F\Phi(-x,v)^\dagger F\Phi(-x',v')^\dagger\,|0\rangle} \\ &\overset{(3.41)}{=} (-1)^{2s}\overline{\langle 0|\,\Phi(-x,v)^\dagger\Phi(-x',v')^\dagger\,|0\rangle} \\ &= (-1)^{2s}\overline{\langle 0|\,(\Phi(-x',v')\Phi(-x,v))^\dagger\,|0\rangle} \\ &= (-1)^{2s}\,\langle 0|\,\Phi(-x',v')\Phi(-x,v)\,|0\rangle \end{aligned}$$

The corresponding result for the two-point function is:

$$\mathcal{W}(x-x',v,v') = (-1)^{2s}\mathcal{W}(x-x',v',v) \tag{3.48}$$

An observation crucial to showing the locality properties of the field $\Phi(x,v)$ is that $W(x-x',v,v')$ permits an analytic extension in the first argument into the tube

$$\mathcal{T} := \mathbb{M} - \mathrm{i}V^+ \tag{3.49}$$

because[16] $\Im(px) = p\Im(x) \leq 0$ for all $p \in \partial V^+$, $x \in \mathcal{T}$. Now if $(x-x')^2 < 0$ it can be w.l.o.g.[17] assumed that $x \in W$ and $x' \in W'$, where W is the standard wedge

$$W = \{x \in \mathbb{M} \mid x^3 > |x^0|\} \tag{3.50}$$

and W' its causal complement[18]

$$W' = \{x' \in \mathbb{M} \mid x'^3 < -|x'^0|\}\ . \tag{3.51}$$

The situation is visualized schematically in Fig. 3.1. The one-parameter group of boosts in the $\vec{e}_3$-direction

$$\Lambda(t) = \begin{pmatrix} \cosh t & & & \sinh t \\ & 1 & & \\ & & 1 & \\ \sinh t & & & \cosh t \end{pmatrix},\ j_0 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{3.52}$$

[15] The fact that $U(j)$ is an antilinear operator has been used in the first step. (3.22 and 3.10)

[16] In this argument the fact that if $x, y \in V+$, it follows that $x^0 \geq |\vec{x}|$ and $y^0 \geq |\vec{y}|$ and hence $xy = x^0y^0 - \vec{x}\cdot\vec{y} \geq |\vec{x}||\vec{y}| \geq 0$ has been used.

[17] This is justified the Poincaré-invariance of $W(x-x',v,v')$.

[18] The interior of the causal complement $V := \{x' \in \mathbb{M} \mid (x-x')^2 < 0\ \forall x \in W\}$ of the standard wedge W is $W' := V^\circ$. Now if $x'^3 > -|x'^0|$, pick $x := x' - (x'^0, 0, 0, -|x'^0|)$ with $x^3 = x'^3 + |x'^0| > 0 = |x^0|$. But $(x-x')^2 = (x'^0, 0, 0, -|x'^0|)^2 = 0$, hence $x' \notin V$. Conversely, if $x'^3 \leq -|x'^0|$ and $x \in W$, it follows $x^3 - x'^3 > |x^0| + |x'^0|$, therefore $(x-x')^2 \leq (x^0 - x'^0)^2 - (x^3 - x'^3)^2 < (x^0 - x'^0)^2 - (|x^0| + |x'^0|)^2 = -2(|x^0x'^0| + x^0x'^0) \leq 0$, hence $x' \in V$. In conclusion, $V = \{x' \in \mathbb{M} \mid x'^3 \leq -|x'^0|\}$, which shows that $W' = \{x' \in \mathbb{M} \mid x'^3 < -|x'^0|\}$.

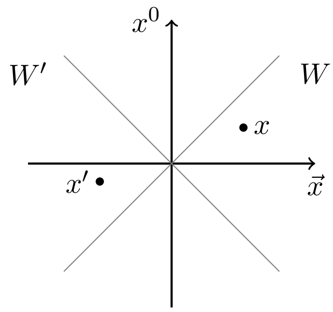

Figure 3.1.: Spacelike separated points x and x'

which preserves W and W' is needed in the following as well as the special reflection j_0, which exchanges them. The covariance and PCT-symmetry of the two-point function $\mathcal{W}(x-x', v, v')$ which have just been derived lead to the following exchange formula:

$$\begin{aligned} \mathcal{W}(x - j_0\Lambda(t)x', v, v') &\overset{(3.48)}{=} (-1)^{2s}\mathcal{W}(x - j_0\Lambda(t)x', v', v) \\ &\overset{(3.46)}{=} (-1)^{2s}\mathcal{W}(x' - j_0\Lambda(-t)x, v', v) \end{aligned} \tag{3.53}$$

The matrix-valued function $\Lambda(t)$ allows an analytic continuation into the whole complex plane. Denoting $t = t' + \mathrm{i}t''$ with $t', t'' \in \mathbb{R}$, it has the form[19]

$$\Lambda(t) = \cos t'' \begin{pmatrix} \cosh t' & & & \sinh t' \\ & 1 & & \\ & & 1 & \\ \sinh t' & & & \cosh t' \end{pmatrix} + \mathrm{i} \sin t'' \begin{pmatrix} \sinh t' & & & \cosh t' \\ & 0 & & \\ & & 0 & \\ \cosh t' & & & \sinh t' \end{pmatrix} . \tag{3.54}$$

[19]This decomposition follows from splitting the formula $\cosh t \pm \sinh t = \mathrm{e}^{\pm t} = \mathrm{e}^{\pm t'}\mathrm{e}^{\pm \mathrm{i}t''} = (\cosh t' \pm \sinh t')(\cos t'' \pm \mathrm{i}\sin t'')$ into the even and odd part.

For t is in the strip $\mathbb{R} + \mathrm{i}[0, \pi]$, where w.l.o.g.[20] $t' = 0$ is used, this implies

$$\begin{aligned}
\Im(j_0\Lambda(t)x') &\overset{(3.54)}{=} \sin t'' \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & 0 & & \\ & & 0 & \\ 1 & & & \end{pmatrix} \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} \\
&= \underbrace{\sin t''}_{\geq 0} \begin{pmatrix} -x'^3 \\ 0 \\ 0 \\ -x'^0 \end{pmatrix} \in V^+ \\
& \quad \text{because (using 3.51)} \; -x'^3 > |x'^0| \geq 0 \text{ and hence } (x'^3)^2 - (x'^0)^2 > 0 \\
\Im(j_0\Lambda(-t)x) &\overset{(3.54)}{=} -\sin t'' \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & 0 & & \\ & & 0 & \\ 1 & & & \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \\
&= -\underbrace{\sin t''}_{\geq 0} \begin{pmatrix} -x^3 \\ 0 \\ 0 \\ -x^0 \end{pmatrix} \in V^+ \\
& \quad \text{because (using 3.50)} \; x^3 > |x^0| \geq 0 \text{ and hence } (x^3)^2 - (x^0)^2 > 0 \; .
\end{aligned}$$

These equations show that for t in the strip the arguments on both sides of 3.53 are in the tube $\mathcal{T}$ from 3.49. By proceeding[21] to $t = \mathrm{i}\pi$, the exchange formula 3.53 extends analytically to

$$\mathcal{W}(x - x', v, v') = (-1)^{2s} \mathcal{W}(x' - x, v', v) \tag{3.55}$$

where $j_0\Lambda(\pm \mathrm{i}\pi) = \mathbf{1}$ has been used, which can directly be seen from 3.52 and 3.54. The process of analytic continuation relies on the fact that x and x' are spacelike separated. This formula implies that the vacuum expectation value of the commutator of the fields

$$\begin{aligned}
\langle 0| \, [\Phi(x, v), \Phi(x', v')]_s \, |0\rangle &\overset{(3.39)}{=} \langle 0| \, (\Phi(x, v)\Phi(x', v') - (-1)^{2s}\Phi(x', v')\Phi(x, v)) \, |0\rangle \\
&\overset{(3.42)}{=} \mathcal{W}(x - x', v, v') - (-1)^{2s}\mathcal{W}(x' - x, v', v) \overset{(3.55)}{=} 0
\end{aligned}$$

vanishes and since it is a multiple of identity, the commutator itself vanishes because of our CCR/CAR choice. Thus the final form for stating the locality of the fields reads

$$(x - x')^2 < 0 \Rightarrow [\Phi(x, v)\Phi(x', v')]_s = 0$$

[20] This is allowed because a change in t' preserves W and W'.

[21] This is possible because $t = \mathrm{i}\pi$ is an element of the strip.

The fact that the correspondence 3.39 between s and the choice between commutator or anticommutator was necessary to establish the locality of the fields leads to the spin-statistics theorem.

A nice property of the field $\Phi(x, v)$ is now evident, namely that the localization properties do not depend on the spin degrees of freedom, encoded in the vector v. In the following chapter, the construction of string-localized fields from [MSY05] will be discussed by starting analogously to the present chapter, only in the case $m = 0$. In that case the (infinite) spin degress of freedom will interfere with the localization properties.

3.4. Currents

Observables which are quadratic in the creation/annihilation operators can conveniently be constructed, at least in the bosonic case ($s \in \mathbb{N}$) by considering the Wick square

$$B(x) := {:}\Phi(x)^2{:}$$

The $: \cdot :$-symbol denotes normal ordering, i.e. the creation operators $a^\dagger$ are moved to the left and the annihilation operators a are moved to the right in the enclosed expression. These operators fulfil canonical commutation relations (CCR) because for $(x - x')^2 < 0$,

$$B(x)B(x') = {:}\Phi(x)^2{:}{:}\Phi(x')^2{:} = {:}({:}\Phi(x)^2{:}{:}\Phi(x')^2{:}){:} + \ldots [\Phi(x), \Phi(x')]_s$$

where in the first term x and x' can be exchanged and the second one vanishes by virtue of the above spacelike commutativity since every contraction between fields yields the corresponding commutator.

Concluding remarks

This chapter has made clear that the spin of the particle, i.e. the choice of a representation, has a great influence on the way the field operators have to be defined in order to satisfy the locality requirement. It has also been shown in how far analyticity considerations can be used to show locality.

The massless case, in particular the *infinite spin* representations, which are discussed in the following chapters, may be considered as even stronger examples of these features.

4. Massless representations with infinite spin

The discussion of the massless case starts analogously to the previous chapter with the construction of the one particle states which transform under the Wigner representations with infinite spin. The corresponding fields are string-localized. Showing their locality properties works like in the massive case to a large extent. Some candidates for observables which are intended to have pointlike localization are then shown in the next chapter.

4.1. Massless single-particle states with infinite spin

In this case there is no preferred choice of the reference momentum. This obstacle can be attributed to the fact that for massless particles, there is no rest frame. The structure of the little group is captured by a new intertwiner, which depends on an additional spacelike direction.

4.1.1. Little group in the massless case

The composition rules of the little group itself are derived and this group is then related to the group of rigid motions in the Euclidean plane via a twofold covering map. The resulting semidirect product structure allows for a characterization of the irreducible representations in a way similiar to the massive case.

Calculation of the little group

Now, $m = 0$. By 2.35, the possible momenta are in $\operatorname{sp} P = \partial V^+$. In the absence of any preferred choice for q it is conventional to pick

$$q = (1/2, 0, 0, 1/2) \Rightarrow \widetilde{q} = \frac{1}{2}(\mathbf{1} + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{4.1}$$

The massless Wigner boost B_p can then by defined by the element

$$R_p = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} p_0 + p_3 & p_1 - \mathrm{i}p_2 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \tag{4.2}$$

such that $\Lambda(R_p) = B_p$. It satisfies the property $q\Lambda(R_p) = p$, i.e. the reference momentum q is mapped to the given momentum $p \in \partial V^+$. Expressing this equation using the covering homomorphism Λ according to 2.17, it reads $R_p^\dagger \widetilde{q} R_p = \widetilde{p}$. Plugging in 4.1, the condition $R^\dagger \widetilde{q} R = \widetilde{q}$ for the little group becomes

$$\left(\begin{array}{cc} \overline{R_{11}} & \overline{R_{21}} \\ \overline{R_{12}} & \overline{R_{22}} \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right)\left(\begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \Rightarrow |R_{11}|^2 = 1 \wedge |R_{12}|^2 = 0 \ ,$$

hence $R_{11} = \mathrm{e}^{\mathrm{i}\varphi}$ with $\varphi \in \mathbb{R}$ and $R_{12} = 0$. The condition $R \in \mathrm{SL}(2,\mathbb{C})$ then gives

$$1 = \det R = R_{11}R_{22} - R_{12}R_{21} = \mathrm{e}^{\mathrm{i}\varphi} R_{22}$$

which gives $R_{22} = \mathrm{e}^{-\mathrm{i}\varphi}$ while R_{21} and hence $a := \mathrm{e}^{\mathrm{i}\varphi}\overline{R_{21}} \in \mathbb{C}$ is still arbitrary. The result of this computation is the following form of the litte group:

$$G_q = \{R \in \mathrm{SL}(2,\mathbb{C}) | R^\dagger \widetilde{q} R = \widetilde{q}\} = \left\{ \underbrace{\left(\begin{array}{cc} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ \mathrm{e}^{\mathrm{i}\varphi}\overline{a} & \mathrm{e}^{-\mathrm{i}\varphi} \end{array}\right)}_{=:[\varphi,a]} | \varphi \in \mathbb{R}, a \in \mathbb{C} \right\} \tag{4.3}$$

The group composition law for the little group G_q can be computed easily in the matrix form

$$\begin{aligned} [\varphi_2, a_2][\varphi_1, a_1] &\overset{(4.3)}{=} \left(\begin{array}{cc} \mathrm{e}^{\mathrm{i}\varphi_2} & 0 \\ \mathrm{e}^{\mathrm{i}\varphi_2}\overline{a_2} & \mathrm{e}^{-\mathrm{i}\varphi_2} \end{array}\right)\left(\begin{array}{cc} \mathrm{e}^{\mathrm{i}\varphi_1} & 0 \\ \mathrm{e}^{\mathrm{i}\varphi_1}\overline{a_1} & \mathrm{e}^{-\mathrm{i}\varphi_1} \end{array}\right) \\ &= \left(\begin{array}{cc} \mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} & 0 \\ \mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}\overline{a_2} + \mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}\overline{a_1} & \mathrm{e}^{-\mathrm{i}(\varphi_1+\varphi_2)} \end{array}\right) \\ &= \left(\begin{array}{cc} \mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} & 0 \\ \mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}\overline{a_2 + \mathrm{e}^{\mathrm{i}2\varphi_2} a_1} & \mathrm{e}^{-\mathrm{i}(\varphi_1+\varphi_2)} \end{array}\right) \\ &\overset{(4.3)}{=} [\varphi_1 + \varphi_2, \mathrm{e}^{\mathrm{i}2\varphi_2} a_1 + a_2] \end{aligned} \tag{4.4}$$

and it shows that G_q is related to the group $E(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$, which is the group of rigid motions (R_φ, a) with

$$R_\varphi := \left(\begin{array}{cc} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{array}\right) \in \mathrm{SO}(2) \Rightarrow R_{\varphi_1} R_{\varphi_2} = R_{\varphi_1+\varphi_2} \tag{4.5}$$

and $a \in \mathbb{R}^2$ in the Euclidean plane with the semidirect multiplication law

$$(R_{\varphi_2}, a_2)(R_{\varphi_1}, a_1) = (R_{\varphi_2} R_{\varphi_1}, R_{\varphi_2} a_1 + a_2) \ , \tag{4.6}$$

where the action of $\mathrm{SO}(2)$ on $\mathbb{R}^2$ is given by matrix multiplication.

Covering of the group of rigid motions in the Euclidean plane

In fact, a group homomorphism λ can be defined by

$$\begin{aligned} \lambda \ : \quad G_q &\to E(2) \\ [\varphi, a] &\mapsto (R_{2\varphi}, a) \end{aligned} \tag{4.7}$$

with the standard identification[1] between $\mathbb{C}$ and $\mathbb{R}^2$ for a. The same symbol is used for the covering of the pure rotations, which reads

$$\begin{aligned} \lambda \; : \; U(1) \subset G_q \; &\to \mathrm{SO}(2) \\ [\varphi, 0] \; &\mapsto R_{2\varphi} \; . \end{aligned}$$

Checking the group homomorphism property is straightforward:

$$\begin{aligned} \lambda([\varphi_2, a_2])\lambda([\varphi_1, a_1]) &\overset{(4.7)}{=} (R_{2\varphi_2}, a_2)(R_{2\varphi_1}, a_1) \overset{(4.6)}{=} (R_{2\varphi_2}R_{2\varphi_1}, R_{2\varphi_2}a_1 + a_2) \\ &\overset{(4.5)}{=} (R_{2(\varphi_1+\varphi_2)}, R_{\varphi_2}a_1 + a_2) \overset{(4.7)}{=} \lambda([\varphi_1 + \varphi_2, \mathrm{e}^{\mathrm{i}2\varphi_2}a_1 + a_2]) \\ &\overset{(4.4)}{=} \lambda([\varphi_2, a_2][\varphi_1, a_1]) \end{aligned}$$

However, since $\lambda^{-1}(\{(R_\varphi, a)\}) = \{[\varphi/2, a], [\varphi/2 + \pi, a]\}$ for all $(R_\varphi, a) \in E(2)$, in other words $\lambda([\varphi, a]) = \lambda([\varphi + \pi, a])$ on all of G_q, the homomorphism $G_q \xrightarrow{\lambda} E(2)$ is in fact a double covering, i.e. $G_q = \widetilde{E}(2)$.[2]

Irreducible representations of the little group

The semidirect product-structure of $\widetilde{E}(2)$ is similar to $\mathcal{P}^c$ whose irreducible representation spaces have already been classified in the general discussion 2.3.1. For the representations D of G_q it is possible to proceed analogously by making the definition $D(a) := U([0, a])$ for the pure translations and writing

$$D([0, a]) = \mathrm{e}^{-\mathrm{i}K\overline{a}} \tag{4.8}$$

with the generators K_i of the translation subgroup, whose properties can be derived in the same way as those for the momenta P_μ before. The standard scalar product in $\mathbb{R}^2$ is used in the expression Ka, as well as an implicit identification between $\mathbb{C}$ and $\mathbb{R}^2$. The complex conjugate is identified with $\overline{(z_1\vec{e}_1 + z_2\vec{e}_2)} := z_1\vec{e}_1 - z_2\vec{e}_2$ correspondingly.

