Conserved quantities in asymptotically de Sitter spacetimes

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Contents

Introduction

The discovery of the accelerated expansion of the universe has led to renewed interest in the cosmological constant. This is due to the fact that a positive cosmological constant is a possible cause for such an expansion. Much of this newfound interest extends to de Sitter space. The reason for this is twofold: First, de Sitter space is the most prominent solution of Einstein's equation with a positive cosmological constant. Second, according to the cosmic no-hair conjecture, every ever-expanding universe with a positive cosmological constant will asymptotically approach de Sitter space.

Spacetimes that asymptotically approach de Sitter space are called asymptotically de Sitter. A noteworthy example of such a spacetime might be our universe: If its expansion is caused by a cosmological constant, it might comply with the cosmic no-hair conjecture. In this thesis, we will have a close look at asymptotically de Sitter spaces. We will try to find conserved quantities for these spacetimes.

Conserved quantities play a major role in all physical theories. They correspond, for instance, to notions of mass and angular momentum. Different methods have been employed to construct such quantities in asymptotically de Sitter spacetimes [1, 2, 3]. But so far, no attempt has been made to approach the construction within a Hamiltonian framework. In this thesis, we will construct conserved quantities for asymptotically de Sitter spacetimes by using the covariant phase space formalism of Wald et al. [4]. It is a Hamiltonian formalism that exploits the fact that Hamiltonians are conserved under certain conditions. This will give us a formula for conserved quantities that is in no obvious way related to any other such formula that has been derived before.

This thesis is structured as follows: In the first chapter, we will briefly recapitulate some definitions and concepts that we will need throughout this thesis. The purpose of chapter 2 is to introduce the term asymptotically de Sitter. To that end, we will review the definitions of certain other classes of spacetimes, we will give the definition of the term asymptotically de Sitter, and we will discuss a few examples of such spacetimes. In the third chapter, we will describe the Hamiltonian framework and Wald's covariant phase space formalism. How these tools can be applied to asymptotically de Sitter spacetimes will be discussed in the second part of chapter 3. In chapter 4, we will present our conserved quantities. We will show how and that they are conserved and we will derive them within the framework of chapter 3. At this point, we will also return to the sample spacetimes that we introduced in chapter 2: We will compute conserved quantities for these spacetimes, which we will then discuss. The last point that we will address in this chapter is the possible existence of positive conserved quantities. These could pave the way to a notion of mass of a spacetime. We will conclude this thesis with a perturbation analysis of de Sitter space. Its purpose is to shed some light on the question of how large the class of asymptotically de Sitter spacetimes is.

Notations and conventions

In general, we will use the symbol M to represent the spacetime manifold. We will assume throughout this thesis that it is smooth, connected, paracompact, orientable, and d-dimensional, where $d \geq 4$. A spacetime is not only comprised of a manifold but of a manifold with a Lorentzian metric. We will denote such a spacetime as (M, g_{ab}). It will be required to be time orientable. The metric g_{ab}, which is assumed to be smooth, has the signature $(-+++\dots)$. The inverse of the metric is denoted as g^{ab}.

Latin indices a, b, c, ... on tensor fields are abstract indices (see e.g. [5]). A $(2,1)$-tensor field, for instance, might be written as $T^{ab}{}_c$. Chapter 3 is an exception to this rule: For the sake of clarity and readability, we will suppress all indices there. Other indices than lower Latin letters from the beginning of the alphabet will denote tensor components. This will in particular be used in chapter 5. Abstract indices on tensor fields will be raised and lowered in the usual way, namely with the metric and its inverse. For example, we have $T_{ac}g^{cb} = T_a{}^b$ and $T_a{}^c g_{cb} = T_{ab}$.

All the derivative operators that we will consider in thesis are torsion-free and associated with a spacetime metric ($\nabla_a g_{ab} = 0$). The Riemann tensor can be defined in terms of derivative operators. We will use the convention $R_{abc}{}^d k_d := 2\nabla_{[a}\nabla_{b]}k_c$, where the brackets represent anti-symmetrization. To denote symmetrization, we will use parantheses. The Ricci tensor is given by $R_{ab} := R_{acb}{}^c$.

An unphysical spacetime manifold that arises from M will be denoted by $\tilde{M}$ and its metric will accordingly be denoted by $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where Ω is a conformal factor. Again, the inverse will be written as $\tilde{g}^{ab}$. Indices on tilde tensor fields will be raised and lowered with the unphysical metric instead of the physical one. In case $\tilde{M}$ is a manifold with boundary, we denote its interior by $\mathrm{int}(\tilde{M})$ and its boundary by $\partial\tilde{M} := \tilde{M} - \mathrm{int}(\tilde{M})$.

Throughout this thesis, we will work extensively with variations of various quantities. The variation δg_0 of a metric g_0 is to be understood as follows: Let g_λ be a smooth one-parameter family of metrics. Then we denote $\mathrm{d}g_\lambda/\mathrm{d}\lambda|_{\lambda=0}$ by δg_0. Similarly, the variation of an arbitrary functional $S[g_\lambda]$ is given by $\delta S[g_0] = \mathrm{d}S[g_\lambda]/\mathrm{d}\lambda|_{\lambda=0}$.

Further note that $c = 1$ in this thesis. For a table of symbols see page 81.

1 Preliminaries

In this chapter, we will shortly introduce some important concepts that we will use in this thesis. The following sections are not to be understood as thorough introductions but rather as short reminders, which emphasize the points that are important for us.

1.1 Isometries and conformal transformations

In general relativity, spacetime is a manifold M with a Lorentzian metric g_{ab}. The metric is a symmetric and non-degenerate $(0,2)$-tensor field on M, which contains all the relevant information about the spacetime.

Should there exist a diffeomorphism between two spacetime manifolds that maps their respective metrics onto each other, both these spacetimes are identical from a physical point of view.

Definition 1. Let M be a manifold with metric g_{ab} and N be a manifold with metric h_{ab}. A diffeomorphism $f : M \to N$ is called an isometry if

$$g_{ab} = (f^*h)_{ab}, \tag{1.1}$$

where $(f^*h)_{ab}$ is the pullback of the metric h_{ab} by f.

Of course, isometries do in general not exist for two arbitrary manifolds. Those that are related via an isometry are called isometric.

Definition 2. Two manifolds M and N with metrics g_{ab} and h_{ab} are called isometric if there exists an isometry $f : M \to N$.

This concept can be easily transferred to a local level:

Definition 3. A manifold M with metric g_{ab} is called locally isometric to a manifold N with metric h_{ab} if there exists an open neighborhood U for every point $p \in M$, such that (U, g_{ab}) is isometric to (V, h_{ab}), where V is an open subset of N.

Consider the following example: Globally, a cylinder clearly differs from $\mathbb{R}^n$ (with their natural metrics). These two spaces are neither isometric nor isomorphic to each other. However, the cylinder is locally isometric to $\mathbb{R}^n$. In this particular example, this means, for instance that the Ricci scalar of the cylinder must vanish everywhere, because it vanishes in $\mathbb{R}^n$.

We mentioned in the beginning that spacetime is a Lorentzian manifold (M, g_{ab}) (with metric signature $- + + \cdots +$) in general relativity. On such manifolds, a vector v^a is said to be timelike if its norm is negative, i.e. $g_{ab}v^av^b < 0$, null if $g_{ab}v^av^b = 0$, or spacelike if $g_{ab}v^av^b > 0$. This concept can easily be carried over to curves on manifolds: A curve is timelike, spacelike or null if the norm of its tangent is everywhere timelike, spacelike or null.

Clearly, isometries preserve this so-called causal structure: If v^a is a timelike, spacelike or null vector, $(f_* v)^a$ is correspondingly also timelike, spacelike or null. Conformal isometries share this property with isometries.[1]

Definition 4. A diffeomorphism $f : M \to N$ between two manifolds M and N with metrics g_{ab} and h_{ab} is called a conformal isometry if

$$g_{ab} = \Omega^2 (f^* h)_{ab}, \tag{1.3}$$

where Ω is a smooth non-vanishing function on M.

A conformal isometry is also an angle preserving map, in the sense that

$$\frac{1}{\sqrt{|v^c v_c w^d w_d|}} g_{ab} v^a w^b = \frac{1}{\sqrt{|v'^c v'_c w'^d w'_d|}} h_{ab} v'^a w'^b, \tag{1.4}$$

where v^a, w^a are not null and $v'^a = (f_* v)^a$, $w'^a = (f_* w)^a$.

Closely related to conformal isometries are conformal transformations.

Definition 5. Let g_{ab} be a metric on M and let Ω be a smooth, positive, and non-vanishing function on M. Then the metric

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \tag{1.5}$$

is said to arise from g_{ab} via a conformal transformation.

Clearly, conformal transformations also preserve angles and the causal structure of a given Lorentz manifold. Some useful relations regarding quantities of conformally tranformed metrics can be found in appendix A.1.

1.2 The Weyl tensor

The Weyl tensor is the tracefree part of the Riemann tensor $R_{abc}{}^d$. It can be written as

$$C_{abcd} = R_{abcd} - \frac{2}{d-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(d-1)(d-2)} R g_{a[c}g_{d]b}, \tag{1.6}$$

where R_{ab} and R are the Ricci tensor and scalar, respectively. The Weyl tensor is a conformally invariant quantity in the following sense: If we denote the Weyl tensor which is associated with a conformally transformed metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ as $\tilde{C}_{abc}{}^d$, the relation

$$\tilde{C}_{abc}{}^d = C_{abc}{}^d \tag{1.7}$$

holds.

[1] See definition 4.

$$g_{ab} v^a v^b = \Omega^2 h_{ab} (f_* v)^a (f_* v)^b \tag{1.2}$$

is satisfied for an arbitrary vector v^a on M. Thus, v^a is timelike if and only if $(f_* v)^a$ is timelike, v^a is null if and only if $(f_* v)^a$ is null and v^a is spacelike if and only if $(f_* v)^a$ is spacelike.

Equation (1.6) implies that the Weyl tensor inherits some of the symmetries of the Riemann tensor. First, it is antisymmetric with respect to commutation of the first pair of indices as well as to commutation of the second pair of indices:

$$C_{abcd} = -C_{abdc} \tag{1.8}$$

$$C_{abcd} = -C_{bacd} \tag{1.9}$$

Second, the antisymmetrization of the Weyl tensor over its first three indices vanishes:

$$C_{[abc]d} = 0 \tag{1.10}$$

From these relations follows that C_{abcd} is symmetric under commutation of the first pair of indices with the second one.

$$C_{abcd} = C_{cdab} \tag{1.11}$$

1.3 Gaussian normal coordinates

First, take notice of the following definition:

Definition 6. Let M be a d-dimensional manifold. An embedded $(d-1)$-dimensional submanifold of M is called a hypersurface.

Now consider a manifold M with metric g_{ab}. Let $\Sigma \subset M$ be (a portion of) a spacelike hypersurface with metric $(h_{ab})_0$ and unit future directed normal field n^a. Then construct the unique geodesics through Σ with tangents n^a. We get a coordinate system in a neighborhood of Σ by choosing coordinates on Σ and by labeling the points in a neighborhood of Σ by the affine parameter t of the geodesics on which they lie and the coordinates of the points of Σ from which the geodesics emanated ($t = 0$ on Σ). In this coordinate system and neighborhood, the metric takes the form

$$g_{ab} = -\nabla_a t \nabla_b t + h_{ab}(t), \tag{1.12}$$

where $h_{ab}(0) = (h_{ab})_0$. $h_{ab}(t)$ is the metric on surfaces of constant t. It satisfies $h_{ab}(t)\nabla^a t = 0$, which means that the geodesics remain orthogonal to the hypersurfaces of constant t.[2] By construction, $n^a \equiv \nabla^a t$ is a geodesic tangent field, i.e. we have (recall that t is an affine parameter)

$$n^a \nabla_a n^b = 0. \tag{1.13}$$

Conversely, if we are given a metric of the form (1.12), where $h_{ab}(t)\nabla^a t = 0$, the vector field $n^a = \nabla^a t$ must be a geodesic tangent field: Its norm clearly satisfies

$$n^a n_a = -1, \tag{1.14}$$

which implies

$$\nabla_a (n^b n_b) = 0. \tag{1.15}$$

Now recall that ∇_a is torsion-free[3], i.e. $\nabla_a \nabla_b t = \nabla_b \nabla_a t$. Therefore, we must have

$$0 = \frac{1}{2}\nabla_b (n^a n_a) = n^a \nabla_b n_a = n^a \nabla_a n_b, \tag{1.16}$$

which proves that n^a is indeed the tangent to a geodesic.

[2] Let X^a be a coordinate basis field of the coordinates on the hypersurfaces of constant t and $t^a = (\partial/\partial t)^a$. X^a satisfies $t^b \nabla_b (t_a X^a) = t_a t^b \nabla_b X^a = t_a X^b \nabla_b t^a = \frac{1}{2} X^b \nabla_b (t^a t_a) = 0$, where the first equality follows from (1.13), the second one from the fact that coordinate basis fields commute, and the last from the norm of t^a being unit.

[3] See chapter "Notations and conventions" on page 6.

2 Asymptotically de Sitter spacetimes

In this chapter, we will give our definition of the term asymptotically de Sitter. But before we do so, we will introduce some concepts which we will need to understand our definition. More precisely, we will discuss the definitions and properties of asymptotically flat and asymptotically anti de Sitter spacetimes. Having introduced the necessary notions, we will then establish the term asymptotically de Sitter. We will conclude this chapter with a few examples of such spacetimes.

2.1 Asymptotically flat spacetimes

Heuristically, asymptotically flat spacetimes are spacetimes that approach Minkowski space at "large distances" from some spacetime region. Now one might be tempted to simply define asymptotically flat spacetimes as manifolds with metrics whose metric can be written as

$$g_{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = \eta_{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + O(1/r) \tag{2.1}$$

in some coordinate system. Here, $r = [(x^1)^2 + (x^2)^2 + \dots]^{1/2}$ and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \dots)$. However, a big drawback of this condition is that it is given in terms of coordinates. For instance, it might be of interest to calculate a quantity at "infinity" in an asymptotically flat spacetime. In a coordinate dependent description, it would be necessary to specify exactly how this is to be done, i.e. how limits ($r \to \infty$) are to be taken.

To evade this and other difficulties, one defines asymptotically flatness in a coordinate independent manner. Roughly speaking, a spacetime is asymptotically flat if a boundary can be attached to the spacetime manifold in a suitable way. This boundary then represents "infinity".

Before we give a precise definition of asymptotic flatness, let us illustrate this concept by constructing such a boundary for d-dimensional Minkowski space. In spherical coordinates, the Minkowski metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\sigma^2_{d-2}, \tag{2.2}$$

where $\mathrm{d}\sigma^2_{d-2}$ is the metric of a $(d-2)$-dimensional sphere. A coordinate transformation to advanced and retarded null coordinates, which are given by

$$v = t + r, \tag{2.3}$$

$$u = t - r, \tag{2.4}$$

casts the metric into the following form:

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + \frac{1}{4}(v-u)^2\mathrm{d}\sigma^2_{d-2} \tag{2.5}$$

Now let us construct a so-called unphysical metric

$$\mathrm{d}\tilde{s}^2 = \Omega^2\mathrm{d}s^2 \tag{2.6}$$

via a conformal transformation (see section 1.1). In our case, an appropriate choice for the conformal factor is

$$\Omega^2 = \frac{4}{(1+v^2)(1+u^2)}. \tag{2.7}$$

If we now attach points with $\Omega = 0$ to the spacetime manifold (representing "infinity"), we get a so-called unphysical manifold. Together, this manifold and the unphysical metric are called the unphysical spacetime. Accordingly, the spacetime manifold with its metric is called the physical spacetime. If we make another coordinate transformation

$$T = \tan^{-1} v - \tan^{-1} u, \tag{2.8}$$

$$R = \tan^{-1} v - \tan^{-1} u, \tag{2.9}$$

we find that the unphysical Minkowski metric can be written as

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R \, \mathrm{d}\sigma^2_{d-2}. \tag{2.10}$$

The ranges of the coordinates that correspond to finite values of t and r, i.e. points of the physical spacetime manifold, are given by

$$-\pi < T + R < \pi, \tag{2.11}$$

$$-\pi < T - R < \pi, \tag{2.12}$$

$$0 \leq R. \tag{2.13}$$

Points with $T + R = \pm\pi$ and/or $T - R = \pm\pi$ correspond to points of the boundary of the unphysical manifold. Equation (2.10) is the metric of the Einstein static universe, which is the manifold $\mathbb{R} \times S^{d-1}$ with its natural Lorentzian metric. Thus, there must exist a conformal isometry that embeds Minkowski spacetime into a finite region of the Einstein static universe.[1]

If we suppress the spherical coordinates of $\mathrm{d}\sigma^2_{d-2}$, we can depict Minkowski space in a conformal (also Penrose or Penrose-Carter) diagram (see figure 2.1). In this diagram, the sets $\mathscr{I}^+$ and $\mathscr{I}^-$ as well as the points i^0, i^+, and i^- correspond to the boundary that we have just constructed. They do not belong to the Minkowski spacetime itself. $\mathscr{I}^+$ and $\mathscr{I}^-$ are null hypersurfaces and are called future and past null infinity. Every null geodesic terminates at the former and emanates at the latter set. The point i^0 is called spatial infinity. It is spacelike related to every point of the interior of the unphysical manifold (i.e. the physical spacetime manifold). Moreover, all the spacelike geodesics of Minkowski space start and end there. The remaining points i^+ and i^- are called future and past timelike infinity, respectively. All the timelike geodesics of the spacetime begin at i^- and end at i^+. Additionally, i^+ and i^- are timelike related to every point of the interior of the unphysical manifold.

Since we suppressed the spherical coordinates in figure 2.1, every point of this diagram represents a $(d-2)$-sphere. Exceptions are the points i^+, i^-, i^0, and points on the leftmost line with $r = 0$. These points do not represent a higher dimensional space. They just correspond to points of the unphysical spacetime manifold.

In the literature, there exist different, inequivalent, notions of asymptotic flatness. Spacetimes can be asymptotically flat at null infinity ($\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$), at spatial infinity ($i^0$) or at spatial as well as null infinity. Ultimately, we want to define what asymptotically de Sitter spacetimes are.

[1] Equivalently: There exists a conformal factor, such that the unphysical metric is isometric (equal) to that of the Einstein static universe.

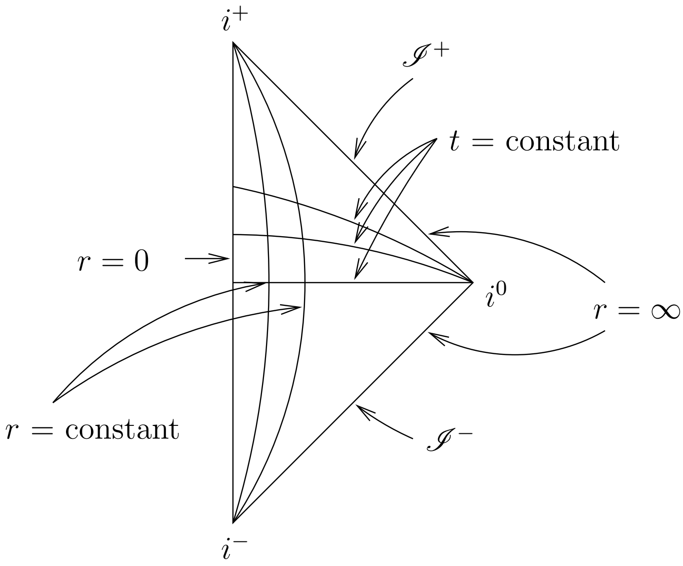

Figure 2.1: Conformal diagram of the Minkowski spacetime. Here, the boundary consists of i^+, i^-, i^0 as well as the hypersurfaces $\mathscr{I}^+$ and $\mathscr{I}^-$.

Such spacetimes, or more precisely their corresponding unphysical spacetimes, cannot possess anything similar to spacelike infinity (or timelike infinity for that matter). There does not exist a point or a set that is spacelike related to every point of the physical manifold. Therefore, we will disregard the notion of asymptotic flatness at spatial infinity. We will only give a definition of asymptotic flatness at null infinity. The definition will be similar to the ones given in [6] and [7] except that we restrict ourselves to vacuum spacetimes. Definitions of asymptotic flatness at null as well as spatial infinity can be found in [8, 5].

Definition 7. Let (M, g_{ab}) be a spacetime. (M, g_{ab}) is called asymptotically flat at null infinity if[2]

(i) g_{ab} satisfies Einstein's vacuum equation

$$R_{ab} - \frac{1}{2} R g_{ab} = 0. \tag{2.14}$$

(ii) One can attach a boundary $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^+$ to M with $\mathscr{I}^+$, $\mathscr{I}^- \cong S^{d-2} \times \mathbb{R}$, such that $\tilde{M} = M \cup \mathscr{I}$ is a manifold with boundary.

[2]Equivalently, definition 7 can be formulated in terms of conformal isometries instead of conformal transformations. Then it would read:
(M, g_{ab}) is called asymptotically flat at null infinity if it satisfies Einstein's vacuum equation and if there exists a manifold with boundary $\tilde{M}$, a smooth metric $\tilde{g}_{ab}$ on $\tilde{M}$ and a conformal isometry $\psi : M \to \psi(M) \subset \tilde{M}$ with conformal factor Ω which satisfy the following conditions:

(i) $\tilde{g}_{ab} = \Omega^2 (\psi^{-1})^* g_{ab}$ in $\psi(M)$ and $\tilde{M} = \psi(M) \cup \mathscr{I}$

(ii) On $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ we have $\Omega = 0$ and $\tilde{\nabla}_a \Omega \neq 0$

(iii) $\mathscr{I}^+$ and $\mathscr{I}^-$ have topology $S^{d-2} \times \mathbb{R}$

(iv) For a smooth function ω on $\tilde{M}$ with $\omega > 0$ which satisfies $\tilde{\nabla}_a (\omega^4 \tilde{\nabla}^a \Omega) = 0$ on $\mathscr{I}^+ \cup \mathscr{I}^+$, the vector field $\omega^{-1} \tilde{\nabla}^a \Omega$ is complete on $\mathscr{I}^+ \cup \mathscr{I}^-$

(iii) On $\tilde{M}$ there is a smooth metric $\tilde{g}_{ab}$ and smooth function Ω, such that $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$ and such that $\Omega = 0$ and $\tilde{n}_a = \tilde{\nabla}_a\Omega \neq 0$ at points of $\mathscr{I}$.

(iv) For a smooth function ω on $\tilde{M}$ with $\omega > 0$ which satisfies $\tilde{\nabla}_a(\omega^4\tilde{\nabla}^a\Omega) = 0$ on $\mathscr{I}^+ \cup \mathscr{I}^+$, the vector field $\omega^{-1}\tilde{\nabla}^a\Omega$ is complete on $\mathscr{I}^+ \cup \mathscr{I}^-$.

Remark 1. If a given spacetime is asymptotically flat, there is not just one unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$ that satisfies the conditions of definition 7. Let Ω be a conformal factor and let $\Omega' := \omega\Omega$, where ω is a smooth and non-vanishing function. In that case, the spacetime $(\tilde{M}', \tilde{g}'_{ab}) = (\tilde{M}, \omega^2\tilde{g}_{ab})$ also satisfies (ii), (iii), and (iv) of the above definition. Hence, there is considerable freedom in the choice of the unphysical spacetime.

The first condition of the definition states that asymptotically flat spacetimes are always solutions of Einstein's vacuum equation. The second and the third condition specify how $\mathscr{I}$ must look like and that it really represents "infinity". The last condition states that "all of $\mathscr{I}$" must be present in the unphysical spacetime and that $\mathscr{I}$ has the global asymptotic structure of Minkowski space (see [7] for details). This means in particular that $\mathscr{I}$ is complete.

Definition 8. $\mathscr{I}$ of an unphysical spacetime manifold $\tilde{M}$ is complete if we cannot find another unphysical spacetime manifold $\tilde{M}'$ that satisfies (ii), (iii), and (iv) of the above definition as well, such that $\tilde{M} \subsetneq \tilde{M}'$

Without this fourth condition, it would, for instance, not be possible to apply the usual criteria (see e.g. [5]) for spacetimes containing black holes to asymptotically flat spaces.

As we have seen, $\mathscr{I}^+$ and $\mathscr{I}^-$ are null surfaces in Minkowski space. Definition 7 implies that this is true for every asymptotically flat spacetime. This can be seen by writing Einstein's equation (2.14) in terms of conformal curvature quantities (see appendix A). For a metric $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$, Einstein's equation takes the form

$$\tilde{S}_{ab} + 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega - \Omega^{-2}\tilde{g}_{ab}\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega = 0, \tag{2.15}$$

where $\tilde{S}_{ab} = 2(d-2)^{-1}\tilde{R}_{ab} - [(d-1)(d-2)]^{-1}\tilde{R}\tilde{g}_{ab}$. Here, $\tilde{R}_{ab}$ and $\tilde{R}$ are the Ricci tensor and scalar of the unphysical metric. Noting that $\tilde{S}_{ab}$ and $\tilde{\nabla}_a\tilde{\nabla}_b\Omega$ are smooth on $\mathscr{I}$ (since $\tilde{g}_{ab}$ and Ω are smooth there) and using the fact that $\Omega = 0$ on $\mathscr{I}$ gives

$$\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega \restriction \mathscr{I} = 0. \tag{2.16}$$

Hence, $\tilde{\nabla}_a\Omega$ is null on $\mathscr{I}$, which implies that $\mathscr{I}^+$ and $\mathscr{I}^-$ are null surfaces.

2.2 Asymptotically anti de Sitter spacetimes

Anti de Sitter space is the maximally symmetric solution of Einstein's equation with a negative cosmological constant. The anti de Sitter metric can be written as

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{\ell'^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{\ell'^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2_{d-2}, \tag{2.17}$$

where $\ell' > 0$ is the anti de Sitter radius and where $\mathrm{d}\sigma^2_{d-2}$ is the natural metric of S^{d-2}. By choosing a suitable conformal factor, it can be shown that (the complete) $\mathscr{I}$ of anti de Sitter is conformally isometric to $\mathbb{R} \times S^{d-2}$. Furthermore, if we make use of Einstein's equation in a way similar to (2.15), we can show that $\mathscr{I}$ of an anti de Sitter spacetime is timelike. An immediate consequence of this is that anti de Sitter spacetimes do not admit Cauchy surfaces.

These are fundamental deviations from the properties of asymptotically flat spacetimes. However, the definition of the term asymptotically anti de Sitter is quite similar to definition 7 [9]:

Definition 9. Let (M, g_{ab}) be a spacetime. (M, g_{ab}) is called asymptotically anti de Sitter if

(i) g_{ab} satisfies Einstein's vacuum equation with a negative cosmological constant Λ

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0. \tag{2.18}$$

(ii) One can attach a boundary, $\mathscr{I} \cong \mathbb{R} \times S^{d-2}$ to M, such that $\tilde{M} = M \cup \mathscr{I}$ is a manifold with boundary.

(iii) On $\tilde{M}$ there is a smooth metric $\tilde{g}_{ab}$ and smooth function Ω such that $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$ and such that $\Omega = 0$ and $\tilde{n}_a = \tilde{\nabla}_a \Omega \neq 0$ at points of $\mathscr{I}$.

(iv) The metric $\tilde{h}_{ab}$ on $\mathscr{I}$ induced by $\tilde{g}_{ab}$ is in the conformal class of the Einstein static universe

$$\tilde{h}_{ab} = e^{\omega}[-(\mathrm{d}t)_a(\mathrm{d}t)_b + \sigma_{ab}], \tag{2.19}$$

where σ_{ab} is the metric of a $(d-2)$-sphere and ω is some smooth function.[3]

The third point completely agrees with the corresponding one of definition 7 and the first one differs only in that the spacetime has to be a solution to Einstein's equation with a negative cosmological constant. The differences in condition (ii) stem from the fact that (the complete) $\mathscr{I}$ of anti de Sitter space consists of only one hypersurface with topology $\mathbb{R} \times S^{d-2}$, whereas $\mathscr{I}$ of Minkowski space is comprised of two hypersurfaces. Only the fourth conditions do not resemble each other at all. This is mainly because definitions 7 and 9 were made with different purposes in mind. As we have already said in section 2.1, the fourth condition in the definition of asymptotic flatness ensures that $\mathscr{I}$ is complete and has the global asymptotic structure of Minkowski space. This enables us to give definitions of asymptotically flat spacetimes containing black holes and tells us, whether a spacetime really resembles Minkowski space everywhere in the asymptotic region. Definition 9, on the other hand, was solely designed to fulfill the needs which arose in the construction of conserved quantities in asymptotically anti de Sitter spacetimes. Ultimately, this fourth condition makes the construction of conserved charges in asymptotically anti de Sitter spacetimes possible [9].

2.3 Asymptotically de Sitter spacetimes

De Sitter space is the maximally symmetric solution of Einstein's equation with a positive cosmological constant. Its metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(\tau)\mathrm{d}\sigma^2_{d-1}, \tag{2.20}$$

where $\mathrm{d}\sigma^2_{d-1}$ is the natural metric of the $(d-1)$-sphere and where the function a is given by $a(\tau) = \ell \cosh(\tau/\ell)$ with $\ell > 0$ being the de Sitter radius. It can be shown that the complete

[3] Equivalently, we could have demanded here that there exists an unphysical spacetime with boundary $\mathscr{I}$ whose metric is exactly that of the Einstein static universe.

$\mathscr{I}$ of de Sitter space consists of two separate parts $\mathscr{I}^+$ and $\mathscr{I}^-$, each of which is conformally isometric to S^{d-1} with its natural metric. Contrary to Minkowski and anti de Sitter space, $\mathscr{I}$ of de Sitter space is spacelike as can be seen as follows: In terms of conformal quantities, Einstein's equation with a positive cosmological constant can be written as

$$\tilde{S}_{ab} + 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega - \Omega^{-2}\tilde{g}_{ab}(\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega + \ell^2) = 0, \tag{2.21}$$

where $\tilde{S}_{ab}$ is as in (2.15). From this we can immediately deduce that $\mathscr{I}$ of de Sitter space is spacelike (see below (2.15)).

