Field Theory

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Contents

Chapter 1

Introduction and Concepts

1.1 Introduction

This lecture is intended as an introduction to the theory of physical fields. The idea of fields as physical objects is an old one. It has always been considered "dual" (in some vague sense) to the idea that point-like objects ("atoms") make up the world. In the times of flourishing of Newtonian mechanics, the reduction of physical theories to the mechanics of point particles was considered the ultimate possibility of understanding nature. Fields (like force fields) were just a tool of description, and did not possess physical reality of their own. Nevertheless there were a number of very useful field theories around (for example hydrodynamics, optics ,..) which produced important results. But the idea of a physical field remained vague and people thought that ultimately all these phenomenological theories could be explained by the mechanics of point particles.

With the advent of electromagnetic theory, most notably with the work of Faraday and Maxwell, fields were back as physical objects, although many people thought, that this was only an intermediary step towards the ultimate, mechanistic explanation of electrodynamics. As we know today, this was a misconception – fields remained. On the other hand, the phenomenon of light, which was thought to be a wave phenomenon beyond doubt after demonstrating interference of light by Fresnel regained some aspects of particles with Einstein's work on the photoelectric effect. The "wave-particle" duality lead to the development of quantum mechanics. The analysis of problems arising from the electrodynamics of moving bodies lead Einstein to modifications of Newtonian mechanics (the theory of special relativity), which – among many other things – implied that a consistent theory of interacting particles requires fields. A disturbance of one particle cannot be felt immediately by another distant particle, because all effects of this disturbance can at most travel with the velocity of light. Therefore there has to be some physical object, which carries the disturbance (its energy, its momentum etc.) in between. Disturbances of charged particles travel as electromagnetic waves. Another most important consequence of the theory of special relativity has been that there is no conservation law of mass, like in Newtonian mechanics. Mass is just a special form of energy, the energy of a body at rest.

Modern physics tries to combine relativity and quantum physics. As we have learned in *Quantum Mechanics II*, one of the first results was that relativistic quantum theories cannot be single particle theories. In relativistic quantum theory, arbitrarily many particles can be generated and destroyed (provided conservation laws of energy, momentum, spin etc. are obeyed). Such processes

are very common in the realm of "elementary particles". Therefore physicists are trying to build theories of elementary particles and elementary interactions as field theories of quantum objects or *quantum field theories*.

Modern physics also considered the old problem of connecting "phenomenological" field theories, like hydrodynamics, with the underlying molecular dynamics. Surprisingly it turned out that this problem is in many important aspects equivalent to the construction of quantum field theories. In fact, the partition function of classical (non-quantum, non-relativistic) fields, which fluctuate due to thermal motion, contains all the information of a relativistic quantum field theory. This connection has become one of the most fruitful "theoretical laboratories" of modern physics, because it allows to transport ideas and findings between two completely different physical regimes. Today you will find identical methods (like the renormalization group) and identical concepts (spontaneous breaking of symmetries, topological defects) both in the theory of condensed matter and in the theory of quantum fields and elementary interactions.

Even this rough scetch must have given you the impression that there is an enormous amount of material to be covered, especially if you lack important pre-knowledge like the theory of special relativity, the phenomenological field theories, an advanced course of electrodynamics and an introductory course on elementary particle physics. It is exactly for this audience that this lecture has been designed. So the aims will be modest. I cannot give you all of the highlights, which have been obtained from field theoretic concepts. Rather I will try to put you in a position, to read many of the excellent textbooks available on the diverse subtopics.

The lectures are divided into two parts. The fist part is about "phenomenological" non-relativistic field theories: Ginzburg-Landau Theory from thermophysics, the theory of deformable media with the most important specializations, i.e. elastic media and hydrodynamics. I do not have the feeling that these theories are "old stuff". They still produce lots of research results and they form a basis, without which you will not be able to grasp elaborate modern theories on quantum gravity, critical phenomena or cosmology, to name but a few.

The second part is about non-quantum relativistic field theories. In fact, it starts out with an introductory part on the theory of special relativity, which – strictly speaking – is not about fields. The aim of this part is twofold: first, we will discuss the most prominent example of a classical, relativistic field theory, which is Maxwell's theory. Here we will start from an undergraduate level and end with the Lagrangian formulation of Maxwell's theory as a manifestly

Lorentz covariant field theory. Second, we will introduce the Lagrangian formulation of field theories and give the important connection between symmetries and conservation laws (Noether's Theorem) on the level of field theories. Equipped with these tools, we will take a look into the theory of gravitation and have a glance at the structures of modern theories of elementary interactions.
You may wonder where the quantum field theories will appear. In fact, they already did appear in the lecture *Quantum Mechanics II*, in particular in the context of the relativistic extensions of Schrödinger's wave theory. There we learned, that "all one has to do" is to replace the wave functions by the proper field operators. More precisely, one can use the Lagrangian formalism for fields to apply the route via Feynman's path integrals to set up a proper *quantum field-theory*. However, any application going beyond the discussion of free particles – which we extensively did last term in *Quantum Mechanics II* – requires a lecture on its own. Therefore, I dropped this subject here to invest the time in a thorough treatment of the classical stuff. Once you have gained a feeling for the concepts here, the step towards quantum fields is possibly cumbersome from a mathematical point of view, but the physical ideas remain the same.