Comparing the following construction of representations D with 2.3.1, there are some changes of signs, which may seem arbitrary and not motivated at this point. They do not have an intrinsic meaning, but have been used to achieve consistency with the string-dependent intertwiners which have been defined in [MSY05].

[1] $z = z_1 + \mathrm{i}z_2 \in \mathbb{C}$ with $z_1, z_2 \in \mathbb{R}$ is identified with $\vec{r} = z_1\vec{e}_1 + z_2\vec{e}_2 \in \mathbb{R}^2$, where $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. Vector arrows for elements of $\mathbb{R}^2$ will be omitted because they are already in use for the spatial part of a four-vector $x = (x^0, \vec{x}) \in \mathbb{M}$.

[2] Both elements of G_q are in fact mapped to the same movement $(R_\varphi, a) \in E(2)$ by λ, which can be seen from the fact that the extra term on the rhs. amounts to an extra angle φ of 2π while $R_{\varphi+2\pi} = R_\varphi$ on the lhs. Taking into account the group structure of G_q and $E(2)$ it would be sufficient to consider only the preimage $\lambda^{-1}(\{(\mathbf{1}, 0)\}) = \{[1, 0], [-1, 0]\}$ of the neutral element in $E(2)$.

For the homogeneous elements $D(\varphi) := D([\varphi, 0])$ the representation property of D implies

$$\begin{aligned} D(\varphi)\mathrm{e}^{-\mathrm{i}K\overline{a}} &\overset{(4.8)}{=} D([\varphi,0])D([0,a]) \overset{(2.50)}{=} D([\varphi,0][0,a]) \overset{(4.4)}{=} D([\varphi,\mathrm{e}^{\mathrm{i}2\varphi}a]) \\ &\overset{(4.4)}{=} D([0,\mathrm{e}^{\mathrm{i}2\varphi}a][\varphi,0]) \overset{(2.50)}{=} D([0,\mathrm{e}^{\mathrm{i}2\varphi}a])D([\varphi,0]) \overset{(4.8)}{=} \mathrm{e}^{-\mathrm{i}k\overline{\lambda(\varphi)a}}D(\varphi)\ , \end{aligned} \tag{4.9}$$

with $\lambda(\varphi) := R_{2\varphi}$. The derivatives w.r.t. the components a_i at $a = 0$ give the transformation law of the K_i:[3]

$$D(\varphi)K = \mathrm{i}\frac{\partial}{\partial\overline{a}}\, D(\varphi)\mathrm{e}^{-\mathrm{i}K\overline{a}}\Big|_{a=0} \overset{(4.9)}{=} \mathrm{i}\frac{\partial}{\partial\overline{a}}\,\mathrm{e}^{-\mathrm{i}K\overline{\lambda(\varphi)a}}D(\varphi)\Big|_{a=0} = K\lambda(-\varphi)D(\varphi) \tag{4.10}$$

The analogous implications for the symmetry of the spectrum of K as well as the construction of the Casimir operator are:

- The joint spectrum of K is invariant under $\lambda(\varphi)$ because for a state vector $|\psi\rangle$ with inner momentum $k \in \mathbb{R}^2$, i.e. the measurement value of K is sharp, the eigenvalue equation

  $$K\,|\psi\rangle = k\,|\psi\rangle \tag{4.11}$$

  together with the transformation law for the operators K implies the new eigenvalue equation

  $$K[D(\varphi)\,|\psi\rangle] \overset{(4.10)}{=} D(\varphi)K\lambda(-\varphi)\,|\psi\rangle \overset{(4.11)}{=} D(\varphi)k\lambda(-\varphi)\,|\psi\rangle = k\lambda(-\varphi)[D(\varphi)\,|\psi\rangle]\ .$$

  This means that the state $|\psi\rangle$ with sharp inner momentum k has been transformed into the state $D(\varphi)\,|\psi\rangle$ of sharp inner momentum $k\lambda(-\varphi)$. Consequently, if k belongs to the spectrum of K, so does the whole circle $k\lambda([0, 4\pi[)$.

- The Casimir operator $K^2 := K_iK_i := K_1^2 + K_2^2$ takes on the role of P^2 because it transforms as a scalar under rotations:

  $$\begin{aligned} D(\varphi)K^2 &= D(\varphi)K_iK_i = K_j\lambda(-\varphi)_{ji}D(\varphi)K_i = K_j\underbrace{\lambda(-\varphi)_{ji}\lambda(-\varphi)_{ki}}_{\delta_{jk}}K_iD(\varphi) \\ &= K^2D(\varphi) \end{aligned}$$

  This transformation rule can be written as

  $$[D(\varphi), K^2] = 0$$

  and by Schur's Lemma it can be concluded that $K^2 = \kappa^2\mathbf{1}$ with $\kappa \in \mathbb{R}_0^+$ in an irreducible representation of G_q.

[3] The identity $\overline{\lambda(\varphi)a} = \lambda(-\varphi)\overline{a}$ has been used in the last step.

Now there are two important classes of possible values for the Pauli Lubanski-parameter[4] κ:

- If $\kappa = 0$, the translation operators are simply $D(a) = \mathbf{1}$, i.e. the representation of the subgroup of translations $[0, a]$ is trivial. In this case only a representation $D(\varphi)$ of the subgroup of rotations $[\varphi, 0]$ needs to be constructed, which can by given by

$$D(\varphi) := [\varphi, 0]^n = \mathrm{e}^{\mathrm{i}n\varphi} \text{ with } n \in \mathbb{Z} \tag{4.12}$$

acting by multiplication on $\mathcal{H}_q = \mathbb{C}$, where the number $n/2$ is called the helicity. These one-dimensional representations can be recovered as a massless limit of 3.5, with $s = n/2$, by using the subspace $\mathbb{C}(1,0)^{\otimes 2s} \subset \mathrm{Sym}((\mathbb{C}^2)^{\otimes 2s})$ of the little Hilbert space for the massive representations. Now the action on a vector $z(1,0)^{\otimes n}$ is

$$D(\varphi)z(1,0)^{\otimes n} \overset{(3.5)}{=} z(D([\varphi,0])(1,0))^{\otimes n} \overset{(4.12)}{=} z(\mathrm{e}^{\mathrm{i}\varphi}(1,0))^{\otimes n} = \mathrm{e}^{\mathrm{i}n\varphi} z(1,0)^{\otimes n} ,$$

which is the representation formula 4.12 from above.

- If $\kappa > 0$, it is possible to proceed analogously to the construction of little Hilbert spaces $\mathcal{H}_p$, which was done in 2.3.2. A brief review of how the construction is modified for the remaining representations of $\widetilde{E}(2)$ is given here. Reference to the corresponding calculations, which have already been done for $\mathcal{P}^c$, is given where appropriate.
$\mathcal{D}_q$ is defined as the subspace of $\mathcal{H}_q$, where the matrix elements $\langle v|D(a)|w\rangle$ are rapidly decreasing for $|a| \to \infty$. Analogously to 2.36, the definition

$$\langle v|w\rangle_\varphi := \frac{1}{(2\pi)^2} \int \mathrm{d}^2 a \, \mathrm{e}^{-\mathrm{i}k(\varphi)\overline{a}} \langle v|U(a)|w\rangle \tag{4.13}$$

is made with $k(\varphi) = \kappa(\cos\varphi, \sin\varphi)$, which is positive semidefinite. For each $k(\varphi)$, using the seminorm $||v||_\varphi := \sqrt{\langle v|v\rangle_\varphi}$ and the corresponding set $\mathcal{N}_{q\varphi} := \{|v\rangle \in \mathcal{D}_q | \; ||v||_\varphi = 0\}$ of isotropic vectors, the little Hilbert space[5] is defined as

$$\mathcal{H}_{q\varphi} := \overline{\mathcal{D}_q/\mathcal{N}_{q\varphi}}^{||\cdot||_\varphi} .$$

Because of the unique mapping $\lambda(\varphi)$, the definition $\mathcal{H}_{qk} := \mathcal{H}_{q\varphi}$ is possible in order to simplify the notation. The notation $|v\rangle(k) := |v\rangle / \mathcal{N}_{q\varphi}$ is used for the equivalence class in $\mathcal{H}_{qk}$ corresponding to the little Hilbert space vector $|v\rangle \in \mathcal{D}_q$. The condition $k^2 = \kappa^2$ allows for a reconstruction of the matrix elements for the translations:

$$\langle v|D(a)|w\rangle = \int \mathrm{d}^2 k \, \mathrm{e}^{\mathrm{i}k\overline{a}} \langle v|w\rangle_\varphi = \int \underbrace{\mathrm{d}^2 k \, \delta(k^2 - \kappa^2)}_{=:\mathrm{d}\nu(k)} \mathrm{e}^{\mathrm{i}k\overline{a}} \underbrace{2\kappa \langle v|w\rangle_\varphi}_{=:\langle v|w\rangle_k}$$

[4] The relation of this parameter to the massive representations is given in [MSY04] and explicitly introduced as another Casimir operator in [Pen04, p. 568]

[5] This is not to be confused with $\mathcal{H}_q$ itself, which had been introduced w.r.t. the spatial translations.

In the first step, the relation $k(\varphi)$ is implicitly used and justifies the integration over all $k \in \mathbb{R}^2$ because $\langle v|w\rangle_\varphi$ vanishes for $k^2 \neq \kappa^2$.

The action of a pure rotation is used to define a transformation between the Hilbert spaces $\mathcal{H}_{qk}$ of sharp inner momentum:

$$\underline{D}(\varphi)\,|v\rangle\,(k) := (D(\varphi)\,|v\rangle)(k\lambda(\varphi)) \tag{4.14}$$

The fact that $\underline{D}$ preserves the scalar product between $\mathcal{H}_{qk}$ and $\mathcal{H}_{qk\lambda(-\varphi)}$ as well as the representation property can be checked in a completely analogous way to 2.41 and 2.42, respectively. Now the construction of a Wigner boost

$$B_k := \lambda(R_k) \text{ with the property } lB_k = k$$

for the fixed inner reference momentum $l := (\kappa, 0)$ is easily done by using the following rotation matrix

$$B_k := \frac{1}{\kappa}\begin{pmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{pmatrix} \overset{(4.7)}{=} \lambda\Bigg(\underbrace{\frac{1}{\sqrt{\kappa}}\begin{pmatrix} \sqrt{k_1 + \mathrm{i}k_2} & \\ & \overline{\sqrt{k_1 + \mathrm{i}k_2}} \end{pmatrix}}_{=:R_k}\Bigg) \tag{4.15}$$

The definition of R_k is made unique by the requirement that $\arg\sqrt{\cdots} \in [0, \pi)$. This Wigner boost admits the identification

$$\begin{aligned} V_q \;:\; \mathcal{H}_q &\;\to\; \mathcal{K}_q := L^2(\mathrm{sp}K) \otimes \mathcal{H}_{ql} \\ |v\rangle &\;\mapsto\; [v \;:\; k \mapsto v(k) := \underline{D}(R_k)^{-1}\,|v\rangle\,(k) = (D(R_k)^{-1}\,|v\rangle)(l)] \end{aligned} \tag{4.16}$$

between the little Hilbert space $\mathcal{H}_q$ and an L^2-space of wavefunctions on the spectrum of K, taking values in $\mathcal{H}_{qk}$. The representation D of G_q on $\mathcal{H}_q$ can be transformed into a representation

$$D_1 := VDV^{-1} \tag{4.17}$$

on $\mathcal{K}_q$, where the inverse of V_q is given by

$$\begin{aligned} V_q^{-1} \;:\; \mathcal{K}_q &\;\to\; \mathcal{H}_q \\ v &\;\mapsto\; |v\rangle := \int_{\mathrm{sp}K}^{\oplus} \mathrm{d}\nu(k)\,\underline{D}(R_k) v(k)\;. \end{aligned} \tag{4.18}$$

The representation property of D_1, which can be checked analogously to 2.47, is

$$D_1([\varphi_1, a_1])D_1([\varphi_2, a_2]) = D_1([\varphi_1, a_1][\varphi_2, a_2]) \tag{4.19}$$

On $\mathcal{K}_q$, concrete formulas can be derived for the action $D_1(a)$:

$$\begin{aligned} [D_1(a)v](k) &\overset{(4.17)}{=} [V_q D(a) V_q^{-1} v](k) \overset{(4.16)}{=} \underline{D}(R_k)^{-1}[D(a)V_q^{-1}v](k) \\ &\overset{(4.8)}{=} \underline{D}(R_k)^{-1}\mathrm{e}^{-\mathrm{i}k\bar{a}}[V^{-1}v](k) \overset{(4.18)}{=} \mathrm{e}^{-\mathrm{i}k\bar{a}}v(k) \end{aligned} \tag{4.20}$$

and $D_1(\varphi)$ on a function $v \in L^2(\mathrm{sp}K)$:

$$\begin{aligned}
[D_1(\varphi)v](k) &\overset{(4.17)}{=} VU(\varphi)V^{-1}v(k) \overset{(4.16)}{=} \underline{D}(R_k)^{-1}[D(\varphi)V^{-1}v](k) \\
&\overset{(4.14)}{=} \underline{D}(R_k)^{-1}\underline{D}(\varphi)[V^{-1}v](k\lambda(-\varphi)) \\
&\overset{(4.18)}{=} \underline{D}(R_k)^{-1}\underline{D}(\varphi)\underline{D}(R_{k\lambda(-\varphi)})v(k\lambda(-\varphi)) \\
&= \underline{D}(R_k^{-1}[\varphi,0]R_{k\lambda(-\varphi)})v(k\lambda(-\varphi))
\end{aligned} \tag{4.21}$$

This is a consistent formula because $\lambda(R_k[\varphi,0]R^{-1}_{k\lambda(-\varphi)}) \in \mathrm{stab}\, l \subset \mathrm{SO}(2)$, which means that the Wigner rotation[6] leaves the reference inner momentum invariant. Just as before, for the case $\underline{D} : \mathcal{H}_{ql} \to \mathcal{H}_{ql}$ only a representation (now called S) of the little group $G_{ql} := \lambda^{-1}(\mathrm{stab}\, l)$ needs to be constructed. Defining

$$R(\varphi, k) := R_k[\varphi, 0]R^{-1}_{k\lambda(-\varphi)} \tag{4.22}$$

the representation of a rotation reads

$$[D_1(\varphi)v](k) = S(R(\varphi, k))v(k\lambda(-\varphi)) \tag{4.23}$$

and hence, for an arbitrary element $[\varphi, a] \in \widetilde{E}(2)$, the action on v is

$$\begin{aligned}
[D_1([\varphi,a])v](k) &\overset{(4.4)}{=} [D_1([0,a][\varphi,0])v](k) \overset{(4.19)}{=} [D_1(a)D_1(\varphi)v](k) \\
&\overset{(4.20)}{=} \mathrm{e}^{-\mathrm{i}k\bar{a}}[D_1(\varphi)v](k) \overset{(4.23)}{=} \mathrm{e}^{-\mathrm{i}k\bar{a}}S(R,\varphi,k)v(k\lambda(-\varphi)) \; .
\end{aligned} \tag{4.24}$$

In order to further investigate the case $\kappa > 0$, a concrete description of the little group

$$\begin{aligned}
G_{ql} &= \lambda^{-1}(\mathrm{stab}\, l) = \{[\varphi, 0] \in G_q \mid l\lambda([\varphi,0]) = l\} \\
&= \left\{ [\varphi,0] \in G_q \mid \begin{pmatrix} \kappa & 0 \end{pmatrix} \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{pmatrix} = \begin{pmatrix} \kappa & 0 \end{pmatrix} \right\} \\
&= \{[0,0],[\pi,0]\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

is given. The irreducible representations of G_{ql} on $\mathcal{H}_{ql} = \mathbb{C}$ are

- the trivial representation with $S(\pm\mathbf{1}) = 1$ and
- the faithful representation with $S(\pm\mathbf{1}) = \pm 1$.

[6] At this point, a remark concerning the terminology is in order: The terms Wigner boost/Wigner rotation have been chosen to reflect the roles these transformations play for the Poincaré group $\mathcal{P}^c$ itself. In the case of $\widetilde{E}(2)$, the Wigner boost is in fact a rotation of the circle and the Wigner rotation is a possible change in the choice of λ^{-1}.

The question in which cases the nontrivial element of G_{ql} occurs can be answered by calculating the Wigner rotation $R(\varphi, k)$. Inserting

$$
\begin{aligned}
k\lambda(-\varphi) &\overset{(4.5)}{=} \begin{pmatrix} k_1 & k_2 \end{pmatrix} \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \\
&= \begin{pmatrix} \Re(\mathrm{e}^{-\mathrm{i}2\varphi}(k_1 + \mathrm{i}k_2)) & \Im(\mathrm{e}^{-\mathrm{i}2\varphi}(k_1 + \mathrm{i}k_2)) \end{pmatrix}
\end{aligned}
$$

into the definition of the Wigner boost yields

$$
\begin{aligned}
R_{k\lambda(-\varphi)} \overset{(4.15)}{=} & \begin{pmatrix} \sqrt{\mathrm{e}^{-\mathrm{i}2\varphi}(k_1 + \mathrm{i}k_2)} & \\ & \overline{\sqrt{\mathrm{e}^{-\mathrm{i}2\varphi}(k_1 + \mathrm{i}k_2)}} \end{pmatrix} \\
& \cdot \begin{cases} +1 & \text{for } \arg(k_1 + \mathrm{i}k_2) - 2\varphi \in [0, 2\pi) + 4\pi\mathbb{Z} \\ -1 & \text{for } \arg(k_1 + \mathrm{i}k_2) - 2\varphi \in [2\pi, 4\pi) + 4\pi\mathbb{Z}\ . \end{cases}
\end{aligned}
$$

The additional factor ± 1 is necessary to ensure that the complex square root evaluates into the correct range, as indicated in the definition of R_k. It makes use of the fact that the domain of definition for the complex square root is a Riemann surface which covers the complex numbers twice with a branch point at 0, where the foil $\arg^{-1}([0, 2\pi))$ is mapped to $\arg^{-1}([0, \pi))$, i.e. the union of open upper half plane with the positive reals. The complete Wigner rotation is

$$
\begin{aligned}
R(\varphi, k) &\overset{(4.22)}{=} R_k[\varphi, 0] R^{-1}_{k\lambda(-\varphi)} \\
&= \pm \frac{1}{\sqrt{\kappa}} \begin{pmatrix} \overline{\sqrt{k_1 + \mathrm{i}k_2}} & \\ & \sqrt{k_1 + \mathrm{i}k_2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & \\ & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \sqrt{\mathrm{e}^{-\mathrm{i}2\varphi}(k_1 + \mathrm{i}k_2)} & \\ & \overline{\sqrt{\mathrm{e}^{-\mathrm{i}2\varphi}(k_1 + \mathrm{i}k_2)}} \end{pmatrix} = \pm \mathbf{1}\ .
\end{aligned}
$$

In other words, the faithful representation S gives an additional sign every time the rotation of k crosses the reference momentum l. Therefore, the representation space $L^2(\{q \in \mathbb{R}^2 | q^2 = k^2\})$ can be used for D. In the following discussion, only the case $S = 1$, the trivial representation of G_{ql}, is considered.