In this thesis, we will focus on the construction of conserved quantities in asymptotically de Sitter spacetimes. The approach that we will pursue in order to achieve that goal is very similar to the approach of Hollands, Marolf, and Ishibashi [9] to conserved charges in asymptotically anti de Sitter spacetimes. Thus, we would expect a useful definition of the term asymptotically de Sitter to resemble the definition of the term asymptotically anti de Sitter (definition 9). Considering the differences between the definitions of the previous sections, we should have to alter definition 9 only minimally. And indeed, apart from one major change in condition (iv), our definition of the notion of asymptotically de Sitter spacetimes is very similar to definition 9.

As already mentioned, (the complete) $\mathscr{I}$ of the de Sitter spacetime is comprised of two separate parts $\mathscr{I}^+$ and $\mathscr{I}^-$, each of which has topology S^{d-1} and is conformally isometric to S^{d-1} with its natural metric. Looking at the topological difference between the boundaries of unphysical spacetimes of asymptotically flat and asymptotically anti de Sitter spaces and their respective definitions, one might be tempted to require $\mathscr{I}^+$ and $\mathscr{I}^-$ of an unphysical asymptotically de Sitter spacetime to have topology S^{d-1}. Unfortunately, this will turn out to be too restrictive. The Schwarzschild de Sitter spacetime, for example, would not be asymptotically de Sitter if we clung to this condition. Therefore, we will not require for a spacetime to be asymptotically de Sitter that it possesses an unphysical spacetime whose $\mathscr{I}^\pm$ has a particular topology. An appropriate unphysical spacetime only needs to have a $\mathscr{I}$ with induced metric $(h_{ab})_0$, such that $(\mathscr{I}^\pm, e^\omega(\tilde{h}_{ab})_0)$ is locally isometric to S^{d-1} with its natural metric for an arbitrary but smooth function ω.

Definition 10. Let (M, g_{ab}) be a spacetime. (M, g_{ab}) is called asymptotically de Sitter if

(i) g_{ab} satisfies Einstein's vacuum equation with a positive cosmological constant[4] Λ

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0. \tag{2.22}$$

(ii) There exists a manifold with boundary $\tilde{M}$, such that $\tilde{M} = M \cup \mathscr{I}$ and $\partial\tilde{M} = \mathscr{I}$ open.

(iii) On $\tilde{M}$ there is a smooth metric $\tilde{g}_{ab}$ and smooth function Ω, such that $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$ and such that $\Omega = 0$ and $\tilde{n}_a = \tilde{\nabla}_a\Omega \neq 0$ at points of $\mathscr{I}$.

(iv) Let $(\tilde{h}_{ab})_0$ be the induced metric on $\mathscr{I}^+$ and $\mathscr{I}^-$, respectively. Then $(\mathscr{I}^\pm, e^\omega(\tilde{h}_{ab})_0)$ is locally isometric to (S^{d-1}, σ_{ab}), where σ_{ab} is the natural metric of S^{d-1} and ω is some smooth function.

(v) $\mathscr{I}^\pm$ satisfies $J^\pm(\mathscr{I}^\pm) = \mathscr{I}^\pm$, i.e. the causal future of $\mathscr{I}^+$ is $\mathscr{I}^+$ itself and the causal past of $\mathscr{I}^-$ is $\mathscr{I}^-$ itself.

[4] Einstein's equation with a positive cosmological constant can also be written as $R_{ab} = g_{ab}(d-1)/\ell^2$, where ℓ is given by (2.25).

Note that this definition can easily be weakened: Instead of demanding all the properties to hold for both $\mathscr{I}^+$ and $\mathscr{I}^-$, we can define a spacetime to be asymptotically de Sitter at $\mathscr{I}^+$ or $\mathscr{I}^-$ if the above definition is satisfied for merely one of these sets.

Definition 11. Let (M, g_{ab}) be a spacetime. (M, g_{ab}) is called future asymptotically de Sitter if

(i) g_{ab} satisfies Einstein's vacuum equation with a positive cosmological constant Λ

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0. \tag{2.23}$$

(ii) There exists a manifold with boundary $\tilde{M}$, such that $\tilde{M} = M \cup \mathscr{I}^+$ and $\partial \tilde{M} = \mathscr{I}^+$ open.

(iii) On $\tilde{M}$ there is a smooth metric $\tilde{g}_{ab}$ and smooth function Ω such that $g_{ab} = \Omega^{-2} \tilde{g}_{ab}$ and such that $\Omega = 0$ and $\tilde{n}_a = \tilde{\nabla}_a \Omega \neq 0$ at points of $\mathscr{I}^+$.

(iv) Let $(\tilde{h}_{ab})_0$ be the induced metric on $\mathscr{I}^+$. Then $(\mathscr{I}^+, e^{\omega} (\tilde{h}_{ab})_0)$ is locally isometric to (S^{d-1}, σ_{ab}), where σ_{ab} is the natural metric of S^{d-1} and ω is some smooth function.

(v) $\mathscr{I}^+$ satisfies $J^+(\mathscr{I}^+) = \mathscr{I}^+$, i.e. the causal future of $\mathscr{I}^+$ is $\mathscr{I}^+$ itself.

We introduced the additional condition (v) in the above definitions to ensure that $\mathscr{I}^+$ and/or $\mathscr{I}^-$ really correspond to future and/or past infinity, respectively. The main analysis of this thesis will be carried out for future asymptotically de Sitter spacetimes. This is mainly for clarity and notational convenience, but it is also somewhat more general: Future asymptotically de Sitter is a weaker condition than asymptotically de Sitter.

Note that we will refer to the conditions (ii), (iii), (iv), and (v) of definition 11 as the asymptotic conditions.

Furthermore, whenever we talk about an unphysical spacetime of a future asymptotically de Sitter space, we will assume that it satisfies the asymptotic conditions.

Remark 2. Consider a future asymptotically de Sitter spacetime M. Definition 11 does not require the boundary (i.e. $\mathscr{I}^+$) of an unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$ to be “maximal” in the following sense: There might exist other manifolds with boundaries N, such that $\mathrm{int}(\tilde{M}) = \mathrm{int}(N)$ but $\partial \tilde{M} \subsetneq \partial N$. It does not even require $\mathscr{I}^+$ to be equal to the “maximal” boundary on which the unphysical metric is smooth (cf. definition 8). $\mathscr{I}^+$ is just a set that satisfies conditions (ii), (iii), and (iv) of definition 11.

For our purposes, this is a very appealing definition. Usually, however, one might rather want to take $\mathscr{I}^+$ to be the “maximal” boundary that satisfies (ii) and (iii) of the above definition. For a spacetime to be future asymptotically de Sitter, this boundary would have to contain a subset that satisfies condition (iv).[5]

Remark 3. Note that some future asymptotically de Sitter spacetimes exist (see below). It is, however, not clear if a wide class of such spacetimes exists.[6]

[5]For our purposes, property (iv) of definition (11) is vital. By using our definition of $\mathscr{I}^+$ rather than the “usual” one, we avoid having to state constantly that an analysis does not hold for all of $\mathscr{I}^+$ but only for subsets that satisfy (iv).

[6]In chapter 5, we will show that definition 11 probably does allow for a large class of spacetimes.

2.4 Examples of asymptotically de Sitter spacetimes

Now let us have a look at a few asymptotically de Sitter spacetimes. We will encounter all of them again in the following chapters. Of particular significance is the de Sitter spacetime. We will refer to it on various occasions throughout this thesis.

2.4.1 De Sitter spacetime

As we have already mentioned, de Sitter space is the maximally symmetric solution of Einstein's equation with a positive cosmological constant. It is a space of constant positive scalar curvature and its Weyl tensor (see section (1.2)) vanishes. Of course, de Sitter space is an asymptotically de Sitter spacetime.

In the global chart, the de Sitter metric can be written as (cf. e.g. [10, 11])

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(\tau)\mathrm{d}\sigma^2_{d-1}. \tag{2.24}$$

$\mathrm{d}\sigma^2_{d-1}$ is the natural metric of the $(d-1)$-sphere, the function a is given by $a(\tau) = \ell\cosh(\tau/\ell)$, and ℓ is the de Sitter radius, which is defined as

$$\ell = \sqrt{\frac{(d-1)(d-2)}{2\Lambda}}. \tag{2.25}$$

Introducing the new coordinate [10]

$$t = 2\arctan(\exp\tau/\ell) - \pi/2 \tag{2.26}$$

gives

$$\mathrm{d}s^2 = a^2(\tau)(-\mathrm{d}t^2 + \mathrm{d}\sigma^2_{d-1}). \tag{2.27}$$

By choosing the conformal factor

$$\Omega = 1/a, \tag{2.28}$$

we find an unphysical metric

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}t^2 + \mathrm{d}\sigma^2_{d-1}, \tag{2.29}$$

which gives an unphysical spacetime if we construct an unphysical manifold by adding points to the spacetime manifold at which the (continuation of) the conformal factor vanishes. A conformal diagram of the de Sitter spacetime is depicted in figure 2.2 [10]. Every hypersurface of constant τ is a Cauchy surface and a $(d-1)$-dimensional sphere. In particular, $\mathscr{I}^+$ and $\mathscr{I}^-$ are $(d-1)$-spheres.

If we choose the conformal factor $\Omega' = 2\exp(-|\tau|/\ell)$ instead of (2.28), we find that the unphysical metric can be written as

$$\mathrm{d}\tilde{s}'^2 = \ell^2\left[-\mathrm{d}\Omega'^2 + \left(1 + \frac{\Omega'^2}{2} + \frac{\Omega'^4}{16}\right)\mathrm{d}\sigma^2_{d-1}\right]. \tag{2.30}$$

It should be noted, however, that (2.30) is not smooth on the $\tau = 0$ hypersurface.

An interesting region of the de Sitter spacetime is the static region [11]. It consists of the region that is denoted by I in figure 2.3. In this region, the metric can be written as

$$\mathrm{d}s^2 = -\Psi\mathrm{d}t^2 + \Psi^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2_{d-2}, \tag{2.31}$$

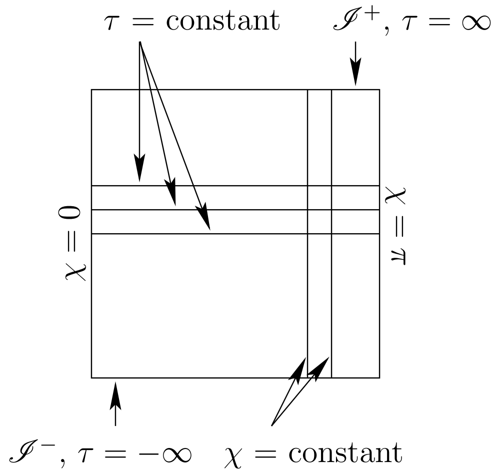

Figure 2.2: Conformal diagram of de Sitter space. χ is a polar angle parametrizing the sphere.

where $\mathrm{d}\sigma_{d-2}^2$ is the natural metric of S^{d-2} and

$$\Psi = 1 - \frac{r^2}{\ell^2}. \tag{2.32}$$

An interesting feature of (2.31) is that it does not contain any functions of t. Hence, $\partial/\partial t$ is a Killing vector field, which is obviously timelike in this region.

In the other regions, it is possible to write the metric in exactly the same way. The only difference is that $r > \ell$ in region II and IV, which implies that Ψ becomes negative there. $\partial/\partial t$ is again a Killing field in these regions, even though it is not necessarily timelike anymore: It is spacelike in region II and IV.

Remark 4. From de Sitter space, we can construct other asymptotically de Sitter spaces in the following way: Consider, for instance, a discrete subgroup of the symmetry group of de Sitter space which maps surfaces of constant τ to itself. Another asymptotically de Sitter space is then given by the quotient of de Sitter space by this group. In that case, $\mathscr{I}^+$ is not a sphere anymore but the quotient of a sphere. However, it is still locally isometric to a sphere.

Remark 5. At this stage, we should point out that there exist different, inequivalent, notions of the term asymptotically de Sitter in the literature. In particular, one often encounters the following definition: Let $\bar{g}_{ab}$ be the de Sitter metric. A spacetime is called asymptotically de Sitter if its metric is of the form

$$g_{ab} = \bar{g}_{ab} + k_{ab}, \tag{2.33}$$

where k_{ab} vanishes at infinity. To see what is meant by "infinity" let us take a look at the cosmological chart of de Sitter space (see figure 2.4). In this chart, the metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \exp(2t/\ell)\mathrm{d}x_{d-1}^2, \tag{2.34}$$

where $\mathrm{d}x_{d-1}^2 = \delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is the $(d-1)$-dimensional flat metric. Now, by "at infinity" usually the limit $r \to \infty$ is meant, where $r = [(x^1)^2 + (x^2)^2 + \dots]^{1/2}$. This limit corresponds to the point that we denoted by i^0 in figure 2.4.[7] Obviously, this definition of asymptotically de Sitter differs drastically from ours.

[7]As in the asymptotically flat case, this point is called spatial infinity.

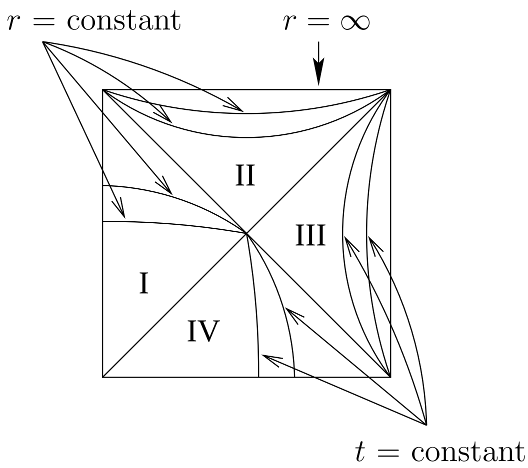

Figure 2.3: Regions I, III and II, IV are symmetric with respect to surfaces of constant t, r. The diagonal lines correspond to $r = \ell$ and are not included in the regions.

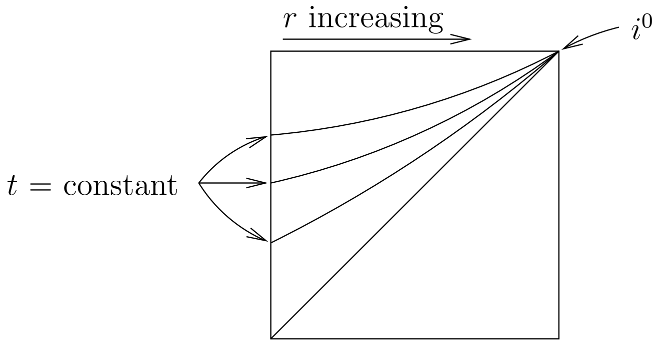

Figure 2.4: The cosmological chart of de Sitter space. It covers only half of de Sitter space. Here: The upper left region.

2.4.2 Schwarzschild de Sitter spacetime

Another asymptotically de Sitter spacetime is the Schwarzschild de Sitter spacetime. It describes the spherically symmetric solutions of Einstein's vacuum equation with a positive cosmological constant. As in the flat case, i.e. as in the usual Schwarzschild solution, this encompasses spherically symmetric black hole solutions.

Let us have a look at the metric [2, 12]

$$\mathrm{d}s^2 = -\Phi \mathrm{d}t^2 + \Phi^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\sigma_{d-2}^2, \tag{2.35}$$

where

$$\Phi = 1 - \frac{C_d}{r^{d-3}} - \frac{r^2}{\ell^2} \tag{2.36}$$

and where $\mathrm{d}\sigma_{d-2}^2$ is the natural metric of the $(d-2)$-sphere and C_d is some constant (the "mass" parameter). This is the metric of the Schwarzschild de Sitter spacetime for values of r and t that do not correspond to $\Phi = 0$. The situation is very similar to the one we discussed in the context

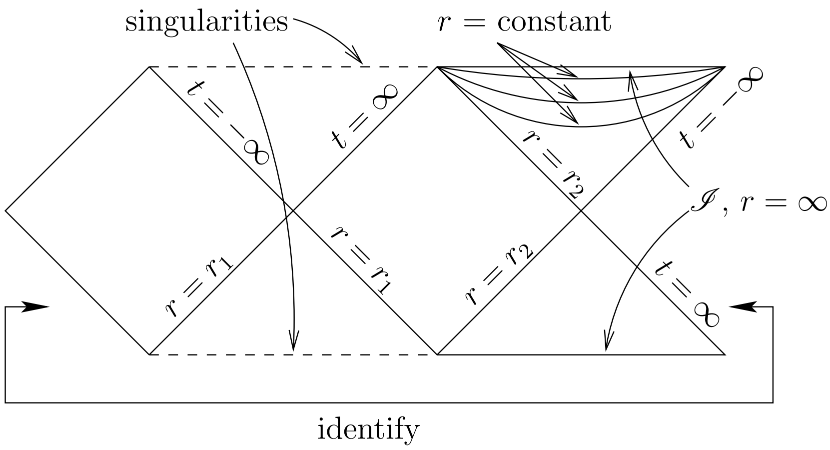

Figure 2.5: Conformal diagram of the Schwarzschild de Sitter spacetime. r_1 and r_2 correspond to the black hole and the cosmological horizon, respectively.

of the static region of de Sitter space: In all the depicted regions in figure 2.5, i.e. between the horizons, it is possible to write the metric as in (2.35). Note that (2.35) does not contain any functions of t. Hence, the vector fields $(\partial/\partial t)^a$ are Killing fields of the Schwarzschild de Sitter spacetime. In particular, $(\partial/\partial t)^a$ is spacelike in a neighborhood of $\mathscr{I}$.

The function Φ possesses up to two positive roots: The larger one corresponds to the position of the cosmological horizon and the smaller one represents the position of the black hole event horizon. This case is depicted in figure 2.5 [13]. If Φ has only one positive root (and $C_d > 0$), the cosmological and the black hole horizon coincide. Conformal diagrams for this degenerate solution can be found in [13, 14]. In case there are no positive roots, the spacetime possesses an initial or final spacelike singularity and (an associated unphysical spacetime possess) correspondingly only $\mathscr{I}^+$ or $\mathscr{I}^-$, respectively.[8] Charts that cover all these horizons can be found in [15].

For our purposes, it will be sufficient to stick to the metric (2.35) to investigate $\mathscr{I}$. Introducing $\chi = 2\arctan(\exp t/\ell)$ and choosing the conformal factor $\Omega = \sin\chi/r$ gives rise to the unphysical metric

$$\begin{aligned}\mathrm{d}\tilde{s}^2 = &-\left(\frac{1}{\ell^2} + \frac{C_d}{\sin^{d-1}\chi}\Omega^{d-1} - \frac{1}{\sin^2\chi}\Omega^2\right)^{-1}(\mathrm{d}\Omega - \Omega\cot\chi\mathrm{d}\chi)^2 \\ &+\left(1 + \ell^2\frac{C_d}{\sin^{d-1}\chi}\Omega^{d-1} - \frac{\ell^2}{\sin^2\chi}\Omega^2\right)\mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}\sigma^2_{d-2} \qquad (2.37)\end{aligned}$$

for $0 < \Omega < \sin\chi/r_i$, where r_i denotes possible horizons. Attaching all the points to the spacetime manifold to which the conformal factor and unphysical metric can be smoothly extended and where then $\Omega = 0$ gives an unphysical spacetime manifold. Note that the scalar

$$\tilde{C}_{abcd}\tilde{C}^{abcd} = (d-1)(d-2)^2(d-3)C_d^2 \cdot \Omega^{2(d-3)}(\sin\chi)^{-2(d-1)} \qquad (2.38)$$

is not well defined at points with $\Omega = 0$, $\chi = 0$ and $\Omega = 0$, $\chi = \pi$ if $C_d \neq 0$. Consequently, the unphysical metric is not smooth there either. Hence, these points are not part of the unphysical spacetime manifold.

[8] This prevents the spacetime from being asymptotically de Sitter. However, it could still be future asymptotically de Sitter.

It follows from (2.37) that the induced metric $\mathrm{d}\tilde{h}^2$ on $\mathscr{I}^+$ and $\mathscr{I}^-$ is given by

$$\mathrm{d}\tilde{h}^2 = \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\sigma^2_{d-2} \tag{2.39}$$

where $\chi \in (0, \pi)$. Therefore, for our choice of the conformal factor, $(\mathscr{I}^+, \mathrm{d}\tilde{h}^2)$ as well as $(\mathscr{I}^-, \mathrm{d}\tilde{h}^2)$ are isometric to $(S^{d-1} \setminus \{p, q\}, \mathrm{d}\sigma^2_{d-1})$, where p and q are antipodal points.[9]

2.4.3 Tolman-Bondi spacetime with a positive cosmological constant

The Tolman-Bondi spacetime [16] is a solution of Einstein's equation with a stress-energy tensor which describes an inhomogeneous spherically symmetric dust sphere. In four dimensions, its metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + X^2(r,t)\mathrm{d}r^2 + Y^2(r,t)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{2.40}$$

where

$$\left(\frac{\partial Y}{\partial t}\right)^2 = W^2 - 1 + \frac{2m(r)}{Y} + \frac{Y^2}{\ell^2}, \tag{2.41}$$

$$X = \frac{\partial Y}{\partial r} W^{-1}(r). \tag{2.42}$$

Here, W is an arbitrary function that corresponds to the binding energy and

$$m(r) = 4\pi \int_0^r \rho W X Y^2 \mathrm{d}r, \tag{2.43}$$

where ρ is the energy density of the dust sphere.

According to our definition, this spacetime is not asymptotically de Sitter, since it possesses a non-vanishing stress-energy tensor. However, its vacuum region can be considered as an asymptotically de Sitter spacetime on its own: Birkhoff's theorem (see e.g. [10]) implies that the metric in the vacuum region outside of the dust sphere must be equal to the Schwarzschild de Sitter metric. Indeed, after a change of coordinates, it can be written as

$$\mathrm{d}s^2 = -\left(1 - \frac{2M_{TB}}{Y} - \frac{Y^2}{\ell^2}\right)\mathrm{d}T^2 + \left(1 - \frac{2M_{TB}}{Y} - \frac{Y^2}{\ell^2}\right)^{-1}\mathrm{d}Y^2 + Y^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{2.44}$$

Remark 6. Note that the entire Tolman-Bondi spacetime is considered asymptotically de Sitter in [16]. This is due to different definitions of the notion of asymptotically de Sitter. (We do not admit non-vacuum spacetimes. But see also remark 5.)

Note that the parameter $2M_{TB}$ corresponds to the "mass" parameter of the Schwarzschild de Sitter metric. This has some interesting consequences if we consider the following result:

Remark 7. Nakao, Shiromizu, and Maeda [16] have shown that it is possible to choose W in such a way that $M_{TB} \equiv m(r \to \infty)$ becomes negative.

If we choose M_{TB} to be negative, the spacetime possesses a timelike singularity in its vacuum region beyond which it cannot be extended (see figure 2.6). This is in sharp contrast to the usual Schwarzschild de Sitter spacetime with a positive parameter $C_{d=4}$ (see (2.35)). All the

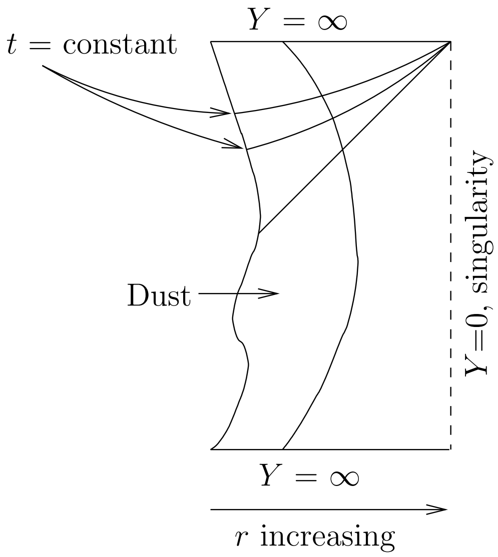

Figure 2.6: Conformal diagram of the Tolman-Bondi spacetime with negative M_{TB}. Taken from [16]. The omitted region on the left side has $m > 0$.

singularities of this spacetime are spacelike. However, if we choose a conformal factor analogous to (2.37) for the vacuum region of the Tolman-Bondi spacetime, $\mathscr{I}^{\pm}_{TB}$ (of the vacuum region) must clearly be isometric to a subset of a sphere. It is "intersected"[10] by the timelike singularity at a single point. As in the Schwarzschild de Sitter case, the contraction of the unphysical Weyl tensors is not smooth at this intersection point. In terms of the coordinates of (2.40), the contraction is not well defined at $\Omega = \sin\chi/Y = 0$, $\chi = 2\arctan r = \pi$. Consequently, in the vacuum region, $(\mathscr{I}^+, \tilde{h}_{ab})$ corresponds to a simply connected subset of a 3-sphere minus one point.

2.5 Asymptotic symmetries

Heuristically, asymptotic symmetries are maps that preserve the intrinsic geometric structure of $\mathscr{I}^+$.[11] For instance, in an asymptotically flat space with unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$, asymptotic symmetries correspond to diffeomorphisms $\psi : \mathscr{I}^+ \to \mathscr{I}^+$ that are conformal isometries with respect to the induced metric $\tilde{h}_{ab}$ on $\mathscr{I}^+$.[12]

Similarly, we give the following definition for future asymptotically de Sitter spacetimes:

Definition 12. Let (M, g_{ab}) be a future asymptotically de Sitter spacetime, let $(\tilde{M}, \tilde{g}_{ab})$ be an associated unphysical spacetime and let $\tilde{h}_{ab}$ be the induced metric on $\mathscr{I}^+$. Then the asymptotic symmetries are the diffeomorphisms $\psi : \mathscr{I}^+ \to \mathscr{I}^+$ which satisfy $\psi^* \tilde{h}_{ab} = \omega^2 \tilde{h}_{ab}$, where ω is an arbitrary but smooth function.

[9] One might suspect that this not being smooth of the metric at $\Omega = 0$, $\chi = 0$ and $\Omega = 0$, $\chi = \pi$ is related to the singularities intersecting $\mathscr{I}^{\pm}$ at these points (for our choice of the conformal factor).

[10] See footnote 9.

[11] One could also include $\mathscr{I}^-$; in this thesis, however, we are only interested in asymptotic symmetries at $\mathscr{I}^+$.

[12] The BMS group.

Remark 8. In the following, we will also use the term asymptotic symmetries for equivalence classes of vector fields on $\tilde{M}$, where each class is comprised of the vector fields that generate the same one-parameter group of asymptotic symmetries.

Remark 9. Note that definition 12 implies that every vector field which represents an asymptotic symmetry is (i) a conformal Killing field on $\mathscr{I}^+$; (ii) spacelike on $\mathscr{I}^+$.

Let ξ^a be a Killing vector field on an future asymptotically de Sitter spacetime (M, g_{ab}). Then we have $\mathcal{L}_\xi g_{ab} = 0$ on all of M. This can be rewritten in terms of an unphysical metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ as

$$\mathcal{L}_\xi \tilde{g}_{ab} = 2\Omega^{-1}(\xi^c \nabla_c \Omega)\tilde{g}_{ab}. \tag{2.45}$$

If $\tilde{g}_{ab}\xi^a\xi^b$ is finite in a neighborhood of $\mathscr{I}^+$, the left hand side of the above equation is finite as well. In that case, (the extension of) ξ^a must be tangent to $\mathscr{I}^+$, because $(\xi^a \nabla_a \Omega)$ has to vanish there to compensate for the Ω^{-1} term. Consequently, the flow generated by ξ^a maps $\mathscr{I}^+$ to $\mathscr{I}^+$. Furthermore, (2.45) implies that ξ^a is a conformal Killing field of the unphysical metric $\tilde{g}_{ab}$. Hence:

Remark 10. A Killing field of a future asymptotically de Sitter spacetime is the generator of a one-parameter group of asymptotic symmetries on $\mathscr{I}^+$ of an associated unphysical spacetime. It is the representative of an asymptotic symmetry in the sense of remark 8.

3 Hamiltonian approach to conserved quantities

Wald and Zoupas [4] have shown how conserved quantities within a Hamiltonian framework can be constructed in theories that arise from a diffeomorphism covariant Lagrangian. In section 3.2, we will apply these ideas to (a subset of) future asymptotically de Sitter spacetimes. But first, following [4], we will describe the general framework that is needed to construct Hamiltonians in general relativity. Details can be found in [17, 18, 19, 20].

3.1 Covariant phase space formalism

3.1.1 Defining the notion of a Hamiltonian

Let M be a d-dimensional manifold and let $\mathcal{F}$ be the space of "kinematically allowed" Lorentizan metrics on M of the theory under consideration. What kinematically allowed means depends on the theory and on what is most suitable for one's purposes. For instance, one could demand all the metrics to have a certain asymptotic behavior: One possibility would be to require the metrics to satisfy the asymptotic conditions of definition 11. For the moment, we will assume that an appropriate field configuration space $\mathcal{F}$ has been chosen, such that the integrals in the formulas below converge.

Let L be a diffeomorphism covariant Lagrangian density d-form[1] on $\mathcal{F}$ which gives rise to the equations of motion $F = 0$ of the theory. Then we can write the variation of L as[2] [4]

$$\delta L = F(g) \cdot \delta g + \mathrm{d}\theta(g, \delta g), \tag{3.1}$$

where θ is the presymplectic potential $(d-1)$-form. This form θ is not uniquely determined by the above equation: We can add an arbitrary closed $(d-2)$-form to θ without (3.1) changing. However, we demand θ to be locally constructed out of g and δg in a covariant manner, which restricts the freedom in the choice of the potential to the addition of exact $(d-2)$-forms [20]. Hence,

$$\theta \to \theta + \mathrm{d}Y(g, \delta g) \tag{3.2}$$

is a valid presymplectic potential as well. θ can be used to define the presymplectic current $(d-1)$-form ω. It is given by[3]

$$\omega(g, \delta_1 g, \delta_2 g) = \delta_1 \theta(g, \delta_2) - \delta_2 \theta(g, \delta_1 g), \tag{3.3}$$

where $\delta_1 g$ and $\delta_2 g$ are two perturbations off of g. Like the presymplectic potential θ, ω is not unique. Its ambiguity stems from the freedom in the choice of θ. Hence,

$$\omega \to \omega + \mathrm{d}[\delta_1 Y(g, \delta_2) - \delta_2 Y(g, \delta_1 g)] \tag{3.4}$$

[1] Diffeomorphism covariant means that for every diffeomorphism f, we have $L(f^* g) = f^* L(g)$. Theories that arise from such Lagrangians are diffeomorphism invariant.