1.1.1 Basic Concepts

The basic idea of field theories is that physical properties are "smoothly" distributed in space and time.
Let us give some examples of classical (non-quantum) fields

a) mass (or charge) density $\rho(\boldsymbol{r}, t)$. Densities can be defined for all additive physical quantities (momentum, energy, ...). Let V be some region in space; then $\int_V \rho(\boldsymbol{r}, t) d^3r = Q(V)$ is the mass (charge or whatever additive quantity) inside V

b) $\boldsymbol{v}(\boldsymbol{r}, t)$, the velocity field of a streaming fluid

c) $h(x, y, t)$, the height profile of e.g. water in a swimming pool

d) $\Psi(\boldsymbol{r}, t)$, the complex wave function of a quantum particle

Although the last example seems to be a "quantum field", it is not. It is only the interpretation of the wave function, which connects it to quantum mechanics. "Quantum fields", on the other hand are – very roughly speaking – fields of quantum operators (see *Quantum Mechanics II*).

The notion of a field implies that there is a"field object space" $\mathcal{F}$ (like $\mathbb{R}$ or a Euclidean vector space $\mathbb{R}^3$, or a group like the rotations $O(3)$ or the complex numbers $\mathbb{C}$ or whatever else) and that you pick an element of this space for each point in "physical space" (or "physical space-time") $\mathcal{P}$. So a field may be considered as a map (mathematicians like that point of view)

$$\mathcal{P} \to \mathcal{F}$$

$\mathcal{F}$ and $\mathcal{P}$ may be completely unrelated or related such that transformations performed in the physical space affect the field map. As an example, think of the density field $\rho(\boldsymbol{r})$ and the velocity field $\boldsymbol{v}(\boldsymbol{r})$ of a fluid. Let us perform a rotation $\hat{R}$ of the physical system (i.e. of the streaming fluid), which changes $\boldsymbol{r} \to \hat{R}\boldsymbol{r}$, and introduce $\rho^R()$ and $\boldsymbol{v}^R()$ the transformed functions (maps), which may or may not be identical to the original functions. For the density you easily see that

$$\rho^R(\hat{R}\boldsymbol{r}) = \rho(\boldsymbol{r})$$

whereas for the velocity

$$\boldsymbol{v}^R(\hat{R}\boldsymbol{r}) = \hat{R}\boldsymbol{v}(\boldsymbol{r})$$

the transformed map has changed, because the direction of the velocity is affected by the rotation. There are also vector fields, which are not affected by rotations. As an example, consider the complex field $\Psi = \Psi_1 + i\Psi_2$ as a two-dimensional vector field with components Ψ_1 and Ψ_2. Obviously, $\Psi^R(\hat{R}\boldsymbol{r}) = \Psi(\boldsymbol{r})$. Physical properties, corresponding to elements of vector spaces, groups etc., which are not affected by space-time transformations are also called *internal*. In internal spaces, there may also be physical transformations. As an example think of the multiplication of Ψ by a phase factor, which is equivalent to a rotation in a two-dimensional, internal space.

If you look at our examples, you see that it is very natural in physics to require the field map $\mathcal{P} \to \mathcal{F}$ to be smooth. If not stated otherwise, we will always assume "smooth fields" in the sense that they are continuously differentiable with respect to the space-time arguments as often as we need it.

The idea, which is "dual" to physical fields is that physical properties are concentrated in point-like objects. In classical physics, point particles may be described as idealizations of smooth densities and vice versa. The passage from a smooth density to a point like object involves a limit of a sequence of smooth densities ρ_n, which become more and more localized around some point $\boldsymbol{r}_0$ as n increases, while the total amount of physical property Q represented by the density (be it mass, charge or whatever) remains constant, i.e. $\int dV \rho_(\boldsymbol{r}) = Q$. Briefly, this can be stated as follows: *The density of a point like object is a delta*

function

$$\rho(\boldsymbol{r}) = Q\delta(\boldsymbol{r} - \boldsymbol{r}_0) \quad .$$

The reverse operation, i.e. constructing smooth fields from point-like objects, can become a very subtle task, but some steps are quite straightforward. You start from the unsmoothed density of point-like objects

$$\rho(\boldsymbol{r}) = q \sum_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$$

A smoothing operation may result from one of several distinct physical mechanisms.

An obvious possibility of smoothing results from the measuring process. The measuring apparatus will in general have a finite resolution and count all particles inside a resolution volume $\Delta V(\boldsymbol{r})$ around $\boldsymbol{r}$. If there are M particles inside the volume, the apparatus attaches the density $\rho(\boldsymbol{r}) = M/\Delta V(\boldsymbol{r})$ to the volume element located at $\boldsymbol{r}$. Replacing a collection of point particles by a density in this way will be an appropriate approximation, if there are many particles in every measuring volume and the number of particles changes little between neighboring volume elements.

There is a more physical smoothing process in condensed matter, which results from the thermal motion of the pointlike constituents. Due to thermal fluctuations, the positions of the particles become random variables and we know from statistical mechanics, that we usually observe thermal averages. Let us denote the averaging by

$$< \cdots >= \int \prod_i d^d \boldsymbol{r}_i p(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)(\cdots)$$

In equilibrium we know that

$$p(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \frac{1}{Z} \exp[-H(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)/kT]$$

where H is the energy of the system. Thus, the observed density will be the smooth function

$$< \rho(\boldsymbol{r}) >= \sum_{i=1}^{N} < \delta(\boldsymbol{r} - \boldsymbol{r}_i) >$$

This is a perfect smoothing operation, because it is accomplished by the system itself. However, it also poses new problems as soon as we consider *nonlinearities*. We have to face the fact, that in general

$$< \rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2 > \neq < \rho(\boldsymbol{r}_1 >< \rho(\boldsymbol{r}_2 > \quad .$$

We can get non-linear, closed equations for the density only if we can neglect these effects of fluctuations. Remember, that statistical physics gives reliable answers to the size of such effects.