4.1.2. Intertwiners for the trivial representation of S

Identity component $\mathcal{L}_+^\uparrow$

The first step is the construction of intertwiners which simplify the transformation behaviour. These are chosen as in [MSY05] and constructed in the following way: As a first step, the parametrization[7]

$$
\xi(z) := \left(\frac{1}{2}(z^2 + 1), z_1, z_2, \frac{1}{2}(z^2 - 1)\right) \overset{(2.6)}{\Rightarrow} \widetilde{\xi(z)} = \begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix} \tag{4.25}
$$

[7]The identification of z with the complex number $z := z_1 + \mathrm{i}z_2$ is implicitly used.

of the G_q-orbit $\Gamma = \{p \in \partial V^+ | pq = 1\}$ is defined.[8] Via

$$
\begin{aligned}
(\xi(z)\Lambda([\varphi,a]))\widetilde{\ } &\overset{(2.17)}{=} [\varphi,a]^\dagger \widetilde{\xi(z)}[\varphi,a] \\
&\overset{(4.3)(4.25)}{=} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi} & \mathrm{e}^{-\mathrm{i}\varphi}a \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} \underbrace{\begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ \mathrm{e}^{\mathrm{i}\varphi}\overline{a} & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix}}_{= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi}(|z|^2 + \overline{za}) & \mathrm{e}^{-\mathrm{i}\varphi}\overline{z} \\ \mathrm{e}^{\mathrm{i}\varphi}(z+\overline{a}) & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix}} \\
&= \begin{pmatrix} |z|^2 + \overline{za} + za + |a|^2 & \mathrm{e}^{-2\mathrm{i}\varphi}(\overline{z}+a) \\ \mathrm{e}^{2\mathrm{i}\varphi}(z+a) & 1 \end{pmatrix} \\
&= \begin{pmatrix} |\mathrm{e}^{2\mathrm{i}\varphi}(z+\overline{a})|^2 & \mathrm{e}^{-2\mathrm{i}\varphi}(\overline{z}+a) \\ \mathrm{e}^{2\mathrm{i}\varphi}(z+a) & 1 \end{pmatrix} \overset{(4.25)}{=} \xi(\widetilde{R_{2\varphi}(z+\overline{a})})
\end{aligned}
\tag{4.26}
$$

the action of the little group G_q on the parameter space $\mathbb{R}^2$ is given by a rigid motion. On the little group orbit Γ, functions v are naturally transformed by Wigner rotations $[\varphi, a]$ via pullback, that is

$$
[\tilde{D}([\varphi,a])v](\xi) := v(\xi\Lambda([\varphi,a])) \ . \tag{4.27}
$$

The parametrization ξ allows for the construction of an intertwiner V with $V\widetilde{D} = \underline{U}V$ by

$$
u(k) := [Vv](k) = \int \mathrm{d}^2 z \, \mathrm{e}^{\mathrm{i}kz} v(\xi(z)) \ , \tag{4.28}
$$

which relates the representation $\widetilde{D}$ with the representation D of G_q from 4.24 in the following way:

$$
\begin{aligned}
& [V[\tilde{D}([\varphi,a])v]](k) \overset{(4.28)}{=} \int \mathrm{d}^2 z \mathrm{e}^{\mathrm{i}kz}[\tilde{D}([\varphi,a])v](\xi(z)) \\
\overset{(4.27)}{=} & \int \mathrm{d}^2 z \mathrm{e}^{\mathrm{i}kz} v(\xi(z)\Lambda([\varphi,a])) \overset{(4.26)}{=} \int \mathrm{d}^2 z \mathrm{e}^{\mathrm{i}kz} v(\xi(R_{2\varphi}(z+\overline{a}))) \\
= & \ \mathrm{e}^{-\mathrm{i}k\overline{a}} \int \mathrm{d}^2 z \mathrm{e}^{\mathrm{i}kz} v(\xi(R_{2\varphi}z)) = \mathrm{e}^{-\mathrm{i}k\overline{a}} \int \mathrm{d}^2 z \mathrm{e}^{\mathrm{i}kR_{-2\varphi}z} v(\xi(z)) \overset{(4.28)}{=} \mathrm{e}^{-\mathrm{i}k\overline{a}}[Vv](kR_{-2\varphi}) \\
= & \ [D([\varphi,a])u](k)
\end{aligned}
\tag{4.29}
$$

4.1.3. Intertwiners on Γ

The wavefunction v is given by a momentum $p \in \partial V^+$ and a spacelike direction vector $e \in H$, where $H := \{e \in \mathbb{M} \mid e^2 = -1\}$ according to the formula

$$
v(p,e)(\xi) := F(\xi\Lambda(R_p)e) \tag{4.30}
$$

[8] Γ is indeed an orbit of G_q because $\Lambda(G_q)q = \{q\}$ and hence $\Lambda(R)pq = p\Lambda(R)^{-1}q = pq = 1$ for all $R \in G_q$, $p \in \Gamma$. Of course, as $\Lambda(G_q)$ is a set of Lorentz transformations, the property $p \in \partial V^+$ is also preserved.

and has the following transformation behaviour:

$$
\begin{aligned}
[\widetilde{D}(R(A,p))v(p\Lambda(A),e)](\xi) &\overset{(4.27)}{=} v(p\Lambda(A),e)(\xi\Lambda(R(a,p))) \\
&\overset{(4.30)}{=} F(\xi\Lambda(\underbrace{R_p A R^{-1}_{p\Lambda(A)}}_{=R(A,p)})\Lambda(R_{p\Lambda(A)})e) \\
&\overset{(2.12)}{=} F(\xi\Lambda(R_p)\Lambda(A)e) \overset{(4.30)}{=} v(p,\Lambda(A)e)(\xi)
\end{aligned}
\tag{4.31}
$$

The intertwiner for string-localized fields is then defined by

$$
\begin{aligned}
u \;:\; \partial V^+ \times H &\rightarrow \mathcal{H}_q \\
(p,e) &\mapsto \left[u(p,e) \;:\; k \mapsto u(p,e)(k) := \int_{\mathbb{R}^2} \mathrm{d}^2 z\, \mathrm{e}^{\mathrm{i}kz} v(p,e)(\xi(z)) \right] ,
\end{aligned}
$$

where at least the structure of the definition is similar to 3.8. The above discussion of the intertwiner V then yields the *intertwiner equation*

$$
\begin{aligned}
D(R(A,p))u(p\Lambda(A),e) &\overset{(4.28)}{=} D(R(A,p))Vv(p\Lambda(A),e) \overset{(4.29)}{=} V\widetilde{D}(R(A,p))v(p\Lambda(A),e) \\
&\overset{(4.31)}{=} Vv(p,\Lambda(A)e) \overset{(4.28)}{=} u(p,\Lambda(A)e) ,
\end{aligned}
\tag{4.32}
$$

which swaps the action of the Wigner rotation $R(A,p)$ with a simple Lorentz transformation of e.

Extension to the proper Lorentz group $\mathcal{L}_+$

The action of $-j_0$, where j_0 is the reflection across the edge of the standard wedge (cf. 3.52), on an arbitrary momentum $p \in \Gamma$, using

$$
\begin{aligned}
(-pj_0)\widetilde{\ } &\overset{(2.6)}{=} \begin{pmatrix} p^0 - p^3 & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & p^0 - p^3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} p^0 - p^3 & -p^1 + \mathrm{i}p^2 \\ -p^1 - \mathrm{i}p^2 & p^0 + p^3 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \overset{(2.6)(2.7)}{=} \sigma_3 \widetilde{p} \sigma_3 ,
\end{aligned}
\tag{4.33}
$$

is implemented as conjugation with σ_3 in the covering homomorphism $\widetilde{\cdot}: \mathbb{M} \to M_{2\times 2}(\mathbb{C})$. This makes sense intuitively, since $-j_0$ is a rotation around the 3-axis about an angle of π, and yields

$$
\begin{aligned}
(-\xi(z)j_0)\widetilde{\ } &= \sigma_3 \widetilde{\xi(z)} \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} |z|^2 & -\overline{z} \\ -z & 1 \end{pmatrix} \\
&= \widetilde{\xi(-z)} ,
\end{aligned}
\tag{4.34}
$$

i.e. the corresponding action on the parametrization ξ of Γ is a reflection of the parameter z. On the other hand, if the Lorentz transformation corresponding to an element $[\varphi, a]$ of

G_q is conjugated with j_0, the result is

$$\begin{aligned}\xi(z)j_0\Lambda([\varphi,a])j_0 &= -\xi(z)j_0\Lambda([\varphi,a])(-j_0) = \xi(-z)\Lambda([\varphi,a])(-j_0)\\ &= -\xi(R_{2\varphi}(-z+\overline{a}))j_0 = \xi(R_{2\varphi}(z-\overline{a})) = \xi(z)\Lambda([\varphi,-a])\end{aligned}$$

which yields the relation[9]

$$j_0\Lambda([\varphi,a])j_0 = \Lambda([\varphi,-a])$$

The corresponding action of the conjugated element $[\varphi,-a] \in G_q$ on $u \in \mathcal{K}_q$ is

$$[D([\varphi,-a])]u(k) = \mathrm{e}^{-\mathrm{i}k\overline{a}}u(k\lambda(\varphi)) = \overline{\mathrm{e}^{\mathrm{i}k\overline{a}}\overline{u(k\lambda(\varphi))}} = \overline{D([\varphi,a])\overline{u(k)}}$$

and hence the conjugation can be factorized in D if $D(j_0)$ is defined to be the complex conjugation:

$$D([\varphi,-a]) = D(j_0)D([\varphi,a])D(j_0) \tag{4.35}$$

In this way, the representation D of G can be extended to an involutive representation of G_q with the additional reflection j_0.

4.1.4. Conjugate intertwiner on $-\Gamma$

To the complex conjugate reflected function $\overline{v(p,e)(-\xi)}$ belongs the conjugate intertwiner

$$u_c(p,e)(k) = \int \mathrm{d}^2z\,\mathrm{e}^{\mathrm{i}kz}\overline{v(p,e)(-\xi(z))}\;. \tag{4.36}$$

Using the conjugation property of the Wigner boost R_p under σ_3

$$\begin{aligned}\sigma_3 R_p \sigma_3 &\overset{(4.2)}{=} \begin{pmatrix} 1 & \\ & -1\end{pmatrix}\frac{1}{\sqrt{p_0+p_3}}\begin{pmatrix} p_0+p_3 & p_1-\mathrm{i}p_2\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & \\ & -1\end{pmatrix}\\ &= \frac{1}{\sqrt{p_0+p_3}}\begin{pmatrix} p_0+p_3 & -p_1+\mathrm{i}p_2\\ 0 & 1\end{pmatrix} \overset{(4.2)}{=} R_{-j_0p}\\ \overset{(4.33)}{\Rightarrow}\; j_0\Lambda(R_p)j_0 &= \Lambda(R_{-j_0p})\end{aligned}$$

the following conjugation formula for the function v can be shown:

$$\begin{aligned}v(p,e)(-\xi(-z)) &\overset{(4.30)}{=} F(-\xi(-z)\Lambda(R_p)e) \overset{(4.34)}{=} F(\xi(z)j_0\Lambda(R_p)e) = F(\xi(z)\Lambda(R_{-pj_0})j_0e)\\ &\overset{(4.30)}{=} v(-pj_0,j_0e)(\xi(z))\end{aligned} \tag{4.37}$$

Using this formula, the conjugate intertwiner is expressed as

$$\begin{aligned}u_c(p,e)(k) &\overset{(4.36)}{=} \overline{\int \mathrm{d}^2z\,\mathrm{e}^{-\mathrm{i}kz}v(p,e)(-\xi(z))}\\ &= \overline{\int \mathrm{d}^2z\,\mathrm{e}^{\mathrm{i}kz}v(p,e)(-\xi(-z))} \overset{(4.37)}{=} \overline{\int \mathrm{d}^2z\,\mathrm{e}^{\mathrm{i}kz}v(-pj_0,j_0e)(\xi(z))}\\ &\overset{(4.28)}{=} D(j_0)u(-pj_0,j_0e)(k)\;.\end{aligned} \tag{4.38}$$

[9]This relation is not immediately obvious from above, but can be shown by applying the Lorentz transformations to a general vector p instead of $\xi(z)$.

In this form of u_c, the preceding remarks concerning the action of j_0 can be used to show that the intertwiner equation for u_c,

$$\begin{aligned}
& D(R(A,p))u_c(p\Lambda(A),e) \overset{(4.38)}{=} D(R(A,p))D(j_0)u(-p\Lambda(A)j_0, j_0 e) \\
\overset{(4.35)}{=} \quad & D(j_0)D([\varphi,-a])u(-pj_0\, \underbrace{j_0\Lambda(A)j_0}_{=\Lambda([\varphi,-a])}, j_0 e) \overset{(4.32)}{=} D(j_0)u(-pj_0, (j_0\Lambda(A)j_0)j_0 e) \\
\overset{(4.38)}{=} \quad & u_c(p, \Lambda(A)e) \ ,
\end{aligned}$$

where $R(A,p) = [\varphi, a]$, is completely analogous to 4.32.

4.1.5. One particle states

The intertwiners u and u_c allow for a definition of wavefunctions $\psi(f,h), \psi_c(f,h)$ on the Hilbert space $\mathcal{H} = L^2(\partial V^+) \otimes L^2(\mathrm{sp}\, K)$

$$\begin{aligned}
\psi(f,h)(p)(k) \quad &:= \quad \int_H \widetilde{\mathrm{d}e}\, h(e)u(p,e)(k)\widetilde{f}(p) \\
\psi_c(f,h)(p)(k) \quad &:= \quad \int_H \widetilde{\mathrm{d}e}\, h(e)u_c(p,e)(k)\widetilde{f}(p) \ ,
\end{aligned} \tag{4.39}$$

where h is a smearing funtion, integrated with the $\mathcal{L}_+$-invariant measure on H. The representation U_1 acts on the functions f and h by pullback for a Poincaré transformation $(A,a) \in \mathcal{P}^c$:[10]

$$\begin{aligned}
& [U_1(A,a)\psi_{(c)}(f,h)](p)(k) \overset{(2.53)}{=} \mathrm{e}^{\mathrm{i}pa}[D(R(A,p))\psi(p\Lambda(A))](k) \\
\overset{(4.39)}{=} \quad & \mathrm{e}^{\mathrm{i}pa} \int_H \widetilde{\mathrm{d}e}\, h(e)[D(R(A,p))u_{(c)}(p\Lambda(A),e)](k)\widetilde{f}(p\Lambda(A)) \\
\overset{(4.32)}{=} \quad & \underbrace{\int_H \widetilde{\mathrm{d}e}\, h(e)u_{(c)}(p,\Lambda(A)e)(k)}_{=\int_H \widetilde{\mathrm{d}e}\, h(\Lambda(A)^{-1}e)u_{(c)}(p,e)(k)} \mathrm{e}^{\mathrm{i}pa}\widetilde{f}(p\Lambda(A)) \\
\overset{(3.19)}{=} \quad & \int_H \widetilde{\mathrm{d}e}\, (\Lambda_* h)(e)u_{(c)}(p,e)(k)((\Lambda(A),a)_* f)\widetilde{\ }(p) \overset{(4.39)}{=} \psi_{(c)}((\Lambda(A),a)_* f, \Lambda_* h)(p)(k)
\end{aligned} \tag{4.40}$$

Just like D, the representation U_1 can also be extended from $\mathcal{L}_+^\uparrow$ to $\mathcal{L}_+$ by putting

$$\begin{aligned}
& U_1(j_0)\psi(f,h)(p)(k) := D(j_0)\psi(f,h)(-pj_0)(k) \\
\overset{(4.39)}{=} \quad & \int_H \widetilde{\mathrm{d}e}\, \overline{h(e)}[D(j_0)u(-pj_0,e)](k)\overline{\widetilde{f}(-pj_0)} \\
\overset{(4.38)}{=} \quad & \int_H \widetilde{\mathrm{d}e}\, \overline{h(e)}u_c(p,j_0 e)(k)\overline{\widetilde{f}(-pj_0)} \overset{(3.20)}{=} \int_H \widetilde{\mathrm{d}e}\, \overline{h(j_0 e)}u_c(p,e)(k)(j_{0*}\overline{f})\widetilde{\ }(p) \\
\overset{(4.39)}{=} \quad & \psi_c(j_{0*}\overline{f}, j_{0*}\overline{h})(p)(k) \ .
\end{aligned} \tag{4.41}$$

[10] (c) means that the equality holds for the original functions ψ and u as well as for the corresponding conjugate functions ψ_c and u_c.

The action of $U_1(j_0)$ also acts on the functions f and h by pullback as well, but also exchanges ψ with the conjugate wavefunction ψ_c:

$$\begin{aligned} & U_1(j_0)\psi_c(f,h)(p)(k) := D(j_0)\psi_c(f,h)(-pj_0)(k) \quad (4.42) \\ \overset{(4.39)}{=} & \int_H \widetilde{\mathrm{d}e}\,\overline{h(e)}[D(j_0)u_c(-pj_0,e)](k)\overline{\tilde{f}(-pj_0)} \\ \overset{(4.38)}{=} & \int_H \widetilde{\mathrm{d}e}\,\overline{h(e)}u(p,j_0e)(k)\overline{\tilde{f}(-pj_0)} \overset{(3.20)}{=} \int_H \widetilde{\mathrm{d}e}\,\overline{h(j_0e)}u(p,e)(k)(j_{0*}\overline{f})\tilde{}(p) \\ \overset{(4.39)}{=} & \psi(j_{0*}\overline{f}, j_{0*}\overline{h})(p)(k) \end{aligned}$$

4.2. Second quantization for string-localized fields

The step of second quantization is carried out analogously to the massive case: The transformation of momentum-space basis vectors is shown and used to define the corresponding quantum field. Afterwards the covariance and string-localization properties of these fields are derived.