[2] Here, the " $\cdot$ " notation means "contract the indices of δg into the first indices of F". For an explanation regarding the variations: See chapter "Notations and conventions" on page 6.

[3] Here and in the following, we will assume that the variations commute, i.e. $\delta_1(\delta_2 g) - \delta_2(\delta_1 g) = 0$.

is a valid presymplectic current form if ω is one.

Now consider a perturbation δg: For any such perturbation, there exists a smooth one-parameter family of metrics g_λ, such that $\delta g = \mathrm{d}g_\lambda/\mathrm{d}\lambda\,|_{\lambda=0}$. In case $g_\lambda \in \mathcal{F}$, this perturbation clearly corresponds to a vector which is tangent to $\mathcal{F}$.

By integrating the presymplectic current (3.3) over a closed spacelike hypersurface[4] without boundary of M, we can construct a map σ_Σ that takes two tangent vectors $\delta_1 g$ and $\delta_2 g$ of $\mathcal{F}$ at a point g into the real numbers. Because of (3.3), this map

$$\sigma_\Sigma(g, \delta_1 g, \delta_2 g) = \int_\Sigma \omega(g, \delta_1 g, \delta_2 g) \tag{3.5}$$

is antisymmetric in the perturbations, which implies that it is a presymplectic form on $\mathcal{F}$. However, it is not a true symplectic form, since it is not non-degenerate [4]. Like the presymplectic current, the presymplectic form inherits the non-uniqueness of the symplectic[5] potential. More precisely, (3.4) gives rise to the ambiguity

$$\sigma_\Sigma(g, \delta_1 g, \delta_2 g) \to \sigma_\Sigma(g, \delta_1 g, \delta_2 g) + \int_{\partial\Sigma} [\delta_1 Y(g, \delta_2 g) - \delta_2 Y(g, \delta_1 g)]. \tag{3.6}$$

By the above integral over $\partial\Sigma$, we mean a limiting process, in which the integral is first taken over the boundary[6] ∂K of a compact region, K of Σ and then K approaches all of Σ in a suitable manner. Of course, the integral over the right hand side will only be well defined if the limit exists and is independent of the way in which K approaches all of Σ.

It is possible to construct a phase space Γ, equipped with a true symplectic form, out of the field configuration space $\mathcal{F}$ with its presymplectic form σ_Σ. But for our purposes, $\mathcal{F}$ and its presymplectic form will be sufficient. Details on the construction of a phase space can be found in [17].

Now we are in a position to define the notion of a Hamiltonian. But first, take notice of the following two remarks:

Remark 11. A complete vector field ξ^a on M can be used to define metric variations $\delta_\xi g \equiv \mathcal{L}_\xi g$. These correspond to a tangent field on $\mathcal{F}$ if the flow ϕ_t generated by $\mathcal{L}_\xi g$ is a diffeomorphism on $\mathcal{F}$ to itself for fixed t.

Remark 12. We will denote the subspace of $\mathcal{F}$ whose elements are solutions to the equations of motion of the theory as $\bar{\mathcal{F}}$. This space $\bar{\mathcal{F}}$ is called the covariant phase space.

Definition 13. Let $\mathcal{F}$ be a field configuration space with covariant phase space $\bar{\mathcal{F}}$ of a diffeomorphism covariant theory on a manifold M. Further, let $\Sigma \subset M$ be a closed spacelike hypersurface without boundary, let ξ^a be a vector field on M, and let σ_Σ be the presymplectic form (3.5) on $\mathcal{F}$. (If there is an ambiguity in σ_Σ due to (3.6), we assume that a particular σ_Σ has been chosen.) Assume that these quantities have been chosen in such a way that the integral $\int_\Sigma \omega(g, \delta g, \mathcal{L}_\xi g)$ converges for all $g \in \bar{\mathcal{F}}$ and all tangent vectors δg to $\bar{\mathcal{F}}$ at g. Then a function $H_\xi : \mathcal{F} \to \mathbb{R}$ is said

[4]We define the orientation of this hypersurface as ${}^{(d-1)}\epsilon := \hat{n} \cdot \epsilon$, where $\hat{n}$ is a future directed timelike vector normal to Σ and " $\cdot$ " represents the contraction of the vector into the first index of the spacetime volume form (i.e. ${}^{(d-1)}\epsilon_{a_1 \ldots a_{d-1}} := \hat{n}^b \epsilon_{b a_1 \ldots a_{d-1}}$).

[5]Note that we will occasionally omit the prefix "pre".

[6]The orientation of ∂K is such that the volume form is given by ${}^{(d-2)}\epsilon = \hat{u} \cdot {}^{(d-1)}\epsilon$, where $\hat{u}$ is an outward pointing vector in compliance with Stokes' theorem.

to be a Hamiltonian conjugate to ξ on a hypersurface Σ if for all $g \in \bar{\mathcal{F}}$ and all field variations δg tangent to $\mathcal{F}$ we have

$$\delta H_\xi = \sigma_\Sigma(g, \delta g, \mathcal{L}_\xi g) = \int_\Sigma \omega(g, \delta g, \mathcal{L}_\xi g). \tag{3.7}$$

Remark 13. In case $\mathcal{L}_\xi g$ is tangent to $\mathcal{F}$, this vector field can be understood as defining a notion of "evolution" on $\mathcal{F}$, similar to the Hamiltonian vector field in classical mechanics.

There does not need to exist a function H_ξ that satisfies (3.7). But in case there is such a function, there are other ones as well: Definition 13 fixes the Hamiltonian only up to terms of vanishing variation. However, under the assumption that $\bar{\mathcal{F}}$ is connected, we can pick out a particular one by requiring the Hamiltonian to vanish for a reference solution: Let g_λ be a path between the reference solution g_0 and an arbitrary other solution g_1. Now, if a Hamiltonian exists, we can write the difference between the Hamiltonian associated to the reference solution and the Hamiltonian associated to g_1 as $(H_\xi)_1 - (H_\xi)_0 = \int_0^1 \mathrm{d}\lambda \delta_\lambda H_\xi$. (If a function H_ξ exists, the right hand side is clearly independent of the choice of the path.) Then, setting $(H_\xi)_0 = 0$ fixes H_ξ uniquely.

Finally, note the following remarks:

Remark 14. $\mathcal{L}_\xi g$ is always a solution of the linearized field equations if g is a solution of the field equations and ξ is a vector field that generates a one-parameter family of diffeomorphisms ϕ_t on the spacetime manifold M. Since (3.1) is diffeomorphism covariant, $\phi_t^* g$ is a solution of the equations of motion as well, i.e. $F(\phi_t^* g) = 0$ holds. Hence it follows that we have $\mathrm{d}F(\phi_t^* g)/\mathrm{d}t = 0$, which corresponds to the linearized field equations with solution $\mathcal{L}_\xi g = \mathrm{d}(\phi_t^* g)/\mathrm{d}t|_{t=0}$. Therefore, $\mathcal{L}_\xi g$ satisfies the linearized equations of motion.

Remark 15. Similarly, if δg is tangent to $\bar{\mathcal{F}}$, it satisfies the linearized field equations. This can be seen as follows: In case δg is tangent to $\bar{\mathcal{F}}$, it corresponds to $\mathrm{d}g_\lambda/\mathrm{d}\lambda|_{\lambda=0}$, where g_λ is a one-parameter family of metrics in $\bar{\mathcal{F}}$. This family satisfies $F(g_\lambda) = 0$, which implies that δg satisfies the linearized field equations, i.e. $\mathrm{d}F(g_\lambda)/\mathrm{d}\lambda|_{\lambda=0} = 0$.

3.1.2 Existence of a Hamiltonian

As mentioned below definition 13, there does not necessarily exist a function H_ξ which satisfies (3.7). It would be helpful to have a condition which implies the existence of a Hamiltonian depending on the theory, the field configuration space, the hypersurface Σ and the vector field ξ. In this section, we will show how such a condition can be found. Again, we will assume throughout this section that all the fields have been chosen in a such a way that the integrals converge.

First, define the Noether current $(d-1)$-form J_ξ. It is given by

$$J_\xi = \theta(g, \mathcal{L}_\xi g) - \xi \cdot L, \tag{3.8}$$

where θ is the presymplectic potential, L is the Lagrangian of the theory, and ξ is some vector field. $\xi \cdot L$ denotes the contraction of ξ into the first index (cf. footnote 4 on page 26) of the differential form L. Note that the ambiguity in the choice of θ (3.2) gives rise to the ambiguity

$$J_\xi \to J_\xi + \mathrm{d}Y(g, \mathcal{L}_\xi g) \tag{3.9}$$

in J_ξ. Equation (3.1) together with the general identity

$$\mathcal{L}_\xi \Lambda = \mathrm{d}(\xi \cdot \Lambda) + \xi \cdot \mathrm{d}\Lambda, \tag{3.10}$$

which holds for any differential form Λ and vector field ξ, implies that the exterior derivative of the Noether current is given by

$$\mathrm{d}J_\xi = -F \cdot \mathcal{L}_\xi g. \tag{3.11}$$

Hence, J_ξ is closed if the equations of motion of the theory (i.e. $F = 0$, see above (3.1)) are satisfied. For the same reasons that restricted the freedom in the choice of the presymplectic potential (see (3.2)) J_ξ is not only closed but exact as well [18, 20]. Consequently, there exists a $(d-2)$-form Q_ξ, called the Noether charge, such that

$$J_\xi = \mathrm{d}Q_\xi \tag{3.12}$$

holds. This form Q_ξ is not unique either. It inherits the ambiguity (3.9) plus an additional exact form[7]:

$$Q_\xi \to Q_\xi + Y(g, \mathcal{L}_\xi g) + \mathrm{d}Z \tag{3.13}$$

If the equations of motion are not satisfied, the Noether current can be written as

$$J_\xi = \mathrm{d}Q_\xi + \xi \cdot C \tag{3.14}$$

[19], where C is a d-form that corresponds to the "constraints" of the theory (since $C = 0$ when $F = 0$).

Now let us calculate the variation of the Noether current J_ξ. To that end, we use (3.8), (3.1), (3.3), and identity (3.10). Then we find that the variation of J_ξ is given by

$$\delta J_\xi = \omega(g, \delta g, \mathcal{L}_\xi g) + \mathrm{d}(\xi \cdot \theta) \tag{3.15}$$

if $g \in \bar{\mathcal{F}}$ and δg tangent to $\mathcal{F}$. Combining this equation with (3.14) gives

$$\omega(g, \delta g, \mathcal{L}_\xi g) = \xi \cdot \delta C + \mathrm{d}(\delta Q_\xi) - \mathrm{d}(\xi \cdot \theta). \tag{3.16}$$

This relation can be used to rewrite (3.7). We find

$$\delta H_\xi = \int_\Sigma \xi \cdot \delta C + \int_{\partial\Sigma} [\delta Q_\xi - \xi \cdot \theta], \tag{3.17}$$

where the integral over $\partial\Sigma$ is to be understood as in (3.6). If the metric perturbation satisfies the linearized equations of motion, (3.17) reduces to

$$\delta H_\xi = \int_{\partial\Sigma} [\delta Q_\xi - \xi \cdot \theta]. \tag{3.18}$$

If a Hamiltonian exists, the right hand side of (3.17) is a variation of some quantity. This clearly implies that

$$0 = (\delta_1\delta_2 - \delta_2\delta_1) H_\xi \tag{3.19}$$

$$= -\int_{\partial\Sigma} \xi \cdot [\delta_1 \theta(g, \delta_2 g) - \delta_2 \theta(g, \delta_1 g)] \tag{3.20}$$

$$= -\int_{\partial\Sigma} \xi \cdot \omega(g, \delta_1 g, \delta_2 g) \tag{3.21}$$

[7] Again, the reason for the form to be exact and not just closed is the same one that led to the ambiguities (3.2) and (3.9).

is satisfied for perturbations $\delta_1 g, \delta_2 g$ that are tangent to $\mathcal{F}$ at $g \in \bar{\mathcal{F}}$. In particular, this holds for $\delta_1 g, \delta_2 g$ tangent to $\bar{\mathcal{F}}$. Even though this condition seems to be only a necessary one, it turns out to also be sufficient for the existence of a Hamiltonian [4]. It is not even necessary to check it for perturbations that are not tangent to $\bar{\mathcal{F}}$. Thus, if

$$\int_{\partial\Sigma} \xi \cdot \omega(g, \delta_1 g, \delta_2 g) = 0 \tag{3.22}$$

is satisfied for all $g \in \bar{\mathcal{F}}$ and all $\delta_1 g$, $\delta_1 g$ tangent to $\bar{\mathcal{F}}$, a Hamiltonian (3.7) exists. This equation will be called the consistency condition.

3.2 Application to asymptotically de Sitter spacetimes

Now let us evaluate the quantities of the previous sections for the Lagrangian (3.23). Since there is an ambiguity in the choice of some of those quantities, the formulas below merely represent convenient choices of the respective quantities to which we will, however, stick in the following chapters.

A diffeomorphism covariant Lagrangian d-form density that gives rise to Einstein's equation with a positive cosmological constant Λ is given by

$$L_{a_1\ldots\epsilon_d} = \frac{1}{16\pi G}(R - 2\Lambda)\epsilon_{a_1\ldots a_d}, \tag{3.23}$$

where R is the Ricci scalar and ϵ is the volume form associated with a metric g_{ab}. It can be shown that this Lagrangian yields the field equations

$$F_{abc_1\ldots c_d} = \frac{1}{16\pi G}\epsilon_{c_1\ldots c_d}\left(R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab}\right) \tag{3.24}$$

and the (or more precisely, a) presymplectic potential

$$\theta_{a_1\ldots a_{d-1}} = \frac{1}{16\pi G}v^c\epsilon_{ca_1\ldots a_{d-1}}, \tag{3.25}$$

where

$$v^a = g^{ac}g^{bd}(\nabla_d\delta g_{bc} - \nabla_c\delta g_{bd}). \tag{3.26}$$

From (3.25), we can derive the associated presymplectic current $(d-1)$-form

$$\omega_{a_1\ldots a_{d-1}} = \frac{1}{16\pi G}w^c\epsilon_{ca_1\ldots a_{d-1}}. \tag{3.27}$$

The vector which appears in the above equation is given by

$$w^a = P^{abcdef}(\delta_1 g_{bc}\nabla_d\delta_2 g_{ef} - \delta_2 g_{bc}\nabla_d\delta_1 g_{ef}), \tag{3.28}$$

where

$$P^{abcdef} = g^{ae}g^{fb}g^{cd} - \frac{1}{2}g^{ad}g^{be}g^{fc} - \frac{1}{2}g^{ab}g^{cd}g^{ef} - \frac{1}{2}g^{bc}g^{ae}g^{fd} + \frac{1}{2}g^{bc}g^{ad}g^{ef}. \tag{3.29}$$

Then, from the definition of the Noether current (3.8), we find

$$(J_\xi)_{a_1\ldots a_{d-1}} = \frac{1}{8\pi G}\nabla_c\nabla^{[c}\xi^{b]}\epsilon_{ba_1\ldots a_{d-1}}. \tag{3.30}$$

Finally, we can use (3.30) to calculate a Noether charge. A possible choice is

$$(Q_\xi)_{a_1\ldots a_{d-2}} = -\frac{1}{16\pi G}(\nabla^b\xi^c)\epsilon_{bca_1\ldots a_{d-2}}. \tag{3.31}$$

3.2.1 Convergence of the Hamiltonian

Some of the formulas in section 3.1 contain metric variations that do not need to be tangent to $\bar{\mathcal{F}}$. These variations play a crucial role in justifying the interpretation of H_ξ as the generator of dynamics conjugate to ξ. But we are mainly interested in the conservation properties of H_ξ. For our purposes, it will therefore be sufficient to investigate (3.18). In contrast to relations (3.7) and (3.17), this equation for the Hamiltonian holds only for perturbations that satisfy the linearized equations of motion. Hence, we will assume in the following that δg is tangent to $\bar{\mathcal{F}}$.[8]

Furthermore, we will restrict our analysis to spacelike hypersurfaces Σ in the physical spacetime manifold M that are closed, without boundary, and extend smoothly to $\mathscr{I}^+$ of an unphysical manifold $\tilde{M}$ for every $g \in \bar{\mathcal{F}}$, such that $\Sigma \cap \mathscr{I}^+$ is a smooth $(d-2)$-dimensional submanifold and $\Sigma \cup \partial\Sigma$ is compact. Note that this means that $\partial\Sigma \subset \mathscr{I}^+$.

Now assume that the vector field ξ is such that $\mathcal{L}_\xi g$ is tangent to the covariant phase space.[9] Then let us take a look at the right hand side of (3.18):

$$I := \int_{\partial\Sigma} [\delta Q - \xi \cdot \theta] \tag{3.32}$$

First, we want to show that this expression is always well defined via the limiting procedure described below (3.6). Let K_i be a nested sequence of compact subsets of Σ such that ∂K_i approaches $\partial\Sigma$ and let

$$I_i = \int_{\partial K_i} [\delta Q - \xi \cdot \theta]. \tag{3.33}$$

According to (3.16), we have

$$\omega(g, \delta g, \mathcal{L}_\xi g) = \mathrm{d}[\delta Q - \xi \cdot \theta], \tag{3.34}$$

which can be used to write the difference between I_i and I_j as

$$I_i - I_j = \int_{\Sigma_{ij}} \mathrm{d}[\delta Q - \xi \cdot \theta] = \int_{\Sigma_{ij}} \omega(g, \delta g, \mathcal{L}_\xi g). \tag{3.35}$$

Here, Σ_{ij} denotes the portion of Σ that lies between ∂K_i and ∂K_j. An important result of chapter 4 will be that $\omega(g, \delta_1 g, \delta_2 g)$ vanishes on $\mathscr{I}^+$ for perturbations $\delta_1 g$, $\delta_2 g$ that are tangent to $\bar{\mathcal{F}}$. Hence it follows that $\omega(g, \delta g, \mathcal{L}_\xi g)$ vanishes on $\mathscr{I}^+$. Together with the compactness of $\Sigma \cup \partial\Sigma$, this implies that I_i is a Cauchy sequence with limit I.

To show that (3.32) is also independent of the choice of the hypersurface, define

$$J_i := \int_{\partial\tilde{K}_i} [\delta Q - \xi \cdot \theta], \tag{3.36}$$

where $\tilde{K}_i$ is a nested sequence of compact subsets of another hypersurface $\tilde{\Sigma}$, such that $\partial\tilde{\Sigma} = \partial\Sigma$. Then, by the same arguments that we employed above, it follows that (3.32) does only depend on the cross section $\partial\Sigma$ with $\mathscr{I}^+$ and not on the particular choice of the hypersurface.[10]

Hence, we have shown that the right hand side of (3.32) or (3.18), respectively, is finite and does not depend on the choice of the hypersurface for $g \in \bar{\mathcal{F}}$ and δg tangent to $\bar{\mathcal{F}}$.

[8] The precise definition of $\bar{\mathcal{F}}$ can be found in section 4.2. However, it is not yet necessary to know what metrics it is comprised of.

[9] The first thing that comes to mind is that ξ is probably a representative of an arbitrary asymptotic symmetry. But in that case (considering our definition of $\bar{\mathcal{F}}$ (definition 14)), we could not immediately conclude that $\mathcal{L}_\xi g$ is tangent to $\bar{\mathcal{F}}$: Let ϕ_t be generated by ζ and let $g \in \bar{\mathcal{F}}$. If we knew that $\phi_t^* g \in \bar{\mathcal{F}}$, $\mathcal{L}_\zeta g = \mathrm{d}\phi_t^* g/\mathrm{d}t|_{t=0}$ would be a vector tangent to $\bar{\mathcal{F}}$. However, we do not know that. Even though ξ is not a representative of an arbitrary asymptotic symmetry, it still is a representative of some asymptotic symmetry.

[10] Note that this argument holds only if the orientations of the boundaries coincide.

3.2.2 Conservation of the Hamiltonian

In the previous section, we have seen that the right hand side of

$$\delta H_\xi = \int_{\partial\Sigma} [\delta Q - \xi \cdot \theta] \tag{3.37}$$

is well defined for appropriate vector fields ξ, suitable hypersurfaces Σ, and our choice of $\bar{\mathcal{F}}$. But we have not yet discussed whether a function H_ξ exists that solves this equation. This can, however, simply be answered by making use of the consistency condition (i.e. equation (3.22)): Together with the fact that $\omega(g, \delta_1 g, \delta_2 g) = 0$ on $\mathscr{I}^+$ for variations $\delta_1 g$, $\delta_2 g$ tangent to $\bar{\mathcal{F}}$ (see previous section), it implies the existence of a function H_ξ (recall that $\partial\Sigma \subset \mathscr{I}^+$). Thus, we can write

$$\delta I_\xi = \delta Q_\xi - \xi \cdot \theta \tag{3.38}$$

for a function I_ξ and

$$H_\xi = \int_{\partial\Sigma} I_\xi. \tag{3.39}$$

Now that we have succeeded in establishing the existence of a Hamiltonian, we can finally begin to investigate as to how H_ξ and δH_ξ are conserved, i.e. depend on the choice of the hypersurface Σ. The hypersurfaces that we considered in the previous section were required to have no boundary in the physical spacetime manifold. But extending them to $\mathscr{I}^+$ of an unphysical manifold had to yield a hypersurface with boundary.

To get an idea of how such hypersurfaces look like, we will resort to de Sitter space as an example spacetime.[11] Cauchy surfaces do not satisfy the above requirements (see figure 3.1(a)), since they do not possess a boundary in an unphysical manifold $\tilde{M}$. The surfaces sketched in 3.1(b), however, do meet these criteria: They are hypersurfaces with boundaries while their restrictions to M are hypersurfaces without boundaries. Consequently, such hypersurfaces can be used to construct Hamiltonians.[12]

Since we want to construct a conserved quantity, H_ξ should be independent of the choice of the hypersurface within a certain class of hypersurfaces for a fixed metric $g \in \bar{\mathcal{F}}$. It follows from (3.1), (3.3), and (3.24) that the exterior derivative of the presymplectic current vanishes, i.e.

$$\mathrm{d}\omega(g, \delta_1 g, \delta_2 g) = 0, \tag{3.40}$$

if $\delta_1 g$ and $\delta_2 g$ satisfy the linearized Einstein equation. Now consider two hypersurfaces Σ_1 and Σ_2 which, together with a portion $\mathscr{I}_{12}$ of $\mathscr{I}^+$, enclose a spacetime volume Σ_{12} (see figure 3.2). Then we can use (3.7) and Stokes' theorem to determine the difference between $\delta H_\xi[\Sigma_1]$ and $\delta H_\xi[\Sigma_2]$. This difference is given by[13,14]

$$\delta H_\xi[\Sigma_1] - \delta H_\xi[\Sigma_2] = \pm \left(\int_{\Sigma_{12}} \mathrm{d}\omega - \int_{\mathscr{I}_{12}} \omega \right) = 0, \tag{3.41}$$

[11] Recall that $\mathscr{I}^\pm$ of every spacetime that satisfies Einstein's equation with a positive cosmological constant is spacelike.

[12] Clearly, such hypersurfaces satisfy all the conditions on hypersurfaces listed above (3.32).

[13] This difference would, for instance, vanish if Σ_1 and Σ_2 were either like the continuous surfaces or like the dashed surfaces in figure 3.1(b). However, due to the orientations of these surfaces, we would have $\sigma_{\text{dashed}} = -\sigma_{\text{continuous}}$.

[14] The sign on the right hand side of (3.41) is determined by the position of the hypersurfaces relative to each other. For the surfaces sketched in figure 3.2 we would get a plus sign.

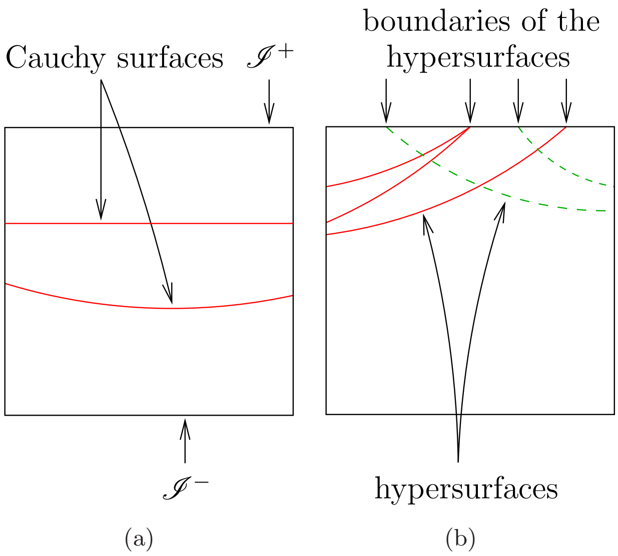

Figure 3.1: Different surfaces in de Sitter spacetime: (a) Cauchy surfaces. (b) Hypersurfaces with boundaries on $\mathscr{I}^+$. Symplectic forms associated with the dashed surfaces are identical, the same holds for the continuous ones (3.41).

where we again made use of the fact that $\omega = 0$ on $\mathscr{I}^+$. This establishes that δH_ξ is independent of the choice of the hypersurface as long as the hypersurfaces together with a portion of $\mathscr{I}^+$ enclose a spacetime volume.

Even though δH_ξ is independent of the choice of Σ, one might suspect that H_ξ is not. However, we have shown below definition 13 that a unique H_ξ can be found for a fixed hypersurface by requiring a solution of (3.7) to vanish for a reference metric in $\bar{\mathcal{F}}$. Now, if the right hand side of (3.7) is independent of the choice of the hypersurface, this requirement clearly ensures that the Hamiltonian H_ξ is independent of the choice as well. Hence, setting

$$H_\xi(g_0) = 0 \tag{3.42}$$

for all asymptotic symmetries in de Sitter space guarantees that H_ξ is independent of the choice of the hypersurfaces within the class of hypersurfaces satisfying (3.41).

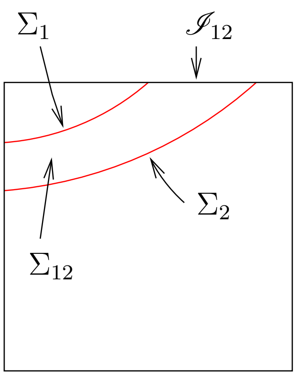

Figure 3.2: Hypersurfaces Σ_1 and Σ_2, the portion $\mathscr{I}_{12}$ of $\mathscr{I}^+$ and the area Σ_{12}.

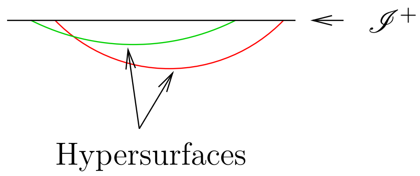

Figure 3.3: Hypersurfaces with disjoint boundaries on $\mathscr{I}^+$.

Remark 16. Our analysis is based on the assumption that the hypersurfaces Σ in the physical spacetime manifold are without boundary. If this is not satisfied, i.e. $\partial\Sigma \cap M \neq \emptyset$, the above formalism might still be applicable if the consistency condition (i.e. equation (3.22)) holds. A Hamiltonian would, however, not necessarily be conserved anymore. This can be seen by considering the following equation:

$$\delta H_\xi[\Sigma_1] - \delta H_\xi[\Sigma_2] = \pm \int_{\Sigma'} \omega \tag{3.43}$$

Here, Σ_1, Σ_2, another hypersurface Σ', and a portion of $\mathscr{I}^+$ enclose a spacetime volume. The non-conservation follows from the fact that the integral on the right hand side is not guaranteed to vanish.

Remark 17. Instead of using hypersurfaces in the construction of H_ξ that behave like the ones that are sketched in figure 3.1(b), we could also use hypersurfaces that behave like the ones in figure 3.3. An H_ξ that is associated with such a surface Σ clearly vanishes if Σ and a portion of $\mathscr{I}^+$ enclose a spacetime volume. Therefore, if H_ξ does not vanish, we can conclude that the situation must be as in figure 3.4: Some part of $\mathscr{I}^+$ must be "missing". For the sake of the argument, let us call this missing part "K". In this scenario, a non-vanishing H_ξ bears some resemblance to a flux of a conserved current that emanates from K through the hypersurface Σ: It is independent of the choice of the hypersurface as long as the surface encloses K.

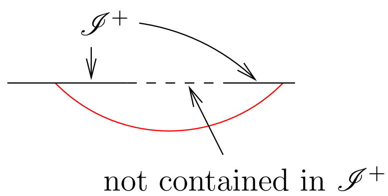

Figure 3.4: Example of a situation in which H_ξ would not necessarily vanish.

4 Conserved quantities in asymptotically de Sitter spacetimes

In this chapter, we will present and derive a formula for conserved quantities in future asymptotically de Sitter spacetimes.

We will begin this chapter by introducing this formula (i.e. (4.1)) and by discussing some of its properties. In section 4.1, we will then investigate as to how conserved quantities arising from this formula are conserved. At the same time, we will show that (4.1) really gives rise to conserved quantities for every future asymptotically de Sitter spacetime. The derivation of the formula for conserved quantities will be presented in section 4.2. By using the Hamiltonian framework of the previous chapter, we will be able to derive a formula that, albeit less general, is identical in form to (4.1). It can only be applied to spacetimes whose metric is in some covariant phase space (see below for the precise requirements), whereas formula (4.1) works for every future asymptotically de Sitter spacetime. Hence, our formula for conserved quantities is a generalization of the one found via the Hamiltonian method. After the construction of the formula, we will apply our results to some sample spacetimes: We will calculate conserved quantities for the Schwarzschild de Sitter and the Tolman-Bondi spacetime. In the last section of this chapter, we will then turn to investigating whether positive conserved quantities exist.