1.1.2 Differentiating Functionals

In a system with pointlike objects, you have to handle functions of the positions, momenta etc. of these objects. These are (usually smooth) functions of finitely many variables. Think of forces on a particle, the energy of a many particle system etc. All the analytic calculations you perform involve functions of finitely many variables

$$F(q_1, q_2, \ldots, q_N)$$

which you have to differentiate, integrate, etc.
In a field theory, the field objects ϕ at all the (continuously many) points in space (or space-time) are degrees of freedom. We have to face the fact, that fields are physical systems with *infinitely many degrees of freedom*. The analogues of functions of finitely many variables are the *functionals*, which map functions into the real or complex numbers. We denote them by

$$F[\phi] \ .$$

The form of the brackets indicates that F maps functions. This notation does not contain information about the nature of the functions ϕ. If we want to indicate, that ϕ are functions mapping, for example, position vectors, we will also use

$$F[\phi(\boldsymbol{r})] \ .$$

You should remember that the notation implies, that the function is mapped and not (what might be suggested) the value of the function at position $\boldsymbol{r}$. If ϕ has several components, for example ϕ_α, we use a list notation: $\{\phi_\alpha\} = \phi_1, \phi_2, \ldots$. A functional may also be a function of one or several additional variables $t_1, \cdots, t_n$. To describe such objects we use the notation

$$F[\{\phi_\alpha\}; t_1, \cdots, t_n] \ .$$

Functional differentiation is the analogue of differentiation of a smooth function. Thus, it has to be performed quite often in physical calculations. The first time you will have encountered a derivative of a functional was in the context of theoretical mechanics. Mechanics can be formulated as an extreme value problem in function space, an approach, which is known as *the principle of least action, Hamilton's principle*. The action functional is a special functional of the trajectories of a system of point particles, $\{q_\alpha(t)\}$:

$$S[\{q_\alpha(t)\}] = \int_{t_1}^{t_2} dt L(\{q_\alpha(t), dq_\alpha/dt\}) \ ,$$

where $L(\{q_\alpha(t), dq_\alpha/dt\})$ is the Lagrange function of the mechanical system.

We define the derivative by considering the change of S under small changes of the trajectories $\{q_\alpha\} \to \{q_\alpha + \epsilon\delta q_\alpha\}$ to linear order in ϵ. We consider S as a functional, which maps trajectories connecting a fixed initial point $\{q_\alpha(t_1)\}$ to a fixed final point $\{q_\alpha(t_2\}$. Therefore the δq_α have to vanish at t_1 and t_2. From the usual Taylor expansion of the Lagrange function L one gets

$$\begin{aligned} L(\{q_\alpha(t) + \epsilon\delta q_\alpha, dq_\alpha/dt + \epsilon d\delta q_\alpha/dt\}) - L(\{q_\alpha(t), dq_\alpha/dt\}) \quad = \\ +\epsilon \sum_\alpha \left[\left(\frac{\partial L}{\partial q_\alpha} \right) \delta q_\alpha + \left(\frac{\partial L}{\partial dq_\alpha/dt} \right) \frac{d\delta q_\alpha}{dt} \right] + O(\epsilon^2) \quad . \end{aligned}$$

Inserting the expansion into S and performing a partial integration (using the "boundary conditions") one finds

$$\delta S[q_\alpha] = \int_{t_1}^{t_2} dt \sum_\alpha \left[\left(\frac{\partial L}{\partial q_\alpha} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial dq_\alpha/dt} \right) \right] \delta q_\alpha \quad .$$

Nothing prevents us to generalize this calculation to functions $\phi_\alpha(\boldsymbol{r})$, which depend on several variables $\boldsymbol{r}$ instead of one variable t. For simplicity of notation, let us consider a one component field $\phi(\boldsymbol{x})$ living on d-dimensional space. We consider functionals of the form

$$F[\phi(\boldsymbol{x})] = \int d^d x f(\phi(\boldsymbol{x}), \boldsymbol{\nabla}\phi)$$

Repeating the above arguments we get the first variation of F

$$\delta F[\phi] = \int d^d x \left[\frac{\partial f}{\partial \phi(\boldsymbol{x})} - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial(\partial\phi/\partial x_i)} \right) \right] \delta\phi(\boldsymbol{x})$$

Note that we have performed a partial integration and have to require that the small deviations $\delta\phi(\boldsymbol{r})$ have to vanish on the boundary of the $\boldsymbol{x}$-integration.
To approach the concept and calculus of *partial differentiation*, which you are used to for functions of finitely many variables, remember that such a calculus is based to a large extent on the linear change of a function given in the form

$$f(\{q_i + \delta q_i\}) = f(\{q_i\}) + \sum_{i=1}^{N} \frac{\partial f}{\partial q_i} \delta q_i$$

Can we generalize partial differentials to functionals, such that

$$F[\phi + \delta\phi] = F[\phi] + \int d^d x \frac{\delta F}{\delta\phi(\boldsymbol{x})} \delta\phi(\boldsymbol{x})?$$

The answer is yes. We are not interested in mathematical details here, rather we are interested in a physical heuristics.