4.2.1. Basis kets

Writing the one particle states in the standard basis of $L^2(\partial V^+) \otimes L^2(\mathrm{sp}\,K)$,

$$\begin{aligned} |\psi_{(c)}(f,h)\rangle &= \int_{\partial V^+} \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k)\, \psi_{(c)}(f,h)(p) \circ |p\rangle \quad (4.43) \\ &\overset{(4.39)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \tilde{f}(p) \int_H \widetilde{\mathrm{d}e}\, h(e) u_{(c)}(p,e) \circ |p\rangle \ , \end{aligned}$$

where $\circ$ stands for integration over the basis of $\mathcal{H}_q = L^2(\mathrm{sp}\,K)$ with the measure $\mathrm{d}\nu(k)$[11], the action of representation U_1 becomes

$$\begin{aligned} & U_1(A,a)\,|\psi_{(c)}(f,h)\rangle \overset{(4.43)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\,[U_1(A,a)\psi_{(c)}(f,h)](p) \circ |p\rangle \quad (4.44) \\ \overset{(4.40)}{=} & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pa} \tilde{f}(p\Lambda(A)) \int_H \widetilde{\mathrm{d}e}\, h(e) u_{(c)}(p, \Lambda(A)e) \circ |p\rangle \\ = & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a} \tilde{f}(p) \int_H \widetilde{\mathrm{d}e}\, h(e) u_{(c)}(p\Lambda(A)^{-1}, \Lambda(A)e) \circ |p\Lambda(A)^{-1}\rangle \end{aligned}$$

for Poincaré transformations. The result is written in a way such that the smearing functions appear with the same argument as in 4.43, and the same is done for $U_1(j_0)$:

$$\begin{aligned} U_1(j_0)\,|\psi(f,h)\rangle &\overset{(4.43)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\,[U_1(j_0)\psi(f,h)](p) \circ |p\rangle \quad (4.45) \\ &\overset{(4.41)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \overline{\tilde{f}(-pj_0)} \int_H \widetilde{\mathrm{d}e}\, \overline{h(e)} u_c(p, j_0e) \circ |p\rangle \\ &= \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \overline{\tilde{f}(p)} \int_H \widetilde{\mathrm{d}e}\, \overline{h(e)} u_c(-pj_0, j_0e) \circ |-pj_0\rangle \end{aligned}$$

[11]This means that $\int_{\partial V^+} \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k)\, f(p)(k)\, |p\rangle \otimes |k\rangle$ is written as $\int_{\partial V^+} \widetilde{\mathrm{d}p}\, f(p) \circ |p\rangle$.

$$
\begin{aligned}
U_1(j_0)\,|\psi_c(f,h)\rangle &\overset{(4.43)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\,[U_1(j_0)\psi_c(f,h)](p)\circ|p\rangle \\
&\overset{(4.42)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\,\overline{\tilde{f}(-pj_0)}\int_H \widetilde{\mathrm{d}e}\,\overline{h(e)}u(p,j_0e)\circ|p\rangle \\
&= \int_{\partial V^+} \widetilde{\mathrm{d}p}\,\overline{\tilde{f}(p)}\int_H \widetilde{\mathrm{d}e}\,\overline{h(e)}u(-pj_0,j_0e)\circ|-pj_0\rangle
\end{aligned} \tag{4.46}
$$

Comparing these expressions with 4.43 and exploting the arbitraryness of f and h yields the transformation behaviour of the intertwiners $u_{(c)}$ at fixed momentum:

$$U_1(A,a)u_{(c)}(p,e)\circ|p\rangle \overset{(4.44)}{=} \mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a}u(p\Lambda(A)^{-1},\Lambda(A)e)\circ|p\Lambda(A)^{-1}\rangle \tag{4.47}$$

$$U_1(j_0)u(p,e)\circ|p\rangle \overset{(4.45)}{=} u_c(-pj_0,j_0e)\circ|-pj_0\rangle \tag{4.48}$$

$$U_1(j_0)u_c(p,e)\circ|p\rangle \overset{(4.46)}{=} u(-pj_0,j_0e)\circ|-pj_0\rangle \tag{4.49}$$

Second quantization, considering the one-particle basis kets $|p\rangle = a^\dagger(p)\,|0\rangle$ in the Fock-space with vacuum vector $|0\rangle$ then gives the corresponding transformation behaviour for the creation operators $a^\dagger(p,k)$ which create a particle with sharp momentum p and sharp "inner momentum" k:

$$U(A,a)u_{(c)}(p,e)\circ a^\dagger(p)U(A,a)^\dagger \overset{(4.47)}{=} \mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a}u_{(c)}(p\Lambda(A)^{-1},\Lambda(A)e)\circ a^\dagger(p\Lambda(A)^{-1}) \tag{4.50}$$

$$U(j_0)u(p,e)\circ a^\dagger(p)U(j_0)^\dagger \overset{(4.48)}{=} u_c(-pj_0,j_0e)\circ a^\dagger(-pj_0) \tag{4.51}$$

$$U(j_0)u_c(p,e)\circ a^\dagger(p)U(j_0)^\dagger \overset{(4.49)}{=} u(-pj_0,j_0e)\circ a^\dagger(-pj_0) \tag{4.52}$$

Since the intertwiner equation holds for u_c as for u itself, the same transformation rules apply:

$$
\begin{aligned}
U(A,a)\overline{u_{(c)}(p,e)}\circ a(p)U(A,a)^\dagger &= (U(A,a)u_{(c)}(p,e)\circ a^\dagger(p)U(A,a)^\dagger)^\dagger \\
&\overset{(4.50)}{=} (\mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a}u_{(c)}u(p\Lambda(A)^{-1},\Lambda(A)e)\circ a^\dagger(p\Lambda(A)^{-1}))^\dagger \\
&= \mathrm{e}^{-\mathrm{i}p\Lambda(A)^{-1}a}\overline{u_{(c)}u(p\Lambda(A)^{-1},\Lambda(A)e)}\circ a(p\Lambda(A)^{-1})
\end{aligned} \tag{4.53}
$$

$$
\begin{aligned}
U(j_0)\overline{u(p,e)}\circ a(p)U(j_0)^\dagger &= (U(j_0)u(p,e)\circ a^\dagger(p)U(j_0)^\dagger)^\dagger \\
&\overset{(4.51)}{=} (u_c(-pj_0,j_0e)\circ a^\dagger(-pj_0))^\dagger = \overline{u_c(-pj_0,j_0e)}\circ a(-pj_0)
\end{aligned} \tag{4.54}
$$

$$
\begin{aligned}
U(j_0)\overline{u_c(p,e)}\circ a(p)U(j_0)^\dagger &= (U(j_0)u_c(p,e)\circ a^\dagger(p)U(j_0)^\dagger)^\dagger \\
&\overset{(4.52)}{=} (u(-pj_0,j_0e)\circ a^\dagger(-pj_0))^\dagger = \overline{u(-pj_0,j_0e)}\circ a(-pj_0)
\end{aligned} \tag{4.55}
$$

4.2.2. String fields

Definition

The quantum field corresponding to the intertwiners $u_{(c)}$ is defined analogously to 3.37 by

$$\Phi(x,e) = \int_{\partial V^+} \widetilde{\mathrm{d}p}\,\mathrm{e}^{\mathrm{i}px}u(p,e)\circ a^\dagger(p) + \mathrm{e}^{-\mathrm{i}px}\overline{u_c(p,e)}\circ a(p) \tag{4.56}$$

and the conjugate field is

$$\Phi(x,e)^\dagger = \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}px} u_c(p,e) \circ a^\dagger(p) + \mathrm{e}^{-\mathrm{i}px} \overline{u(p,e)} \circ a(p) \tag{4.57}$$

where u and u_c are exchanged like in 3.38. The canonical commutation relations (CCR) w.r.t. the measures $\widetilde{\mathrm{d}p}$ and $\mathrm{d}\nu(k)$ read

$$[a(p)(k), a^\dagger(p')(k')] = 2p_0 \delta(\vec{p} - \vec{p}')2\delta(\varphi_k - \varphi_{k'})\mathbf{1}$$

with φ_k the polar angle of k. This choice can be justified by comparing the construction of the Lorentz-invariant measure on the upper mass-shell H_m^+ or the mantle of the forward light cone ∂V^+ to the rotation invariant measure on sp$K$:

- In the first case a manifestly covariant measure which is localized on the joint spectrum of the translation generators P^μ on $\mathbb{R}^4$, i.e. $\mathrm{d}^4p\, \delta(p^2 - m^2)\Theta(p^0)$ is integrated out in the time direction to obtain $\mathrm{d}^3p/2p^0$ with $p^0 = \sqrt{m^2 + \vec{p}^2}$.

- For the invariant measure on the "internal mass shell" the manifestly rotation-invariant measure $\mathrm{d}^2k\, \delta(k^2 - \kappa^2)$ is localized on the joint spectrum of the translation generators G_i on $\mathbb{R}^2$. The expression for this measure reads $\mathrm{d}k\, k\mathrm{d}\varphi_k\, \delta(k^2 - \kappa^2)$ in polar coordinates. An integration over the radial direction reduces this to $\mathrm{d}\varphi_k/2$.

Covariance

Using the transformation properties of the creation and annihilation operators $a^\dagger(p)$ and $a(p)$ obtained above, it follows

$$\begin{aligned}
& U(A,a)\Phi(x,e)U(A,a)^\dagger \\
\overset{(4.56)}{=}\; & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}px} U(A,a) u(p,e) \circ a^\dagger(p) U(A,a)^\dagger \\
& \qquad\qquad + \mathrm{e}^{-\mathrm{i}px} U(A,a) \overline{u_c(p,e)} \circ a(p) U(A,a)^\dagger \\
\overset{(4.50)(4.53)}{=}\; & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}p(x+\Lambda(A)^{-1}a)} u(p\Lambda(A)^{-1}, \Lambda(A)e) \circ a^\dagger(p\Lambda(A)^{-1}) \\
& \qquad\qquad + \mathrm{e}^{-\mathrm{i}p(x+\Lambda(A)^{-1}a)} \overline{u_c(p\Lambda(A)^{-1}, \Lambda(A)e)} \circ a(p\Lambda(A)^{-1}) \\
=\; & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}p(\Lambda(A)x+a)} u(p, \Lambda(A)e) \circ a^\dagger(p) + \mathrm{e}^{-\mathrm{i}p(\Lambda(A)x+a)} \overline{u_c(p, \Lambda(A)e)} \circ a(p) \\
\overset{(4.56)}{=}\; & \Phi(\Lambda(A)x + a, \Lambda(A)e)\ ,
\end{aligned} \tag{4.58}$$

i.e. the localization point x transforms like a four-vector under the Poincaré transformation $(\Lambda(A), a) \in \mathcal{P}$, while e only acquires a Lorentz transformation $\Lambda(A) \in \mathcal{L}_+^\uparrow$.

PCT-symmetry

The action of j_0 in the representation U_1

$$
\begin{aligned}
& U(j_0)\Phi(x,e)U(j_0)^\dagger && (4.59)\\
\overset{(4.56)}{=}\quad & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}px} U(j_0)u(p,e)\circ a^\dagger(p)U(j_0^\dagger) + \mathrm{e}^{\mathrm{i}px}U(j_0)\overline{u_c(p,e)}\circ a(p)U(j_0)^\dagger \\
\overset{(4.51)(4.55)}{=}\quad & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}px} u_c(-pj_0, j_0 e)\circ a^\dagger(-pj_0) + \mathrm{e}^{\mathrm{i}px}\overline{u(-pj_0, j_0 e)}\circ a(-pj_0) \\
=\quad & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{\mathrm{i}pj_0x} u_c(p, j_0 e)\circ a^\dagger(p) + \mathrm{e}^{-\mathrm{i}pj_0x}\overline{u(p, j_0 e)}\circ a(p) \\
\overset{(4.57)}{=}\quad & \Phi(j_0x, j_0e)^\dagger
\end{aligned}
$$

transforms both x and e as expected and gives an additional hermitean conjugation of the field.

Locality

Since Φ is a linear expression in the creation/annihilation operators, the commutator of fields will be a multiple of identity. Unlike for currents, it is therefore sufficient to consider the two-point (or rather the two-string) function, following [MSY04]:

$$
\langle 0|\,\Phi(x,e)\Phi(x',e')\,|0\rangle = \int \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p(x-x')}\,\underbrace{\overline{u_c(p,e)}\circ u(p,e')}_{=:M(p,e,e')} =: \mathcal{W}(x-x',e,e') \qquad (4.60)
$$

If the strings $x+\mathbb{R}^+e$ and $x'+\mathbb{R}^+e'$ are spacelike separated, according to [BGL02, p. 8],

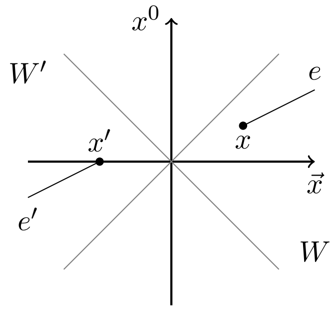

Figure 4.1.: Spacelike separated strings $x+\mathbb{R}^+e$ and $x'+\mathbb{R}^+e'$

there is a wedge W such that $x+\mathbb{R}^+e\subset W$ and $x'+\mathbb{R}^+e'\subset W'$. The situation is shown in Fig. 4.1. Let $\Lambda(t)$ be the additive subgroup of $\mathcal{P}$ consisting of Lorentz-boosts along a fixed direction perpendicular to the edge of W. As in the massive case, it is possible to assume w.l.o.g. that $\Lambda(t)=\Lambda(\mathrm{e}^{\mathrm{i}\sigma_3 t})$ and $W'=j_0W$, i.e. W is the standard wedge, as defined in 3.50. From the definition of M in 3.42 the covariance formula

$$\begin{aligned} M(p\Lambda(A),e,e') &= \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \overline{u_c(p\Lambda(A),e)} \circ u(p\Lambda(A),e') \\ &\overset{(4.24)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \overline{D(R(A,p))u_c(p\Lambda(A),e)} \circ D(R(A,p))u(p\Lambda(A),e') \\ &\overset{(4.32)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \overline{u_c(p,e\Lambda(A))} \circ u(p,e'\Lambda(A)) \\ &= M(p,e\Lambda(A),e'\Lambda(A)) \end{aligned} \tag{4.61}$$

can be derived, which implies

$$\begin{aligned} \langle 0|\,\Phi(x,e)\Phi(x',e')\,|0\rangle &\overset{(4.60)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}p(x-x')} M(p,e,e') \\ &= \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}p\Lambda(A)(x-x')}\ \underbrace{M(p\Lambda(A),e,e')}_{=M(p,e\Lambda(A),e'\Lambda(A))} \\ &\overset{(4.60)}{=} \langle 0|\,\Phi(\Lambda(A)x+a,\Lambda(A)e)\Phi(\Lambda(A)x'+a,\Lambda(A)e')\,|0\rangle\ . \end{aligned}$$

This formula can be derived more systematically, using $U(A,a)\,|0\rangle = |0\rangle$:

$$\begin{aligned} \langle 0|\,\Phi(x,e)\Phi(x',e')\,|0\rangle &= \langle 0|\,U(A,a)\Phi(x,e)U(A,a)^\dagger U(A,a)\Phi(x',e')U(A,a)^\dagger\,|0\rangle \\ &\overset{(4.58)}{=} \langle 0|\,\Phi(\Lambda(A)x+a,\Lambda(A)e)\Phi(\Lambda(A)x'+a,\Lambda(A)e')\,|0\rangle \end{aligned}$$

In contrast to 3.46, the transformations of the localization points x, x' and the string-directions e, e' in these expressions are not independent of each other. This is already a hint that the latter influence the localization properties. Either way, using 3.42, the result can be stated in the form

$$\mathcal{W}(x-x',e,e') = \mathcal{W}(\Lambda(x-x'),\Lambda e,\Lambda e')\ . \tag{4.62}$$

On the other hand, since $D(j_0)$ is defined as the complex conjugation, M has the reflection property

$$\begin{aligned} M(p,e,e') &= \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \overline{u_c(p,e)} \circ u(p,e') \\ &= \int_{\partial V^+} \widetilde{\mathrm{d}p}\, D(j_0)u_c(p,e) \circ \overline{D(j_0)u(p,e')} \\ &\overset{(4.38)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, u(-pj_0,j_0e) \circ \overline{u_c(-pj_0,j_0e')} \\ &= M(-pj_0,j_0e',j_0e) \overset{(4.61)}{=} M(p,-e',e)\ , \end{aligned}$$

which gives

$$
\begin{aligned}
\langle 0|\,\Phi(x,e)\Phi(x',e')\,|0\rangle &\overset{(3.42)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}p(x-x')} M(p,e,e') \\
&= \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}p\Lambda(A)(-x'-(-x))} \underbrace{M(p\Lambda(A),e,e')}_{=M(p,-e',e)} \\
&\overset{(3.42)}{=} \langle 0|\,\Phi(-x',-e')\Phi(-x,-e)\,|0\rangle \ .
\end{aligned}
$$

Using $U(j_0)\,|0\rangle = |0\rangle$, the more abstract calculation reads

$$
\begin{aligned}
\langle 0|\,\Phi(x,e)\Phi(x',e')\,|0\rangle &= \overline{\langle 0|\,U(j_0)\Phi(x,e)U(j_0)^\dagger U(j_0)\Phi(x',e')U(j_0)^\dagger\,|0\rangle} \\
&\overset{(4.59)}{=} \overline{\langle 0|\,\Phi(j_0x,j_0e)^\dagger\Phi(j_0x',j_0e')^\dagger\,|0\rangle} \\
&= \overline{\langle 0|\,(\Phi(j_0x',j_0e')\Phi(j_0x,j_0e))^\dagger\,|0\rangle} \\
&= \langle 0|\,\Phi(j_0x',j_0e')\Phi(j_0x,j_0e)\,|0\rangle \\
&\overset{(4.62)}{=} \langle 0|\,\Phi(-x',-e')\Phi(-x,-e)\,|0\rangle
\end{aligned}
$$