Let (M, g_{ab}) be an arbitrary future asymptotically de Sitter spacetime with unphysical spacetime $(\tilde{M}, \tilde{g}_{ab} = \Omega^2 g_{ab})$ and let C be a cut[1] of $\mathscr{I}^+$. Then a conserved quantity conjugate to a vector field ξ^a which represents an asymptotic symmetry is given by

$$H_\xi = \frac{\ell}{8\pi G}\int_C \tilde{E}_{ab}\tilde{u}^a\xi^b \mathrm{d}\tilde{S}. \tag{4.1}$$

$\mathrm{d}\tilde{S}$ is the integration element on C, $\tilde{u}^a$ is a unit[2] spacelike normal to C within $\mathscr{I}^+$, and

$$\tilde{E}_{ab} = \frac{\ell^2}{d-3}\Omega^{3-d}\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d \tag{4.2}$$

is the normalized electric part of the unphysical Weyl tensor.[3] The definition of the Weyl tensor can be found in section 1.2 and the vector $\tilde{n}^a$ is given by $\tilde{n}^a = \tilde{\nabla}^a\Omega$, where $\tilde{\nabla}_a$ is the derivative operator that is associated with the unphysical metric. Note that the vector $\tilde{u}^a$ is not uniquely defined by the above condition. Being a unit spacelike normal to C within $\mathscr{I}^+$ fixes $\tilde{u}^a$ only up to sign. Equation (4.1) is consequently only fixed up to sign as well.

It is worth noticing that it can explicitly be seen that H_ξ is independent of the choice of the conformal factor. Expressing H_ξ in terms of another conformal factor $\Omega' = \omega\Omega$, where ω is a smooth and strictly positive or negative function on C, does not change the form of (4.1).

As we mentioned at the beginning of this chapter, H_ξ is foremost a conserved quantity. It is a Hamiltonian in the sense of chapter 3 only if the following conditions are satisfied: (i) the metric

[1] A cut is a smooth $(d-2)$-dimensional submanifold of $\mathscr{I}^+$.

[2] Normalized with respect to the unphysical metric.

[3] Despite the inverse powers of Ω, $\tilde{E}_{ab}$ is finite on $\mathscr{I}^+$ (cf. sections 4.1 and 4.2).

g_{ab} is an element of a covariant phase space $\bar{\mathcal{F}}$ over M (see definition 14); (ii) the cut C is the boundary of a suitable hypersurface (see section 3.2.1); (iii) $\mathcal{L}_\xi g$ is tangent to $\bar{\mathcal{F}}$. Hence, if H_ξ is a Hamiltonian, the sign of $\tilde{u}^a$ is fixed by the induced orientation on C (see footnotes 4 and 6 on page 26 for our orientation conventions).

Remark 18. It should be pointed out that the physical interpretation of H_ξ remains unclear at this point. In particular, we cannot relate any conserved quantity or Hamiltonian to the mass of a spacetime: To calculate the mass, we would first turn to a representative of an asymptotic symmetry ξ^a that is timelike on $\mathscr{I}^+$. But according to section 2.5, such symmetries do not exist in future asymptotically de Sitter spacetimes.

Remark 19. Note that $H_\xi[C]$ can be understood as the flux of (the conserved current (cf. (4.15)))

$$\tilde{J}_a := \frac{\ell}{8\pi G}\tilde{E}_{ab}\xi^b \tag{4.3}$$

through C. In terms of $\tilde{J}_a$, H_ξ can be written as

$$H_\xi = \int_C \tilde{J}_a \mathrm{d}\tilde{S}^a. \tag{4.4}$$

This property is related to the one portrayed in remark 17.

Remark 20. Note that most of the conformal quantities that we will encounter in the following (unphysical Riemann tensor, unphysical Ricci tensor and scalar, unphysical Weyl tensor, unphysical extrinsic curvature tensor, ...) are smooth on $\mathscr{I}^+$. These quantities depend smoothly on the (smooth) unphysical metric, the conformal factor, and smooth derivatives of thereof.

4.1 Conservation

Before we derive the formula for H_ξ within the framework that we outlined in chapter 3, we will explicitly show how and that H_ξ is conserved for all future asymptotically de Sitter spacetimes. But first note that our formula for H_ξ really gives rise to a finite quantity: This follows from the fact that the electric part of the Weyl tensor of an arbitrary unphysical spacetime of a future asymptotically de Sitter spacetime behaves like $\tilde{C}_{abcd}\tilde{n}^b\tilde{n}^d = O(\Omega^{d-3})$ in a neighborhood of $\mathscr{I}^+$.[4] This implies that $\tilde{E}_{ab}$ is smooth on $\mathscr{I}^+$ and that H_ξ is finite.

Now let us turn to the conservation properties of H_ξ. By using Einstein's equation and the contracted Bianchi identity, it can be shown that

$$\tilde{\nabla}^a(\Omega^{3-d}\tilde{C}_{acbd}) = 0, \tag{4.5}$$

where Ω is an arbitrary conformal factor. A derivation of this equation can be found in appendix A. There we also give a derivation of

$$R_{ab} = \tilde{R}_{ab} + (d-2)\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \tilde{g}_{ab}\Omega^{-1}\tilde{\nabla}^c\tilde{\nabla}_c\Omega - \tilde{g}_{ab}(d-1)\Omega^{-2}\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega. \tag{4.6}$$

[4] We will derive this relation in the next section (see (4.41)). Even though this derivation will be done for particular unphysical spacetimes, $\tilde{C}_{abcd}\tilde{n}^b\tilde{n}^d = O(\Omega^{d-3})$ holds for any unphysical spacetime satisfying the conditions of definition 11. This can easily be checked by making use of the conformal invariance of the Weyl tensor.

This equation describes the relation between the Ricci tensors R_{ab} and $\tilde{R}_{ab}$. The former is associated with some future asymptotically de Sitter spacetime and the latter with a respective unphysical spacetime. We can rewrite (4.6) as (cf. (A.9))

$$\tilde{\nabla}_a \tilde{n}_b = \frac{1}{d-2} \left[(d-1)\Omega^{-1}(\tilde{n}^c \tilde{n}_c + \ell^{-2})\tilde{g}_{ab} - (\tilde{\nabla}^c \tilde{n}_c)\tilde{g}_{ab} - \Omega \tilde{R}_{ab} \right], \tag{4.7}$$

where we again made use of Einstein's equation and where $\tilde{n}^a = \tilde{\nabla}^a \Omega$. If we contract $\tilde{n}^c \tilde{n}^d$ into (4.5) and use (4.7) and the symmetries of the Weyl tensor as well as its tracelessness, we find

$$\tilde{\nabla}^a (\Omega^{3-d} \tilde{C}_{acbd} \tilde{n}^c \tilde{n}^d) + \frac{1}{d-2} \Omega^{4-d} \tilde{C}_{acbd} \tilde{n}^c \tilde{R}^{ad} = 0. \tag{4.8}$$

Now we again need to refer to the asymptotic behavior of unphysical Weyl tensors in future asymptotically de Sitter spacetimes. As we will show in the next section, $\tilde{C}_{abcd} \tilde{n}^d = O(\Omega^{d-3})$ holds (cf. (4.43)). This implies that

$$\Omega^{4-d} \tilde{C}_{acbd} \tilde{n}^c \tilde{R}^{ad} = 0 \tag{4.9}$$

on $\mathscr{I}^+$ and we can conclude that

$$\tilde{\nabla}^a \tilde{E}_{ab} = 0 \tag{4.10}$$

on $\mathscr{I}^+$. If we denote the metric on surfaces of constant Ω by $\tilde{h}_{ab}$ and its associated derivative operator by $\tilde{D}_a$, we can write

$$\tilde{D}^a \tilde{E}_{ab} = \tilde{h}^a{}_c \tilde{h}_a{}^d \tilde{h}_b{}^e \tilde{\nabla}^c \tilde{E}_{de}, \tag{4.11}$$

from which follows that

$$\tilde{D}^a \tilde{E}_{ab} = 0 \tag{4.12}$$

on $\mathscr{I}^+$. To obtain this last equation, we expressed the unphysical metric as $\tilde{g}_{ab} = -f(p)\tilde{n}_a \tilde{n}_b + \tilde{h}_{ab}$ in a neighborhood of $\mathscr{I}^+$ and used relation (4.7) and the symmetries of the Weyl tensor. Here, f is a smooth positive function with $f(p) \restriction \mathscr{I}^+ = \ell^2$.

Hence we have shown that $\tilde{E}_{ab}$ is divergence-free on $\mathscr{I}^+$. We can use this as follows: According to Stokes' theorem, the difference between conserved quantities associated to cuts C_1, C_2 of $\mathscr{I}^+$ which are the only boundaries of a set $\mathscr{I}_{12}$ can be written as either

$$H_\xi[C_1] - H_\xi[C_2] = \pm \frac{\ell}{8\pi G} \int_{\mathscr{I}_{12}} \tilde{D}^a (\tilde{E}_{ab} \xi^b) \mathrm{d}\tilde{s} \tag{4.13}$$

or

$$H_\xi[C_1] + H_\xi[C_2] = \pm \frac{\ell}{8\pi G} \int_{\mathscr{I}_{12}} \tilde{D}^a (\tilde{E}_{ab} \xi^b) \mathrm{d}\tilde{s} \tag{4.14}$$

depending on the specific choice of the vector field $\tilde{u}^a$ on the cuts. The right hand sides of these equations vanish, since

$$\tilde{D}^a (\tilde{E}_{ab} \xi^b) = \tilde{E}^{ab} \tilde{D}_{(a} \xi_{b)} = \frac{1}{d-1} \tilde{E}_a{}^a \tilde{D}_b \xi^b = 0. \tag{4.15}$$

Here we used (4.12) and the fact that representatives of asymptotic symmetries are conformal Killing fields on $\mathscr{I}^+$ (cf. section 2.5). Hence, conservation of H_ξ means the following: If two cuts can be smoothly deformed into each other, i.e. are homotopic, the respective H_ξ's will differ at most by sign. If H_ξ is a Hamiltonian that is associated with some hypersurface, conservation in the sense of (4.13) corresponds to the conservation properties outlined in section 3.2.2 (cf. (3.41)).

Let us now investigate how the conserved quantities H_ξ are affected by the structure of $\mathscr{I}^+$. To that end, consider an arbitrary unphysical spacetime of a future asymptotically de Sitter space. Then consider a subset $\mathscr{I}_C$ of $\mathscr{I}^+$ that is diffeomorphic to the disc D^{d-1}. A conserved quantity H_ξ associated to the boundary C of $\mathscr{I}_C$ can then be expressed as

$$H_\xi = \frac{\ell}{8\pi G}\int_{\mathscr{I}_C} \tilde{D}^a(\tilde{E}_{ab}\xi^b)\mathrm{d}\tilde{s} = 0, \tag{4.16}$$

where the last equality follows from the fact that $\tilde{D}^a(\tilde{E}_{ab}\xi^b) = 0$ on $\mathscr{I}^+$. Thus, every cut $C \subset \mathscr{I}^+$ of an arbitrary unphysical spacetime of a future asymptotically de Sitter space which is the boundary of a subset of $\mathscr{I}^+$ that is diffeomorphic to D^{d-1} can only have vanishing H_ξ's associated to it.

Hence, starting from a sphere, the minimal condition which allows for the possibility of non-vanishing conserved quantities is the following one: There does not exist an unphysical spacetime of a given spacetime whose $\mathscr{I}^+$ is diffeomorphic to a $(d-1)$-sphere or a $(d-1)$-sphere with one point (or a connected set) removed.

4.2 Construction

In this section, we will, in a manner similar to [9], derive equation (4.1) within the Hamiltonian framework of chapter 3. The necessary steps are to show that: (i) the variation of H_ξ is given by (3.18) for perturbations tangent to $\bar{\mathcal{F}}$; (ii) H_ξ vanishes for de Sitter space. The second property is certainly satisfied, since the Weyl tensor vanishes in de Sitter space. To verify the first property, we need to give a definition of the covariant phase space $\bar{\mathcal{F}}$. But prior to that, we will analyze certain consequences of our definition of the term future asymptotically de Sitter. We will do so to ensure that the conservation of H_ξ is guaranteed on account of the arguments of the previous section for all future asymptotically de Sitter spacetimes and to motivate our definition of $\bar{\mathcal{F}}$. Hence, up to (4.48), the following analysis holds for all future asymptotically de Sitter spacetimes and not just for spacetimes in a covariant phase space.

According to condition (iv) of definition 11, it is always possible to find a conformal factor Ω and unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$ of a future asymptotically de Sitter spacetime (M, g_{ab}) such that the induced metric $(\tilde{h}_{ab})_0$ on $\mathscr{I}^+$ is locally isometric to the natural metric on S^{d-1}. Since this condition fixes Ω only on $\mathscr{I}^+$, it is further possible to choose the conformal factor in such a way that (see section 1.3)

$$\tilde{g}_{ab} = -\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{h}_{ab} \tag{4.17}$$

in a neighborhood of $\mathscr{I}^+$.[5] Here, $\tilde{h}_{ab}$ is the metric on surfaces of constant Ω and $(\tilde{h}_{ab})_0 \equiv \tilde{h}_{ab}(\Omega = 0)$. In the following and in the related appendices, we will assume that the unphysical metric and conformal factor have been chosen, such that (4.17) is satisfied.

For de Sitter space, the conformal factor (see section 2.4.1)

$$\Omega = 2\exp(-|\tau|) \tag{4.18}$$

can be used to get an unphysical metric of the form (4.17). Expression (4.18) is not smooth everywhere, but it certainly is smooth in a neighborhood of $\mathscr{I}^+$, which is all that is needed.

[5] See [9] for as to how such a particular conformal factor can be found.

Using this conformal factor, we find

$$\tilde{\tilde{g}}_{ab} = -\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \left(1 + \frac{\Omega^2}{2} + \frac{\Omega^4}{16}\right)\sigma_{ab} \tag{4.19}$$

for the unphysical de Sitter metric $\tilde{\tilde{g}}$ and

$$\tilde{\tilde{h}}_{ab} = \left(1 + \frac{\Omega^2}{2} + \frac{\Omega^4}{16}\right)\sigma_{ab} \tag{4.20}$$

for the induced unphysical metric on hypersurfaces of constant Ω. Here, σ_{ab} is the standard metric of S^{d-1}.

Now let us have a look at Einstein's equation to analyze the behavior of future asymptotically de Sitter metrics more closely at $\mathscr{I}^+$. The unphysical Ricci tensor and scalar can be used to define the tensor

$$\tilde{S}_{ab} := \frac{2}{d-2}\tilde{R}_{ab} - \frac{1}{(d-1)(d-2)}\tilde{R}\tilde{g}_{ab}. \tag{4.21}$$

Then we can write Einstein's equation with a positive cosmological constant as (appendix A.1)

$$\tilde{S}_{ab} + 2\Omega^{-1}\tilde{\nabla}_a\tilde{n}_b - \Omega^{-2}\tilde{g}_{ab}(\tilde{n}^c\tilde{n}_c + \ell^{-2}) = 0, \tag{4.22}$$

where $\tilde{n}^a = \tilde{\nabla}^a\Omega$ and where ℓ is the de Sitter radius (cf. (2.25)). For the sake of simplicity, we set

$$\ell = 1 \tag{4.23}$$

for the rest of this section and in the related appendices. The de Sitter radius ℓ can always be restored by dimensional arguments. Then it can be read off (4.22) that

$$\tilde{n}^a\tilde{n}_a \restriction \mathscr{I}^+ = -1, \tag{4.24}$$

i.e. $\tilde{n}^a$ is unit and timelike on $\mathscr{I}^+$. By making use of (4.23) and the specific form (4.17) of the unphysical metric, we can reduce (4.22) to

$$\tilde{S}_{ab} = -2\Omega^{-1}\tilde{\nabla}_a\tilde{n}_b. \tag{4.25}$$

Due to the nature of (4.17) this clearly holds only in a neighborhood of $\mathscr{I}^+$. This equation can be split up with respect to the hypersurfaces of constant Ω (see appendix A.2 and A.3): First, by contracting $\tilde{n}^a\tilde{n}^b$ and $\tilde{h}_c{}^a\tilde{n}^b$ into (4.25), we find

$$\tilde{\mathcal{R}} + \tilde{K}^2 - \tilde{K}_{ab}\tilde{K}^{ab} - 2(d-2)\Omega^{-1}\tilde{K} = 0, \tag{4.26}$$

$$\tilde{D}_b\tilde{K}_a{}^b - \tilde{D}_a\tilde{K} = 0. \tag{4.27}$$

These equations are called the constraint equations. The remaining information that is contained in (4.25) can be extracted by contracting $\tilde{h}_c{}^a\tilde{h}_d{}^b$ into this equation. The resulting relation, which we will call the first evolution equation, can be expressed as

$$-\frac{d}{d\Omega}\tilde{K}_a{}^b = \tilde{\mathcal{R}}_a{}^b + \tilde{K}_a{}^b\tilde{K} - (d-2)\Omega^{-1}\tilde{K}_a{}^b - \delta_a{}^b\Omega^{-1}\tilde{K}. \tag{4.28}$$

Here, $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}_{ab}$ are the intrinsic Ricci scalar and tensor of the surfaces of constant Ω and $\tilde{D}_a$ is the derivative operator that is associated with $\tilde{h}_{ab}$. The tensor $\tilde{K}_{ab} = -\tilde{\nabla}_c\tilde{n}_d$ is the extrinsic curvature of these surfaces. It is symmetric and purely spatial[6], i.e. it satisfies $\tilde{K}_{ab} = -\tilde{h}_a{}^c \tilde{h}_b{}^d \tilde{\nabla}_c \tilde{n}_d$. Hence, we can raise and lower its indices with the spatial metric $\tilde{h}_{ab}$. We denote its trace by $K = K_a{}^a$. The second evolution equation is given by

$$\frac{d}{d\Omega}\tilde{h}_{ab} = 2\tilde{h}_{bc}\tilde{K}_a{}^c. \tag{4.29}$$

It just captures the fact that

$$\mathcal{L}_{\tilde{n}}\tilde{h}_{ab} = -2\tilde{K}_{ab} \tag{4.30}$$

(cf. appendix A.3).

In order to investigate the consequences implied by (4.28) and (4.29), we perform a Taylor expansion in Ω, such that

$$\tilde{h}_{ab} = \sum_{j=0}^{\infty}(\tilde{h}_{ab})_j\Omega^j, \qquad \tilde{p}_a{}^b = \sum_{j=0}^{\infty}(\tilde{p}_a{}^b)_j\Omega^j, \qquad \tilde{K} = \sum_{j=0}^{\infty}(\tilde{K})_j\Omega^j, \tag{4.31}$$

where $\tilde{p}_a{}^b$ is the traceless part of $\tilde{K}_a{}^b$ and where the expansion coefficients are independent of Ω. If we similarly expand the intrinsic Ricci tensor and scalar, we get

$$(-d+2+j)(\tilde{p}_a{}^b)_j = -(\tilde{\mathcal{R}}_a{}^b)_{j-1} + \frac{1}{d-1}(\tilde{\mathcal{R}})_{j-1}\delta_a{}^b - \sum_{m=0}^{j-1}(\tilde{K})_m(\tilde{p}_a{}^b)_{j-1-m} \tag{4.32}$$

$$(-2d+3+j)(\tilde{K})_j = -(\tilde{\mathcal{R}})_{j-1} - \sum_{m=0}^{j-1}(\tilde{K})_m(\tilde{K})_{j-1-m} \tag{4.33}$$

$$j(\tilde{h}_{ab})_j = 2\sum_{m=0}^{j-1}\left[(\tilde{h}_{bc})_m(\tilde{p}_a{}^c)_{j-1-m} + \frac{1}{d-1}(\tilde{h}_{ab})_m(\tilde{K})_{j-1-m}\right] \tag{4.34}$$

from the evolution equations.

If we knew $(\tilde{p}_a{}^b)_0$, $(\tilde{K})_0$ and $(\tilde{h}_{ab})_0$, the above relations would uniquely determine $(\tilde{K}_a{}^b)_j$ and $(\tilde{h}_{ab})_j$ up to order $d-2$ and $d-1$, respectively. All the information we need about these "initial conditions" can be found in the definition of the term future asymptotically de Sitter (definition 11). First, from Einstein's equation follows that (to be more precise: from multiplying (4.28) by Ω)

$$\tilde{K}_a{}^b = 0 \tag{4.35}$$

on $\mathscr{I}^+$, which implies

$$(\tilde{p}_a{}^b)_0 = (\tilde{K})_0 = 0. \tag{4.36}$$

Second, the necessary information about $(\tilde{h}_{ab})_0$ is provided by the fourth condition of definition 11. We have already seen the consequences of this condition in (4.17): With our choice of the conformal factor, $(\mathscr{I}^+, (\tilde{h}_{ab})_0)$ is locally isometric to (S^{d-1}, σ_{ab}).

Now recall that $(\mathscr{I}^+, (\tilde{h}_{ab})_0)$ of an unphysical de Sitter spacetime is not only locally but really isometric (equal) to a $(d-1)$-sphere with its natural metric for our choice of the conformal factor. This implies that we recover the (unphysical) spatial metric of de Sitter space and its extrinsic curvature (up to the respective orders) if we insert the metric of a sphere into the recursion

[6] Symmetry follows from the fact that $\tilde{\nabla}_a$ is torsion-free. Being spatial follows from that fact that $\tilde{n}^a$ is a unit geodesic tangent field.

relations. But this also means that $(\mathscr{I}^+, (\tilde{h}_{ab})_0)$ of an arbitrary future asymptotically de Sitter space is locally isometric to that of de Sitter space. Consequently, if we insert a metric into the recursion relations that is locally isometric to the metric of a sphere, $(\tilde{h}_{ab})_j$ and $(\tilde{K}_a{}^b)_l$ must be locally diffeomorphic to their de Sitter counterparts (up to the respective orders) and the maps that relate these quantities to the respective de Sitter quantities must be equal to the ones that relate $(\tilde{h}_{ab})_0$ to its de Sitter counterpart.[7,8] Hence, if we recall the form of the induced unphysical metric on surfaces of constant Ω of de Sitter space (i.e. (4.20)), we can conclude that

$$\tilde{h}_{ab} = \left(1 + \frac{\Omega^2}{2} + \frac{\Omega^4}{16}\right)(\tilde{h}_{ab})_0 + O(\Omega^{d-1}) \tag{4.37}$$

must hold for any future asymptotically de Sitter spacetime in a neighborhood of $\mathscr{I}^+$.[9]

At order $j = d-2$, the left hand side of equation (4.32) is equal to $0 \cdot (\tilde{p}_a{}^b)_{d-2}$. Therefore, we cannot calculate this coefficient and all the successive ones without further information. But if $(\tilde{p}_a{}^b)_{d-2}$ is given, the coefficients $(\tilde{h}_{ab})_j$, $(\tilde{K})_j$, and $(\tilde{p}_a{}^b)_j$ are uniquely determined for $j \geq d-1$.[10] Consequently, this tensor carries all the information about the metric that is not supplied by the asymptotic conditions.

Now we will show that this tensor is related to the leading order electric part of the unphysical Weyl tensor. To that end, consider the following equation (see appendix A.4):

$$\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d = \mathcal{L}_{\tilde{n}}\tilde{K}_{ab} + \tilde{K}_{ac}\tilde{K}_b{}^c + \Omega^{-1}\tilde{K}_{ab} \tag{4.38}$$

Expanding equation (4.38) in powers of Ω gives

$$-\frac{1}{d-3}(\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d)_{d-3} = (\tilde{K}_{ab})_{d-2} - \frac{1}{d-3}\sum_{m=0}^{d-3}(\tilde{K}_{ac})_m(\tilde{K}_b{}^c)_{d-3-m} \tag{4.39}$$

for the coefficients at order Ω^{d-3}. The second term on the right hand side of this equation vanishes[11] for $d = 4$ as well as $d \geq 6$ and is equal to $-1/8(\tilde{h}_{ab})_0$ for $d = 5$. Hence, we can rewrite (4.39) as

$$-\frac{1}{d-3}(\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d)_{d-3} = \begin{cases} (\tilde{K}_{ab})_{d-2} & \text{for } d = 4, d \geq 6 \\ (\tilde{K}_{ab})_3 - \frac{1}{8}(\tilde{h}_{ab})_0 & \text{for } d = 5, \end{cases} \tag{4.40}$$

which relates the leading order of the electric part of the Weyl tensor to the coefficient $(\tilde{K}_{ab})_{d-2}$. Of course, this also relates $(\tilde{p}_a{}^b)_{d-2}$ to the electric part of the Weyl tensor, because $\tilde{p}_a{}^b$ is just the tracefree part of $\tilde{K}_a{}^b$.

[7] More precisely, for every point $p \in \mathscr{I}^+$, there must exist a neighborhood $U \subset \mathscr{I}^+$ of p and a diffeomorphism f, such that $(f^{-1})^*(\tilde{h}_{ab})_j = (\tilde{\tilde{h}}_{ab})_j$ and $(f^{-1})^*(\tilde{K}_a{}^b)_l = (\tilde{\tilde{K}}_a{}^b)_l$ on U for $j < d-1$ and $l < d-2$, where $(\tilde{\tilde{h}}_{ab})_j$ and $(\tilde{\tilde{K}}_a{}^b)_l$ are the respective quantities of de Sitter space.

[8] In coordinates, we can locally write the spatial metric of $\mathscr{I}^+$ of an arbitrary future asymptotically de Sitter spacetime in the form of the metric of the sphere. Then, obviously, the recursion relations yield identical quantities (locally) for de Sitter space and other future asymptotically de Sitter spaces.

[9] This extends to quantities that depend only on $(\tilde{h}_{ab})_j$ and $(\tilde{K}_a{}^b)_l$ for $j < d-1$ and $l < d-2$: They must be locally diffeomorphic to their de Sitter counterparts.

[10] Note that we can get $(K)_{2d-3}$ from the constraint equations.

[11] The vanishing in $d \geq 6$ follows from the fact that we have $(\tilde{h}_{ab})_{d-1} = 0$ in de Sitter space: This implies $(\tilde{K}_{ab})_{d-2} = 0$ via equation (4.30), which means that the first term on the right hand side of (4.39) vanishes. This is also true for the term on the left hand side, because the Weyl tensor vanishes in de Sitter space. Consequently, the sum in (4.39) must vanish for de Sitter space. From this follows that this sum has to vanish in any future asymptotically de Sitter spacetime, because it is locally diffeomorphic to the respective de Sitter expression.

Remark 21. The quantity $\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d$ (see (4.38)) depends only on the extrinsic curvature (and the spatial metric) and derivatives thereof. More precisely, up to order $d-4$, its expansion coefficients can be expressed through[12] $(\tilde{K}_a{}^b)_l$ and $(\tilde{h}_{ab})_l$, where $l < d-2$. Hence, $(\tilde{C}_{abcd}\tilde{n}^b\tilde{n}^d)_j$ must be locally isometric to the corresponding quantity in de Sitter space for $j < d-3$, from which

$$\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d = O(\Omega^{d-3}) \tag{4.41}$$

follows, since the Weyl tensor vanishes in pure de Sitter space. We can draw the same conclusion for (cf. appendix A.4)

$$\tilde{C}_{abcd}\tilde{n}^d = 2\tilde{n}_{[a}\tilde{C}_{b]ecd}\tilde{n}^d\tilde{n}^e - \tilde{D}_a\tilde{K}_{bc} + \tilde{D}_b\tilde{K}_{ac}. \tag{4.42}$$

The asymptotic behavior of the first term on the right hand side of this equation clearly agrees with the one of (4.41). Up to order $d-4$, the second and third term contain $(\tilde{h}_{ab})_j$ and $(\tilde{K}_a{}^b)_j$ as well as ordinary derivatives to first order of these quantities to no higher order than $j = d-4$. Hence,

$$\tilde{C}_{abcd}\tilde{n}^d = O(\Omega^{d-3}), \tag{4.43}$$

i.e. the asymptotic behavior of this quantity is identical to the one of (4.41).

To proceed with our analysis, we now relate $(\tilde{h}_{ab})_{d-1}$ to the electric part of the Weyl tensor. According to (4.30), the coefficient $(h_{ab})_{d-1}$ is simply given by

$$(\tilde{h}_{ab})_{d-1} = \frac{2}{d-1}(\tilde{K}_{ab})_{d-2}. \tag{4.44}$$

Then we can use the relation between the leading order of the electric part of the Weyl tensor and the coefficient $(\tilde{K}_{ab})_{d-2}$ that we have just derived (i.e. relation (4.40)): By definition, we have (see (4.2))

$$(\tilde{E}_{ab})_0 = \frac{1}{d-3}(\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d)_{d-3}, \tag{4.45}$$

which means that (4.44) can be rewritten as

$$(\tilde{h}_{ab})_{d-1} = \begin{cases} -\frac{2}{d-1}(\tilde{E}_{ab})_0 & \text{for } d=4, d \geq 6 \\ -\frac{1}{2}(\tilde{E}_{ab})_0 + \frac{1}{16}(\tilde{h}_{ab})_0 & \text{for } d=5 \end{cases}. \tag{4.46}$$

Now recall equation (4.37). This equation and the above result imply that we can write the unphysical metric of an arbitrary future asymptotically de Sitter spacetime as

$$\tilde{g}_{ab} = -\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \left(1 + \frac{1}{2}\Omega^2\right)(\tilde{h}_{ab})_0 - \frac{2}{3}\Omega^3\tilde{E}_{ab} + O(\Omega^4) \quad \text{for } d=4 \tag{4.47}$$

or as

$$\tilde{g}_{ab} = -\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \left(1 + \frac{\Omega^2}{2} + \frac{\Omega^4}{16}\right)(\tilde{h}_{ab})_0 - \frac{2}{d-1}\Omega^{d-1}\tilde{E}_{ab} + O(\Omega^d) \quad \text{for } d>4 \tag{4.48}$$

in a neighborhood of $\mathscr{I}^+$.

Now let us restrict our analysis to the covariant phase space $\bar{\mathcal{F}}$.

[12]Recall equations (A.25) and (A.27) and that $\tilde{K}_{ab} = \tilde{K}_a{}^c\tilde{h}_{bc}$.