Imagine that you divide $\boldsymbol{x}$ space into small cells of volume $\Delta V(\boldsymbol{x}_i)$, centered at the discrete positions $\boldsymbol{x}_i$. Then from the functional F we may define a *function* $\tilde{F}$ such that

$$\tilde{F}(\{\phi_i\}) = F[\phi]$$

where

$$\phi_i = \int\limits_{\boldsymbol{x}\in\Delta V(\boldsymbol{x}_i)} \phi(\boldsymbol{x}) d^d x.$$

If we consider sequences of refined cells, we can approximate any reasonable field configuration to any desired precision. For the function $\tilde{F}$ we can perform a conventional Taylor expansion

$$\begin{aligned}\tilde{F}(\{\phi_i + \delta\phi_i\} &= \tilde{F}(\{\phi_i\}) + \sum_i \frac{\partial \tilde{F}}{\partial \phi_i}\delta\phi_i \\ &+ \frac{1}{2}\sum_{i,j} \frac{\partial^2 \tilde{F}}{\partial\phi_i \partial\phi_j}\delta\phi_i\delta\phi_j + \cdots\end{aligned}$$

Now we want to consider the limit of increasing refinements $\Delta V \to 0$. If everything is nicely behaved we can view the sums in the terms of the Taylor expansion as approximants of integrals and $\tilde{F}$ as our original functional F, thus replacing

$$\sum_i \Delta V \to \int d^d x$$

For the first order term we then obtain

$$\sum_i \Delta V \frac{1}{\Delta V}\frac{\partial \tilde{F}}{\partial \phi_i}\delta\phi_i \to \int d^d x \frac{\delta F}{\delta\phi(\boldsymbol{x})}\delta\phi(\boldsymbol{x})$$

with

$$\frac{\delta F}{\delta\phi(\boldsymbol{x})} = \lim_{\Delta V\to 0}\lim_{\delta\phi_i\to 0}\frac{\tilde{F}(\phi_1,\ldots,\phi_i+\delta\phi_i,\ldots) - \tilde{F}(\{\phi_i\})}{\Delta V \delta\phi_i} \tag{1.1}$$

This is our (heuristic) generalization of partial differentials of functions to functionals. The object (1.1) is called *functional derivative*. The most important thing about our heuristics is that it allows to transfer many well-known rules of usual calculus to functional calculus. The most important rule, which you will use over and over again is

$$\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{y})} = \delta^d(\boldsymbol{x}-\boldsymbol{y}) \qquad \left(\text{from } \frac{\partial\phi_i}{\partial\phi_j} = \delta_{ij}\right)$$

Furthermore, from the chain rule of differentiation we get

$$\frac{\partial\chi(\phi(\boldsymbol{x}))}{\partial\phi(\boldsymbol{y})} = \frac{\partial\chi}{\partial\phi}\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{y})} = \frac{\partial\chi}{\partial\phi}\delta^d(\boldsymbol{x}-\boldsymbol{y})$$

Finally, linearity of functional differentiation is obvious, in particular

$$\frac{\delta}{\delta\phi(\boldsymbol{y})}\int d^dx G[\phi;\boldsymbol{x}] = \int d^dx \frac{\delta G[\phi;\boldsymbol{x}]}{\delta\phi(\boldsymbol{y})}$$

These rules are sufficient for a powerful calculus. As an example, consider our functional $F[\phi] = \int d^dx f(\phi, \nabla\phi)$. By mechanistic application of the 2 rules, you easily recover the result from above

$$\frac{\delta F}{\delta\phi(\boldsymbol{y})} = \int d^dx \left(\frac{\partial f}{\partial\phi(\boldsymbol{x})}\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{y})} + \sum_i \frac{\partial f}{\partial(\partial_i\phi(\boldsymbol{x}))}\partial_i\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{y})}\right)$$

Replacing the functional derivatives of $\phi(\boldsymbol{x})$ by δ-functions and performing a partial integration (boundary terms required to vanish), we can finally integrate over $\boldsymbol{x}$ to get

$$\frac{\delta F}{\delta\phi(\boldsymbol{y})} = \frac{\partial f}{\partial\phi(\boldsymbol{y})} - \boldsymbol{\nabla}\cdot\frac{\partial f}{\partial(\boldsymbol{\nabla}\phi(\boldsymbol{y}))}$$

Note that it is almost trivial to extend the definition of the first functional derivative to higher order functional derivatives. So we may also transfer the whole idea of a Taylor expansion to functionals, where it is called *Volterra expansion*. The first terms look as follows

$$F[\phi+\delta\phi] = F[\phi] + \int d^dx \frac{\delta F}{\delta\phi(\boldsymbol{x})}\delta\phi(\boldsymbol{x}) + \frac{1}{2}\int d^dx d^dy \frac{\delta^2 F}{\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})}\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})$$

Part I

Non-Relativistic Field Theories

Some common aspects of all non-relativistic field theories physics are:

- *Physical space* is a space of geometrical points, which has the structure of a 3-dimensional, affine, Euclidean space $\mathcal{E}_3$. The associated vectorspace V^3 is isomorphic to $\mathbb{R}^3$.

- a field configuration ϕ takes on values in a set F and may be considered as a map
$$\phi : \mathcal{E}_3 \longrightarrow F$$
with $\phi(\boldsymbol{r})$ denoting the value of this map at a point $\boldsymbol{r} \in \mathcal{E}_3$.

- Time dependent field configurations are maps $\phi : \mathcal{E}_3 \times \mathbb{R} \longrightarrow F$ with values $\phi(\boldsymbol{r}, t)$.

- If $F \subset \mathbb{R}$, the field is called a *(real) scalar field*. Examples are temperature or pressure fields.

- If F is a vector space, ϕ is called a *vector field*. Sometimes, the term "vector field" is only used in the narrower sense: if the vector space F equals V^3 (the vector space associated to the affine space of geometric points). Think of a force field $\boldsymbol{F}(\boldsymbol{r})$ or an electric or magnetic field. The vectors of such fields have a direction in the physical geometric space!