The result obtained in both ways can again be written as a property of the function $\mathcal{W}$:

$$
\mathcal{W}(x-x',e,e') = \mathcal{W}(x'-x,-e',-e) \tag{4.63}
$$

Combining this with the covariance property yields the exchange formula

$$
\begin{aligned}
\mathcal{W}(x-j_0\Lambda(t)x',e,j_0\Lambda(t)e') &\overset{(4.63)}{=} \mathcal{W}(x-j_0\Lambda(t)x',-j_0\Lambda(t)e',-e) \\
&\overset{(4.62)}{=} \mathcal{W}(x'-j_0\Lambda(-t)x,e',j_0\Lambda(-t)e)
\end{aligned} \tag{4.64}
$$

Now $\mathcal{W}(x-x',e,e')$ permits analytic continuation in the first argument into the tube by the same argument as in the massive case. In [MSY05], it also is shown that for $F(z) = z^\alpha$, $\alpha \in \mathbb{C}\backslash\mathbb{N}_0$,[12] the function is analytic for e' in $\Im(e') \in V^+$ and for e in $\Im(e) \in -V^+$. This is due to the fact that the function z^α is defined with a cut along $\Re(z) < 0$ and continuous in the closed upper half plane $\Im(z) \geq 0$. Now if $\Im(e) \in V^+$, then $\Im(\xi(z)\Lambda(R_p)e) = \xi(z)\Lambda(R_p)\Im(e) \geq 0$, since $\xi(z) \in \partial V^+$. In [MSY05] it is shown that the inclusions $x + \mathbb{R}^+e \subset W$ and $x' + \mathbb{R}^+e'$ imply $x, e \in W$ and $x', e' \in W'$. Hence for $t \in \mathbb{R} + \mathrm{i}[0,\pi]$, the imaginary parts of the involved four-vectors fulfil $\Im(j_0\Lambda(t)x') \in V^+$, $\Im(j_0\Lambda(-t)x) \in V^+$, as well as $\Im(j_0\Lambda(t)e' \in V^+$ and $\Im(j_0\Lambda(-t)e \in V^+$. Therefore, the analytic continuation to $t = \mathrm{i}\pi$ as possible, where the exchange formula 4.64 becomes

$$
\mathcal{W}(x-x',e,e') = \mathcal{W}(x'-x,e',e) \tag{4.65}
$$

[12] The natural numbers are excluded because the Fourier transform in 4.28 together with the polynomial form of 4.25 would imply that $u(k)$ is concentrated at $k = 0$ as a distribution. By $k^2 = \kappa^2 > 0$ this would in turn imply that the fields $\Phi(x,e)$ vanish.

and the vacuum expectation value of the commutator of the fields vanishes:

$$\begin{aligned} \langle 0| \left[\Phi(x,e)\Phi(x',e')\right] |0\rangle &= \langle 0| \Phi(x,e)\Phi(x',e') - \Phi(x',e')\Phi(x,e) |0\rangle \\ &\overset{(4.60)}{=} \mathcal{W}(x-x',e,e') - \mathcal{W}(x'-x,e',e) \overset{(4.65)}{=} 0 \end{aligned}$$

It should be emphasized that the process of analytic continuation which has been used to derive this formula relies on the spacelike separation between the strings, explicitly involving the string-directions e and e'. This is in contrast to the massive exchange formula 3.53, which merely involves the localization points x and x'. Because this commutator is a multiple of identity, the string-locality can be restated in the form

$$(x + \mathbb{R}^+ e - (x' + \mathbb{R}^+ e'))^2 < 0 \Rightarrow [\Phi(x,e), \Phi(x',e')] = 0$$

A detailed construction of the fields $\Phi(x,e)$ and their properties in the framework of modular localization is given in [**?**].

5. Current fields

> In physics, your solution should convince a reasonable person. In math, you have to convince a person who's trying to make trouble. Ultimately, in physics, you're hoping to convince Nature. And I've found Nature to be pretty reasonable.
>
> FRANK WILCZEK

5.1. Construction

From a physical point of view, it is reasonable to look for observables which form a pointlike localized subalgebra of the algebra given by the string-localized fields. The following construction is motivated in [MSY05] by the fact that the Lorentz indices for the directions of two strings can be contracted and then inserted into a numerical function instead. In this way, the resulting operators are quadratic in the creation and annihilation operators, sidestepping the No-Go Theorem [Yng69].

However, the mutual locality of these operators as well as their relation to the string fields can so far be shown on a formal level only, which is described in this chapter. The existence of a numerical function which allows the analytic continuations, which are necessary for showing the various locality properties, is not yet fully clear.

5.1.1. Two-particle intertwiners

The construction of observables which are intended to have pointlike localization properties in [MSY05] and [Sch08] starts from a numerical function v_2 on $\Gamma \times \Gamma$, which is given by two momenta p and $\tilde{p}$ and has the form

$$v_2(p,\tilde{p})(\xi,\tilde{\xi}) := F(\xi\Lambda(R_p) \cdot \tilde{\xi}\Lambda(R_{\tilde{p}})) \ . \tag{5.1}$$

Here, F is a numerical function which still is to be specified. Its argument is worked out explicitly in Appendix C. In the representation $\widetilde{D}$ from 4.27, this function has the transformation behaviour

$$\begin{aligned}
&[\widetilde{D}(R(A,p))\otimes\widetilde{D}(R(A,\tilde{p}))v_2(p\Lambda(A),\tilde{p}\Lambda(A))](\xi,\tilde{\xi}) \qquad (5.2)\\
\overset{(4.27)}{=}\;& v_2(p\Lambda(A),\tilde{p}\Lambda(A))(\xi\Lambda(R(A,p)),\tilde{\xi}\Lambda(R(A,\tilde{p})))\\
\overset{(5.1)}{=}\;& F(\xi\underbrace{\Lambda(R(A,p))\Lambda(R_{p\Lambda(A)})}_{=\Lambda(R_p)\Lambda(A)}\cdot\tilde{\xi}\underbrace{\Lambda(R(A,\tilde{p}))\Lambda(R_{\tilde{p}\Lambda(A)})}_{=\Lambda(R_{\tilde{p}})\Lambda(A)})\\
=\;& F(\xi\Lambda(R_p)\cdot\tilde{\xi}\Lambda(R_{\tilde{p}}))\overset{(5.1)}{=}v_2(p,\tilde{p})(\xi,\tilde{\xi})
\end{aligned}$$

and gives rise to the two-particle intertwiner

$$u_2(k,\tilde{k}) := \int \mathrm{d}^2z\int\mathrm{d}^2\tilde{z}\,\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}}v_2(\xi(z),\xi(\tilde{z}))\overset{(4.28)}{=}[V^{\otimes 2}v_2](k,\tilde{k})\ , \qquad (5.3)$$

which fulfils the *two-particle intertwiner equation*

$$\begin{aligned}
&D(R(A,p))\otimes D(R(A,\tilde{p}))u_2(p\Lambda(A),\tilde{p}\Lambda(A))\\
\overset{(5.3)}{=}\;&(D(R(A,p))V)\otimes(D(R(A,\tilde{p}))V)v_2(p\Lambda(A),\tilde{p}\Lambda(A))\\
\overset{(4.29)}{=}\;&(V\widetilde{D}(R(A,p)))\otimes(V\widetilde{D}(R(A,\tilde{p})))v_2(p\Lambda(A),\tilde{p}\Lambda(A))\\
\overset{(5.2)}{=}\;&V^{\otimes 2}v_2(p,\tilde{p})=u_2(p,\tilde{p})\ .
\end{aligned}$$

5.1.2. Current operator

In the following definition of the operator $B(x,\tilde{x})$, $\circ u_2$ denotes integration over k and $u_2\circ$ integration over $\tilde{k}$. It is motivated by the expressions for the pure creation part, which can be found in [MSY05] and [Sch08]:

$$\begin{aligned}
B(x,\tilde{x}) = \int_{\partial V^+}\widetilde{\mathrm{d}p}\int_{\partial V^+}\widetilde{\mathrm{d}\tilde{p}}\,:\Big(\;&\mathrm{e}^{\mathrm{i}px}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}}a^\dagger(p)\circ u_2(p,\tilde{p})\circ a^\dagger(\tilde{p}) \qquad (5.4)\\
+\;&\mathrm{e}^{\mathrm{i}px}\mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}}a^\dagger(p)\circ u_2'(p,\tilde{p})\circ a(\tilde{p})\\
+\;&\mathrm{e}^{-\mathrm{i}px}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}}a(p)\circ u_2'(p,\tilde{p})\circ a^\dagger(\tilde{p})\\
+\;&\mathrm{e}^{-\mathrm{i}px}\mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}}a(p)\circ u_2(p,\tilde{p})\circ a(\tilde{p})\quad\Big):
\end{aligned}$$

Here u_2' is defined like u_2, but with $F(-z)$ instead of $F(z)$, which does not affect the transformation properties. The fact that $B(x,\tilde{x})$ is a quadratic expression in the creation and annihilation operators, leads to a certain form of gauge invariance, which is described in Appendix A.

5.1.3. Covariance

Inserting the intertwiner equation 4.32 with momentum $p\lambda(A)^{-1}$

$$D(R(A,p\Lambda(A)^{-1}))u(p,e)=u(p\Lambda(A)^{-1},\Lambda(A)e)$$

into the covariant transformation rules 4.50 and 4.53 for the creation and annihilation operators, it is possbile to extract the transformation rules for these operators themselves as

$$
\begin{aligned}
U(A,a)a^\dagger(p)U(A,a)^\dagger &= \mathrm{e}^{\mathrm{i}p\Lambda(A)^{-1}a}D(R(A,p\Lambda(A)^{-1}))^T a^\dagger(p\Lambda(A)^{-1}) && (5.5)\\
U(A,a)a^\dagger(p)U(A,a) &= \mathrm{e}^{-\mathrm{i}p\Lambda(A)^{-1}a}D(R(A,p\Lambda(A)^{-1}))^T a(p\Lambda(A)^{-1})\ . && (5.6)
\end{aligned}
$$

The superscript ^{T} indicates a transposition of indices with respect to the $\circ$-integration with the measure $\mathrm{d}\nu(k)$. To check the covariance properties of B, only the first term is written out explicitly. The calculation works analogously for the other terms, denoted by "...":

$$
\begin{aligned}
& U(A,a)[B(x,\tilde{x}) - \ldots]U(A,a)^\dagger && (5.7)\\
=& \int_{\partial V^+} \widetilde{\mathrm{d}p} \int_{\partial V^+} \widetilde{\mathrm{d}\tilde{p}}\, \mathrm{e}^{\mathrm{i}px}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}} U(A,a)a^\dagger(p)U(A,a)^\dagger \circ u_2(p,\tilde{p}) \circ U(A,a)a^\dagger(\tilde{p})U(A,a)^\dagger \\
=& \int_{\partial V^+} \widetilde{\mathrm{d}p} \int_{\partial V^+} \widetilde{\mathrm{d}\tilde{p}}\, \mathrm{e}^{\mathrm{i}p(x+\Lambda(A)^{-1}a)}\mathrm{e}^{\mathrm{i}\tilde{p}(\tilde{x}+\Lambda(A)^{-1}a)} \\
& D(R(A,p\Lambda(A)^{-1}))^T a^\dagger(p\Lambda(A)^{-1}) \circ u_2(p,\tilde{p}) \circ D(R(A,\tilde{p}\Lambda(A)^{-1}))^T a^\dagger(\tilde{p}\Lambda(A)^{-1}) \\
=& \int_{\partial V^+} \widetilde{\mathrm{d}p} \int_{\partial V^+} \widetilde{\mathrm{d}\tilde{p}}\, \mathrm{e}^{\mathrm{i}p(\Lambda(A)x+a)}\mathrm{e}^{\mathrm{i}\tilde{p}(\Lambda(A)\tilde{x}+a)} \\
& \qquad a^\dagger(p) \circ \underbrace{D(R(A,p)) \otimes D(R(A,\tilde{p}))u_2(p\Lambda(A),\tilde{p}\Lambda(A))}_{=u_2(p,\tilde{p})} \circ a^\dagger(\tilde{p}) \\
=& B(\Lambda(A)x+a, \Lambda(A)\tilde{x}+a)
\end{aligned}
$$

Both x and $\tilde{x}$ pick up a Poincaré transformation.

5.2. Locality

The crucial problem of locality is adressed in this section by first showing that the current operator $B(x,\tilde{x})$, especially the pattern of the chosen signs, gives rise to matrix elements of the commutator that fit together in such a way that - at least on a formal level - the fields are local.

Relative locality between the current and string fields is discussed next, which works equally well on a formal level, but is subject to similar analyticity problems.

These can be traced back to the function F, which occurs in 5.1. It is discussed in which sense its choice is problematic in contrast to the case of string-localized fields $\Phi(x,e)$ themselves.

5.2.1. Locality of the currents

For an investigation of the localization properties of the current field $B(x,\tilde{x})$, it is convenient to insert the definitions of u_2 from 5.3 and v_2 from 5.1, and to write out the $\circ$-integrations over k and $\tilde{k}$ explicitly:

$$
\begin{aligned}
B(x,\tilde{x}) \stackrel{(5.4)}{=} \int_{\partial V^+} \widetilde{\mathrm{d}p} \int_{\partial V^+} \widetilde{\mathrm{d}\tilde{p}} \int \mathrm{d}\nu(k) \int \mathrm{d}\nu(\tilde{k}) \int \mathrm{d}^2 z \int \mathrm{d}^2 \tilde{z} \\
:\big[F(+\xi(z)B_p \cdot \xi(\tilde{z})B_{\tilde{p}})\big(\mathrm{e}^{\mathrm{i}px}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}}a^\dagger(p,k)a^\dagger(\tilde{p},\tilde{k}) \\
+\mathrm{e}^{-\mathrm{i}px}\mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{-\mathrm{i}kz}\mathrm{e}^{-\mathrm{i}\tilde{k}\tilde{z}}a(p,k)a(\tilde{p},\tilde{k})\big) \\
+F(-\xi(z)B_p \cdot \xi(\tilde{z})B_{\tilde{p}})\big(\mathrm{e}^{\mathrm{i}px}\mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{-\mathrm{i}\tilde{k}\tilde{z}}a^\dagger(p,k)a(\tilde{p},\tilde{k}) \\
+\mathrm{e}^{-\mathrm{i}px}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{-\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}}a(p,k)a^\dagger(\tilde{p},\tilde{k})\big)\big]:
\end{aligned}
$$

Again, $:\cdot:$ denotes normal ordering. Also, the Wigner boosts $B_p = \Lambda(R_p)$ from 4.2 have been inserted. In [MSY04], $F \in \mathcal{S}(\mathbb{R})$ is required, and it is already shown that the fields B are local in the vacuum state. This is a consequence of the fact that the two-point function $\langle 0| [B(x,\tilde{x}), (x',\tilde{x}')] |0\rangle$ is a Lorentz invariant distribution, odd in the differences between primed and unprimed variables, and that spacelike vectors in $\mathbb{M}$ and their negatives are connected by a Lorentz transformation. In order to achieve a consistent notation with [MSY04], the various Lorentz transformations are written as left multiplications in the following.

Since in an expression like $B(x,\tilde{x})B(x',\tilde{x}')$, not all contractions are multiples of unity, the vacuum expectation value is not the only matrix-element to be considered for locality. For a better overview, all integration signs are suppressed in the following. Wick's Theorem [IZ05, p. 180] yields:

$$
\begin{aligned}
B(x,\tilde{x})B(x',\tilde{x}') &= :B(x,\tilde{x})B(x',\tilde{x}'): + :\mathrm{SC}_1: + \mathrm{FC}_1 \\
B(x',\tilde{x}')B(x,\tilde{x}) &= :B(x',\tilde{x}')B(x,\tilde{x}): + :\mathrm{SC}_2: + \mathrm{FC}_2
\end{aligned}
\tag{5.8}
$$

SC and FC stand for the simply contracted and fully contracted parts, respectively. Since under the normal ordering sign $:\cdot:$ the operators commute anyway, it has to be checked that x and x' can be exchanged for each matrix element in $:\mathrm{SC}:$ and FC individually. If $(x,\tilde{x}) >< (x',\tilde{x}')$ (componentwise), the wedge W with wedge-preserving boost $\mathcal{P}$-subgroup $\Lambda(t)$, reflection across the edge j_0 and $x,\tilde{x} \in W$, while $x',\tilde{x}' \in W'$ can again be chosen as the standard wedge 3.50 because of the covariance property 5.7. This is visualized in Fig. 5.1.