Definition 14. Let M be a manifold, let $\mathscr{I}^+$ be a (future[13]) boundary, such that $\tilde{M} = M \cup \mathscr{I}^+$ and let Ω be a smooth conformal factor on $\tilde{M}$. A covariant phase space $\bar{\mathcal{F}}$ over M satisfies the following properties: (i) it is connected; (ii) $\bar{g} \in \bar{\mathcal{F}}$, where $\bar{g}$ is the de Sitter metric; (iii) $g_{ab} \in \bar{\mathcal{F}}$ is smooth on M and $\tilde{g}_{ab} = \Omega^2 g_{ab}$ extends smoothly to $\mathscr{I}^+$; (iv) in a neighborhood of $\mathscr{I}^+$, g_{ab} is of the form $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$, where $\tilde{g}_{ab}$ is given by (4.47) or (4.48) and $(\tilde{h}_{ab})_0$ is the same for all $g_{ab} \in \bar{\mathcal{F}}$; (v) g_{ab} satisfies Einstein's equation (2.23).

Remark 22. Note that (M, g_{ab}) is future asymptotically de Sitter for all $g_{ab} \in \bar{\mathcal{F}}$. The opposite is not true.

All the metrics in $\bar{\mathcal{F}}$ are of the form $g_{ab} = \Omega^{-2}\tilde{g}_{ab}$, where $\tilde{g}_{ab}$ is as in (4.47) or (4.48) with $(h_{ab})_0$ fixed. Hence, every vector tangent to $\bar{\mathcal{F}}$ must asymptotically (on the spacetime manifold) be of the form

$$\delta g_{ab} = \gamma_{ab} + \mathcal{L}_\eta g_{ab}, \tag{4.49}$$

where

$$\gamma_{ab} = -\frac{2}{d-1}\Omega^{d-3}\delta\tilde{E}_{ab} + O(\Omega^{d-2}) \tag{4.50}$$

and

$$\mathcal{L}_\eta g_{ab} = -\frac{2}{d-1}\Omega^{d-3}\mathcal{L}_\eta\tilde{E}_{ab} + O(\Omega^{d-2}). \tag{4.51}$$

$\mathcal{L}_\eta g_{ab}$ corresponds to the gauge freedom in perturbations (see e.g. [5]). It is an infinitesimal diffeomorphism generated by a vector field η^a. With our choice of the covariant phase space $\bar{\mathcal{F}}$, the vector field η^a is not arbitrary, but must be in compliance with definition 14. This means that

$$\mathcal{L}_\eta\bar{g} = O(\Omega^{d-2}), \tag{4.52}$$

for the de Sitter metric $\bar{g}$, which leads to (4.51). Calculating the symplectic current $(d-1)$-form (3.3) for perturbations (4.49) yields (cf. appendix A.5)

$$\omega(g, \delta_1 g, \delta_2 g) = 0 \tag{4.53}$$

on $\mathscr{I}^+$. Therefore, as argued in chapter 3, a Hamiltonian H_ξ exists.[14] Now recall that

$$\delta H_\xi = \int_C [\delta Q_\xi - \xi \cdot \theta(g, \delta g)] \tag{4.54}$$

for variations δg that are tangent to $\bar{\mathcal{F}}$. Here, $C \subset \mathscr{I}^+$ is the boundary of a suitable hypersurface Σ (cf. chapter 3). To evaluate this expression for (4.49), we analyze (4.50) and (4.51) separately. We begin with $\delta g_{ab} = \gamma_{ab}$.

The Noether charge Q_ξ and its variation under (4.50) can be written as[15]

$$(Q_\xi)_{a_1\ldots a_{d-2}} = \frac{1}{8\pi G}\Omega^{1-d}\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\tilde{n}^b\xi^c - \frac{1}{16\pi G}\Omega^{2-d}\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\tilde{g}^{be}\tilde{\nabla}_e\xi^c \tag{4.55}$$

and

$$(\delta Q_\xi)_{a_1\ldots a_{d-2}} = \frac{1}{8\pi G}\tilde{\epsilon}_{a_1\ldots a_{d-2}bc}\tilde{n}^b\delta\tilde{E}^c{}_d\xi^d + O(\Omega), \tag{4.56}$$

[13]See condition (v) of definition 11.
[14]Recall that ξ^a is such that $\mathcal{L}_\xi g$ is tangent to $\bar{\mathcal{F}}$. See footnote 9 on page 30.
[15]Recall that we have chosen the particular Noether charge (3.31).

respectively (see appendix A.6). To further reexpress this equation, let us take a look at the volume form ${}^{(d-2)}\tilde{\epsilon}$ on C. It is fully determined by the volume forms of the physical spacetime: The volume form of Σ is $\hat{n}^b\epsilon_{ba_1\ldots a_{d-1}}$, where $\hat{n}^a$ is a future directed unit vector normal to Σ.[16] The volume forms of boundaries of a compact subsets of Σ are $\hat{u}^c\hat{n}^b\epsilon_{bca_1\ldots a_{d-2}}$, where $\hat{u}^a$ is an outward pointing unit vector. In terms of conformal quantities, the volume forms of these boundaries can be written as

$$\Omega^{d-2}\hat{u}^c\hat{n}^b\epsilon_{bca_1\ldots a_{d-2}} = \tilde{\hat{u}}^c\tilde{\hat{n}}^b\tilde{\epsilon}_{bca_1\ldots a_{d-2}}, \tag{4.57}$$

where $\tilde{\hat{u}}^c = \Omega^{-1}\hat{u}^c$ and $\tilde{\hat{n}}^b = \Omega^{-1}\hat{n}^b$. On $\mathscr{I}^+$, we find[17]

$$\tilde{\hat{u}}^c\tilde{\hat{n}}^b\tilde{\epsilon}_{bca_1\ldots a_{d-2}} \restriction \mathscr{I}^+ = \tilde{u}^c\tilde{n}^b\tilde{\epsilon}_{bca_1\ldots a_{d-2}} \restriction \mathscr{I}^+ = {}^{(d-2)}\tilde{\epsilon}_{a_1\ldots a_{d-2}}, \tag{4.58}$$

where $\tilde{u}^a$ and $\tilde{n}^a$ are as defined below (4.1). This condition fixes the vector $\tilde{u}^a$ uniquely. The volume form $\tilde{\epsilon}$ can be written as

$$\tilde{\epsilon} = \tilde{u} \wedge \tilde{n} \wedge {}^{(d-2)}\tilde{\epsilon}. \tag{4.59}$$

Inserting this into (4.56) and restricting[18] to C yields

$$(\delta Q_\xi)_{a_1\ldots a_{d-2}} \restriction C = \frac{1}{8\pi G}\delta\left[{}^{(d-2)}\tilde{\epsilon}_{a_1\ldots a_{d-2}}\tilde{E}^c{}_d\tilde{u}_c\xi^d\right]. \tag{4.60}$$

We still need to calculate the second term on the right hand side of (4.54). However, it turns out that (see appendix A.7)

$$\theta(g,\gamma) = 0 \tag{4.61}$$

on $\mathscr{I}^+$. Therefore, we can write (4.54) as

$$\delta H_\xi = \int_C \delta Q_\xi = \frac{1}{8\pi G}\delta\int_C \tilde{E}_{ab}\tilde{u}^b\xi^a\mathrm{d}\tilde{S} \tag{4.62}$$

for perturbations of type (4.50). Restoring ℓ in this equation already gives formula (4.1), even though we still need to investigate the second part of the variation (4.49). However, if we repeat the above analysis for (4.51), we find the exact same result, namely

$$\delta_\eta H_\xi = \int_C \delta_\eta Q_\xi = \frac{1}{8\pi G}\delta_\eta\int_C \tilde{E}_{ab}\tilde{u}^b\xi^a\mathrm{d}\tilde{S}. \tag{4.63}$$

Thus, the variation of H_ξ under (4.49) indeed satisfies (4.54) (i.e. (3.18)) and is consequently the correct expression for the Hamiltonian.

4.3 Examples

In this section, we will evaluate the conserved quantities H_ξ in some examples. In particular, we will calculate conserved quantities for the Schwarzschild de Sitter spacetime in d dimensions.

[16] See footnotes 4 and 6 on page 26.
[17] The first equality in this equation can always be satisfied: $\tilde{\epsilon}$ is antisymmetric.
[18] Note: This means also restricting the resulting forms to vectors tangent to C.

4.3.1 Schwarzschild de Sitter space

In section 2.4.2, we have shown that there exists an unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$ of the Schwarzschild de Sitter spacetime whose $(\mathscr{I}^+, \tilde{h}_{ab})$ is isometric to a $(d-1)$-sphere minus two antipodal points. Therefore, it might be possible to construct non-vanishing conserved quantities (cf. section 4.1).

Let us take a look at the Schwarzschild de Sitter metric (2.35). For simplicity, we choose the conformal factor $\Omega = 1/r$, which gives rise to the unphysical metric

$$\mathrm{d}\tilde{s}^2 = -1/\left(\frac{1}{\ell^2} + C_d\Omega^{d-1} - \Omega^2\right)\mathrm{d}\Omega^2 + \left(\frac{1}{\ell^2} + C_d\Omega^{d-1} - \Omega^2\right)\mathrm{d}t^2 + \mathrm{d}\sigma^2_{d-2}. \tag{4.64}$$

To calculate a conserved quantity, we need a representative of an asymptotic symmetry, the representation of the vector field $\tilde{n}^a$ (on $\mathscr{I}^+$) in the above coordinates, and a suitable cut with vector field $\tilde{u}^a$. We choose the Killing field $t^a \equiv (\partial/\partial t)^a$ as the representative of an asymptotic symmetry (see section 2.5) and we note that the vector field $\tilde{n}^a$ is given by $\tilde{n}^a = -\ell^{-2}(\partial/\partial\Omega)^a$ at points of $\mathscr{I}^+$. A generic cut that does not necessarily have only vanishing quantities associated to it (again, see section 4.1), is given by the set whose points satisfy $\Omega = 0$ and $t = t_0$, where $t_0 \in \mathbb{R}$. Finally, in compliance with the cut, we choose $\tilde{u}^a$ to be $\tilde{u}^a = -\ell \cdot t^a$. Then we find $\tilde{E}_{ab}\tilde{u}^a t^b = (d-2)\cdot C_d/(2\ell)$ and consequently

$$H_t = \frac{(d-2)}{16\pi G} A_{d-2} C_d, \tag{4.65}$$

where A_d is the surface area of a unit d-sphere. In ordinary flat d-dimensional Schwarzschild spacetimes, one usually chooses $C_d = \frac{16\pi G}{(d-2)A_{d-2}} M$ [21], where M is the generalization of the ADM-mass (see e.g. [5]) to higher dimensions. If we use this, we find

$$H_t = M. \tag{4.66}$$

H_t is a non-vanishing quantity for $d \geq 4$, which means that there cannot exist an unphysical spacetime of Schwarzschild de Sitter whose $\mathscr{I}^+$ is diffeomorphic to S^{d-1} or $S^{d-1} \setminus \{p\}$, where $p \in S^{d-1}$.

Another conserved quantity can be calculated by replacing the Killing field t^a with the Killing field $\phi^a \equiv (\partial/\partial\phi)^a$.[19] We find

$$H_\phi = 0. \tag{4.67}$$

Let us now have a look at a few suggestions for the mass of the Schwarzschild de Sitter spacetime:

(i) The most prominent suggestion for masses of asymptotically de Sitter spacetimes is arguably the AD-mass (Abbott and Deser, [1]). Interestingly, H_t agrees with the AD-mass for the Schwarzschild de Sitter spacetime in 4 dimensions. Unfortunately, we are not aware of any attempt to calculate the AD-mass in higher dimensions. A general comparison of

[19] This Killing field represents (part of) the rotational symmetry of the Schwarzschild de Sitter spacetime. Usually, such fields are used to calculate the total angular momentum of a given spacetime. For instance: A field theory on a fixed background spacetime with rotational symmetry represented by a Killing field ϕ^a. In that case, if Σ is a Cauchy surface, $\int_\Sigma T^a{}_b \phi^b \mathrm{d}\Sigma_a$ is a conserved quantity and is usually taken to be the total angular momentum. In our case, this interpretation of H_ϕ might also be possible: The angular momentum of the Schwarzschild de Sitter spacetime should vanish.

our formalism with the one of Abbott and Deser would be extremely interesting: It is the most widely known approach and often considered as reference in the literature. Since our result agrees with the AD-mass in the Schwarzschild de Sitter case, one could hope that these expressions agree in general for appropriate choices of the symmetry and the cut.

(ii) Another approach to conserved quantities in asymptotically de Sitter spaces has been given by Balasubramanian, Boer, and Minic [2]. They also used their approach to calculate the mass of Schwarzschild de Sitter spacetimes. More precisely, they calculated the mass in 4 and 5 dimensions. But contrary to our formula, their construction assigns non-vanishing conserved quantities to de Sitter space, which causes the mass of the Schwarzschild de Sitter spacetime to deviate from H_t by this de Sitter contribution. There is also a difference in sign, but that can be attributed to conventions. This deviation from our result is not completely unexpected in light of some results of [9]: In that paper, an analysis similar to the one found in this thesis has been carried out for asymptotically anti de Sitter spacetimes. They compared their conserved quantities (also [22]), which are formally identical to our H_ξ's, to charges that were constructed with the so-called counterterm subtraction method (see e.g. [23]). Since this method was also employed by Balasubramanian et al. in the construction of their charges and since the form of these charges is identical to their anti de Sitter counterparts, the comparison of [9] can probably be carried over to the case at hand with only minor changes. It showed that the charges differ by an integral over a cut of $\mathscr{I}$, which, in our case, probably corresponds to the de Sitter contribution.

(iii) A third approach to conserved quantities has been given by Kastor and Traschen [3]. They also applied their method to the Schwarzschild de Sitter spacetime. But the mass they computed differs completely from H_t. We mention this approach, because Kastor and Traschen's expressions for spacetime masses (the Spinor charge) have the interesting property of always being positive. In section 4.4, we will to some extent look into the relation between these expressions and our conserved quantities.

But note that within our formalism we have no reason to consider H_t as the spacetime mass. The vector fields t^a and $\tilde{u}^a$ as well as the cut were arbitrarily chosen.

4.3.2 Tolman-Bondi space with a cosmological constant

In general, our formalism cannot be applied to spacetimes with non-vanishing stress-energy tensors. But, as already mentioned in section 2.4.3, the vacuum region of the Tolman-Bondi spacetime can be considered as a future asymptotically de Sitter spacetime on its own. Since we know by Birkhoff's theorem that the spacetime is Schwarzschild de Sitter outside of the dust sphere, we can immediately give an expression for a conserved quantity which is associated with a vector field that equals $(\partial/\partial t)^a$ of Schwarzschild de Sitter on $\mathscr{I}^+$ (see previous section). It is given by (4.65), where $C_{d=4} < 0$ if $M_{TB} < 0$. Similarly, we can transfer the result for the field ϕ^a of the previous section to the Tolman-Bondi case.

4.4 Are there positive conserved quantities?

It has been shown that the masses of asymptotically flat and asymptotically anti de Sitter spacetimes are positive (e.g. the ADM-mass). This is known as the positive mass theorem. We would expect a similar result for future asymptotically de Sitter spacetimes. That is, if we find a condition that fixes a representative of an asymptotic symmetry ξ^a, the vector field $\tilde{u}^a$, and the cut C in such a way that H_ξ is always positive, then this quantity H_ξ might be related to a viable expression for the spacetime mass.

Remark 23. The quantity H_t that we have calculated for the Schwarzschild de Sitter spacetime seems to be a viable expression for the spacetime mass (recall that H_t is equal to the AD-mass of Schwarzschild de Sitter). But as we have already mentioned, within our formalism we have no reason to regard H_t as the spacetime mass: We arbitrarily chose the symmetry t^a and the vector field $\tilde{u}^a$ in the computation of H_t. Even if the mass of a general future asymptotically de Sitter spacetime could be calculated in the same way[20] as H_t, we would be faced with the following problem: Which representative of an asymptotic symmetry should we use? ξ^a or $-\xi^a$? A similar problem stems from the definition of the vector field $\tilde{u}^a$. It is only defined up to sign.

To find such a condition, we will try to relate H_ξ to a manifestly positive expression, the Spinor charge Q_ψ [3]. This approach worked very well in asymptotically anti de Sitter spacetimes: The Spinor charge can also be defined there and it was successfully related to the charges $\mathcal{H}_\zeta$ of [9], which are formally identical to our H_ξ's. This implied the positivity of $\mathcal{H}_\zeta$ for suitable vector fields ζ^a. Apart from [9], analyzes regarding the relation between the Spinor charge and $\mathcal{H}_\zeta$ can be found in [24, 25]. For simplicity, we will again set $\ell = 1$ in the following. Details and notations regarding this section can be found in appendix B.

On a future asymptotically de Sitter spacetime (M, g_{ab}) that satisfies (4.75) and admits a spinor bundle and curved spacetime gamma matrices with $\gamma_{(a}\gamma_{b)} = g_{ab}$, define the Nester 2-form by

$$B_{ab} := \bar{\psi}\gamma_{[a}\gamma_b\gamma_{c]}\hat{\nabla}^c\psi. \tag{4.68}$$

Here, $\bar{\psi}$ is the adjoint spinor to ψ and $\hat{\nabla}_a$ is the super-covariant derivative operator which is given by

$$\hat{\nabla}_a\psi = \nabla_a\psi + \frac{i}{2}\gamma_a\psi. \tag{4.69}$$

Let Σ be a spacelike hypersurface without boundary in the physical spacetime that can be smoothly extended to $\mathscr{I}^+$ in the unphysical spacetime. If we write the metric on the hypersurface Σ as

$$g_{ab} = -\hat{\eta}_a\hat{\eta}_b + q_{ab}, \tag{4.70}$$

where $\hat{\eta}^a$ are the future pointing geodesic normals to Σ and q_{ab} is its intrinsic metric, we can define the Spinor charge

$$Q_\psi := \frac{1}{2}\int_{\partial\Sigma}(B_{ab} + B^*_{ab})\hat{u}^b\hat{\eta}^a \mathrm{d}S. \tag{4.71}$$

In this equation, the star denotes complex conjugation, $\partial\Sigma$ is to be understood as in (3.6), and $\hat{u}^a$ is an outward pointing vector which is orthogonal to $\hat{\eta}^a$ as well as $\partial\Sigma$. Equation (4.71) can then be reexpressed as [16]

$$\begin{aligned} Q_\psi = \int_\Sigma [q^{ab}(\hat{\nabla}_a\psi)^\dagger\hat{\nabla}_b\psi - (q^{ab}\gamma_a\hat{\nabla}_b\psi)^\dagger(q^{cd}\gamma_c\hat{\nabla}_d\psi) \\ - 2i(\psi^\dagger q^{ab}\gamma_a\hat{\nabla}_b\psi) - (q^{ab}\gamma_a\hat{\nabla}_b\psi)^\dagger\psi)]\,\mathrm{d}s. \end{aligned} \tag{4.72}$$

[20] By choosing a vector field as the representative of an asymptotic symmetry that approaches the Killing field $(\partial/\partial t)^a$ (see below (2.32)) of de Sitter in the asymptotic region. (Note that $-(\partial/\partial t)^a$ is also a Killing field of de Sitter. It is not possible to distinguish between $(\partial/\partial t)^a$ and $-(\partial/\partial t)^a$ in a canonical way.) This requires, of course, that the metric can be written as $g_{ab} = \bar{g}_{ab} + k_{ab}$, where $\bar{g}_{ab}$ is the de Sitter metric and where k_{ab} is a deviation which vanishes at $\mathscr{I}^+$.

If the Witten equation

$$q^{ab}\gamma_b\hat{\nabla}_a\psi = 0 \tag{4.73}$$

is satisfied, (4.72) reduces to

$$Q_\psi = \int_\Sigma q^{ab}(\hat{\nabla}_a\psi)^\dagger\hat{\nabla}_b\psi\,\mathrm{d}s, \tag{4.74}$$

which is a manifestly positive expression or zero.

Now let us try to find a relation between H_ξ and the Spinor charge Q_ψ. We consider, however, only spacetimes (M, g_{ab}) whose metric is of the form

$$g_{ab} = \bar{g}_{ab} + k_{ab}, \tag{4.75}$$

where $\bar{g}_{ab}$ is the de Sitter metric and where k_{ab} is a tensor which vanishes sufficiently fast as we go to $\mathscr{I}^+$. Additionally, we will assume that the metric of the unphysical spacetime $(\tilde{M}, \tilde{g}_{ab} = \Omega^2 g_{ab})$ satisfies (4.47) or (4.48). First, let us reexpress the super-covariant derivative of a spinor in terms of a background and a deviation part:

$$\begin{aligned}\hat{\nabla}_a\psi &= \hat{\bar{\nabla}}_a\psi + (\hat{\nabla}_a - \hat{\bar{\nabla}}_a)\psi \\ &= \hat{\bar{\nabla}}_a\psi + (\tilde{\nabla}_a - \tilde{\bar{\nabla}}_a)\psi + \frac{1}{2}\Omega^{-1}(-\tilde{\gamma}_a\tilde{\gamma}_b\tilde{n}^b + \tilde{\bar{\gamma}}_a\tilde{\bar{\gamma}}_b\tilde{\bar{n}}^b + i\tilde{\gamma}_a - i\tilde{\bar{\gamma}}_a)\psi.\end{aligned} \tag{4.76}$$

Here, the bar quantities are associated with the de Sitter metric $\bar{g}_{ab}$, the tilde quantities are associated with the unphysical metric $\tilde{g}_{ab}$, and the tilded bar quantities are associated with the unphysical de Sitter metric. The unphysical gamma matrices[21] satisfy $\tilde{g}_{ab} = \tilde{\gamma}_{(a}\tilde{\gamma}_{b)}$ and $\tilde{\gamma}_a = \Omega\gamma_a$. To further simplify (4.76), note that

$$e^\mu{}_a = \bar{e}^\mu{}_a + \frac{1}{2}k_{ab}\bar{g}^{bc}\bar{e}^\mu{}_c + O(k^2) \tag{4.77}$$

constitutes an orthonormal basis with respect to g_{ab} if $\bar{e}^\mu{}_a$ is an orthonormal basis with respect to the pure de Sitter metric. The tensor k_{ab} in (4.77) is the deviation from the de Sitter metric (see (4.75)). In a neighborhood of $\mathscr{I}^+$, it can be written as (see (4.47) and (4.48))

$$\Omega^2 k_{ab} = \tilde{k}_{ab} = -\frac{1}{d-1}\Omega^{d-1}\tilde{E}_{ab} + O(\Omega^d). \tag{4.78}$$

For the respective conformal metrics, we have

$$\tilde{e}^\mu{}_a = \tilde{\bar{e}}^\mu{}_a + \frac{1}{2}\tilde{k}_{ab}\tilde{\bar{g}}^{bc}\tilde{\bar{e}}^\mu{}_c + O(k^2). \tag{4.79}$$

Now let us define

$$\tilde{\psi} := \Omega^{1/2}\psi, \tag{4.80}$$

which we assume to be smooth on $\mathscr{I}^+$. Together with relation (B.9), we can use (4.79) to calculate the difference between two derivative operators which act on a spinor and are associated with $\tilde{\bar{g}}_{ab}$ and $\tilde{g}_{ab}$, respectively. This difference appears in the second term on the right hand side of (4.76) and is given by

$$(\tilde{\nabla}_a - \tilde{\bar{\nabla}}_a)\psi = \frac{1}{4}\tilde{g}_{bc}\bar{C}^b{}_{ad}\tilde{\gamma}^{cd}\psi + O(\Omega^{d-3/2}) = -\frac{1}{2}i\Omega^{d-5/2}\tilde{E}_{ab}\tilde{\gamma}^b\tilde{\psi} + O(\Omega^{d-3/2}), \tag{4.81}$$

[21]Note that $\tilde{\gamma}_a$ is clearly smooth on $\mathscr{I}^+$.

where $\bar{C}^a{}_{bc} = \frac{1}{2}\tilde{g}^{ad}(\tilde{\bar{\nabla}}_b\tilde{h}_{dc} + \tilde{\bar{\nabla}}_c\tilde{h}_{bd} - \tilde{\bar{\nabla}}_d\tilde{h}_{bc})$. Furthermore, we can use (4.79) to show how the gamma matrices $\tilde{\gamma}_a$ and $\tilde{\bar{\gamma}}_a$ are related to each other.

$$\tilde{\gamma}_a - \tilde{\bar{\gamma}}_a = \frac{1}{2}\tilde{h}_a{}^b\tilde{\gamma}_b + O(h^2) \tag{4.82}$$

Then, utilizing (4.81) and (4.82), we can rewrite (4.76) as

$$\hat{\nabla}_a\psi = \hat{\bar{\nabla}}_a\psi - \frac{1}{2}i\Omega^{d-5/2}\tilde{E}_{ab}\tilde{\gamma}^b\tilde{\psi} + \frac{1}{2(d-1)}\Omega^{d-5/2}\tilde{E}_a{}^b\tilde{\gamma}_b(\tilde{\gamma}_c\tilde{n}^c - i)\tilde{\psi} + O(\Omega^{d-3/2}). \tag{4.83}$$

Alternatively, this equation can be written as

$$\hat{\nabla}_a\psi = \Omega^{-1/2}\tilde{\bar{\nabla}}_a\tilde{\psi} - \frac{1}{2}i\Omega^{d-5/2}\tilde{E}_{ab}\tilde{\gamma}^b\tilde{\psi} - \frac{1}{2}\Omega^{-3/2}\tilde{\gamma}_a(\tilde{\gamma}_b\tilde{n}^b - i)\tilde{\psi} + O(\Omega^{d-3/2}). \tag{4.84}$$

To get to (4.84), we reexpressed the first term on the right hand side of (4.83) and made use of (4.82).

In [3], Kastor and Traschen have calculated a spinor ψ_0 that satisfies

$$\hat{\bar{\nabla}}_a\psi_0 = 0. \tag{4.85}$$

Such spinors are called Killing spinors. Here, ψ_0 is a Killing spinor of the de Sitter spacetime. It yields a vector ξ^a (via (B.8)), which is normal to $\mathscr{I}^+$ and satisfies $\tilde{g}_{ab}\xi^a\xi^a \restriction \mathscr{I}^+ < \infty$. Let us try to use this particular spinor in our attempt to reexpress Q_ψ.[22]

If $\psi_0 = \Omega^{-1/2}\tilde{\psi}_0$ is a Killing Spinor of the de Sitter spacetime, we can immediately conclude from (4.84) that $\tilde{\bar{\gamma}}_a(\tilde{\bar{\gamma}}_b\tilde{\bar{n}}^b - i)\tilde{\psi}_0 = O(\Omega)$, from which $\tilde{\gamma}_a(\tilde{\gamma}_b\tilde{n}^b - i)\tilde{\psi}_0 = O(\Omega)$ follows (cf. (4.82)). Hence, (4.83) reduces to

$$\hat{\nabla}_a\psi_0 = -\frac{1}{2}i\Omega^{d-2}\tilde{E}_{ab}\tilde{\gamma}^b\psi_0 + O(\Omega^{d-3/2}) \tag{4.86}$$

for a Killing spinor of the pure de Sitter spacetime.

Before we insert this expression into equation (4.71), note that we can certainly write (recall that B_{ab} is antisymmetric)

$$B_{ab}\hat{u}^b\hat{\eta}^a = B_{ab}\bar{u}^b\bar{n}^a, \tag{4.87}$$

where $\bar{n}^a \to \Omega^{-1}\tilde{n}^a$ and $\bar{u}^a \to \Omega^{-1}\tilde{u}^a$ in the limit $\Omega \to 0$. $\tilde{n}^a$ and $\tilde{u}^a$ correspond to the vectors that we defined below (4.1) only that $\tilde{u}^a$ is uniquely fixed by this condition.

If we make use of the relation $\tilde{\gamma}_{[a}\tilde{\gamma}_b\tilde{\gamma}_{c]} = \tilde{\gamma}_{[a}\tilde{\gamma}_{b]}\tilde{\gamma}_c + 2\tilde{g}_{c[a}\tilde{\gamma}_{b]}$, we can proceed to show that

$$B_{ab}\hat{u}^b\hat{\eta}^a = -\frac{1}{2}\Omega^{d-2}\bar{\tilde{\psi}}_0\tilde{\gamma}_a\tilde{\psi}_0\tilde{E}_b{}^a\tilde{u}^b, \tag{4.88}$$

which leads to

$$Q_\psi = -\int_{\partial\Sigma}\bar{\tilde{\psi}}_0\tilde{\gamma}^a\tilde{\psi}_0\tilde{E}_{ab}\tilde{u}^b\Omega^{d-2}\mathrm{d}S. \tag{4.89}$$

Now recall that $\xi^a = -\bar{\psi}_0\gamma^a\psi_0$, where the relation

$$\bar{\tilde{\psi}}_0\tilde{\gamma}^a\tilde{\psi}_0 = \bar{\psi}_0\gamma^a\psi_0 \tag{4.90}$$

[22] $\tilde{g}_{ab}\xi^a\xi^a \restriction \mathscr{I}^+ < \infty$ can be written as $\Omega^2\tilde{g}_{ab}\bar{\psi}_0\tilde{\gamma}^a\psi_0\bar{\psi}_0\tilde{\gamma}^b\psi_0 \restriction \mathscr{I}^+ < \infty$ (B.8), which implies that ψ_0 is of order $\Omega^{-1/2}$. This means that our assumption in the above paragraph was justified.

holds, since $\tilde{\bar{\psi}} = \Omega^{1/2}\bar{\psi}$. Using $\mathrm{d}S = \Omega^{2-d}\mathrm{d}\tilde{S}$, we can then rewrite (4.89) as

$$Q_\psi = \int_{\partial\Sigma} \tilde{E}_{ab}\xi^a \tilde{u}^b \mathrm{d}\tilde{S}. \tag{4.91}$$

This result seems to be very promising, because it looks like our formula for H_ξ (i.e. (4.1)) (apart from a constant prefactor). But actually, it is very unfortunate: Since the vector ξ^a is normal to $\mathscr{I}^+$, $\tilde{E}_{ab}\xi^a$ vanishes and we have

$$Q_\psi = 0. \tag{4.92}$$

At this point, our entire approach stalls. Even though we have shown that Q_ψ can be written in the same way as our formula for H_ξ (4.91), we cannot take advantage of this fact. If Q_ψ did not vanish, i.e. was positive, we could infer a positivity condition for H_ξ from (4.91). With Q_ψ vanishing, this is, however, not possible. There is no new information (regarding our problem) contained in (4.91).