- Vector fields in the narrower sense have special transformation rules under rotations and translations. Suppose we actively rotate all points of the space $\boldsymbol{r} \to \hat{R}\boldsymbol{r}$. (For example, if there is only a charged capacitor in space, we take this capacitor and rotate it.) Then we get a new field configuration $\boldsymbol{\varphi}^R$, such that
$$\boldsymbol{\varphi}^R(\hat{R}\boldsymbol{r}) = \hat{R}\boldsymbol{\varphi}(\boldsymbol{r})$$
This means that the field at the image of $\boldsymbol{r}$ under rotation ($\hat{R}\boldsymbol{r}$) is calculated by rotating the value of the field at $\boldsymbol{r}$. If we insert $\boldsymbol{y} = \hat{R}\boldsymbol{r}$ we can also write this as
$$\boldsymbol{\varphi}^R(\boldsymbol{y}) = \hat{R}\boldsymbol{\varphi}(\hat{R}^{-1}\boldsymbol{y})$$
So the new field map is constructed from the old one by the composition
$$\boldsymbol{\varphi}^R = \hat{R} \circ \boldsymbol{\varphi} \circ \hat{R}^{-1}$$
This way of writing transformation of fields should remind you of the way transformations were implemented in Quantum Mechanics!

Chapter 2

Ginzburg-Landau Theory

2.1 Ordered States of Matter

A ferromagnet is an ordered state of matter. It is characterized by a *spontaneous magnetization*, i.e. a magnetic moment, which appears at vanishing magnetic field in thermal equilibrium. If the temperature is increased above the so called *Curie temperature* T_c, the spontaneous magnetization vanishes and the material is in the *paramagnetic* or disordered state. In a real, ferromagnetic material, the magnetic moment varies in space (forming magnetic domains etc). Let $\boldsymbol{M}(\boldsymbol{r})$ be the thermal average of the total magnetic moment of a small volume element $\Delta V(\boldsymbol{r})$ located at $\boldsymbol{r}$ and let $\boldsymbol{m}(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r})/\Delta V(\boldsymbol{r})$. This field is an example of a (smoothed) *order parameter field*. Its presence signals a particular ordered state of the material. Other examples of such ordered states are

- the ferroelectric state, signaled by a non-vanishing spontaneous electric moment
- the anti-ferromagnetic state, signaled by a non-vanishing periodic structure of the microscopic magnetic moments.
- the crystalline state, signaled by a non-vanishing periodic density on atomic scales.
- the superconducting state, signaled by a non-vanishing *condensate order parameter*, which indicates a special electronic correlation (pairing).

In general, the order parameter of a ferromagnet is a *vector* $\boldsymbol{M}$. There are variants of a ferromagnetic state with simpler order parameters. For example, in an *easy axis ferromagnet*, the magnetization is always directed parallel to a fixed axis ($\boldsymbol{e}_z$), so that the order parameter is completely characterized by a single number $M = \boldsymbol{M} \cdot \boldsymbol{e}_z$.
Studying ordered states of matter (and discovering new ones) makes up a large part of modern condensed matter physics. Therefore a question of central importance is:

> How can we determine $\boldsymbol{m}(\boldsymbol{r})$?

Ginzburg and Landau remarked that we have a very powerful theorem (from thermodynamics) at hand, which provides an excellent starting point to answer the question:

> If x is an unconstrained variable and $F(x, \ldots)$ is a thermodynamic potential at *fixed* x, the equilibrium value of x will be the one, which minimizes $F(x, \ldots)$.

So we can find $\boldsymbol{m}(\boldsymbol{r})$, if we know the form of the (constrained) thermodynamic potential $F[\boldsymbol{m}(\boldsymbol{r}); T, \boldsymbol{H}]$, i.e. the magnetic free energy, *calculated at a fixed profile of the magnetization*[1].

2.2 The Ginzburg Landau Expansion at Work

Two very powerful assumptions and a symmetry constraint (item 3.) lead Ginzburg and Landau to general principles fixing the form of $F[\boldsymbol{m}(\boldsymbol{r}); T, \boldsymbol{H}]$ in terms of very few phenomenological parameters:

1.) $F[\boldsymbol{m}; T, \boldsymbol{H}]$ is an *analytic function* of $\boldsymbol{m}$, so that F may be expanded in powers of the order parameter, and may be approximated by the first few terms of the expansion whenever the order parameter is small. In particular, this is the case in the vicinity of a continuous phase transition.

2.) For short range interactions, field configurations $\boldsymbol{m}(\boldsymbol{r})$ with small F are smooth and thus we may expand F in terms of increasing orders of spatial derivatives $\partial_i m_j$ (*gradient expansion*).

3.) F is a scalar quantity and thus the possible terms of the expansions of F are limited by symmetry.

First, let us illustrate the Ginzburg Landau expansion for the simple case of an easy axis ferromagnet. We only consider magnetic fields, which are parallel to the easy axis, $\boldsymbol{H} = H\boldsymbol{e}_z$. In general $F[m(\boldsymbol{r}), T, H]$ is a *functional* of the field $m(\boldsymbol{r})$.
The analyticity assumption (1.) tells us, that the *Volterra expansion* of F makes sense:

$$\begin{aligned} F[m(\boldsymbol{r}), T] \;=\; & F_0(T,H) + \int d^d r_1 a_1(\boldsymbol{r}_1, T, H) m(\boldsymbol{r}_1) + \\ & + \frac{1}{2}\int d^d r_1 \int d^d r_2 a_2(\boldsymbol{r}_1, \boldsymbol{r}_2, T, H) m(\boldsymbol{r}_1) m(\boldsymbol{r}_2) + \\ & \ldots + \frac{1}{n!}\int d^d r_1 \cdots \int d^d r_n a_n(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n, T, H) m(\boldsymbol{r}_1)\cdots m(\boldsymbol{r}_n) + \\ & + \text{higher order terms} \end{aligned} \tag{2.1}$$