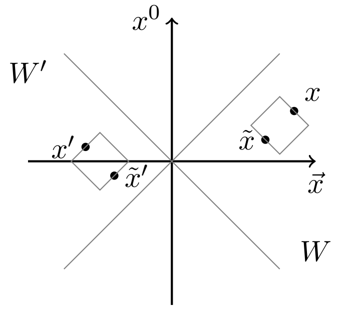

Figure 5.1.: Spacelike separated pairs of points $x, \tilde{x}$, $x', \tilde{x}'$

Then the function

$$\begin{aligned}
\mathrm{SC}(t) \quad := \quad & F(-B_p\xi(z)\cdot B_{\tilde{p}}\xi(\tilde{z}))F(+B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\
& \left(\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}}a^\dagger(\tilde{p},\tilde{k})\mathrm{e}^{\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{\mathrm{i}\tilde{k}'\tilde{z}'}a^\dagger(\tilde{p}',\tilde{k}') + \begin{array}{c} x\leftrightarrow\tilde{x} \\ x'\leftrightarrow\tilde{x}'\end{array}\right) \\
+ \quad & F(+B_p\xi(z)\cdot B_{\tilde{p}}\xi(\tilde{z}))F(-B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\
& \left(\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{-\mathrm{i}\tilde{k}\tilde{z}}a(\tilde{p},\tilde{k})\mathrm{e}^{-\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{-\mathrm{i}\tilde{k}'\tilde{z}'}a(\tilde{p}',\tilde{k}') + \begin{array}{c} x\leftrightarrow\tilde{x} \\ x'\leftrightarrow\tilde{x}'\end{array}\right) \\
+ \quad & F(+B_p\xi(z)\cdot B_{\tilde{p}}\xi(\tilde{z}))F(+B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\
& \left(\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{-\mathrm{i}\tilde{k}\tilde{z}}a(\tilde{p},\tilde{k})\mathrm{e}^{\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{\mathrm{i}\tilde{k}'\tilde{z}'}a^\dagger(\tilde{p}',\tilde{k}') + \begin{array}{c} x\leftrightarrow\tilde{x} \\ x'\leftrightarrow\tilde{x}'\end{array}\right) \\
+ \quad & F(-B_p\xi(z)\cdot B_{\tilde{p}}\xi(\tilde{z}))F(-B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\
& \left(\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}}\mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}}a^\dagger(\tilde{p},\tilde{k})\mathrm{e}^{-\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{-\mathrm{i}\tilde{k}'\tilde{z}'}a(\tilde{p}',\tilde{k}') + \begin{array}{c} x\leftrightarrow\tilde{x} \\ x'\leftrightarrow\tilde{x}'\end{array}\right)
\end{aligned} \tag{5.9}$$

allows an analytic continuation to $t = \mathrm{i}\pi$, where due to $j_0\Lambda(\pm\mathrm{i}\pi) = \mathbf{1}$ it coincides with SC_1. In the first two lines, quadratic and mixed terms have been contracted to yield pure creation and annihilation terms. In the last two lines, only quadratic and then only mixed terms have been contracted to give mixed terms. Using

$$\begin{aligned}
B_{-j_0\Lambda(t)p}\xi(z) \quad &= \quad -j_0 B_{\Lambda(t)p}(-j_0)\xi(z) = -j_0 B_{\Lambda(t)p}\xi(-z) = -j_0\Lambda(t)B_p R(\Lambda(-t),p)\xi(-z) \\
&=: \quad -j_0\Lambda(t)B_p\xi(-R(t)z + c(t))
\end{aligned}$$

and substituting integration variables $p \to -j_0\Lambda(-t)$, $z \to -R(t)^{-1}(z+c)$, $z' \to -R(t)^{-1}(z'+c)'$ (where the shift part cancels), and finally absorbing the rotation part via $k \to R(t)^{-1}k$

leads to another form

$$\begin{aligned}
\mathrm{SC}(t) \;=\;& F(+j_0\Lambda(t)B_p\xi(z)\cdot B_{\tilde p}\xi(\tilde z))F(-j_0\Lambda(t)B_p\xi(z')\cdot B_{\tilde p'}\xi(\tilde z')) \\
&\left(\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}k(z'-z)}\mathrm{e}^{\mathrm{i}\tilde p\tilde x}\mathrm{e}^{\mathrm{i}\tilde k\tilde z}a^\dagger(\tilde p,\tilde k)\mathrm{e}^{\mathrm{i}\tilde p'\tilde x'}\mathrm{e}^{\mathrm{i}\tilde k'\tilde z'}a^\dagger(\tilde p',\tilde k') + \begin{array}{c} x\leftrightarrow\tilde x \\ x'\leftrightarrow\tilde x'\end{array}\right) \\
+\;& F(-j_0\Lambda(t)B_p\xi(z)\cdot B_{\tilde p}\xi(\tilde z))F(+j_0\Lambda(t)B_p\xi(z')\cdot B_{\tilde p'}\xi(\tilde z')) \\
&\left(\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}k(z'-z)}\mathrm{e}^{-\mathrm{i}\tilde p\tilde x}\mathrm{e}^{-\mathrm{i}\tilde k\tilde z}a(\tilde p,\tilde k)\mathrm{e}^{-\mathrm{i}\tilde p'\tilde x'}\mathrm{e}^{-\mathrm{i}\tilde k'\tilde z'}a(\tilde p',\tilde k') + \begin{array}{c} x\leftrightarrow\tilde x \\ x'\leftrightarrow\tilde x'\end{array}\right) \\
+\;& F(-j_0\Lambda(t)B_p\xi(z)\cdot B_{\tilde p}\xi(\tilde z))F(-j_0\Lambda(t)B_p\xi(z')\cdot B_{\tilde p'}\xi(\tilde z')) \\
&\left(\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}k(z'-z)}\mathrm{e}^{-\mathrm{i}\tilde p\tilde x}\mathrm{e}^{-\mathrm{i}\tilde k\tilde z}a(\tilde p,\tilde k)\mathrm{e}^{\mathrm{i}\tilde p'\tilde x'}\mathrm{e}^{\mathrm{i}\tilde k'\tilde z'}a^\dagger(\tilde p',\tilde k') + \begin{array}{c} x\leftrightarrow\tilde x \\ x'\leftrightarrow\tilde x'\end{array}\right) \\
+\;& F(+j_0\Lambda(t)B_p\xi(z)\cdot B_{\tilde p}\xi(\tilde z))F(+j_0\Lambda(t)B_p\xi(z')\cdot B_{\tilde p'}\xi(\tilde z')) \\
&\left(\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}k(z'-z)}\mathrm{e}^{\mathrm{i}\tilde p\tilde x}\mathrm{e}^{\mathrm{i}\tilde k\tilde z}a^\dagger(\tilde p,\tilde k)\mathrm{e}^{-\mathrm{i}\tilde p'\tilde x'}\mathrm{e}^{-\mathrm{i}\tilde k'\tilde z'}a(\tilde p',\tilde k') + \begin{array}{c} x\leftrightarrow\tilde x \\ x'\leftrightarrow\tilde x'\end{array}\right)
\end{aligned} \tag{5.10}$$

Now analytic continuation to $t \to \mathrm{i}\pi$ makes the function equal SC_2. In summary, the simply contracted terms are related via

$$:\mathrm{SC}_1: \overset{(5.9)}{=} :\mathrm{SC}(\mathrm{i}\pi): \overset{(5.10)}{=} :\mathrm{SC}_2: . \tag{5.11}$$

Furthermore, the function

$$\begin{aligned}
\mathrm{FC}(t) \;=\;& F(+B_p\xi(z)\cdot B_{\tilde p}\xi(\tilde z))F(+B_p\xi(z')\cdot B_{\tilde p}\xi(\tilde z')) \\
&\left(\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}\tilde p(\tilde x-j_0\Lambda(t)\tilde x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{-\mathrm{i}\tilde k(\tilde z-\tilde z')} + \begin{array}{c} x\leftrightarrow\tilde x \\ x'\leftrightarrow\tilde x'\end{array}\right)
\end{aligned} \tag{5.12}$$

coincides with FC_1 at $t = \mathrm{i}\pi$: Only the pure creation and annihilations terms give contributions to the full contraction. With the above notations, using

$$\begin{aligned}
B_{-j_0\Lambda(t)p}\xi(z)\cdot B_{-j_0\Lambda(t)\tilde p}\xi(\tilde z) &= B_{\Lambda(t)p}(-j_0)\xi(z)\cdot B_{\Lambda(t)\tilde p}(-j_0)\xi(\tilde z) \\
&= B_{\Lambda(t)p}\xi(-z)\cdot B_{\Lambda(t)\tilde p}\xi(-\tilde z) \\
&= B_pR(\Lambda(-t),p)\xi(-z)\cdot B_{\tilde p}R(\Lambda(-t),\tilde p)\xi(-\tilde z) \\
&= B_p\xi(-R(t)z+c(t))\cdot B_{\tilde p}\xi(-\tilde R(t)z+\tilde c(t)) \;,
\end{aligned}$$

and performing the same substitutions, gives

$$\begin{aligned}
\mathrm{FC}(t) \;=\;& F(+B_p\xi(z)\cdot B_{\tilde p}\xi(\tilde z))F(+B_p\xi(z')\cdot B_{\tilde p}\xi(\tilde z')) \\
&\left(\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}\tilde p(\tilde x'-j_0\Lambda(-t)\tilde x)}\mathrm{e}^{-\mathrm{i}k(z'-z)}\mathrm{e}^{-\mathrm{i}\tilde k(\tilde z'-\tilde z)} + \begin{array}{c} x\leftrightarrow\tilde x \\ x'\leftrightarrow\tilde x'\end{array}\right)
\end{aligned} \tag{5.13}$$

which after analytic continuation to $t = \mathrm{i}\pi$ agrees with FC_2. Combining these results yields

$$\mathrm{FC}_1 \overset{(5.12)}{=} \mathrm{FC}(\mathrm{i}\pi) \overset{(5.13)}{=} \mathrm{FC}_2 \tag{5.14}$$

Using 5.8, the partial results :SC_1: = :SC_2: and $\mathrm{FC}_1 = \mathrm{FC}_2$ can be combined to

$$[B(x,\tilde{x}), B(x',\tilde{x}')] \overset{(5.11)(5.14)}{=} 0 \tag{5.15}$$

which is the desired locality property. However, great care must be taken when performing the necessary analytic continuations because it is still not clear whether they are compatible with the integrations over the variables z. This is described in the last section.

5.2.2. Relative locality between String fields and currents

If a string $x+\mathbb{R}^+e \in W$ is spacelike separated from the pair of localization points $x', \tilde{x}' \in W'$ for the current, as shown in Fig. 5.2, it can again be assumed w.l.o.g. that W is the standard wedge 3.50.

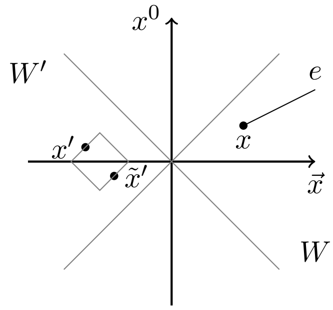

Figure 5.2.: A string $x+\mathbb{R}^+e$, spacelike separated from a pair of points $x', \tilde{x}'$

Wick's theorem yields

$$\begin{aligned} \Phi(x,e)B(x',\tilde{x}') &= :\Phi(x,e)B(x',\tilde{x}'): + \mathrm{SC}_1' \\ B(x',\tilde{x}')\Phi(x,e) &= :B(x',\tilde{x}')\Phi(x,e): + \mathrm{SC}_2' \,, \end{aligned} \tag{5.16}$$

where SC_1 is the value of the function

$$\begin{aligned} \mathrm{SC}'(t) &= (-e\cdot B_p\xi(z))^\alpha F(+B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\ &\quad [\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{\mathrm{i}\tilde{k}'\tilde{z}'}a^\dagger(\tilde{p}',\tilde{k}') + x' \leftrightarrow \tilde{x}'] \\ &+ (-e\cdot B_p\xi(z))^\alpha F(-B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\ &\quad [\mathrm{e}^{-\mathrm{i}p(x-j_0\Lambda(t)x')}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{-\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{-\mathrm{i}\tilde{k}'\tilde{z}'}a(\tilde{p}',\tilde{k}') + x' \leftrightarrow \tilde{x}'] \end{aligned} \tag{5.17}$$

at $t = \mathrm{i}\pi$. Just like for the SC-terms before, the substitution of the integration variables yields

$$\begin{aligned} \mathrm{SC}'(t) &= (+j_0\Lambda(-t)e\cdot B_p\xi(z))^\alpha F(-j_0\Lambda(t)B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\ &\quad [\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{\mathrm{i}\tilde{k}'\tilde{z}'}a^\dagger(\tilde{p}',\tilde{k}') + x' \leftrightarrow \tilde{x}'] \\ &+ (+j_0\Lambda(-t)e\cdot B_p\xi(z))^\alpha F(+j_0\Lambda(t)B_p\xi(z')\cdot B_{\tilde{p}'}\xi(\tilde{z}')) \\ &\quad [\mathrm{e}^{-\mathrm{i}p(x'-j_0\Lambda(-t)x)}\mathrm{e}^{-\mathrm{i}k(z-z')}\mathrm{e}^{-\mathrm{i}\tilde{p}'\tilde{x}'}\mathrm{e}^{-\mathrm{i}\tilde{k}'\tilde{z}'}a(\tilde{p}',\tilde{k}') + x' \leftrightarrow \tilde{x}'] \end{aligned} \tag{5.18}$$

which at $t = \mathrm{i}\pi$ agrees with SC_2. In sum,

$$\mathrm{SC}_1' \overset{(5.17)}{=} \mathrm{SC}'(\mathrm{i}\pi) \overset{(5.18)}{=} \mathrm{SC}_2' \tag{5.19}$$

and combining this result with 5.16 gives

$$[\Phi(x,e), B(x', \tilde{x}')] \overset{(5.19)}{=} 0,$$

i.e. the relative locality property. Since in 5.18 the argument of F is a complex number, the same warnings and restrictions as for the locality among the currents apply. In [Sch08, p. 12] the same problem of relative locality is considered and it is pointed out that the Fourier transforms, where integrations over the variables z need to be carried out, are hints for a non-polynomial occurence of p which leads to fuzzy support properties w.r.t. $x - x'$ in general.

5.3. Analytic problems of the currents' locality

For $t \in \mathbb{R} + \mathrm{i}[0, \pi]$, the arguments of F can take any complex value in contrast to the expressions $\xi(z)B_p \cdot e$ for the string-localized fields. This can be understood from the fact that the choice of the boost $\Lambda(t)$ for a string field is made in such a way that the imaginary part of $\Lambda(t)e$ is always in the interior of the forward lightcone V^+, hence $\xi(z)B_p \cdot \Lambda(t)e$ is in the upper half-plane. On the other hand, in the argument $\xi(z)B_p \cdot \xi(z')B_p'$ of F, both vectors are lightlike, but the imaginary part of the boosted vector $\xi(z)B_p\Lambda(t)$ can become spacelike, which implies that the argument of F is not restricted to a half-plane. This can also be seen from the explicit calculation in Appendix C.

F should therefore be an analytic function on the whole complex plane, which is not compatible with any strong decay assumption in z as it would violate Liouville's theorem. Choosing a polynomial for F would render $B = 0$, due to the same reasons given for the choice $\alpha \in \mathbb{C}\backslash\mathbb{N}$ in 4.64.

However, these difficulties only arise for the :SC:-terms because in FC no analytic continuation in the argument of F is needed. Hence the vacuum expectation value of the commutator 5.15 is local as expected, as was also proven in [MSY05] and [Sch08].

In conclusion, there is still the possibility that no numerical function F exists which allows a localization proof of the present type. An alternative method to check the localization properties of B would be the Jost-Lehmann-Dyson representation, which can be found in [IZ05, p. 249] and [BLOT90, p. 170]. It is introduced in Appendix B

As an outlook how one might proceed to tame the :SC:-terms, one idea concerning the possibilities how the definition of the operators B could be modified to in order to regularize their analytic behaviour is outlined in Appendix E.

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A. Gauge invariance

Because $B(x,\tilde{x})$ is a quadratic expression in the creation and annihilation operators, this operator does not mix even and odd particle numbers: Using the particle number operator

$$N = \int_{\partial V^+} \widetilde{\mathrm{d}p} \int \mathrm{d}\mu(k)\, a^\dagger(p,k) a(p,k) \ ,$$

this property is a form of gauge invariance w.r.t the unitary operator

$$U := (-1)^N$$

which has the properties

$$U a^\dagger(p,k) U = -a^\dagger(p,k) \text{ and } U a(p,k) U = -a(p,k)$$

and therefore gives a change of sign if applied to the field

$$U \Phi(x,e) U^\dagger = -\Phi(x,e) \ .$$

Because these signs cancel if U is applied to a product of two creation or annihilation operators, for example

$$U a^\dagger(p,k) a^\dagger(\widetilde{p},\widetilde{k}) U^\dagger = a^\dagger(p,k) a^\dagger(\widetilde{p},\widetilde{k}) \ ,$$

there is a gauge invariance property:

$$U B(x,\widetilde{x}) U = B(x,\widetilde{x})$$

B. Jost-Lehmann-Dyson representation

This representation, which is motivated from a physical point of view in [IZ05] and proven in [**?**], is based on the fact that if a function $f\ :\ \mathbb{M} \to \mathbb{C}$ satisfies $x^2 < 0 \Rightarrow f(x) = 0$, the Fourier transform has the general form

$$\tilde{f}(p) = \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}px} f(x) = \int_0^\infty \mathrm{d}\mu \int_{V^+} \widetilde{\mathrm{d}q}\, \rho(\mu, q) \mathrm{sgn}(p^0 - q^0) \delta((p-q)^2 - \mu^2) \tag{B.1}$$

where $\rho(\mu, q)$ is called the spectral function. For example, $\rho(\mu, q) = \delta(\mu - m)\delta(q)$ yields the usual commutator function of a free scalar field with mass $m \geq 0$. The spectral function can be chosen such that

$$\exists\, p \in q + H_\mu^\pm :\ \tilde{f}(p) = 0 \Rightarrow \rho(\mu, q) = 0 \ .$$

In [MSY05] it is also suggested to investigate the problem of locality for $B(x, \tilde{x})$ from 5.4 by comparing it to the form of the Jost Lehmann Dyson Representation. The Fourier transforms of the functions

$$\begin{aligned}
f_c(x) &= \langle 0|\, a(p,k) a(\tilde{p}, \tilde{k})[B(x,x), B(-x,-x)]\, |0\rangle \\
f_a(x) &= \langle 0|\, [B(x,x), B(-x,-x)] a^\dagger(p,k) a^\dagger(\tilde{p}, \tilde{k})\, |0\rangle = -\overline{f_c(x)} \\
f_n(x) &= \langle 0|\, a(p,k)[B(x,x), B(-x,-x)] a^\dagger(\tilde{p}, \tilde{k})\, |0\rangle \ ,
\end{aligned}$$

accounting for the creation-, annihilation-[1] and particle number preserving parts of the commutator, respectively, thus have to take the above form.

[1]In fact, due to the symmetry between the creation and annhilation part, stated in the second equation, only either one of f_c or f_a has to be examined explicitly.