Remark 24. Note that we used a Killing spinor from equation (4.85) up to equation (4.91). This spinor does not satisfy the Witten equation, which is required to hold for Q_ψ to be positive. Since the above analysis failed, we refrained from showing that a spinor can be found that satisfies the Witten equation and asymptotically approaches the Killing spinor we used in such a way that (4.91) remains unchanged.

5 Perturbation analysis of de Sitter space

In the previous chapter, we introduced a formula for conserved quantities in future asymptotically de Sitter spacetimes. But apart from the sample spacetimes we looked at in sections 2.4 and 4.3, we do not know if there really exists a wide class of future asymptotically de Sitter spacetimes. The first condition of definition 11 could in general be incompatible with the asymptotic conditions. Unfortunately, due to the non-linearities of Einstein's equation, it is difficult to analyze this issue in a straightforward manner. It is, however, possible to address it in the context of perturbation theory. In the following, we will investigate whether perturbations off of de Sitter space that satisfy the linearized Einstein equation are compatible with the asymptotic conditions.

Note that the results of this chapter were not derived by me alone. Originally, I carried the following analysis out using the form (2.31) of the metric (see appendix E). However, for the purposes of this thesis it turned out to be advantageous to work in the global chart. A key part of this analysis rests on formulas derived by Prof. Dr. Ishibashi, whom I had asked for advice.[1] He derived (5.43) and (5.47) for ψ_S and solved (5.49). I subsequently derived the differential equations (5.44) and (5.45), which can be cast into (5.47), and solved (5.50).

5.1 Weyl curvature perturbations

Note that the derivation of the master equation resembles a similar derivation in [9]. In the following, we will frequently use the global coordinates of de Sitter space (cf. section 2.4.1). In these coordinates, the de Sitter metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(\tau)(\mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\sigma^2_{d-2}), \tag{5.1}$$

where $\mathrm{d}\sigma^2_{d-2}$ is the metric of the unit $(d-2)$-sphere and $a(\tau) = \ell\cosh(\tau/\ell)$. Here, ℓ is the de Sitter radius.

Instead of working directly with metric perturbations, it is more convenient to use Weyl tensor perturbations instead. These have three advantages over metric perturbations: The Weyl tensor is conformally and gauge invariant and it is the key quantity in our formula for H_ξ (cf. (4.1)). Conformal invariance of the Weyl tensor refers to (1.7) and gauge invariance simply means that ${C_{abc}}^d = 0$ for de Sitter space. That it is the key quantity in our formula for H_ξ will enable us to immediately conclude whether a generic metric perturbation gives rise to a perturbed spacetime for which conserved quantities can be defined.

By using Einstein's equation and the Bianchi identity, we can show that Weyl perturbations in de Sitter space satisfy

$$\left(\nabla^e\nabla_e - \frac{2(d-1)}{\ell^2}\right)\delta C_{abcd} = 0, \tag{5.2}$$

$$\nabla^a \delta C_{abcd} = 0, \tag{5.3}$$

[1] Due to a miscalculation around equation (5.24), I got stuck in the derivation of (5.37)

where ∇_a is the covariant derivative operator that is compatible with the de Sitter metric g_{ab}. Now let us define

$$Y := -\sinh(\tau/\ell), \tag{5.4}$$
$$Z_a := \nabla_a Y, \tag{5.5}$$

and

$$\mathcal{E}_{ab} := \delta C_{acbd} Z^c Z^d. \tag{5.6}$$

Then, if we make use of the relations

$$\nabla_a Z_b = -\frac{Y}{\ell^2} g_{ab}, \tag{5.7}$$
$$\nabla_a \nabla_b Z_c = -\frac{1}{\ell^2} Z_a g_{bc}, \tag{5.8}$$

we can show that (5.2) and (5.3) imply

$$\left(\nabla^c \nabla_c - \frac{2(d-2)}{\ell^2}\right) \mathcal{E}_{ab} = 0, \tag{5.9}$$
$$\nabla^a \mathcal{E}_{ab} = 0. \tag{5.10}$$

We now introduce a new notation: Coordinates with capital Latin letters denote the coordinates τ and χ and coordinates with lowercase Latin letters $i, j, \ldots$ denote the angular coordinates of the $(d-2)$-dimensional subspace on which $\mathrm{d}\sigma_{d-2}^2$ acts. Then we have

$$g_{AB} \mathrm{d}y^A \mathrm{d}y^B = -\mathrm{d}\tau^2 + a^2(\tau) \mathrm{d}\chi^2 \tag{5.11}$$

and

$$\mathrm{d}\sigma_{d-2}^2 = \sigma_{ij} \mathrm{d}\theta^i \mathrm{d}\theta^j. \tag{5.12}$$

In contrast to abstract indices, these new indices denote tensor components and not the tensors themselves. We denote the derivative operator associated with g_{AB} by D_A and the derivative operator associated with σ_{ij} by $\hat{D}_i$. Then we find the following relations between these derivative operators and the derivative operator ∇_a which is associated with the de Sitter metric g_{ab}:

$$\nabla_A t_B = D_A t_B - X^C{}_{AB} t_C - X^i{}_{AB} t_i, \tag{5.13}$$
$$\nabla_A t_i = D_A t_i - X^C{}_{Ai} t_C - X^k{}_{Ai} t_k, \tag{5.14}$$
$$\nabla_i t_j = \hat{D}_i t_j - \hat{X}^C{}_{ij} t_C - \hat{X}^k{}_{ij} t_k, \tag{5.15}$$
$$\nabla_i t_A = \hat{D}_i t_A - \hat{X}^C{}_{iA} t_C - \hat{X}^k{}_{iA} t_k. \tag{5.16}$$

The connection coefficients are given by

$$X^C{}_{AB} = 0, \quad X^k{}_{ij} = 0, \quad X^i_{AB} = 0, \quad X^B{}_{Aj} = \hat{X}^B{}_{Aj} = 0, \tag{5.17}$$
$$\hat{X}^A{}_{ij} = -\frac{D^A r}{r} g_{ij}, \quad X^i{}_{Aj} = \hat{X}^i{}_{Aj} = \frac{D_A r}{r} \delta^i{}_j, \tag{5.18}$$

where

$$r = a(\tau) \sin\chi. \tag{5.19}$$

Then, if we define

$$\hat{\Delta} := \hat{D}^i \hat{D}_i, \tag{5.20}$$

we can rewrite (5.9) as

$$\begin{aligned} D^C D_C \mathcal{E}_{AB} + (d-2)\frac{D^C r}{r} D_C \mathcal{E}_{AB} - (d-2)\frac{D^C r}{r}\left(\frac{D_A r}{r}\mathcal{E}_{BC} + \frac{D_B r}{r}\mathcal{E}_{CA}\right) - \frac{2(d-2)}{\ell^2}\mathcal{E}_{AB} \\ + \frac{\hat{\Delta}}{r^2}\mathcal{E}_{AB} - 2\frac{D_A r}{r}\hat{D}_m \mathcal{E}^m{}_B - 2\frac{D_B r}{r}\hat{D}_m \mathcal{E}^m{}_A + 2\frac{(D_A r) D_B r}{r^2}\mathcal{E}^m{}_m = 0, \end{aligned} \tag{5.21}$$

$$\begin{aligned} D^C D_C \mathcal{E}_{Ai} + (d-4)\frac{D^C r}{r} D_C \mathcal{E}_{Ai} - \left(\frac{D^C D_C r}{r} + (d-3)\frac{(D^C r)(D_C r)}{r^2}\right)\mathcal{E}_{Ai} \\ - d\frac{(D_A r) D^C r}{r^2}\mathcal{E}_{Ci} + \frac{\hat{\Delta}}{r^2}\mathcal{E}_{Ai} - \frac{2(d-2)}{\ell^2}\mathcal{E}_{Ai} - 2\frac{D_A r}{r}\hat{D}_m \mathcal{E}^m{}_i + 2\frac{D^C r}{r}\hat{D}_i \mathcal{E}_{CA} = 0, \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} D^C D_C \mathcal{E}_{ij} + (d-6)\frac{D^C r}{r} D_C \mathcal{E}_{ij} - 2\left(\frac{D^C D_C r}{r} + (d-4)\frac{(D^C r)(D_C r)}{r^2}\right)\mathcal{E}_{ij} + \frac{\hat{\Delta}}{r^2}\mathcal{E}_{ij} \\ - \frac{2(d-2)}{\ell^2}\mathcal{E}_{ij} + 2\frac{D^C r}{r}(\hat{D}_i \mathcal{E}_{Cj} + \hat{D}_j \mathcal{E}_{Ci}) + 2\frac{(D^C r) D_A r}{r^2}\mathcal{E}_{CA} g_{ij} = 0. \end{aligned} \tag{5.23}$$

Similarly, (5.10) can be reexpressed as

$$D_C \mathcal{E}^C{}_A + (d-2)\frac{D_C r}{r}\mathcal{E}^C{}_A + \frac{D_A r}{r}\mathcal{E}^C{}_C + \hat{D}_m \mathcal{E}^m{}_A = 0, \tag{5.24}$$

$$\hat{D}_m \mathcal{E}^m{}_j + \frac{1}{r^{d-2}} D_A (r^{d-2}\mathcal{E}^A{}_j) = 0. \tag{5.25}$$

Now define

$$\chi^a := \left(\frac{\partial}{\partial \chi}\right)^a, \qquad \tau^a := \left(\frac{\partial}{\partial \tau}\right)^a, \tag{5.26}$$

and[2]

$$\mathcal{E} := \mathcal{E}_{AB}\chi^A \chi^B, \tag{5.27}$$

$$\mathcal{E}_i := \mathcal{E}_{Ai}\chi^A. \tag{5.28}$$

By making use of

$$D_A \chi_B = \frac{\dot{a}}{a}(\chi_A \tau_B - \tau_A \chi_B), \tag{5.29}$$

$$\frac{D_A r}{r} = -\frac{\dot{a}}{a}\tau_A + \frac{1}{a^2}\frac{1}{\tan\chi}\chi_A, \tag{5.30}$$

[2]Note that $\mathcal{E}_{AB}\tau^A = 0$. This is because $Z^a = (a/\ell^2)\tau^a$.

where " $\dot{}$ " denotes the derivative with respect to τ, we can show that (5.24) implies

$$\chi^A \hat{D}_m \mathcal{E}^m{}_A = -\frac{1}{a^2}\left[\chi^C D_C \mathcal{E} + (d-1)\frac{1}{\tan\chi}\mathcal{E}\right], \tag{5.31}$$

$$\mathcal{E}^C{}_C = \frac{1}{a^2}\mathcal{E}, \tag{5.32}$$

and (5.25) implies

$$\hat{D}_m \mathcal{E}^m{}_i + (d-2)\frac{1}{a^2}\frac{1}{\tan\chi}\mathcal{E}_i + \frac{1}{a^2}\chi^C D_C \mathcal{E}_j = 0. \tag{5.33}$$

Furthermore, note that

$$\chi^A \chi^B D_C \mathcal{E}_{AB} = D_C \mathcal{E} + 2\frac{\dot{a}}{a}\tau_C \mathcal{E}, \tag{5.34}$$

$$\chi^A D^C D_C \mathcal{E}_{Aj} = D^C D_C \mathcal{E}_j + 2\frac{\dot{a}}{a}\tau^C D_C \mathcal{E}_j + \frac{1}{\ell^2}\mathcal{E}_j, \tag{5.35}$$

$$\chi^A \chi^B D^C D_C \mathcal{E}_{AB} = D^C D_C \mathcal{E} + 4\frac{\dot{a}}{a}\tau^C D_C \mathcal{E} + \frac{2}{a^2}\mathcal{E}. \tag{5.36}$$

Using these equations and inserting (5.31) and (5.32) into (5.21) gives

$$\begin{aligned} D^C D_C \mathcal{E} &+ \left[-(d-6)\frac{\dot{a}}{a}\tau^C + (d+2)\frac{1}{a^2}\frac{1}{\tan\chi}\chi^C\right] D_C \mathcal{E} \\ &+ \left[-2\frac{\dot{a}}{a^2} + 2(d-2)\frac{\dot{a}^2}{a^2} + \frac{2(d-1)}{a^2}\frac{1}{\tan^2\chi} - \frac{2(d-2)}{\ell^2} + \frac{\hat{\Delta}}{r^2}\right]\mathcal{E} = 0. \end{aligned} \tag{5.37}$$

Similarly, by inserting (5.33) into (5.22), we obtain

$$\begin{aligned} D^C D_C \mathcal{E}_i &+ \left[-(d-6)\frac{\dot{a}}{a}\tau^C + (d-2)\frac{1}{a^2}\frac{1}{\tan\chi}\chi^C\right] D_C \mathcal{E}_i \\ &+ \left[(d-4)\frac{\dot{a}^2}{a^2} + \frac{\hat{\Delta}-(d-3)}{r^2} + (d-4)\frac{1}{a^2}\frac{1}{\tan^2\chi} - 2\frac{(d-2)}{\ell^2} + \frac{d}{\ell^2}\right]\mathcal{E}_i \\ &= -2\frac{1}{a^2}\frac{1}{\tan\chi}\hat{D}_i \mathcal{E}. \end{aligned} \tag{5.38}$$

Just simplifying (5.23) gives

$$\begin{aligned} D^C D_C \mathcal{E}_{ij} &+ (d-6)\left[-\frac{\dot{a}}{a}\tau^C + \frac{1}{a^2}\frac{1}{\tan\chi}\chi^C\right] D_C \mathcal{E}_{ij} + \frac{\hat{\Delta}-2(d-4)}{r^2} \\ &= -2\frac{1}{a^2}\frac{1}{\tan\chi}(\hat{D}_i \mathcal{E}_j + \hat{D}_j \mathcal{E}_i) - 2\frac{1}{a^4}\frac{1}{\tan^2\chi}\mathcal{E} g_{ij}. \end{aligned} \tag{5.39}$$

5.2 Master equation

The above differential equations are coupled to each other. It is possible to decouple them if we make use of certain results of [9] (also [26] and [27]). Consider the harmonic functions $\mathbb{S}_{\mathbf{k}}$, vectors $\mathbb{V}_{\mathbf{k}i}$, and symmetric tensors $\mathbb{T}_{\mathbf{k}ij}$ which are defined by the eigenvalue equations

$$\begin{aligned}
&(\hat{\Delta} + \mathbf{k}_S^2)\mathbb{S}_{\mathbf{k}} = 0, \\
&(\hat{\Delta} + \mathbf{k}_V^2)\mathbb{V}_{\mathbf{k}j} = 0, \quad \hat{D}_i \mathbb{V}_{\mathbf{k}}{}^i = 0, \\
&(\hat{\Delta} + \mathbf{k}_T^2)\mathbb{T}_{\mathbf{k}ij} = 0, \quad \hat{D}_i \mathbb{T}_{\mathbf{k}}{}^i{}_j = 0, \quad \mathbb{T}_{\mathbf{k}}{}^i{}_i = 0.
\end{aligned} \tag{5.40}$$

We can then expand $\mathcal{E}$, $\mathcal{E}_i$, and $\mathcal{E}_{ij}$ in terms of these harmonics as follows [26, 9]:

$$\begin{aligned}
\mathcal{E} &= \psi_S \mathbb{S}, \\
\mathcal{E}_i &= \phi_S \hat{D}_i \mathbb{S} + \psi_V \mathbb{V}_i, \\
\mathcal{E}_{ij} &= \mathcal{E}_L \sigma_{ij} \mathbb{S} + \mathcal{E}_T \left(\hat{D}_i \hat{D}_j - \frac{1}{d-2} \hat{\Delta} \sigma_{ij} \right) \mathbb{S} + \mathcal{E}_V \hat{D}_{(i} \mathbb{V}_{j)} + \psi_T \mathbb{T}_{ij}.
\end{aligned} \tag{5.41}$$

Here we omitted the indices $\mathbf{k}$ which label the different eigenvalues of the different harmonics and the summation symbol $\Sigma_{\mathbf{k}}$ over them. It can be shown that there exist solutions of (5.40) for eigenvalues $\mathbf{k}_S^2 = l(l+d-3)$, $l = 0, 1, \ldots$ and $\mathbf{k}_V^2 = l(l+d-3) - 1$, $l = 1, 2, \ldots$ and $\mathbf{k}_T^2 = l(l+d-3) - 2$, $l = 2, 3, \ldots$ [26].

The coefficients ψ_S, ϕ_S, ψ_V, $\mathcal{E}_L$, $\mathcal{E}_T$, $\mathcal{E}_V$, ψ_T are not independent of each other: They are related by (5.31), (5.32), and (5.33). It can be shown that (i) $\mathcal{E}_L$ is described by ψ_S; (ii) ϕ_S is described in terms of (ψ_S, $\partial_\chi \psi_S$); (iii) $\mathcal{E}_T$ is described by ($\mathcal{E}_L$, ϕ_S, $\partial_\chi \phi_S$); (iv) $\mathcal{E}_V$ is described in terms of (ψ_V, $\partial_\chi \psi_V$) (cf. appendix C.2). Consequently, if we know ψ_S, ψ_V, and ψ_T, we know all the expansion coefficients.[3]

Noting that

$$D^C D_C \psi = -\frac{\partial^2}{\partial \tau^2}\psi - \frac{\dot{a}}{a}\frac{\partial}{\partial \tau}\psi + \frac{1}{a^2}\frac{\partial^2}{\partial \chi^2}\psi, \tag{5.42}$$

we find the following decoupled differential equations for ψ_S, ψ_V, and ψ_T by inserting (5.41) into (5.37), (5.38), and (5.39):

$$\left[-\frac{\partial^2}{\partial \tau^2} - (d-5)\frac{\dot{a}}{a}\frac{\partial}{\partial \tau} - \frac{4(d-2)}{a^2} \right] \psi_S + \frac{1}{a^2}\left[\frac{\partial^2}{\partial \chi^2} + (d+2)\frac{\cos\chi}{\sin\chi}\frac{\partial}{\partial \chi} + \frac{2(d-1) - \mathbf{k}_S^2}{\sin^2\chi} \right] \psi_S = 0, \tag{5.43}$$

$$\left[-\frac{\partial^2}{\partial \tau^2} - (d-5)\frac{\dot{a}}{a}\frac{\partial}{\partial \tau} \right] \psi_V + \frac{1}{a^2}\left[\frac{\partial^2}{\partial \chi^2} + (d-2)\frac{\cos\chi}{\sin\chi}\frac{\partial}{\partial \chi} - \frac{1 + \mathbf{k}_V^2}{\sin^2\chi} - 2(d-4) \right] \psi_V = 0, \tag{5.44}$$

and

$$\left[-\frac{\partial^2}{\partial \tau^2} - (d-5)\frac{\dot{a}}{a}\frac{\partial}{\partial \tau} \right] \psi_T + \frac{1}{a^2}\left[\frac{\partial^2}{\partial \chi^2} + (d-6)\frac{\cos\chi}{\sin\chi}\frac{\partial}{\partial \chi} - \frac{2(d-4) + \mathbf{k}_T^2}{\sin^2\chi} \right] \psi_T = 0. \tag{5.45}$$

Now let us define

$$\begin{aligned}
\psi_S &=: a(\tau)^{-(d-5)/2} (\sin\chi)^{-(d+2)/2} \Psi_S, \\
\psi_V &=: a(\tau)^{-(d-5)/2} (\sin\chi)^{-(d-2)/2} \Psi_V, \\
\psi_T &=: a(\tau)^{-(d-5)/2} (\sin\chi)^{-(d-6)/2} \Psi_T.
\end{aligned} \tag{5.46}$$

[3] The tensor harmonics $\mathbb{T}_{ij}$ have $d(d-4)/2$ independent components, the vector harmonics $\mathbb{V}_i$ have $d-3$ independent components, and the scalar harmonics $\mathbb{S}$ have one independent component. This corresponds to the number of dynamical degrees of freedom for gravitational radiation in d-dimensional spacetimes. Therefore, ψ_S, ψ_V, and ψ_T describe all the dynamical modes of gravitational perturbations [9].

If we use these relations to rewrite (5.42), (5.43), and (5.44), we find that these equations are cast into the same form. This master equation is given by

$$a(\tau)^2\left[\frac{\partial^2}{\partial\tau^2}-\frac{(d-5)^2}{4\ell^2}\right]\Psi=\left[\frac{\partial^2}{\partial\chi^2}-\left\{\frac{(d-2)(d-4)}{4}+l(l+d-3)\right\}\frac{1}{\sin^2\chi}+\frac{1}{4}\right]\Psi \tag{5.47}$$

where Ψ_S, Ψ_V, and Ψ_V are collectively denoted by Ψ.

Before we set out to solve this equation, let us recapitulate what we have done so far. We began by deriving two equations ((5.2) and (5.3)), which govern the behavior of perturbed Weyl tensors that obey the linearized Einstein equation. These equations implied the simpler equations (5.9) and (5.10), which we subsequently rewrote in terms of particular derivative operators and global coordinates. This finally yielded (5.37), (5.38), and (5.39). However, these were still coupled differential equations, which were too complicated to be solved directly. By expanding the fields $\mathcal{E}$, $\mathcal{E}_i$, and $\mathcal{E}_{ij}$ in terms of certain harmonic functions, we were, however, able decouple these differential equations. The last step of our analysis was to show that these decoupled equations could be written as a single master equation (5.47). This means that the solutions of this master equation provide solutions for the linearized equations (5.2) and (5.3) we started with.

5.3 General solutions of the master equation

Now let us try to find solutions of the master equation (5.47). To that end, we make the following ansatz: Let

$$\Psi=T(\tau)X(\chi) \tag{5.48}$$

and insert this into the master equation. This yields two ordinary differential equations, namely

$$\left[-\frac{\partial^2}{\partial\chi^2}+\left\{\frac{(d-2)(d-4)}{4}+l(l+d-3)\right\}\frac{1}{\sin^2\chi}\right]X=\omega^2X, \tag{5.49}$$

$$\left[-\frac{\partial^2}{\partial\tau^2}+\frac{-\omega^2+1/4}{a^2}+\frac{(d-5)^2}{4\ell^2}\right]T=0, \tag{5.50}$$

where ω^2 is a separation constant.

If we define $X =: (\sin\chi)^{\nu+1/2}(\cos\chi)\bar{X}$, $z := \cos^2\chi$, and

$$\zeta^{\omega}_{\nu,\sigma} := \frac{\omega+\nu+\sigma+1}{2}, \tag{5.51}$$

(5.49) can be cast into a hypergeometric differential equation. More precisely, it can be written as

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{X}+\left[\frac{3}{2}-(\zeta^{\omega}_{\nu,1/2}+\zeta^{-\omega}_{\nu,1/2}+1)z\right]\frac{\mathrm{d}}{\mathrm{d}z}\bar{X}-\zeta^{\omega}_{\nu,1/2}\zeta^{-\omega}_{\nu,1/2}\bar{X}=0, \tag{5.52}$$

where

$$\nu := l+\frac{d-3}{2}. \tag{5.53}$$

Hence, any solution of (5.49) is a linear combination of the following two equations [28]:

$$X_1=(\sin\chi)^{\nu+1/2}(\cos\chi)\cdot F(\zeta^{\omega}_{\nu,1/2},\zeta^{-\omega}_{\nu,1/2};3/2;\cos^2\chi), \tag{5.54}$$

$$X_2=(\sin\chi)^{\nu+1/2}\cdot F(\zeta^{\omega}_{\nu,-1/2},\zeta^{-\omega}_{\nu,-1/2};1/2;\cos^2\chi) \tag{5.55}$$

Here, F is the hypergeometric function (see appendix D). The hypergeometric functions in the equations above are not well defined[4] at $\chi = 0$ and $\chi = \pi$, which can be easily shown with the methods outlined in appendix D. However, it is possible that the functions X_1 and X_2 are well defined. If the first two arguments of the hypergeometric functions in (5.54) and (5.55) are not negative integers, we can rewrite these equations as (see (D.4))

$$X_1 = (\sin\chi)^{1/2-\nu}(\cos\chi) \cdot F(\zeta^{-\omega}_{-\nu,1/2}, \zeta^{\omega}_{-\nu,1/2}; 3/2; \cos^2\chi), \tag{5.56}$$

$$X_2 = (\sin\chi)^{1/2-\nu} \cdot F(\zeta^{-\omega}_{-\nu,-1/2}, \zeta^{\omega}_{-\nu,-1/2}; 1/2; \cos^2\chi). \tag{5.57}$$

Then it turns out that X_1 as well as X_2 do not converge at 0 and π: The hypergeometric functions in these expressions converge for all $\chi \in [0, \pi]$ but we have[5] $\nu > 1/2$, which implies that the sine terms diverge. Hence, X_1 is only normalizable, i.e. well defined for all $\chi \in [0, \pi]$, if $\zeta^{\omega}_{\nu,1/2}$ or $\zeta^{-\omega}_{\nu,1/2}$ is a negative integer, which is equivalent to

$$\omega = \pm(2m + \nu + 3/2), \tag{5.58}$$

where $m \in \mathbb{N}_0$. Similarly, X_2 is normalizable only if

$$\omega = \pm(2m + \nu + 1/2). \tag{5.59}$$

A normalizable solution of (5.49) for other values of ω and ν might be given by a linear combination of (5.54) and (5.55). And indeed, another solution of (5.49) is given by [28]

$$X_3 = (\sin\chi)^{\nu+1/2}(\cos\chi) \cdot F(\zeta^{\omega}_{\nu,1/2}, \zeta^{-\omega}_{\nu,1/2}; 1 + \nu; \sin^2\chi), \tag{5.60}$$

which can be rewritten as (again, cf. (D.4))

$$X_3 = (\sin\chi)^{\nu+1/2} \cdot F(\zeta^{-\omega}_{\nu,-1/2}, \zeta^{\omega}_{\nu,-1/2}; 1 + \nu; \sin^2\chi), \tag{5.61}$$

in case the first two arguments of the hypergeometric function in (5.60) are not negative integers (recall that $\nu > 0$). Equation (5.60) is normalizable for all possible values of ω and ν. It clearly is finite if $\zeta^{\omega}_{\nu,1/2}$ or $\zeta^{-\omega}_{\nu,1/2}$ are negative integers, and, as we can read off (5.61), it is also finite for arbitrary other values of ω and ν.

Now let us turn our attention to solving (5.50). Again, it is possible to cast this equation into the form of the hypergeometric differential equation. This time, we define $T =: (\sinh\tau/\ell)(\cosh\tau/\ell)^{\omega+1/2}\bar{T}$, which enables us to write (5.50) as

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{T} + [1 + \omega - (1 + \zeta^{\omega}_{\mu,1/2} + \zeta^{\omega}_{-\mu,1/2})z]\frac{\mathrm{d}}{\mathrm{d}z}\bar{T} - \zeta^{\omega}_{\mu,1/2}\zeta^{\omega}_{\mu,1/2}\bar{T} = 0, \tag{5.62}$$

where $z := \cosh^2\tau/\ell$ and

$$\mu := \frac{d-5}{2}. \tag{5.63}$$

[4]We are not interested in solutions that are not well defined for all $\chi \in [0, \pi]$, because these cannot give rise to asymptotically de Sitter spacetimes (to first order).

[5]This does not hold for the mode which is associated with $\mathbf{k}_S = 0$ (see (5.40)).

The solutions of this equation depend drastically on the dimension d. In the following, we have to distinguish between three different cases. If d is even, two linearly independent solutions of equation (5.50) are

$$T_1 = (\tanh\tau/\ell)(\cosh\tau/\ell)^{-\mu}\cdot F\left(\zeta^{\omega}_{\mu,1/2},\zeta^{-\omega}_{\mu,1/2},1+\mu;\frac{1}{\cosh^2\tau/\ell}\right), \tag{5.64}$$

$$T_2 = (\tanh\tau/\ell)(\cosh\tau/\ell)^{+\mu}\cdot F\left(\zeta^{\omega}_{-\mu,1/2},\zeta^{-\omega}_{-\mu,1/2},1-\mu;\frac{1}{\cosh^2\tau/\ell}\right), \tag{5.65}$$

because neither $1+\mu$ nor $1-\mu$ can be a negative integer or zero. However, if d is odd, T_2 is either not well defined or not linearly independent of T_1. Thus, we need other second solutions in these cases.

First, let us have a look at $d=5$. In this case, we have $1-\mu=1+\mu=1$, which implies that $T_1=T_2$. Hence, we are left with only one solution, namely

$$T_1 = (\tanh\tau/\ell)\cdot F\left(\zeta^{\omega}_{0,1/2},\zeta^{-\omega}_{0,1/2},1;\frac{1}{\cosh^2\tau/\ell}\right). \tag{5.66}$$

We can show that a linearly independent second solution is given by

$$\begin{aligned}T_2 = (\tanh\tau/\ell)\Bigg[&-2F\left(\zeta^{\omega}_{0,1/2},\zeta^{-\omega}_{0,1/2},1;\frac{1}{\cosh^2\tau/\ell}\right)\log(\cosh\tau/\ell)\\&+\sum_{n=1}^{\infty}\frac{(\zeta^{\omega}_{0,1/2})_n(\zeta^{-\omega}_{0,1/2})_n}{(n!)^2}\left(\frac{1}{\cosh\tau/\ell}\right)^{2n}\Big\{\psi(\zeta^{\omega}_{0,1/2}+n)-\psi(\zeta^{\omega}_{0,1/2})\\&+\psi(\zeta^{-\omega}_{0,1/2}+n)-\psi(\zeta^{-\omega}_{0,1/2})-2\psi(n+1)+2\psi(1)\Big\}\Bigg],\end{aligned} \tag{5.67}$$

where $(\alpha)_n=\Gamma(\alpha+n)/\Gamma(\alpha)$ is the Pochhamer symbol (see (D.2)) and where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function.