Now let us explore the consequences of symmetry. Note that all the coefficient functions of the Volterra expansion are functional derivatives of F at vanishing

[1] Don't get confused by the fact that this function depends both on magnetization and on magnetic field. If you do get confused, consult a good textbook on thermodynamics!

magnetic moment (i.e. in the paramagnetic state), for example:

$$a_2(\mathbf{r}_1, \mathbf{r}_2, T, H) = \left. \frac{\delta^2 F}{\delta m(\mathbf{r}_1)\delta m(\mathbf{r}_2)} \right|_{m=0}$$

Therefore the coefficients must respect the symmetries of the paramagnetic state, in particular

1.) translation symmetry

2.) $\boldsymbol{e}_z \to -\boldsymbol{e}_z$ combined with $H \to -H$,

3.) rotations around z-axis

The following statements thus follow from symmetry arguments:

- Due to translation symmetry, the coefficients a_n can only depend upon differences of position vectors. In particular, a_1 has to be independent of $\boldsymbol{r}_1$.
- If $H = 0$, symmetry 2) implies that the expansion can only contain *even powers* of m (because F has to be invariant under $m \to -m$).

Now we turn to the gradient expansion. a_2 is the first coefficient, which depends on position vectors, so we consider the corresponding contribution to F, which we write in the form, which already makes use of translation invariance:

$$\int d^d r \int d^d x a_2(\boldsymbol{x}, T, H) m(\mathbf{r}) m(\boldsymbol{r} + \boldsymbol{x})$$

For simplicity, let us first restrict the discussion to $H = 0$.
What is the physical meaning of a_2? Suppose you change the magnetization around $\boldsymbol{r}_1$ a little bit. This will lead to a change in free energy

$$\Delta F \propto \frac{\delta F}{\delta m(\boldsymbol{r}_1)}$$

a_2 contains the following information: How is the change ΔF modified further by an additional change of the magnetization around $\boldsymbol{r}_2$? Now imagine a paramagnet being divided into a number of volume elements containing enough degrees of freedom for a sensible thermal average (e.g. 100-1000 out of 10^{23} spins). Then the properties of each volume element are statistically independent of the state of the other volume elements[2]. Stated otherwise:

> In a disordered state of matter, correlations between order parameter fluctuations are of microscopic range.

[2] In case you forgot: This a cornerstone of statistical mechanics

This implies that $a_2(\boldsymbol{x})$ decays on "microscopic" scales, whereas the interesting (low F!) configurations of the order parameter field are smooth and only change significantly on much larger length scales. As a consequence, we may expand

$$m(\boldsymbol{r}+\boldsymbol{x}) = m(\boldsymbol{r}) + \sum_i \partial_i m(\boldsymbol{r}) x_i + \frac{1}{2}\sum_{i,j} \partial_i \partial_j m(\boldsymbol{r}) x_i x_j + O(x^3)$$

and insert the expansion into the contribution to F. Note that the x integrations then no longer involve the magnetization. The term

$$\int d^d x x_i a_2(\boldsymbol{x}, T)$$

has to vanish and

$$\int d^d x x_i x_j a_2(\boldsymbol{x}, T) = -\xi^2 \delta_{i,j}$$

due to symmetry (change $x_i \to -x_i$ within the integration). In addition let us define

$$a(T) = \int d^d x a_2(\boldsymbol{x}, T)$$

Thus the contribution to F is (up to second order of spatial derivatives):

$$\int d^d r \frac{a(T)}{2} m^2 - \frac{\xi^2}{2} m \nabla^2 m$$

This term is the leading contribution to $F[m; T, H = 0]$ for weak spatial variation of $m(\boldsymbol{r})$ and small $m(\boldsymbol{r})$ (the 2 small parameters, which control the Ginzburg Landau expansion). The next terms will be $O(m^4)$ and $O(\nabla^4 m^2)$.
Let us slightly rewrite the leading term by performing a partial integration:

$$F(T) - F_0(T) \approx \int d^d r \frac{a(T)}{2} m^2 + \frac{\xi^2}{2} (\nabla m)^2$$

The boundary terms vanish for a piece of ferromagnet embedded in free space.

2.3 Continuous Phase Transition

Now you have seen the Ginzburg Landau expansion at work and you can work out all the higher order terms for yourself. But an important question remains: when shall we stop?
Let us examine the leading order term more closely and try to find the configuration of minimal F. Here we have to face a problem: If the coefficient $a < 0$ or $\xi^2 < 0$ F is not bounded from below. You can get arbitrarily small F by either making m larger ($a < 0$) or by making m rougher (increasing ∇m if $\xi^2 < 0$).

On the other hand, if both a and ξ^2 are positive, the absolute minimum of F is the trivial solution $m = 0$ (corresponding to the paramagnetic state). Thus, the leading order terms are obviously not sufficient to study the ferromagnetic phase.
Both from physical plausibility and from results on correlations of model systems we know that $\xi^2 > 0$ for simple ferromagnets: If ξ^2 would be negative, the minimum of F would be a non-homogeneous structure, which is not compatible with a simple ferromagnet[3]. Thus we only need additional terms of higher orders in m, but without spatial derivatives of m. Without magnetic field (i.e. the symmetry $m \to -m$) the next contribution comes from the fourth-order term in (2.1). According to the introductory remarks, derivatives of $m(\boldsymbol{r})$ can be neglected here, i.e. we may replace

$$m(\boldsymbol{r}_1)m(\boldsymbol{r}_2)m(\boldsymbol{r}_3)m(\boldsymbol{r}_4) \to m(\boldsymbol{r}_1)^4 \ .$$