C. Explicit form of the two-particle intertwiner

The purpose of this section is to give an explicit form of the argument of F, as it appears in 5.1 and is then used in 5.3, i.e. to evaluate the expression $\xi(z)\Lambda(R_p)\cdot\xi(\tilde z)\Lambda(R_{\tilde p})$. The first step is (using the abbreviations $p_\pm := p^0 \pm p^3$ and $\mathrm{p} := p_1 + \mathrm{i}p_2$) to calculate one factor:

$$
\begin{aligned}
(\xi(z)\Lambda(R_p))^{\sim} &\overset{(2.17)}{=} R_p^\dagger \widetilde{\xi(z)} R_p \\
&\overset{(4.2)(4.25)}{=} \frac{1}{p_0+p_3}\begin{pmatrix} p_0+p_3 & 0 \\ p_1+\mathrm{i}p_2 & 1\end{pmatrix}\begin{pmatrix} |z|^2 & \overline{z} \\ z & 1\end{pmatrix}\begin{pmatrix} p_0+p_3 & p_1-\mathrm{i}p_2 \\ 0 & 1\end{pmatrix} \\
&= \frac{1}{p_-}\begin{pmatrix} p_- & 0 \\ \mathrm{p} & 1\end{pmatrix}\begin{pmatrix} |z|^2 & \overline{z} \\ z & 1\end{pmatrix}\begin{pmatrix} p_- & \overline{\mathrm{p}} \\ 0 & 1\end{pmatrix} \\
&= \frac{1}{p_-}\begin{pmatrix} p_- & 0 \\ \mathrm{p} & 1\end{pmatrix}\begin{pmatrix} |z|^2 p_- & |z|^2\overline{\mathrm{p}}+\overline{z} \\ z p_- & z\overline{\mathrm{p}}+1\end{pmatrix} \\
&= \frac{1}{p_-}\begin{pmatrix} |z|^2 p_-^2 & |z|^2\overline{\mathrm{p}}p_- + \overline{z}p_- \\ |z|^2 p_-\mathrm{p} + z p_- & |z|^2|\mathrm{p}|^2 + \overline{z}\mathrm{p} + z\overline{\mathrm{p}} + 1\end{pmatrix} \\
&= \begin{pmatrix} |z|^2 p_- & \overline{z}(1+z\overline{\mathrm{p}}) \\ z(1+\overline{z}\mathrm{p}) & |1+\overline{z}\mathrm{p}|^2/p_-\end{pmatrix}
\end{aligned}
$$

Then the Minkowski product between two such expressions becomes

$$
\begin{aligned}
\xi(z)\Lambda(R_p)\cdot\xi(\tilde z)\Lambda(R_{\tilde p}) &\overset{(2.10)}{=} \frac{1}{2}\mathrm{Tr}[\xi(z)\Lambda(R_p)]_{\sim}[\xi(\tilde z)\Lambda(R_{\tilde p})]^{\sim} \\
&= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} |1+\overline{z}\mathrm{p}|^2/p_- & -\overline{z}(1+z\overline{\mathrm{p}}) \\ -z(1+\overline{z}\mathrm{p}) & |z|^2 p_-\end{pmatrix}\begin{pmatrix} |\tilde z|^2\tilde p_- & \overline{\tilde z}(1+\tilde z\overline{\tilde{\mathrm{p}}}) \\ \tilde z(1+\overline{\tilde z}\tilde{\mathrm{p}}) & |1+\overline{\tilde z}\tilde{\mathrm{p}}|^2/\tilde p_-\end{pmatrix} \\
&= \frac{1}{2p_-\tilde p_-}\left|\tilde p_-(1+\overline{z}\mathrm{p})\tilde z - p_-(1+\overline{\tilde z}\tilde{\mathrm{p}})z\right|^2 \\
&= \frac{1}{2}|(1+\overline{z}\mathrm{p})\tilde z|^2\frac{\tilde p_-}{p_-} + \frac{1}{2}|(1+\overline{\tilde z}\tilde{\mathrm{p}})z|^2\frac{p_-}{\tilde p_-} - \Re[(1+z\overline{\mathrm{p}})(1+\overline{\tilde z}\tilde{\mathrm{p}})\overline{\tilde z}z]\ .
\end{aligned}
$$

D. Infinite spin vs continuous spin

The representations of $\widetilde{E}(2)$ on the space $L^2(\mathrm{sp}K)$ of functions on the circle where mentioned as representations of *infinite spin* throughout the thesis. This terminology can be understood intuitively from the fact that arbitrary Fourier modes are allowed. As indicated in [Sch08, p. 7], the continuous parameter κ^2 suggests the term *continuous spin* instead.

In the following discussion, the relation between the massive and massless representations of $\mathcal{P}^c$ is worked out in more detail by studying another Casimir operator, which can be expressed in terms of m and s in any massive representation and in terms of κ for any massless representation. This will justify the description of representations with $\kappa^2 > 0$ as those of infinite spin.

It is is already clear from the explicit formulas 2.19 and 2.22, describing Lorentz boosts and spatial rotations, respectively, that the elements of the Lie group $\mathrm{SL}(2, \mathbb{C})$ can be written as

$$A = \mathrm{e}^{m_i \omega^i} \ , \tag{D.1}$$

where i enumerates the ordered set $\mathcal{B} := (\sigma_1, \sigma_2, \sigma_3, \mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3)$, which can be thought of as a basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, $m_i \in \mathcal{B}$ and ω is a set of parameters. This yields an infinitesimal version of the action on $\mathbb{M}$:

$$2(p\Lambda(m_i))\widetilde{\ } := \frac{\partial}{\partial \omega^i} \left(p\Lambda(\mathrm{e}^{m_i\omega^i})\right)\widetilde{\ }\Big|_{\omega^i=0} \overset{(2.17)}{=} \frac{\partial}{\partial \omega^i} (\mathrm{e}^{m_i\omega^i})^\dagger \widetilde{p} \mathrm{e}^{m_i\omega^i}\Big|_{\omega^i=0} = m_i^\dagger \widetilde{p} + \widetilde{p} m_i \tag{D.2}$$

The map $\Lambda \ : \ \mathfrak{sl}(2, \mathbb{C}) \mapsto \mathcal{L}_+^\uparrow$ defined herein is $\mathbb{R}$-linear since

$$\begin{aligned} 2(p\Lambda(m_i\Gamma_{ij}))\widetilde{\ } &\overset{(2.17)}{=} m_i^\dagger \Gamma_{ij} \widetilde{p} + \widetilde{p} m_i \Gamma_{ij} = (m_i^\dagger \widetilde{p} + \widetilde{p} m_i)\Gamma_{ij} \overset{(2.17)}{=} 2(p\Lambda(m_i))\widetilde{\ }\Gamma_{ij} \\ &\overset{(2.6)}{=} 2(p\Lambda(m_i)\Gamma_{ij})\widetilde{\ } \end{aligned} \tag{D.3}$$

and p is arbitrary. Furthermore, it satisfies

$$\begin{aligned} 2(p\Lambda(BmB^{-1}))\widetilde{\ } &\overset{(2.17)}{=} (BmB^{-1})^\dagger \widetilde{p} + \widetilde{p} BmB^{-1} = (B^{-1})^\dagger (m^\dagger B^\dagger \widetilde{p} B + B^\dagger \widetilde{p} B m) B^{-1} \\ &\overset{(2.17)}{=} (B^{-1})^\dagger (m^\dagger (p\Lambda(B))\widetilde{\ } + (p\Lambda(B))\widetilde{\ } m) B^{-1} \\ &\overset{(D.2)}{=} 2(B^{-1})^\dagger (p\Lambda(B)\Lambda(m))\widetilde{\ } B^{-1}) \overset{(2.17)}{=} 2(p\Lambda(B)\Lambda(m)\Lambda(B^{-1}))\widetilde{\ }. \end{aligned}$$

Denoting the basis vectors of $\mathbb{M}$ by e_μ ($\mu = 0, \ldots, 3$), i.e. $\widetilde{e_0} = \mathbf{1}$ and $\widetilde{e_i} = \sigma_i$, the infinitesimal action of the Lorentz transformations can be calculated using the covering map in the form (2.17) again, as well as the multiplication formula 2.8 for the Pauli matrices. This gives

$$\begin{aligned}
2(e_0\Lambda(\sigma_i))\breve{} &= \sigma_i\mathbf{1}+\mathbf{1}\sigma_i = 2\sigma_i\\
2(e_i\Lambda(\sigma_j))\breve{} &= \sigma_j\sigma_i+\sigma_i\sigma_j = \{\sigma_i,\sigma_j\} = 2\mathbf{1}\delta_{ij}\\
2(e_0\Lambda(\mathrm{i}\sigma_i))\breve{} &= -\mathrm{i}\sigma_i\mathbf{1}+\mathrm{i}\mathbf{1}\sigma_i = 0\\
2(e_i\Lambda(\mathrm{i}\sigma_j))\breve{} &= -\mathrm{i}\sigma_j\sigma_i+\mathrm{i}\sigma_i\sigma_j = \mathrm{i}[\sigma_i,\sigma_j] = -2\epsilon_{ijk}\sigma_k
\end{aligned}$$

which determines the components of each $\Lambda_{\mu\nu}(\sigma_\lambda)$, of which the nonvanishing ones are stated explicitly:

$$\Lambda_{0i}(\sigma_j) = -\Lambda_{i0}(\sigma_j) = \delta_{ij} \text{ and } \Lambda_{ij}(\mathrm{i}\sigma_k) = \epsilon_{ijk} \tag{D.4}$$

In the following, any $\mathrm{SL}(2,\mathbb{C})$-transformation of the generators $m_i \in \mathfrak{sl}(2,\mathbb{C})$ will be encoded by the map[1]

$$\begin{aligned}
\Gamma\ :\ \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{End}(\mathfrak{sl}(2,\mathbb{C}))\\
B &\mapsto (m \mapsto BmB^{-1})
\end{aligned} \tag{D.5}$$

whose definition in terms of $\mathfrak{sl}(2,\mathbb{C})$-indices reads

$$\Gamma_{ij}(B)m_j \quad := \quad \Gamma(B)(m_i) \overset{(\mathrm{D.5})}{=} Bm_iB^{-1} \tag{D.6}$$

$$\begin{aligned}
\Rightarrow \Gamma_{ij}(B) &\overset{(2.8)}{\hat{=}} \frac{1}{2}\mathrm{Tr}(\Gamma(B)(m_i)m_j) \overset{(\mathrm{D.5})}{=} \frac{1}{2}\mathrm{Tr}(Bm_iB^{-1}m_j) = \frac{1}{2}\mathrm{Tr}(B^{-1}m_jBm_i)\\
&\overset{(\mathrm{D.5})}{=} \frac{1}{2}\mathrm{Tr}(\Gamma(B^{-1})(m_j)m_i) \overset{(2.8)}{\hat{=}} \Gamma_{ji}(B^{-1})\ ,
\end{aligned} \tag{D.7}$$

where $\hat{=}$ means that if both $m_i, m_j \in \mathcal{B}$ are Pauli matrices, the sides are equal, while the trace has to be divided by one or two prefactors i which occur in front of the Pauli matrices in all of the other cases.

The representation of A (given in the form D.1) on $\mathcal{H}$ is also written in exponential form[2]

$$U(A) \overset{(\mathrm{D.1})}{=} U(\mathrm{e}^{m_i\omega^i}) = \mathrm{e}^{\mathrm{i}M_i\omega^i} \Rightarrow M_i = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}U(\mathrm{e}^{m_i\omega})\ , \tag{D.8}$$

corresponding to D.1, with M_i a set of hermitean operators on $\mathcal{H}$, which generate the unitary representation of $\mathrm{SL}(2,\mathbb{C})$ in the sense of Stone's theorem. They transform in the

[1]This map can bee viewed as the exponentiation of the adjoint action.

[2]All derivatives w.r.t. ω which are used in this chapter are implicitly meant to be evaluated at $\omega = 0$.

following way under the representation of an arbitrary element $B \in \mathrm{SL}(2,\mathbb{C})$:[3]

$$\begin{aligned}
U(B)\mathrm{e}^{\mathrm{i}M_i\omega^i} &\overset{(D.8)}{=} U(B)U(A) \overset{(D.1)}{=} U(B)U(\mathrm{e}^{m_i\omega^i}) \overset{(2.2)}{=} U(B\mathrm{e}^{m_i\omega^i}B^{-1})U(B) \\
&= U(B\sum_{n=0}^{\infty}\frac{1}{n!}(m_i\omega^i)^n B^{-1})U(B) = U(\sum_{n=0}^{\infty}\frac{1}{n!}(Bm_iB^{-1}\omega^i)^n)U(B) \\
&= U(\mathrm{e}^{Bm_iB^{-1}\omega^i})U(B) \overset{(D.6)}{=} U(\mathrm{e}^{\Gamma_{ij}(B)m_j\omega^i})U(B) \overset{(D.8)}{=} \mathrm{e}^{\mathrm{i}M_j\Gamma_{ij}(B)\omega^i}U(B) \\
\Rightarrow U(B)M_i &\overset{(D.8)}{=} -\mathrm{i}\frac{\partial}{\partial\omega^i}\,U(B)\mathrm{e}^{\mathrm{i}M_i\omega^i}\Big|_{\omega^i=0} = -\mathrm{i}\frac{\partial}{\partial\omega^i}\,\mathrm{e}^{\mathrm{i}M_j\Gamma_{ij}(B)\omega^i}U(B)\Big|_{\omega^i=0} \\
&\overset{(D.8)}{=} M_j\Gamma_{ij}(B)U(B)
\end{aligned} \tag{D.9}$$

The *Pauli-Lubanski spin vector* is defined as the following operator

$$S^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}\Lambda_{\nu\lambda}(m_i)M_iP_\kappa \tag{D.10}$$

and transforms like a four-vector:[4]

$$\begin{aligned}
2U(B)S^\mu U^\dagger(B) &\overset{(D.10)}{=} U(B)\epsilon^{\mu\nu\lambda\kappa}\Lambda_{\nu\lambda}(m_i)M_iP_\kappa U^\dagger(B) \qquad\qquad (D.11)\\
&= \epsilon^{\mu\nu\lambda\kappa}\Lambda_{\nu\lambda}(m_i)\underbrace{U(B)M_iU^\dagger(B)}_{\overset{(D.9)}{=}M_j\Gamma_{ij}(B)}\underbrace{U(B)P_\kappa U^\dagger(B)}_{\overset{(2.33)}{=}P_\sigma\Lambda^\sigma{}_\kappa(B)} \\
&\overset{(D.3)}{=} \epsilon^{\mu\nu\lambda\kappa}\Lambda_{\nu\lambda}(m_i\underbrace{\Gamma_{ij}(B)}_{\overset{(D.7)}{=}\Gamma_{ji}(B^{-1})})M_jP_\sigma\Lambda^\sigma{}_\kappa(B) \\
&\overset{(D.6)}{=} \epsilon^{\mu\nu\lambda\kappa}\Lambda_{\nu\lambda}(B^{-1}m_jB)M_jP_\sigma\Lambda^\sigma{}_\kappa(B) \\
&\overset{(2.12)}{=} \epsilon^{\mu\nu\lambda\kappa}\Lambda_\nu{}^\tau(B^{-1})\Lambda_{\tau\rho}(m_j)\Lambda^\rho{}_\lambda(B)M_jP_\sigma\Lambda^\sigma{}_\kappa(B) \\
&= \epsilon^{\mu\nu\lambda\kappa}\Lambda^\tau{}_\nu(B)\Lambda^\rho{}_\lambda(B)\Lambda^\sigma{}_\kappa(B)\Lambda_{\tau\rho}(m_j)M_jP_\sigma \\
&= \epsilon^{\sigma\nu\lambda\kappa}\Lambda_{\nu\lambda}(m_i)M_iP_\kappa\Lambda_\sigma{}^\mu(B) \overset{(D.10)}{=} 2S^\nu\Lambda_\nu{}^\mu(B)
\end{aligned}$$

Hence the squared spin vector

$$\mathcal{S}^2 := S_\mu S^\mu \tag{D.12}$$

commutes with the representation $U(B)$, since

$$\begin{aligned}
U(B)\mathcal{S}^2 &\overset{(D.12)}{=} U(B)S^\mu S_\mu \overset{(D.11)}{=} S^\nu\Lambda(B)_\nu{}^\mu U(B)S_\mu \overset{(D.11)}{=} S^\nu\underbrace{\Lambda(B)_\nu{}^\mu\Lambda(B)^\lambda{}_\mu}_{=\delta_\nu{}^\lambda}S_\lambda U(B) \\
&\overset{(D.12)}{=} \mathcal{S}^2U(B) \Rightarrow [U(B),\mathcal{S}^2] = 0\ ,
\end{aligned}$$

[3]In the second line, each term is rewritten in the following way: $BA^nB^{-1} = B\underbrace{AA\cdots A}_{n\text{ factors}}B^{-1} = \underbrace{BAB^{-1}BAB^{-1}\cdots BAB^{-1}}_{n\text{ factors}} = (BAB^{-1})^n$

[4]It is used in the step to the last line, that $\epsilon^{\mu\nu\lambda\kappa}$ transforms like a 4-tensor.

and is therefore a Casimir operator, which in an irreducible representation means that $\mathcal{S}^2 = \lambda\mathbf{1}$ with $\lambda \in \mathbb{R}$.[5] This operator can be applied to both the massive representations, specified by the mass m and spin s as well as to the massless representations of infinite spin, characterised by the Pauli-Lubanski parameter κ. Since $\mathcal{S}^2$ is a multiple of identity, it is sufficient to calculate its eigenvalue on any vector $v \in \mathcal{H}_q$, where q is the reference momentum chosen in 3.1 and 4.1, respectively. The generators M_i correspond to the representation D on $\mathcal{H}_q$ in this case. It is also used that $\mathcal{H}_q$ is defined as the subspace of $\mathcal{H}$ where P has the sharp joint eigenvalue q.

- For a massive representation with $q = (m, 0, 0, 0)$, S^μ can therefore be simplified to

$$\begin{aligned} S^\mu &\overset{\text{(D.10)}}{=} \frac{m}{2}\epsilon^{\mu\nu\lambda 0}\Lambda(m_i)_{\nu\lambda}M_i \overset{\text{(D.8)}}{=} \frac{m}{2}\epsilon^{\mu\nu\lambda 0}\Lambda(\mathrm{i}\sigma_k)_{\nu\lambda}\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}D(\mathrm{e}^{\mathrm{i}\omega\sigma_k}) \\ &\overset{\text{(D.4)}}{=} \frac{m}{2}\underbrace{\epsilon^{\mu\nu\lambda 0}\epsilon_{\nu\lambda k}}_{=2\delta^\mu{}_k}\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}D(\mathrm{e}^{\mathrm{i}\omega\sigma_k}) = m\delta^\mu{}_k\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}D(\mathrm{e}^{\mathrm{i}\omega\sigma_k}) \end{aligned} \tag{D.13}$$

  because none of the indices $\mu\nu\lambda$ can be 0 for nonvanishing terms. Hence S^μ only has spatial components.