For odd d with $d>5$, $1-\mu$ becomes a negative integer or zero, in which case (5.65) is indeterminate. Consequently, we again need another second solution, which can be shown to be

$$\begin{aligned}T_2 = (\tanh\tau/\ell)(\cosh\tau/\ell)^{-\mu}\Bigg[&-2F\left(\zeta^{\omega}_{\mu,1/2},\zeta^{-\omega}_{\mu,1/2},1+\mu;\frac{1}{\cosh^2\tau/\ell}\right)\log(\cosh\tau/\ell)\\&+\sum_{n=1}^{\infty}\frac{(\zeta^{\omega}_{\mu,1/2})_n(\zeta^{-\omega}_{\mu,1/2})_n}{(1+\mu)_n n!}\left(\frac{1}{\cosh\tau/\ell}\right)^{2n}\Big\{\psi(\zeta^{\omega}_{\mu,1/2}+n)-\psi(\zeta^{\omega}_{\mu,1/2})+\psi(\zeta^{-\omega}_{\mu,1/2}+n)\\&-\psi(\zeta^{-\omega}_{\mu,1/2})-\psi(\mu+n+1)+\psi(\mu+1)-\psi(n+1)+\psi(1)\Big\}\\&-\sum_{n=1}^{\mu}\frac{(n-1)!(-\mu)_n}{(1-\zeta^{\omega}_{\mu,1/2})_n(1-\zeta^{-\omega}_{\mu,1/2})_n}(\cosh\tau/\ell)^{2n}\Bigg].\end{aligned} \tag{5.68}$$

The solutions T_1 are smooth on the entire physical spacetime manifold. This is also true for T_2 in even dimensions. The second solutions in odd dimensions are smooth on the entire physical spacetime manifold as well, unless $\zeta^{\pm\omega}_{\mu,1/2}$ evaluates to zero or a negative integer. In that case, the solutions are nowhere well defined.

We are mainly interested in solutions that result in normalizable Ψ's. In the following, we will therefore disregard the non-normalizable solutions and we will assume that Ψ is smooth on the spacetime manifold.

5.4 Asymptotic behavior of the solutions of the master equation

For our purposes, we do not need the exact behavior of the solutions of the previous section on the entire spacetime manifold (as long as they are smooth there). What matters to us is the behavior of these solutions on $\mathscr{I}^+$. The asymptotic behavior[6] of Ψ, which we denote by the symbol "$\sim$", can be immediately read off the above equations. Note that we will disregard prefactors when we talk about the asymptotic behavior. For $d \neq 5$, we have

$$\Psi \sim \begin{cases} a^{-\mu} \\ a^{+\mu} \end{cases} \tag{5.69}$$

whereas

$$\Psi \sim \begin{cases} \text{const.} \\ \tau \end{cases} \tag{5.70}$$

in $d = 5$ dimensions. (Recall that $a = \ell \cosh(\tau/\ell)$.) The curly braces in the above formulas denote the two possible distinct asymptotic behaviors of the solutions of the master equation. Now, since (see (5.46))

$$\psi_S,\, \psi_V,\, \psi_T \propto a^{-(d-5)/2}\Psi, \tag{5.71}$$

the asymptotic behavior of Ψ results in the following asymptotic behavior of the expansion coefficients: For $d \neq 5$, we have

$$\psi_S,\, \psi_V,\, \psi_T \sim \begin{cases} \text{const.} \\ \Omega^{d-5} \end{cases} \tag{5.72}$$

and for $d = 5$, we find

$$\psi_S,\, \psi_V,\, \psi_T \sim \begin{cases} \text{const.} \\ \log\Omega \end{cases}, \tag{5.73}$$

where $\Omega = a^{-1}$. Ω corresponds to the conformal factor (2.28) of the de Sitter metric.

From this we get the asymptotic behavior of certain Weyl tensor components via (5.41).[7] If we recall that $Z^a = (a/\ell^2)\tau^a$, we first find

$$\delta C_{\chi\tau\chi\tau} \propto \Omega^2 \mathcal{E}_{ab}\chi^a\chi^b = \Omega^2\mathcal{E} = \Omega^2\psi_S\mathbb{S}. \tag{5.74}$$

Now note that the other expansion coefficients that we introduced in (5.41) exhibit the same asymptotic behavior as (5.72) and (5.73) (see appendix C.2). Hence, we find

$$\delta C_{\chi\tau i\tau} \propto \Omega^2 \mathcal{E}_{ab}\chi^a\theta_i^b = \Omega^2\mathcal{E}_i \sim \Omega^2\psi_V\mathbb{V}_i \tag{5.75}$$

as well as

$$\delta C_{i\tau j\tau} \propto \Omega^2 \mathcal{E}_{ab}\theta_i^a\theta_j^b = \Omega^2\mathcal{E}_{ij} \sim \Omega^2\psi_T\mathbb{T}_{ij}. \tag{5.76}$$

Now it is of course possible that, for instance, the asymptotic behavior of ψ_S dominates the one of ψ_V, in which case we would have $\delta C_{\chi\tau i\tau} \sim \psi_S\hat{D}_i\mathbb{S}$. But this does not give us new possible asymptotic behaviors, which is why we did not explicitly write this down. For the same reason, we wrote the asymptotic behavior of $\delta C_{i\tau j\tau}$ as in (5.76).

[6]That is the behavior as $\tau \to \infty$.

[7]Recall that $\mathcal{E}_{ab}$ is related to the perturbed Weyl tensor by (5.6).

With these results, we can finally analyze the asymptotic behavior of (4.1), i.e.

$$H_\xi = \frac{\ell}{8\pi G}\int_C \tilde{E}_{ab}\tilde{u}^a\xi^b \mathrm{d}\tilde{S} \tag{5.77}$$

for perturbations off of de Sitter space that satisfy the linearized Einstein equation. In our case and for our choice of the conformal factor $\Omega = a^{-1}$, the electric part of the unphysical Weyl tensor $\tilde{E}_{ab}$ can be written as

$$\tilde{E}_{ab} = \frac{\ell^2}{d-3}\Omega^{3-d}\delta\tilde{C}_{acbd}\tilde{n}^b\tilde{n}^d = \frac{1}{d-3}\Omega^{3-d}\delta C_{a\tau b\tau}, \tag{5.78}$$

where $\tilde{n}^a = (\Omega\ell)^{-1}\tau^a$. Both vector fields ξ^a and $\tilde{u}^a$ of the integrand $\tilde{E}_{ab}\tilde{u}^a\xi^b$ of (5.77) are tangent to $\mathscr{I}^+$ and of finite norm. Thus, they must be given by a linear combination of χ^a and $\theta^a_i, i = 1, \ldots, d-2$, which implies that the integrand is of the form $\lambda\tilde{E}_{\chi\chi} + \mu^i\tilde{E}_{\chi i} + \nu^i_1\nu^j_2\tilde{E}_{ij}$, where the coefficients are smooth functions on $\mathscr{I}^+$. It follows from (5.74) - (5.76) and (5.78) that

$$\tilde{E}_{\chi\chi}\,\tilde{E}_{\chi i},\,\tilde{E}_{ij} \sim \begin{cases}\text{const.}\\ \Omega^{-(d-5)}\end{cases} \tag{5.79}$$

in $d \neq 5$ dimensions and

$$\tilde{E}_{\chi\chi}\,\tilde{E}_{\chi i},\,\tilde{E}_{ij} \sim \begin{cases}\text{const.}\\ \log\Omega\end{cases} \tag{5.80}$$

in $d = 5$ dimensions.

Remark 25. Note that it is possible to calculate all the components of the (perturbed) Weyl tensor from $\delta C_{\chi\tau\chi\tau}$, $\delta C_{\chi\tau i\tau}$, and $\delta C_{i\tau j\tau}$ by making use of equation (C.3) (the Bianchi identity combined with Einstein's equation) and the symmetries of the Weyl tensor (see section 1.2). Then it can be shown that the unphysical Weyl tensor exhibits the same asymptotic behavior as $\delta\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d$ (see [9]).

5.5 Conclusion

We will assume here that $\tilde{E}_{ab}$ is smooth on the physical spacetime manifold (cf. section 5.3). As we can read off (5.79) and (5.80), different perturbations may then have different asymptotic behaviors.

In the most interesting case, $d = 4$, $\tilde{E}_{ab}$ is smooth on $\mathscr{I}^+$ for both asymptotic behaviors of (5.79). This means that a generic metric perturbation in 4 dimensions which is smooth on the physical spacetime manifold gives rise to spacetimes for which quantities H_ξ can be defined. Considering that the smoothness of $\tilde{E}_{ab}$ implies the behavior (4.41) of the Weyl tensor (cf. remark 25), H_ξ is a conserved quantity in the sense of section 4.1.[8]

In $d \geq 5$, $\tilde{E}_{ab}$ can be either smooth on $\mathscr{I}^+$ or not well defined there. As in 4 dimensions, the solutions that result in smooth electric Weyl tensors correspond to spacetimes that admit quantities H_ξ. On the other hand, if $\tilde{E}_{ab}$ is not smooth on $\mathscr{I}^+$, we can immediately preclude the associated spacetimes from being future asymptotically de Sitter. This is due to the fact that

[8] We have not explicitly shown that spacetimes which admit quantities H_ξ are necessarily future asymptotically de Sitter. Here, it is, however, a very fair assumption that this holds.

this behavior is not consistent with a direct consequence of the definition of future asymptotically de Sitter: According to (4.41), an unphysical Weyl tensor of a future asymptotically de Sitter spacetime behaves like $O(\Omega^{d-3})$ in a neighborhood of $\mathscr{I}^+$.[9]

Consequently, we have shown that, for any dimension, there exist metric perturbations whose corresponding spacetimes allow for the definition of H_ξ's. In $d = 4$, this is even possible for all our solutions. This indicates that there probably exists a wide class of spacetimes (solutions to the full Einstein equation) for which quantities H_ξ can be defined and which are future asymptotically de Sitter.

However, we need to keep in mind that our results hold only to first order: We worked with the linearized field equation instead of the full one, which means that the question of linearization stability remains open. Moreover, we must not forget that we had to consider solutions with distinct asymptotic behaviors. In the non-linearized case, we would expect a generic solution (that corresponds to our linearized solutions) to have an asymptotic behavior that is a mixture of the behaviors of (5.79) (or (5.80)). This means, in particular in $d \geq 5$, that a generic $\tilde{E}_{ab}$ will probably not be smooth everywhere on a boundary of the spacetime. However, $\tilde{E}_{ab}$ will be smooth on subsets of such a boundary, which then might correspond to a $\mathscr{I}^+$.

[9]However, $\tilde{E}_{ab}$ is not necessarily not smooth on all of $\mathscr{I}^+$. Appropriate superpositions of solutions (5.48) could, for instance, lead to perturbations that are compactly supported on $\mathscr{I}^+$. In that case, our formula might again be applicable to the respective spacetimes, because $\mathscr{I}^+$ could be chosen to not include the set on which $\tilde{E}_{ab}$ is not smooth.

Summary and outlook

In this thesis, we used a Hamiltonian framework - Wald's covariant phase space formalism - to construct conserved quantities for (future) asymptotically de Sitter spacetimes.[10] We investigated these quantities by explicitly deriving their conservation properties and we demonstrated them by applying our method to the Schwarzschild de Sitter and Tolman-Bondi spacetime.

We also tried to show that there exists a particular prescription for calculating conserved quantities within our formalism that always yields positive expressions. Our hope was that such a prescription would lead to a viable notion of mass of a spacetime. However, our analysis stalled at a certain point and we failed in finding such a prescription.

The last part of this thesis was concerned with the question of whether our definition of the term (future) asymptotically de Sitter is too strict to admit a wide class of spacetimes. We approached this question with a perturbative analysis of de Sitter space. Even though this analysis did not imply the existence of a wide class of (future) asymptotically de Sitter spacetimes, it considerably supported the assumption that such a wide class exists.

To further extend the understanding and validity of the conserved quantities (4.1) and the approach of this thesis, the following things could be done:

- The question of whether there exists a prescription for positive conserved quantities could be further pursued. As already said, this may lead to a viable notion of mass.
- Second, our definition of asymptotically de Sitter excludes spacetimes with non-vanishing stress-energy tensors. It should, however, be possible to extend the framework of this thesis to certain non-vacuum spacetimes.
- Third, it is not clear how our asymptotic conditions are related to the ones that have been used in other publications (cf. e.g. remark 5). This information is necessary to make an extensive comparison between H_ξ and other conserved quantities that rely on these asymptotic conditions possible. It would be particularly interesting to compare our conserved quantities to the ones of Abbott and Deser [1]. Their quantities are probably the most prominent and referenced ones.

[10] Strictly speaking, we only used this framework to construct conserved quantities for some covariant phase space. The extension to all (future) asymptotically de Sitter spacetimes was done by explicitly checking the conservation properties of these quantities.

Appendices

A Formulas needed in the construction of the Hamiltonian

A.1 Curvature quantities of conformally transformed metrics

(i) The difference between two derivative operators acting on a one form ω_c can be written as[1]

$$\tilde{\nabla}_b\omega_c - \nabla_b\omega_c = C^d{}_{bc}\omega_d. \tag{A.1}$$

If ∇_a is associated with g_{ab} and $\tilde{\nabla}_a$ is associated with $\tilde{g}_{ab} = \Omega^2 g_{ab}$, we find

$$C^c{}_{ab} = -\Omega^{-1}(\delta_b{}^c\tilde{\nabla}_a\Omega + \delta_a{}^c\tilde{\nabla}_b\Omega - \tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_d\Omega). \tag{A.2}$$

Applying another derivative operator ∇_a to (A.1) gives

$$\begin{aligned}\nabla_a\nabla_b\omega_c = \tilde{\nabla}_a\tilde{\nabla}_b\omega_c - C^d{}_{ab}\tilde{\nabla}_d\omega_c - C^d{}_{ac}\tilde{\nabla}_b\omega_d - (\tilde{\nabla}_a C^d{}_{bc})\omega_d \\ + C^e{}_{ab}C^d{}_{ec}\omega_d + C^e{}_{ac}C^d{}_{be}\omega_d - C^d{}_{bc}\tilde{\nabla}_a\omega_d.\end{aligned} \tag{A.3}$$

Now recall the definition of the Riemann tensor[2] (see chapter "Notations and conventions" on page 6):

$$R_{abc}{}^d\omega_d = \nabla_a\nabla_b\omega_c - \nabla_b\nabla_a\omega_c \tag{A.4}$$

We can insert (A.3) into this equation, which yields

$$R_{abc}{}^d\omega_d = (\tilde{R}_{abc}{}^d - 2\tilde{\nabla}_{[a}C^d{}_{b]c} + 2C^e{}_{c[a}C^d{}_{b]e})\omega_d. \tag{A.5}$$

Contracting over b and d and using (A.2) gives the Ricci tensor

$$R_{ac} = \tilde{R}_{ac} + (d-2)\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_c\Omega + \tilde{g}_{ac}\Omega^{-1}\tilde{\nabla}^d\tilde{\nabla}_d\Omega - \tilde{g}_{ac}(d-1)\Omega^{-2}\tilde{\nabla}^d\Omega\tilde{\nabla}_d\Omega, \tag{A.6}$$

and contracting over the remaining indices gives the Ricci scalar

$$R = \Omega^2\tilde{R} + 2(d-1)\Omega^1\tilde{\nabla}^c\tilde{\nabla}_c\Omega - d(d-1)\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega. \tag{A.7}$$

These two equations relate the Ricci tensor and Ricci scalar of some spacetime to the respective quantities of an associated unphysical spacetime.

---

[1] For tensors of higher order, we have

$$\nabla_a T^{b_1\ldots b_k}{}_{c_1\ldots c_l} = \tilde{\nabla}_a T^{b_1\ldots b_k}{}_{c_1\ldots c_l} - \sum_{i=1}^{l} C^d{}_{ac_i}T^{b_1\ldots b_k}{}_{c_1\ldots d\ldots c_l} + \sum_{j=1}^{k} C^{b_j}{}_{ad}T^{b_1\ldots d\ldots b_k}{}_{c_1\ldots c_l}$$

[2] and

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)T^{c_1\ldots c_k}{}_{d_1\ldots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i}T^{c_1\ldots e\ldots c_k}{}_{d_1\ldots d_l} + \sum_{j=1}^{l} R_{abd_j}{}^{e}T^{c_1\ldots c_k}{}_{d_1\ldots e\ldots d_l}$$

(ii) Inserting the Ricci scalar $R = d(d-1)/\ell^2$ (i.e. the trace of Einstein's equation with a positive cosmological constant (see (2.22))) into (A.7) gives

$$\tilde{R} = -2(d-1)\Omega^{-1}\tilde{\nabla}^a\tilde{\nabla}_a\Omega + d(d-1)\Omega^{-2}(\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega + \ell^{-2}), \tag{A.8}$$

where the de Sitter radius ℓ is given by (2.25). For spacetimes without a cosmological constant, the term containing ℓ drops out. Similarly, if we insert Einstein's equation (2.22) into (A.6), we find

$$\tilde{R}_{ac} = -(d-2)\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_c\Omega - \tilde{g}_{ac}\Omega^{-1}\tilde{\nabla}^d\tilde{\nabla}_d\Omega + \tilde{g}_{ac}(d-1)\Omega^{-2}(\tilde{\nabla}^d\Omega\tilde{\nabla}_d\Omega + \ell^{-2}). \tag{A.9}$$

Now let us define

$$\tilde{S}_{ab} := \frac{2}{d-2}\tilde{R}_{ab} - \frac{1}{(d-1)(d-2)}\tilde{R}\tilde{g}_{ab}. \tag{A.10}$$

Inserting this into (A.9) and using (A.8) gives the following form of Einstein's equation:

$$\tilde{S}_{ab} + 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega - \Omega^{-2}\tilde{g}_{ab}(\tilde{\nabla}^c\Omega\tilde{\nabla}_c\Omega + \ell^{-2}) = 0 \tag{A.11}$$

For vacuum spacetimes without a cosmological constant, the term containing ℓ drops out.

(iii) The Bianchi identity

$$\nabla_{[a}R_{bc]de} = 0 \tag{A.12}$$

implies

$$\nabla^a R_{adbc} - \nabla_b R_{cd} + \nabla_c R_{bd} = 0. \tag{A.13}$$

In spacetimes that satisfy Einstein's vacuum equation with a positive cosmological constant, the Ricci tensor is proportional to the spacetime metric. This means that $\nabla_a R_{bc} = 0$ holds and it follows from (A.13) that

$$\nabla^a C_{abcd} = 0 \tag{A.14}$$

(see (1.6) for the definition of the Weyl tensor). The conformal invariance of the Weyl tensor (1.7) then implies

$$\nabla_a \tilde{C}^a{}_{bcd} = 0. \tag{A.15}$$

We can rewrite this equation in terms of an unphysical derivative operator as (cf. footnote 1 on page 67)

$$\tilde{\nabla}^a\tilde{C}_{abcd} + (3-d)\Omega^{-1}\tilde{n}^a\tilde{C}_{abcd} = 0 \tag{A.16}$$

if we make use of the symmetries of the Weyl tensor. Hence it follows that

$$\tilde{\nabla}^a(\Omega^{3-d}\tilde{C}_{abcd}) = 0. \tag{A.17}$$

A.2 Constraint equations

Let us define

$$\tilde{G}_{ab} := \tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab}. \tag{A.18}$$

Then we can write (4.25) as

$$\tilde{G}_{ab} = \tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} = (d-2)\Omega^{-1}(\tilde{K}_{ab} - \tilde{g}_{ab}\tilde{K}). \tag{A.19}$$

Consequently, the first constraint equation can be obtained by contracting $\tilde{n}^a\tilde{n}^b$ into (A.19). We find

$$\begin{aligned} \tilde{G}_{ab}\tilde{n}^a\tilde{n}^b &= \frac{1}{2}(\tilde{\mathcal{R}} + \tilde{K}^2 - \tilde{K}_{ab}\tilde{K}^{ab}) \\ &= (d-2)\Omega^{-1}\tilde{K}, \end{aligned} \tag{A.20}$$

where $\tilde{\mathcal{R}}$ is the Ricci scalar of surfaces of constant Ω and where $\tilde{K}_{ab}$ is the extrinsic curvature of these surfaces. K is the trace of $\tilde{K}_{ab}$. A derivation of the first line of the above equation is given in [5]. The second constraint equation can be calculated by contracting $\tilde{h}_a{}^c\tilde{n}^b$ into (A.19). This calculation yields

$$\tilde{G}_{cb}\tilde{h}_a{}^c\tilde{n}^b = \tilde{D}_b\tilde{K}_a{}^b - \tilde{D}_a\tilde{K} = 0. \tag{A.21}$$

Again, a derivation of this can be found in [5].

A.3 Evolution equations

First, consider the following two relations:

- We can show that

$$\begin{aligned} \tilde{h}_a{}^e\tilde{h}_c{}^f\tilde{R}_{ebfd}\tilde{n}^b\tilde{n}^d &= \tilde{h}_a{}^e\tilde{h}_c{}^f\tilde{n}^b(\tilde{\nabla}_e\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}_e)\tilde{n}_f \\ &= \mathcal{L}_n\tilde{K}_{ac} + \tilde{K}_{bc}\tilde{K}_a{}^b, \end{aligned} \tag{A.22}$$

  where $\mathcal{L}_{\tilde{n}}\tilde{K}_{ab} = \tilde{n}^c\tilde{\nabla}_c\tilde{K}_{ab} + \tilde{K}_{cb}\tilde{\nabla}_a\tilde{n}^c + \tilde{K}_{ac}\tilde{\nabla}_b\tilde{n}^c$ is the Lie derivative of $\tilde{K}_{ab}$ with respect to $\tilde{n}^a$.

- According to [5],

$$\tilde{h}_a{}^f\tilde{h}_b{}^g\tilde{h}_c{}^k\tilde{h}_j{}^d\tilde{R}_{fgk}{}^j = \tilde{\mathcal{R}}_{abc}{}^d + \tilde{K}_{ac}\tilde{K}_b{}^d - \tilde{K}_{bc}\tilde{K}_a{}^d \tag{A.23}$$

  holds, where $\tilde{\mathcal{R}}_{abc}{}^d$ is the Riemann tensor of surfaces of constant Ω.

We again consider (4.25) in the form (A.19). The first evolution equation is then given by

$$\begin{aligned} \tilde{G}_{cd}\tilde{h}_a{}^c\tilde{h}_b{}^d &= \tilde{R}_{ckd}{}^e\tilde{h}_l{}^k\tilde{h}_a{}^c\tilde{h}_b{}^d\tilde{h}_e{}^l - \tilde{R}_{ckdj}\tilde{n}^k\tilde{n}^j\tilde{h}_a{}^c\tilde{h}_b{}^d - \frac{1}{2}\tilde{h}_{ab}\tilde{R} \\ &= (d-2)\Omega^{-1}(\tilde{K}_{cd} - \tilde{g}_{cd}\tilde{K})h_a{}^ch_b{}^d. \end{aligned} \tag{A.24}$$

Inserting (A.22) and (A.23) into this equation and using $\tilde{R} = 2(d-1)\Omega^{-1}\tilde{K}$ as well as

$$\mathcal{L}_{\tilde{n}}\tilde{K}_{ab} = \mathcal{L}_{\tilde{n}}\tilde{K}_a{}^c\tilde{h}_{bc} - 2\tilde{K}_a{}^c\tilde{K}_{bc} \tag{A.25}$$

gives

$$\mathcal{L}_{\tilde{n}}\tilde{K}_a{}^b = \tilde{\mathcal{R}}_a{}^b + \tilde{K}_a{}^b\tilde{K} - (d-2)\Omega^{-1}\tilde{K}_a{}^b - \tilde{\delta}_a{}^b\Omega^{-1}\tilde{K}. \tag{A.26}$$

The second evolution equation can be obtained by simply calculating the Lie derivative of $\tilde{h}_{ab}$:

$$\mathcal{L}_{\tilde{n}}\tilde{h}_{ab} = -2\tilde{K}_{ab} = -2\tilde{h}_{bc}\tilde{K}_a{}^c \tag{A.27}$$

Note that

$$\tilde{n}^a = \tilde{\nabla}^a \Omega = -\left(\frac{\partial}{\partial\Omega}\right)^a, \tag{A.28}$$

which implies

$$\mathcal{L}_{\tilde{n}} \tilde{K}_a{}^b = -\frac{d}{d\Omega} \tilde{K}_a{}^b, \qquad \mathcal{L}_{\tilde{n}} \tilde{h}_{ab} = -\frac{d}{d\Omega} \tilde{h}_{ab}. \tag{A.29}$$

If we insert these relations into (A.26) and (A.27), we find the evolution equations as given in (4.28) and (4.29).

A.4 Relation between the extrinsic curvature and the electric part of the Weyl tensor

It is straightforward to show that

$$\tilde{R}_{abcd} = \tilde{C}_{abcd} + \tilde{g}_{a[c}\tilde{S}_{d]b} - \tilde{g}_{b[c}\tilde{S}_{d]a}. \tag{A.30}$$

The latter terms on the right hand side of this equation satisfy

$$\tilde{h}_a{}^e \tilde{h}_c{}^f \tilde{n}^b \tilde{n}^d (\tilde{g}_{e[f}\tilde{S}_{d]b} - \tilde{g}_{b[f}\tilde{S}_{d]e}) = -\Omega^{-1}\tilde{K}_{ac}, \tag{A.31}$$

which, together with (A.30) and (A.22), implies

$$\tilde{C}_{abcd}\tilde{n}^b\tilde{n}^d = \tilde{h}_a{}^e \tilde{h}_c{}^f \tilde{C}_{ebfd}\tilde{n}^b\tilde{n}^d = \mathcal{L}_{\tilde{n}}\tilde{K}_{ac} + \tilde{K}_{ab}\tilde{K}^b{}_c + \Omega^{-1}\tilde{K}_{ac}. \tag{A.32}$$

Additionally, let us try to find a relation between $\tilde{C}_{abcd}\tilde{n}^d$ and the extrinsic curvature. Note that

$$\begin{aligned} \tilde{C}_{abcd}\tilde{n}^d &= -\tilde{C}_{aecd}\tilde{n}^d\tilde{n}^e\tilde{n}_b + \tilde{C}_{aecd}\tilde{n}^d\tilde{h}_b{}^e \\ &= 2\tilde{n}_{[a}\tilde{C}_{b]ecd}\tilde{n}^d\tilde{n}^e + \tilde{h}_a{}^f\tilde{h}_c{}^g\tilde{h}_b{}^e\tilde{C}_{fegd}\tilde{n}^d. \end{aligned} \tag{A.33}$$

Together with (A.30) and (A.4), this yields

$$\tilde{C}_{abcd}\tilde{n}^d = 2\tilde{n}_{[a}\tilde{C}_{b]ecd}\tilde{n}^d\tilde{n}^e - \tilde{D}_a\tilde{K}_{bc} + \tilde{D}_b\tilde{K}_{ac}, \tag{A.34}$$

where we used that $\tilde{h}_a{}^d\tilde{h}_b{}^e\tilde{h}_c{}^f\tilde{\nabla}_d\tilde{K}_{ef} = \tilde{D}_a\tilde{K}_{bc}$ and $\tilde{S}_{ab}\tilde{n}^a = 0$ (4.25). By inserting (A.32) into this equation, we obtain the relation we were looking for.

A.5 The presymplectic current on $\mathscr{I}^+$

A general presymplectic current ω satisfies (3.3). In our case, ω is given by (3.27). For perturbations $\delta_1 g, \delta_2 g$ that satisfy (4.49),

$$\begin{aligned} 16\pi G \cdot \omega_{a_1\ldots a_{d-1}} &= \Omega^{6-d}\tilde{\epsilon}_{ca_1\ldots a_{d-1}}\tilde{P}^{cabdef}(\delta_1 g_{ab}\nabla_d\delta_2 g_{ef} - \delta_2 g_{ab}\nabla_d\delta_1 g_{ef}) \\ &= \Omega^{6-d}\tilde{\epsilon}_{ca_1\ldots a_{d-1}}\tilde{P}^{cabdef}\cdot O(\Omega^{2d-7}) \\ &= O(\Omega^{d-1}) \end{aligned} \tag{A.35}$$

implies that

$$\omega(g;\delta_1 g, \delta_2 g) = 0 \tag{A.36}$$

on $\mathscr{I}^+$.

A.6 The Noether charge and its variation

(i) The particular Noether charge we want to work with is given by (3.31). It can be reexpressed in terms of conformal quantities:

$$\begin{aligned}(Q_\xi)_{a_1\dots a_{d-2}} &= -\frac{1}{16\pi G}\epsilon_{a_1\cdots a_{d-2}bc}\nabla^b\xi^c \\ &= -\frac{1}{16\pi G}\Omega^{2-d}\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\tilde{g}^{be}\tilde{\nabla}_e\xi^c - \frac{1}{16\pi G}\Omega^{2-d}\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\tilde{g}^{be}C^c{}_{ef}\xi^f \\ &= -\frac{1}{16\pi G}\Omega^{2-d}\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\tilde{g}^{be}\tilde{\nabla}_e\xi^c + \frac{1}{8\pi G}\Omega^{1-d}\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\tilde{n}^b\xi^c\end{aligned} \tag{A.37}$$

The quantity $C^c{}_{ef}$ in the second line corresponds to (A.2).