If we now substitute $\boldsymbol{r}_1 = \boldsymbol{r}$, $\boldsymbol{r}_2 = \boldsymbol{r} + \boldsymbol{x}_2$, $\boldsymbol{r}_3 = \boldsymbol{r} + \boldsymbol{x}_3$, $\boldsymbol{r}_4 = \boldsymbol{r} + \boldsymbol{x}_4$ and use translational invariance, the next order term can be written as

$$\frac{u(T)}{4!} \int d^d r m(\mathbf{r})^4$$

with

$$u = \int d^d x_2 \int d^d x_3 \int d^d x_4 a_4(\mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) \ .$$

The standard Ginzburg Landau equation of the easy axis ferromagnet thus becomes:

$$F_{GL} = F - F_0 = \frac{1}{2} \int d^d r [a(T)m^2 + \frac{u}{12} m^4 + \xi^2 (\nabla m)^2] \tag{2.2}$$

The temperature dependence of $a(T)$ is the important feature, which selects paramagnetic or ferromagnetic state as the minimum of F.
Let now assume that there exists a *critical Temperature* T_c with $m(T > T_c) = 0$. Provided that $u(T) > 0$ (which actually is required by global stability[4]) we necessarily must have $a(T > T_c) > 0$ to stabilize the paramagnetic state. When we lower the temperature, the paramagnetic state looses its stability at T_c and a *homogeneous* ferromagnetic state appears. The transition is called *continuous*, because the order parameter m vanishes continuously for $T \nearrow T_c$. As the state of minimal F is homogeneous on both sides of T_c, we conclude that ξ^2 stays positive. On the other hand $a(T)$ has to change sign.

[3] Even if we allow for magnetic *domains* like in real ferromagnets, the magnetization within a macroscopic domain usually is homogeneous.

[4] Global stability of F is required because we consider stable systems.

To see this clearly, we note that from minimizing (2.2), homogeneous states have to obey the relation

$$[a(T) + \frac{u}{6}m^2]m = 0$$

Clearly, a solution $m \neq 0$ requires $a(T) < 0$.
For $T > T_c$, $m = 0$ is the absolute minimum of F (see Fig. 2.1). For $T < T_c$, $m = 0$ is a local maximum, whereas $\bar{m} = \pm\sqrt{6|a|/u}$ are the absolute minima.

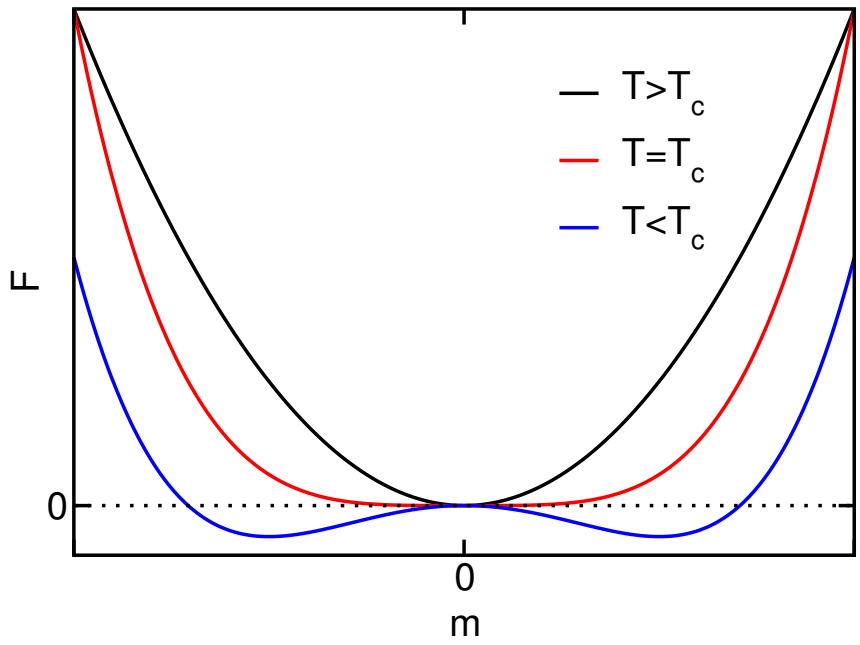

Figure 2.1: Schematic behavior of $F[m]$ for $T > T_c$, $T = T_c$ and $T < T_c$.

As $a(T)$ is a property of the paramagnetic system we may safely assume that it is smooth in the vicinity of T_c and thus, close to T_c it should have the form

$$a(T) = a_0(T - T_c) + O([T - T_c]^2)$$

This implies a very interesting and quantitative result: *Within Ginzburg-Landau theory, the spontaneous magnetization in the vicinity of T_c behaves like*

$$m(T) \propto \Theta(T_c - T)(T_c - T)^{1/2}$$

Thus m vanishes with a square root singularity (infinite slope!). Many more interesting and quantitative thermodynamic results can be obtained from Ginzburg-Landau theory, which you may look up in [11].