  For simplicity, the operator S^2 will be applied to the vector $v = |{\uparrow}\rangle^{\otimes 2s} \in \mathcal{H}_q$:

$$\begin{aligned} -S^2 v &\overset{\text{(D.13)}}{=} \sum_{k=0}^{3} m^2\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}D(\mathrm{e}^{\mathrm{i}\omega\sigma_k})\right)^2 |{\uparrow}\rangle^{\otimes 2s} \\ &\overset{\text{(3.5)}}{=} m^2\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}D(\mathrm{e}^{\mathrm{i}\omega\sigma_k})\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}(\mathrm{e}^{\mathrm{i}\omega\sigma_k}|{\uparrow}\rangle)^{\otimes 2s} \\ &= m^2\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}D(\mathrm{e}^{\mathrm{i}\omega\sigma_k})(\sigma_k|{\uparrow}\rangle \otimes \cdots \otimes |{\uparrow}\rangle + \ldots) \\ &\overset{\text{(3.5)}}{=} m^2\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}(\mathrm{e}^{\mathrm{i}\omega\sigma_k}\sigma_k|{\uparrow}\rangle \otimes \cdots \otimes \mathrm{e}^{\mathrm{i}\omega\sigma_k}|{\uparrow}\rangle + \ldots) \\ &= m^2(\sigma_k^2|{\uparrow}\rangle \otimes \cdots \otimes |{\uparrow}\rangle + \ldots + \sigma_k|{\uparrow}\rangle \otimes \sigma_k|{\uparrow}\rangle \otimes \cdots \otimes |{\uparrow}\rangle + \ldots) \\ &= m^2(3 \cdot 2s + 2s(2s-1))v = 4m^2 s(s+1) \end{aligned}$$

  To obtain the last line, it has been used that $\sigma_k^2 = 3\mathbf{1}$ for the $2s$ quadratic terms, while for the mixed terms, only those with $k = 3$ contribute, where $\sigma_3|{\uparrow}\rangle = |{\uparrow}\rangle$, since the terms with $\sigma_1|{\uparrow}\rangle = |{\downarrow}\rangle$ and $\sigma_2|{\uparrow}\rangle = \mathrm{i}|{\downarrow}\rangle$ cancel. Then there are $2s$ possibilities to place the first σ_3 and $2s-1$ remaining possibilities to place the second one.

  In conclusion, the square of the Pauli-Lubanski spin vector is essentially the product of the squares of mass m and spin s in a massive representation.

[5]This statement is due to the fact that each S^μ and therefore also $\mathcal{S}^2$ is a hermitean operator.

- In a massless representation with $q = (1/2, 0, 0, 1/2)$, S^μ becomes

$$
\begin{aligned}
S^\mu &\overset{(D.10)}{=} \frac{1}{2}(\epsilon^{\mu\nu\lambda 0} + \epsilon^{\mu\nu\lambda 3})\Lambda(m_i)_{\nu\lambda} M_i \qquad (D.14)\\
\Rightarrow S^0 &\overset{(D.8)}{=} \frac{1}{2} \underbrace{\epsilon^{0\nu\lambda 3}\Lambda(\mathrm{i}\sigma_k)_{\nu\lambda}}_{=2\Lambda(\mathrm{i}\sigma_k)_{12} \overset{(D.4)}{=} 2\epsilon_{12k}} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_k}) = \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_3})\\
S^1 &\overset{(D.8)}{=} \frac{1}{2} \underbrace{\epsilon^{1\nu\lambda 0}\Lambda(\mathrm{i}\sigma_k)_{\nu\lambda}}_{=-2\Lambda(\mathrm{i}\sigma_k)_{23} \overset{(D.4)}{=} -2\epsilon_{23k}} \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_k})\\
&+ \frac{1}{2} \underbrace{\epsilon^{1\nu\lambda 3}\Lambda(\sigma_k)_{\nu\lambda}}_{=-2\Lambda(\mathrm{i}\sigma_k)_{02} \overset{(D.4)}{=} -2\delta_{02}} \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\omega\sigma_k})\\
&= -\frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_1}) - \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\omega\sigma_2}) \overset{(D.8)}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega} D(\mathrm{e}^{-\mathrm{i}\omega(\sigma_1 - \mathrm{i}\sigma_2)})\\
S^2 &\overset{(D.8)}{=} \frac{1}{2} \underbrace{\epsilon^{2\nu\lambda 0}\Lambda(\mathrm{i}\sigma_k)_{\nu\lambda}}_{=-2\Lambda(\mathrm{i}\sigma_k)_{31} \overset{(D.4)}{=} -2\epsilon_{31k}} \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_k})\\
&+ \frac{1}{2} \underbrace{\epsilon^{2\nu\lambda 3}\Lambda(\sigma_k)_{\nu\lambda}}_{=2\Lambda(\mathrm{i}\sigma_k)_{01} \overset{(D.4)}{=} 2\delta_{01}} \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\omega\sigma_k})\\
&= -\frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_2}) + \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\omega\sigma_1}) \overset{(D.8)}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\omega(\sigma_1 - \mathrm{i}\sigma_2)})\\
S^3 &\overset{(D.8)}{=} \frac{1}{2} \underbrace{\epsilon^{3\nu\lambda 0}\Lambda(\mathrm{i}\sigma_k)_{\nu\lambda}}_{=-2\Lambda(\mathrm{i}\sigma_k)_{12} \overset{(D.4)}{=} -2\epsilon_{12k}} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega} D(\mathrm{e}^{\mathrm{i}\omega\sigma_k}) = \frac{1}{i}\frac{\partial}{\partial\omega} D(\mathrm{e}^{-\mathrm{i}\omega\sigma_3})
\end{aligned}
$$

For S_1 and S_2, it has been used in the respective last step that the exponential representation of U and hence D can involve linear combinations of the generators M_i. It follows that the representations of the following $\mathrm{SL}(2,\mathbb{C})$-matrices have to be considered:

$$
\mathrm{e}^{\pm\mathrm{i}\omega\sigma_3} = \exp\begin{pmatrix} \pm\mathrm{i}\omega & \\ & \mp\mathrm{i}\omega \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\pm\mathrm{i}\omega\sigma_3} & \\ & \mathrm{e}^{\mp\mathrm{i}\omega\sigma_3} \end{pmatrix} \overset{(4.3)}{=} [\pm\omega, 0] \qquad (D.15)
$$

$$
\mathrm{e}^{(-\mathrm{i})\omega(\sigma_1 - \mathrm{i}\sigma_2)} = \exp\begin{pmatrix} 0 & \\ (-\mathrm{i})2\omega & 0 \end{pmatrix} = \begin{pmatrix} 1 & \\ (-\mathrm{i})2\omega & 1 \end{pmatrix} \overset{(4.3)}{=} [0, 2(\mathrm{i})\omega] \qquad (D.16)
$$

Let now $v \in \mathcal{H}_q = L^2(\operatorname{sp} K)$ be an arbitrary vector in the little Hilbert space $\mathcal{H}_q$, i.e.

a function on the circle. The action on v by the various components S^μ becomes

$$\begin{aligned}
\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}[D(\mathrm{e}^{\pm\mathrm{i}\omega\sigma_3})v](k) &\overset{(D.15)}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}[D([\pm\omega,0])v](k) \overset{(4.23)}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}v(k\lambda(\mp\omega)) \\
&\overset{(4.7)}{=} \pm\mathrm{i}v'(k) \\
\frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}[D(\mathrm{e}^{(-\mathrm{i})\omega(\sigma_1-\mathrm{i}\sigma_2)})]v(k) &\overset{(D.16)}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}[D([0,2(\mathrm{i})\omega])]v(k) \\
&\overset{(4.23)}{=} \frac{1}{\mathrm{i}}\frac{\partial}{\partial\omega}\mathrm{e}^{-\mathrm{i}k[(-\mathrm{i})2\omega]}v(k) \\
&= -2k_1 \text{ or } 2k_2\text{, respectively.}
\end{aligned}$$

Substituting into the generators S^μ in D.14 yields

$$\begin{aligned}
[(S^0)^2v](k) &= [(S^3)^2v](k) = -v''(k) \\
[(S^1)^2v](k) &= 4k_1^2v(k) \\
[(S^2)^2v](k) &= 4k_2^2v(k) \\
\Rightarrow [\mathcal{S}^2v](k) &= [((S^0)^2-(S^1)^2-(S^2)^2-(S^3)^2)v](k) \\
&= -4(k_1^2+k_2^2)v(k) = -4\kappa^2v(k)
\end{aligned}$$

Summarizing these results, the Pauli-Lubanski parameter is $\lambda = -4m^2s(s+1)$ in the massive and $\lambda = -4\kappa^2$ in the massless case. Therefore the name infinite spin can be justified in the following way: Proceeding from a massive to to the corresponding (in the sense of keeping λ constant) massless representation, $m \to 0$ implies $s \to \infty$. In each case, $\lambda = 0$ would mean $s = 0$ or $\kappa = 0$, respectively, i.e. the choice of the trivial representation of the little group G_q, up to possible helicity phases, as given in 4.12.

E. Modified two-particle intertwiner

One approach one can try to cure the difficulties involved in the analytic continuations could be to define a modified version of the numerical function v_2 from 5.1 by

$$\tilde{v}_2(p,\tilde{p})(\xi,\tilde{\xi}) := \frac{1}{p\cdot\tilde{p}} F\left(\frac{\xi B_p \cdot \tilde{\xi} B_{\tilde{p}}}{p\cdot\tilde{p}}\right) ,$$

where the extra factor including $p\cdot\tilde{p}$ does not alter the transformation under $\widetilde{D}$ corresponding to 5.2 for $\tilde{v}_2$. The modified two-particle intertwiner is defined analogously to 5.3:

$$\tilde{u}_2(k,\tilde{k}) := \int \mathrm{d}^2 z \int \mathrm{d}^2 \tilde{z}\, \mathrm{e}^{\mathrm{i}kz} \mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}} \tilde{v}_2(\xi(z),\tilde{\xi}(\tilde{z}))$$

The modified current operator is defined as

$$\tilde{B}(x,\tilde{x}) = \int_{\partial V^+} \widetilde{\mathrm{d}p} \int_{\partial V^+} \widetilde{\mathrm{d}\tilde{p}} :(\mathrm{e}^{\mathrm{i}px} a^\dagger(p) - \mathrm{e}^{-\mathrm{i}px} a(p)) \circ \tilde{u}_2(p,\tilde{p}) \circ (\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}} a^\dagger(\tilde{p}) - \mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}} a(\tilde{p})): .$$

Expanding the $\circ$-product clearly gives the same sign pattern as in 5.4, but with respect to the prefactor $(p\cdot\tilde{p})^{-1}$, while the argument of F always appears with the same sign. The SC-part of $\tilde{B}(x,\tilde{x})\tilde{B}(x',\tilde{x}')$, which is known to be problematic for the locality property, can be given in the following way:[1] Define

$$\begin{aligned}
& f(p_-,b_1,b_2,d) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{E.1})\\
:= \quad & \mathrm{e}^{-\mathrm{i}pd} \frac{1}{p\cdot\tilde{p}} F\left(\frac{\frac{1}{2}|(1+\overline{z}\mathrm{p})\tilde{z}|^2 \frac{\tilde{p}_-}{p_-} b_1 + \frac{1}{2}|(1+\overline{\tilde{z}}\tilde{\mathrm{p}})z|^2 \frac{p_-}{\tilde{p}_-} b_2 - \Re[(1+z\overline{\mathrm{p}})(1+\overline{\tilde{z}}\tilde{\mathrm{p}})\overline{\tilde{z}}z]}{\tilde{p}_- \frac{|\mathrm{p}|^2}{p_-} b_1 + \tilde{p}_+ p_- b_2 - 2\Re(\overline{\mathrm{p}}\tilde{\mathrm{p}})}\right)\\
& \frac{1}{p\cdot\tilde{p}'} F\left(\frac{\frac{1}{2}|(1+\overline{z'}\mathrm{p})\tilde{z}|^2 \frac{\tilde{p}'_-}{p_-} b_1 + \frac{1}{2}|(1+\overline{\tilde{z}'}\tilde{\mathrm{p}}')z'|^2 \frac{p_-}{\tilde{p}'_-} b_2 - \Re[(1+z'\overline{\mathrm{p}})(1+\overline{\tilde{z}'}\tilde{\mathrm{p}}')\overline{\tilde{z}'}z']}{\tilde{p}'_- \frac{|\mathrm{p}|^2}{p_-} b_1 + \tilde{p}'_+ p_- b_2 - 2\Re(\overline{\mathrm{p}}\tilde{\mathrm{p}}')}\right) ,
\end{aligned}$$

where $d := x - x'$. For $b_1 = b_2 = 1$ and comparing to Appendix C, this function gives the specific part of

$$\begin{aligned}
\mathrm{SC} \quad = \quad & \int_{\partial V^+} \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}p(x-x')} \mathrm{e}^{-\mathrm{i}k(z-z')} \frac{1}{p\cdot\tilde{p}} F\left(\frac{\xi(z)B_p\cdot\xi(\tilde{z})B_{\tilde{p}}}{p\cdot\tilde{p}}\right) \frac{1}{p\cdot\tilde{p}'} F\left(\frac{\xi(z')B_p\cdot\xi(\tilde{z})B_{\tilde{p}'}}{p\cdot\tilde{p}'}\right)\\
& (\mathrm{e}^{\mathrm{i}\tilde{p}\tilde{x}} \mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}} a^\dagger(\tilde{p},\tilde{k}) + \mathrm{e}^{-\mathrm{i}\tilde{p}\tilde{x}} \mathrm{e}^{-\mathrm{i}\tilde{k}\tilde{z}} a(\tilde{p},\tilde{k}))(\mathrm{e}^{\mathrm{i}\tilde{p}'\tilde{x}'} \mathrm{e}^{\mathrm{i}\tilde{k}'\tilde{z}'} a^\dagger(\tilde{p}',\tilde{k}') + \mathrm{e}^{-\mathrm{i}\tilde{p}'\tilde{x}'} \mathrm{e}^{-\mathrm{i}\tilde{k}'\tilde{z}'} a(\tilde{p}',\tilde{k}'))
\end{aligned}$$

[1] For a simpler notation, the symmetrizations $x \leftrightarrow \tilde{x}$ and $x' \leftrightarrow \tilde{x}'$ are omitted in the following description. All other notations are the same as in the above sections.

which is needed to carry out the integration over the coordinate p_-. Using the coordinates $p_\pm = p^0 \pm p^3$ with

$$\begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^3 \end{pmatrix} \Rightarrow \mathrm{d}p_+ \mathrm{d}p_- = 2\mathrm{d}p^0 \mathrm{d}p^3$$

results in the following form for the Lorentz invariant measure on $\mathcal{L}_+^\uparrow$:

$$\mathrm{d}^4 p \delta(p^2)\Theta(p^0) = \frac{1}{2}\mathrm{d}p_+ \mathrm{d}p_- \mathrm{d}^2 \mathrm{p}\delta(p_+ p_- - |\mathrm{p}|^2)\Theta(p_+ + p_-)$$

Together with the form of the W-preserving boost $\Lambda(t)$ in these coordinates

$$\begin{aligned} p\Lambda(t) &= \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} \cosh t & & & \sinh t \\ & 1 & & \\ & & 1 & \\ \sinh t & & & \cosh t \end{pmatrix} \\ &= \begin{pmatrix} p_0 \cosh t + p_3 \sinh t & p_1 & p_2 & p_0 \sinh t + p_3 \cosh t \end{pmatrix} \\ \Rightarrow (p\Lambda(t))_\pm &= p_0 \cosh t + p_3 \sinh t \mp (p_0 \sinh t + p_3 \cosh t) \\ &= (p_0 \mp p_3)\mathrm{e}^{\mp t} = \mathrm{e}^{\mp t} p_\pm \end{aligned}$$

and integrating over p_+, the analytic continuation of t into the strip $\mathbb{R} + \mathrm{i}[0, \pi]$ of the equation

$$\int_0^\infty \frac{\mathrm{d}p_-}{p_-} f(p_-, 1, -\mathrm{e}^t, x - j_0\Lambda(t)x') = \int_0^\infty \frac{\mathrm{d}p_-}{p_-} f(p_-, \mathrm{e}^t, -1, -j_0(x' - j_0\Lambda(-t)x))$$

(by substitution) and evaluation at $t = \mathrm{i}\pi$ gives an identity which can be integrated over the remaining variables p, $\tilde{p}$, $\tilde{p}'$, k, $\tilde{k}$, $\tilde{k}'$, z, $\tilde{z}$, z' and $\tilde{z}'$ to the locality of the SC-part because the signs in front of $p_\pm$ will be different on both sides of the equation. As can be seen by the last argument of f, the substitution of p by $-pj_0$ is still necessary on the r.h.s., which will also flip the sign of p.

The analytic behaviour can thus be improved in the following sense: Put $F(z) := \exp(-z^2)$, for example. Now $F \in \mathcal{S}(\mathbb{R})$, as demanded in [MSY05]. For $p \cdot \tilde{p} \to 0$, F also decreases faster than $(p \cdot \tilde{p})^{-1}$ increases. For almost all choices of the remaining variables stated above, the prefactors in front of b_1 and b_2 are nonvanishing in the definition E.1 of f. Therefore, the problematic limits $p_- \to 0$ and $p_- \to \infty$ will lead to an asymptotically constant argument of F, while the factor $(p_-(p \cdot \tilde{p})(p \cdot \tilde{p}'))^{-1}$ vanishes like p_-^{-3} for $p_- \to \infty$ and can diverge like p_-^{-1} for $p_- \to 0$. This can be seen as an artifact of the terms in the commutator being of a distributional nature. After smearing in d with a suitable testfunction by considing matrix elements between normalizable states, the exponential in f is replaced by a function that decreases faster than p_-^{-1} increases because $p_- \to 0$ implies $p \to \infty$ for almost all $\mathrm{p} \neq 0$. In principle, it could still happen that the denominator $p \cdot \tilde{p}$

becomes 0 inside F, which could lead to an exponential divergence. However, since one of b_1 and b_2 is always equal to 1 on each side of the equation, the corresponding other term will always be contained inside the upper halfplane, as long as t is in the interior of the strip $\mathbb{R} + \mathrm{i}[0, \pi]$, thus avoiding this divergence. Appraching any boundary of the strip, where b_1, b_2 both become real again, will return the divergence, but in the real direction, where F decreases like a Gaussian.

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