(ii) The variation of the Noether charge under (4.50) can be written as

$$\begin{aligned}(\delta Q_\xi)_{a_1\dots a_{d-2}} = \frac{1}{8\pi G}\Omega^{1-d}\Big[&(\delta\tilde{\epsilon}_{a_1\dots a_{d-2}bc})\tilde{n}^b\xi^c + \tilde{\epsilon}_{a_1\dots a_{d-2}bc}(\delta\tilde{n}^b)\xi^c + \tilde{\epsilon}_{a_1\dots a_{d-2}bc}\tilde{n}^b\xi^c \\ &-\frac{1}{2}\Omega(\delta\tilde{\epsilon}_{a_1\cdots a_{d-2}bc})\tilde{g}^{be}\tilde{\nabla}_e\xi^c - \frac{1}{2}\Omega\tilde{\epsilon}_{a_1\cdots a_{d-2}bc}\delta(\tilde{g}^{be}\tilde{\nabla}_e\xi^c)\Big].\end{aligned} \tag{A.38}$$

If we make use of the following relations

$$\delta\tilde{\epsilon}_{ab\dots c} = -\frac{1}{d-1}\Omega^{d-1}\delta\tilde{E}_d{}^d\tilde{\epsilon}_{ab\dots c} + O(\Omega^d) = O(\Omega^d), \tag{A.39}$$

$$\delta\tilde{n}^a = \frac{2}{d-1}\Omega^{d-1}\delta\tilde{E}^{ab}\tilde{n}_b + O(\Omega^d) = O(\Omega^d), \tag{A.40}$$

$$\delta(\tilde{g}^{be}\tilde{\nabla}_e\xi^c) = -2\Omega^{d-2}\tilde{n}^{[b}\delta\tilde{E}^{c]}{}_d\xi^d - \Omega^{d-2}\delta\tilde{E}^{(bc)}\tilde{n}_d\xi^d + O(\Omega^{d-1}), \tag{A.41}$$

we find

$$(\delta Q_\xi)_{a_1\dots a_{d-2}} = \frac{1}{8\pi G}\tilde{\epsilon}_{a_1\dots a_{d-2}bc}\tilde{n}^b\delta\tilde{E}^c{}_d\xi^d + O(\Omega). \tag{A.42}$$

A.7 The presymplectic potential on $\mathscr{I}^+$

Before we investigate the behavior of θ on $\mathscr{I}^+$ for variations $\delta g_{ab} = \gamma_{ab}$ (see (4.50)), let us take a look at the asymptotic behavior of some other quantities. We have

$$\delta\tilde{E}_{ab}\tilde{g}^{ab} = -\tilde{E}_{ab}\delta\tilde{g}^{ab} = \tilde{E}^{ab}\delta\tilde{g}_{ab} = O(\Omega^{d-1}) \tag{A.43}$$

and

$$\tilde{n}^a\delta\tilde{E}_{ab} = -\tilde{E}_{ab}\delta n^a = O(\Omega^d), \tag{A.44}$$

where we used (A.40) in (A.44). Together with (A.1), these equations can be used to obtain

$$\begin{aligned}\tilde{g}^{ac}\nabla_a\delta\tilde{E}_{bc} &= \tilde{g}^{ac}[\tilde{\nabla}_a\delta\tilde{E}_{bc} - C^e{}_{ab}\delta\tilde{E}_{ec} - C^e{}_{ac}\delta\tilde{E}_{be}] = -\tilde{g}^{ac}[C^d{}_{ab}\delta\tilde{E}_{ec} + C^e{}_{ac}\delta\tilde{E}_{be}] + O(\Omega^0) \\ &= \frac{1}{\Omega}\Big[\delta\tilde{E}_{bc}n^c + \delta\tilde{E}_{ac}\tilde{g}^{ac}n_b - \delta\tilde{E}_{be}n^e + \delta\tilde{E}_{bc}n^c + \delta\tilde{E}_{be}n^e - d\cdot\delta\tilde{E}_{be}n^e\Big] + O(\Omega^0) \\ &= O(\Omega^0),\end{aligned} \tag{A.45}$$

where $C^c{}_{ab}$ is given by (A.2). Equipped with these results, we can turn our attention to the actual calculation of the asymptotic behavior of the presymplectic potential (3.25) for perturbations (4.50):

$$\begin{aligned} 16\pi G \cdot \theta_{a_1 \ldots a_{d-1}} &= \Omega^{4-d} \tilde{\epsilon}_{ca_1 \ldots a_{d-1}} \tilde{g}^{cb} \tilde{g}^{de} (\nabla_e \delta g_{bd} - \nabla_b \delta g_{de}) \\ &= -\frac{2}{d-1} \tilde{\epsilon}_{ca_1 \ldots a_{d-1}} \tilde{g}^{cb} (\tilde{n}^d \delta \tilde{E}_{bd} + \Omega \tilde{g}^{de} \nabla_e \delta \tilde{E}_{bd}) + O(\Omega) \\ &= O(\Omega) \end{aligned} \tag{A.46}$$

Hence,

$$\theta(g; \gamma) = 0 \tag{A.47}$$

on $\mathscr{I}^+$.

B Spinors and curved spacetime gamma matrices

Let (M, g_{ab}) be a Lorentzian manifold that admits a spinor bundle S and curved spacetime gamma matrices. These gamma matrices satisfy the Clifford algebra relation

$$\gamma_{(a}\gamma_{b)} = g_{ab}. \tag{B.1}$$

This is identical to the usual gamma matrix relation

$$\gamma_{(\mu}\gamma_{\nu)} = \eta_{\mu\nu}. \tag{B.2}$$

Note that we denote flat gamma matrices by Greek indices. We raise and lower these Greek indices with $\eta_{\mu\nu}$ while we raise and lower the indices on the curved spacetime gamma matrices with the spacetime metric g_{ab}. Now assume that we have an orthonormal basis in some region U of the spacetime which satisfies

$$g_{ab}e_{\mu}{}^{a}e_{\nu}{}^{b} = \eta_{\mu\nu}, \qquad \eta_{\mu\nu}e^{\mu}{}_{a}e^{\nu}{}_{b} = g_{ab}. \tag{B.3}$$

According to this equation, we can construct curved spacetime gamma matrices by

$$\gamma_a = e^{\mu}{}_{a}\gamma_\mu, \qquad \gamma^a = e_{\mu}{}^{a}\gamma^\mu \tag{B.4}$$

in U.

Let $(\tilde{M}, \tilde{g}_{ab})$ be an unphysical spacetime which is associated with $(M, g_{ab} = \Omega^{-2}\tilde{g}_{ab})$. This spacetime also admits a spinor bundle $\tilde{S}$ with

$$\tilde{\gamma}_{(a}\tilde{\gamma}_{b)} = \tilde{g}_{ab}. \tag{B.5}$$

A curved spacetime gamma matrix γ_a is related to the respective matrix $\tilde{\gamma}_a$ by

$$\tilde{\gamma}_a = \Omega\gamma_a, \qquad \gamma^a = \Omega\tilde{\gamma}^a. \tag{B.6}$$

A spinor field ψ on M is a section in the spinor bundle S. Let us define the Dirac conjugate (or adjoint) spinor of a spinor ψ by

$$\bar{\psi} := \psi^\dagger\gamma^0, \tag{B.7}$$

where γ^0 is the zeroth flat spacetime gamma matrix. Then the quantity

$$\xi^a = -\bar{\psi}\gamma^a\psi \tag{B.8}$$

is a vector, which is future directed and timelike for all spinors ψ. The covariant derivative operator on spinor fields is given by

$$\nabla_a\psi = \partial_a\psi + \frac{1}{4}\omega_a{}^{\mu\nu}\gamma_{[\mu}\gamma_{\nu]}\psi, \tag{B.9}$$

where ${\omega_a}^{\mu\nu}$ is the spin connection. It is defined by

$$ {\omega_a}^{\mu\nu} = e^{\mu b}\nabla_a {e^\nu}_b = e^{\mu b}(\partial_a {e^\nu}_b - {\Gamma^c}_{ab}{e^\nu}_c). \tag{B.10} $$

Therefore, the difference between two derivative operators which are associated with a conformally transformed metric $\tilde{g}_{ab}$ and the spacetime metric g_{ab} is given by

$$ \begin{aligned} (\tilde{\nabla}_a - \nabla_a)\psi &= \frac{1}{4}({\tilde{\omega}_a}{}^{\mu\nu} - {\omega_a}^{\mu\nu})\gamma_{[\mu}\gamma_{\nu]}\psi \\ &= -\frac{1}{4}\tilde{g}_{bc}{C^b}_{ad}\tilde{\gamma}^{[c}\tilde{\gamma}^{d]} = \frac{1}{2}\Omega^{-1}(\tilde{\gamma}_a\tilde{\gamma}_b\tilde{n}^b - \tilde{n}_a), \end{aligned} \tag{B.11} $$

where ${C^c}_{ab} = -\Omega^{-1}({\delta_b}^c\tilde{\nabla}_a\Omega + {\delta_a}^c\tilde{\nabla}_b\Omega - \tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_d\Omega)$ (see (A.2)).

C Formulas needed in the perturbation analysis of de Sitter space

C.1 Weyl perturbations

Inserting Einstein's equation with a positive cosmological constant (see e.g. (2.22)) into (1.6) gives

$$R_{abcd} = C_{abcd} + \frac{1}{\ell^2}(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{C.1}$$

Then we can immediately conclude from the Bianchi identity

$$\nabla_{[a}R_{bc]de} = 0 \tag{C.2}$$

that

$$\nabla_{[a}C_{bc]de} = 0, \tag{C.3}$$

which in turn implies

$$\nabla^a C_{bcae} = 0. \tag{C.4}$$

If we apply ∇^a to (C.3), we find

$$\nabla_a\nabla^a C_{bcde} + \nabla^a\nabla_b C_{cade} + \nabla^a\nabla_c C_{abde} = 0. \tag{C.5}$$

Now we can show that

$$\begin{aligned}\nabla^a\nabla_b C_{cade} &= \nabla_b\nabla^a C_{cade} + R^a{}_{bc}{}^f C_{fade} + R^a{}_{ba}{}^f C_{cfde} + R^a{}_{bd}{}^f C_{cafe} + R^a{}_{be}{}^f C_{cadf} \\ &= C^a{}_{bc}{}^f C_{fade} + C^a{}_{bd}{}^f C_{cafe} + C^a{}_{be}{}^f C_{cadf} + \frac{1}{\ell^2}(d-1)C_{cbde}\end{aligned} \tag{C.6}$$

holds, where we used the generalization of (A.4) to tensors of higher order (cf. footnote 2 on page 67), the symmetries of the Weyl tensor as well as (C.1) and (C.4). Combining (C.6) with (C.5) yields

$$\left(\nabla^a\nabla_a - \frac{2(d-1)}{\ell^2}\right)C_{bcde} = \text{terms quadratic in } C. \tag{C.7}$$

Now note that

$$\delta(\nabla_a C_{bcde}) = \bar\nabla_a \delta C_{bcde}, \tag{C.8}$$

for perturbations off of de Sitter space, where $\bar\nabla_a$ is the derivative operator associated with the de Sitter metric.[1] Hence it follows from (C.4) that

$$\bar\nabla^a \delta C_{abcd} = 0 \tag{C.9}$$

and from (C.7) that

$$\left(\bar\nabla^e\bar\nabla_e - \frac{2(d-1)}{\ell^2}\right)\delta C_{abcd} = 0. \tag{C.10}$$

The last equations holds, since variations of terms that are quadratic in the Weyl tensor must clearly vanish for perturbations off of de Sitter space.

[1]Recall that the Weyl tensor vanishes in de Sitter space.

C.2 Relations between the coefficients of the harmonic expansion

The relation between $\mathcal{E}_L$ and ψ_S can be found by making use of the tracelessness of $\mathcal{E}_{ab}$. We have $\mathcal{E}^a{}_a = 0$, which implies

$$\mathcal{E}_L = -\frac{1}{d-2}(\sin^2\chi)\psi_S \tag{C.11}$$

if we take (5.32) as well as (5.41) into account. The next relation follows from (5.31). Inserting (5.41) into this equation gives

$$\phi_S = \frac{1}{k_S^2}\Big[(\sin^2\chi)\chi^C D_C\psi_S + (d-1)(\sin\chi)(\cos\chi)\psi_S\Big]. \tag{C.12}$$

For the two remaining relations, we take a look a (5.33). This equation implies

$$\mathcal{E}_V = \frac{2}{k_V^2}\Big[(d-2)(\sin\chi)(\cos\chi)\psi_V + (\sin^2\chi)\chi^C D_C\psi_V\Big] \tag{C.13}$$

as well as

$$\mathcal{E}_T = \frac{1}{k_S^2}\frac{d-2}{d-3}\Big[\mathcal{E}_L + (d-2)(\sin\chi)(\cos\chi)\phi_S + (\sin^2\chi)\chi^C D_C\phi_S\Big]. \tag{C.14}$$

The derivatives of the coefficients in the above equations must have the same asymptotic behavior as the base quantities themselves. Hence, the asymptotic behavior of the coefficients is given by

$$\mathcal{E}_L \sim \psi_S, \qquad \phi_S \sim \psi_S, \tag{C.15}$$

$$\mathcal{E}_V \sim \psi_V, \qquad \mathcal{E}_T \sim \psi_S. \tag{C.16}$$

D A short note on hypergeometric functions

The Gauss hypergeometric series, also called the hypergeometric function, is given by

$$F(a,b;c;z) \equiv {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \tag{D.1}$$

where

$$\begin{aligned} &(a)_0 = 1, \\ &(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)} = a(a+1)(a+2)\ldots(a+n-1) \end{aligned} \tag{D.2}$$

is the Pochhammer symbol. The circle of convergence of (D.1) is given by $|z| = 1$, assuming that neither a, b or c are negative integers or zero. On the circle, the series diverges if $\mathrm{Re}(c-a-b) \leq -1$, it converges if $\mathrm{Re}(c-a-b) > 0$, and it converges, except at the point $z = 1$, if $-1 < \mathrm{Re}(c-a-b) \leq 0$. If a or b are negative integers or zero and c is either not a negative integer or zero or smaller than the negative integer a or b, the series becomes a finite sum. If c is a negative integer or zero and neither a or b are negative integers or zero with $a > c$ or $b > c$, (D.1) is indeterminate.

In general, it is non-trivial to calculate $F(a,b;c;z)$ for fixed z. However, for $z = 0$, the hypergeometric series (D.1) simply reduces to

$$F(a,b;c;0) = 1 \tag{D.3}$$

if $c \neq 0$.

There exist a lot of relations between hypergeometric functions with different parameters. In chapter 5, we will use the following one:

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \tag{D.4}$$

This holds only if a, b, and c are not negative integers of zero.

Most prominently, the hypergeometric function appears in solutions of the hypergeometric differential equation

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} w + [c - (a+b+1)z] \frac{\mathrm{d}}{\mathrm{d}z} w - abw = 0. \tag{D.5}$$

For instance, if a and b and c are such that $F(a,b;c;z)$ is well defined for $|z| < 1$, a solution of (D.5) in that region is given by $w = F(a,b;c;z)$. For further solutions and solutions in other domains see e.g. [28, 29].

E Perturbation analysis in another chart

Originally, I did not conduct the perturbation analysis in the global chart. Instead, I used the form (2.31) of the metric. However, I ran into certain problems that I will briefly describe at the end of this section. I will not go into much detail here but will only give a short summary of (some of) the progress I made within this approach. A lot of the following is quite similar to chapter 5. Consequently, we will repeatedly refer to said chapter and the notations used therein.

Let us take a look at figure 2.3. We will consider region II of de Sitter space with metric

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - 1\right)^{-1} \mathrm{d}r^2 + \left(\frac{r^2}{\ell^2} - 1\right) \mathrm{d}t^2 + r^2 \mathrm{d}\sigma_{d-2}^2. \tag{E.1}$$

As in chapter 5, we define

$$Y := V(r)\cosh(t/\ell), \tag{E.2}$$

$$Z_a := \nabla_a Y, \tag{E.3}$$

and

$$\mathcal{E}_{ab} := \delta C_{acbd} Z^c Z^d, \tag{E.4}$$

where

$$V(r) = \left(\frac{r^2}{\ell^2} - 1\right)^{1/2}. \tag{E.5}$$

Note that $V(r) > 0$ in region II.

This gives us the following differential equations

$$\left(\nabla^c \nabla_c - \frac{2(d-2)}{\ell^2}\right) \mathcal{E}_{ab} = 0, \tag{E.6}$$

$$\nabla^a \mathcal{E}_{ab} = 0 \tag{E.7}$$

if we make use of (5.2) and (5.3). $\mathcal{E}_{ab}$'s that solve these give rise to Weyl tensors that satisfy the linearized Einstein equation. Introducing a component notation[1] as in section 5.1, we find that we can write the above differential equations in exactly the same form as (5.21) - (5.25), only that r is no longer given by (5.19) but corresponds to the r of (E.1).

From there, we can proceed to the following, equivalent, set of equations:

$$D^C D_C \mathcal{E} + \left[(d-2)\frac{D^C r}{r} - 4\frac{Z^C}{Y}\right] D_C \mathcal{E} + \left[\frac{1}{r^2}\hat{\Delta} - \frac{6}{\ell^2} Y^2\right] \mathcal{E} = 0, \tag{E.8}$$

$$D^C D_C \mathcal{E}_i + \left[(d-4)\frac{D^C r}{r} - 2\frac{Z^C}{Y}\right] D_C \mathcal{E}_i + \left[\frac{1}{r^2}\hat{\Delta} - \frac{2}{\ell^2} Y^2 - \frac{d-3}{r^2}\right] \mathcal{E}_i = 2\frac{\ell^2}{r^2}\frac{\dot{Y}}{Y}\hat{D}_i \mathcal{E}, \tag{E.9}$$

[1] See above (5.11); capital Latin letters represent t or r indices, whereas lowercase Latin indices i, j, ... represent angular indices.

$$D^C D_C \mathcal{E}_{ij} + (d-6)\frac{D^C r}{r} D_C \mathcal{E}_{ij} + \left[\frac{1}{r^2}\hat{\Delta} - \frac{2(d-4)}{r^2}\right]\mathcal{E}_{ij}$$
$$= \frac{2\ell^2}{r^2}\frac{\dot{Y}}{Y}(\hat{D}_i \mathcal{E}_j + \hat{D}_j \mathcal{E}_i) - 2\left(\frac{\ell^2}{r^2}\frac{\dot{Y}}{Y}\right)^2 g_{ij}\mathcal{E}, \quad \text{(E.10)}$$

where

$$t^a = \left(\frac{\partial}{\partial t}\right)^a, \qquad \mathcal{E} = \mathcal{E}_{AB} t^A t^B, \qquad \mathcal{E} = \mathcal{E}_{Ai} t^A, \qquad \dot{Y} = \frac{\partial Y}{\partial t}. \quad \text{(E.11)}$$

We can decouple these equations in exactly the same way that we used in chapter 5 (see at the beginning of section 5.2). If we stick to the notation we used there, we find

$$\left(\frac{\partial^2}{\partial t^2} - 4\frac{\dot{Y}}{Y}\frac{\partial}{\partial t} - \frac{6}{\ell^2}\frac{V^2}{Y^2}\right)\psi_S = \left[\frac{V^2}{r^{d-2}}\frac{\partial}{\partial r}\left(r^{d-2}V^2\frac{\partial}{\partial r}\right) - 4V^4\frac{Y'}{Y}\frac{\partial}{\partial r} + \frac{V^2}{r^2}\mathbf{k}_S^2\right]\psi_S, \quad \text{(E.12)}$$

$$\left(\frac{\partial^2}{\partial t^2} - 2\frac{\dot{Y}}{Y}\frac{\partial}{\partial t} - \frac{2}{\ell^2}\frac{V^2}{Y^2}\right)\psi_V$$
$$= \left[\frac{V^2}{r^{d-4}}\frac{\partial}{\partial r}\left(r^{d-4}V^2\frac{\partial}{\partial r}\right) - 4V^4\frac{Y'}{Y}\frac{\partial}{\partial r} + \{(d-3) + \mathbf{k}_V^2\}\frac{V^2}{r^2}\right]\psi_V, \quad \text{(E.13)}$$

and

$$\frac{\partial^2}{\partial t^2}\psi_T = \left[\frac{V^2}{r^{d-6}}\frac{\partial}{\partial r}\left(r^{d-6}V^2\frac{\partial}{\partial r}\right) + \{2(d-4) + \mathbf{k}_T^2\}\frac{V^2}{r^2}\right]\psi_T, \quad \text{(E.14)}$$

where Y' denotes the derivative of Y with respect to r. If we define

$$\begin{aligned} \psi_S &=: Y^2 \cdot r^{-(d-2)/2}\Psi_S, \\ \psi_V &=: Y \cdot r^{-(d-4)/2}\Psi_V, \\ \psi_T &=: r^{-(d-6)/2}\Psi_T, \end{aligned} \quad \text{(E.15)}$$

and introduce

$$\frac{r}{\ell} =: \frac{1}{\tanh x}, \quad \text{(E.16)}$$

we can again reduce these equations to one master equation:

$$\ell^2\frac{\partial^2}{\partial t^2}\Psi = \left[\frac{\partial^2}{\partial x^2} + \left(\frac{(d-2)(d-4)}{4} + l(l+d-3)\right)\frac{1}{\cosh^2 x} - \frac{(d-4)(d-6)}{4}\frac{1}{\sinh^2 x}\right]\Psi \quad \text{(E.17)}$$

With the ansatz

$$\Psi = R(r)T(t), \quad \text{(E.18)}$$

we find

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\omega^2}{\ell^2}\right)T = 0, \quad \text{(E.19)}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left[\sigma^2 - \frac{1}{4}\right]\frac{1}{\cosh^2 x} + \left[\nu^2 - \frac{1}{4}\right]\frac{1}{\sinh^2 x}\right)R = \omega^2 R, \quad \text{(E.20)}$$

where $\sigma = l + (d-3)/2$ and $\nu = (d-5)/2$.

The solutions of the first equation are simply given by

$$T(t) = A\exp(i\omega t/\ell) + B\exp(-i\omega t/\ell). \tag{E.21}$$

To solve the second equation, we cast it into the form of a hypergeometric differential equation. Then it reads

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{R} + \left[1+\sigma-(\zeta^{\omega}_{\nu,\sigma}+\zeta^{-\omega}_{\nu,\sigma}+1)z\right]\frac{\mathrm{d}}{\mathrm{d}z}\bar{R} - \zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}\bar{R} = 0, \tag{E.22}$$

where

$$R = (\sinh x)^{\nu+1/2}(\cosh x)^{\sigma+1/2}\bar{R}, \tag{E.23}$$

$$z = \cosh^2 x, \tag{E.24}$$

and ζ is as defined in (5.51). Then it is possible to find the general solutions of this equation.

Before we discuss why this entire approach is problematic for our purposes, let us recall what the coordinates r and t describe. First of all, $(\partial/\partial r)$ is timelike in region II (cf. figure 2.3) while $(\partial/\partial t)$ is spacelike there. Hypersurfaces of constant r are spacelike and $r \to \infty$ corresponds to $\mathscr{I}^+$. Conversely, hypersurfaces of constant t are timelike. If we keep r fixed and let $t \to \pm\infty$, we end up at points[2] of $\mathscr{I}^+$ (upper corners of figure 2.3).

There are several reasons, why the used coordinate system is not the best possible choice for our purposes. A very significant one is that (E.21) is not well defined for $t \to \pm\infty$ (i.e. on $\mathscr{I}^+$) if $\omega \neq 0$. To resolve this, one could try to find a class of superpositions of solutions (E.18) that exhibit a suitable decay behavior for $t \to \pm\infty$ or have compact support in t. However, it turned out that both these solutions are quite difficult to implement. This is supplemented by the fact that even though we can find solutions of (E.22) that are smooth for $x = 0$ (which corresponds to $r \to \infty$), it turned out that their behavior for $x \to \infty$ depends on the separation parameter ω in such a way that the solutions might not be well defined there. This is problematic, because perturbations that are not smooth in the interior of the spacetime manifold cannot give rise to asymptotically de Sitter spacetimes. Hence, we would have to be very careful which mode to use when trying to construct superpositions of (E.18).

Another property of the coordinate system that interfered with our analysis is related to the fact that it does not cover the entire spacetime manifold. By using these coordinates, we cannot find out what happens in the other regions (i.e. region I, III, and IV; cf. figure 2.3) of the spacetime. This is of considerable interest, because, as mentioned above, perturbations that are not smooth in the interior cannot give rise to asymptotically de Sitter spacetimes.

These problems can be evaded by using the global coordinate system. Its constant time hypersurfaces do not end at $\mathscr{I}^+$ and it covers the entire spacetime. This led us to redo the analysis in this coordinate system (see chapter 5).

We were able to carry a lot of the above analysis over to the one in the global chart. The results of this appendix might also be useful in other contexts.

[2]Note that the coordinates are not smooth at these points: The horizon "intersects" $\mathscr{I}^+$ there and we may end up at the the same point if we take $t \to \pm\infty$ for different values of r.

Symbols

Here we list some of the symbols that appear repeatedly throughout the thesis.

Symbol	Description
M	Spacetime manifold
g_{ab}	Spacetime metric; in chapter 5: the de Sitter metric
d	Dimension of the spacetime manifold
$\mathrm{d}s^2$	Spacetime metric without abstract indices
$\tilde{M}$	Unphysical spacetime manifold
$\partial\tilde{M}$	$\partial\tilde{M} = \tilde{M} - M$
$\tilde{g}_{ab}$	Unphysical metric
Ω	Conformal factor
$\mathscr{I}^+$	Future infinity of $\tilde{M}$
$\mathscr{I}^-$	Past infinity of $\tilde{M}$
$\mathscr{I}$	$\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$
∇_a	Derivative operator associated with the spacetime metric; in chapter 5: associated with the de Sitter metric
$\tilde{\nabla}_a$	Derivative operator associated with the unphysical metric
$\eta_{\mu\nu}$	$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \dots)$
$J^{\pm}(U)$	Causal future/past of U
$\epsilon_{a_1 \dots a_d}$	Volume form
S^d	d-dimensional sphere
σ_{ab}	Natural metric of the sphere
$\mathrm{d}\sigma_d^2$	Metric of the sphere; given in terms of coordinates without abstract indices
C_d	"Mass" parameter of the Schwarzschild de Sitter spacetime, cf. (2.35)
$\mathcal{F}$	Field configuration space
$\bar{\mathcal{F}}$	Covariant phase space
$\theta_{a_1 \dots a_{d-1}}$	Presymplectic potential form, cf. (3.1) and (3.25)
$\omega_{a_1 \dots a_{d-1}}$	Presymplectic current form, cf. (3.3) and (3.27)
$(J_\xi)_{a_1 \dots a_{d-1}}$	Noether current with respect to vector field ξ^a, cf. (3.8) and (3.30)
$(Q_\xi)_{a_1 \dots a_{d-2}}$	Noether charge with respect to vector field ξ^a, cf. (3.12) and (3.31)
$\bar{g}_{ab}$	De Sitter metric
$\bar{\nabla}_a$	Derivative operator associated with the de Sitter metric
$\mathcal{L}_\xi$	Lie derivative with respect to the vector ξ^a
$\tilde{n}_a$	$\tilde{n}_a = \tilde{\nabla}_a \Omega$
Λ	Cosmological constant
ℓ	De Sitter radius, $\ell = \sqrt{(d-1)(d-2)/(2\Lambda)}$
$C^c{}_{ab}$	Difference between two derivative operators that are associated with a spacetime metric and a corresponding unphysical metric, i.e. $(\nabla_a - \tilde{\nabla}_a)\omega_b = C^c{}_{ab}\omega_c$

${R_{abc}}^d$	Riemann tensor
${C_{abc}}^d$	Weyl tensor
${\tilde{R}_{abc}}{}^d$	Riemann tensor with respect to the unphysical metric
${\tilde{C}_{abc}}{}^d$	Weyl tensor with respect to the unphysical metric
$\tilde{E}_{ab}$	Normalized electric part of the unphysical Weyl tensor; $\tilde{E}_{ab} = \frac{\ell^2}{d-3}\Omega^{3-d}\tilde{C}_{acbd}\tilde{n}^c\tilde{n}^d$
R_{ab}	Ricci tensor
$\tilde{R}_{ab}$	Ricci tensor with respect to the unphysical metric
R	Ricci scalar
$\tilde{R}$	Ricci scalar with respect to the unphysical metric
h_{ab}	Metric on some spacelike hypersurface; in chapter 4: metric on surfaces of constant Ω
$\tilde{\mathcal{R}}_{ab}$	Unphysical intrinsic Ricci tensor of surfaces of constant Ω
$\tilde{\mathcal{R}}$	Unphysical intrinsic Ricci scalar of surfaces of constant Ω
$\tilde{K}_{ab}$	Unphysical extrinsic curvature of surfaces of constant Ω; $\tilde{K}_{ab} = -\tilde{\nabla}_a\tilde{n}_b$
$\tilde{K}$	Trace of $\tilde{K}_{ab}$
$\tilde{D}_a$	Derivative operator on surfaces of constant Ω
k_{ab}	Deviation from the de Sitter metric
${e_\mu}^a$	Basis vector of an orthonormal basis with respect to the physical metric
${\tilde{e}_\mu}{}^a$	Basis vector of an orthonormal basis with respect to the unphysical metric
${\bar{e}_\mu}{}^a$	Basis vector of an orthonormal basis with respect to the de Sitter metric
q_{ab}	Metric on a spacelike hypersurface, cf. (4.70)
ψ	Dirac spinor
$\tilde{\psi}$	$\tilde{\psi} = \Omega^{1/2}\psi$
$\bar{\psi}$	Dirac adjoint, $\bar{\psi} = \psi^\dagger\gamma^0$
γ^μ	Flat spacetime gamma matrix
γ^a	Curved spacetime gamma matrix
B_{ab}	The Nester form, cf. (4.68)
$\hat{\nabla}_a$	The super-covariant derivative operator, cf. (4.69)
a	$a(\tau) = \ell\cosh(\tau/\ell)$
Y	$Y = -\sinh(\tau/\ell)$
Z	$Z_a = \nabla_a Y$
$\mathcal{E}_{ab}$	$\mathcal{E}_{ab} = \mathcal{E}_{ab} = \delta C_{acbd}Z^cZ^d$
$\mathcal{E}$	$\mathcal{E} = \mathcal{E}_{ab}\chi^a\chi^b$
$\mathcal{E}_i$	$\mathcal{E}_i = \mathcal{E}_{ab}\chi^a\theta_i^b$, where θ_i^a is as in (5.12)
$\mathcal{E}_{ij}$	$\mathcal{E}_{ij} = \mathcal{E}_{ab}\theta_i^a\theta_j^b$
$\psi_{S,V,T}$	See (5.40)
ϕ_S	See (5.40)
$\mathcal{E}_{L,T,V}$	See (5.40)
$\mathbb{N}_0$	$\mathbb{N}\cup\{0\}$
Ψ	See (5.46)
$F(a,b;c;z)$	$F = {}_2F_1$, the Gauss hypergeometric function

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