2.4 Interface Solution

Let us now consider spatially inhomogeneous states. They have to obey the *Ginzburg Landau (field) equation*:

$$\frac{\delta F}{\delta m(\mathbf{r})} = 0 = a(T)m(\mathbf{r}) + \frac{u}{6}m^3(\mathbf{r}) - \xi^2\nabla^2 m(\mathbf{r})$$

As specific example consider a *magnetic wall* or *interface*. Suppose that in a large system $\bar{m} = -\sqrt{6|a|/u}$ for $z \to -\infty$, whereas $\bar{m} = +\sqrt{6|a|/u}$ for $z \to \infty$ is enforced. This situation idealizes magnetic domains. Asymptotically, the values of the spontaneous magnetization thus correspond to the two possible equilibrium values. Now we ask for the profile $m(z)$ of the order parameter field, which is the structure of the interface between coexisting ordered states. Due to symmetry, the profile will only depend on z and the GL equation simplifies somewhat:

$$\xi^2\frac{d^2m}{dz^2} = a(T)m + \frac{u}{6}m^3$$

Before we proceed to actually solve this equation, let us point out an illuminating analogy, which helps you to find wall-like solutions in many (more complicated) situations. Note that the equation for the order parameter field looks like a Newtonian equation of one-dimensional motion, if we identify z with time, ξ^2 with mass and the right-hand side with a force. The force is conservative with a potential

$$U = -\frac{a}{2}m^2 - \frac{u}{24}m^4$$

this analogy tells us that a wall like solution corresponds to a motion from one maximum of U to the other maximum in infinite time. You should have enough experience with classical mechanics by now to draw a rough sketch of this particular motion immediately.

If you look for solutions of a field equation, it is always a good idea to first rescale variables into a dimensionless form. So let us rescale m with the positive equilibrium solution:

$$\frac{\xi^2}{|a|}\frac{d^2(m/\bar{m})}{dz^2} = -\left[1 - \frac{m^2}{\bar{m}^2}\right]\frac{m}{\bar{m}}$$

Next we rescale lengths by introducing $y = z/(\xi/\sqrt{|a|}) = z/\xi(T)$. With $\hat{m} = m/\bar{m}$ the rescaled equation takes on the form

$$\frac{d^2\hat{m}}{dy^2} = -\left[1 - \hat{m}^2\right]\hat{m}$$

It is easy to check that

$$\hat{m}(y) = \tanh(y/\sqrt{2})$$

is the solution, which obeys the required boundary conditions for $z \to \pm\infty$. This corresponds to the unscaled solution

$$m(z) = \bar{m} \tanh\left(\frac{z}{\sqrt{2}\xi(T)}\right)$$

The length scale

$$\xi(T) \approx \xi_0 \cdot (T_c - T)^{-1/2} \text{ near } T_c$$

characterizes the width of the interface. This width diverges as the temperature approaches the critical temperature T_c from below.

Chapter 3

Kinematics of Deformable Media

3.1 Displacement, Distortion and Strain Tensor

Consider a deformable medium in an undistorted state. In a spatially fixed Cartesian coordinate system (independent of the medium) a specific material point of this medium is characterized by the position vector $\boldsymbol{R}$. Note that we may use the position vectors in the undistorted state to index the material points of the medium.[1] Now suppose that we distort the medium, so that the location of the material point $\boldsymbol{R}$ is transformed into $\boldsymbol{x}(t, \boldsymbol{R})$ at time[2] t. After the distortion is completed (at time T) the new position of the material point is

$$\boldsymbol{x}(T, \boldsymbol{R}) =: \boldsymbol{x}(\boldsymbol{R}) = \boldsymbol{R} + \boldsymbol{u}(\boldsymbol{R})$$

$\boldsymbol{u}$ is called the *displacement vector*. A homogeneous displacement $\boldsymbol{u}(\boldsymbol{R}) = \boldsymbol{u}$ corresponds to a rigid translation of the entire body and will not change the internal state of the medium (in particular its internal energy) at all.
Now consider two neighboring material points $\boldsymbol{R}$ and $\boldsymbol{R} + d\boldsymbol{R}$ in the undistorted state, so that their distance is $dR = \sqrt{(d\boldsymbol{R}) \cdot (d\boldsymbol{R})}$. In the distorted state, the distance is changed to $dx = \sqrt{(d\boldsymbol{x}) \cdot (d\boldsymbol{x})}$ with $d\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{R} + d\boldsymbol{R}) - \boldsymbol{x}(\boldsymbol{R})$. In Cartesian components

$$dx_i = dR_i + \sum_j \frac{\partial u_i}{\partial R_j} dR_j + O(dR_i^2)$$

The field

$$D_{ij} = \partial_j u_i = \frac{\partial u_i}{\partial R_j}$$

is called *distortion tensor*. It contains the information about *infinitesimal* local distortions in the sense, that vectors between neighboring material points in the distorted state are considered as linear functions of the vectors in the undistorted reference state.

Note: In the following we will use a Cartesian summation convention. It says, that every Cartesian index appearing twice in an expression has to be summed over, for example

$$(\partial_i a_{ij}) b_{jl} = \sum_i \sum_j (\frac{\partial a_{ij}}{\partial x_i}) b_{jl}$$

This saves a lot of $\sum$ signs.

[1] Imagine, that a particular material point, located at $\boldsymbol{r}$ is marked by a little spot of colored ink. Even if the position of the material point will change later on, we will always refer to the marked point as "the material point that once was at $\boldsymbol{R}$". In this sense, position vectors are indices for material points.

[2] Note that $\boldsymbol{x}(t = 0, \boldsymbol{R}) = \boldsymbol{R}$

3.1.1 Physical Interpretation of Distortion

The distortion tensor contains information about those small displacements of a deformable medium, which are not homogeneous translations:

$$du_i = dx_i - dR_i = D_{ij} dR_j$$

We can learn more about the distortion tensor if we split it into its *symmetric* and its *anti-symmetric* part

$$D_{ij} = \frac{D_{ij} - D_{ji}}{2} + \frac{D_{ij} + D_{ji}}{2} = \omega_{ij} + \epsilon_{ij}$$

The action of the anti-symmetric part may be written as a cross product

$$\omega_{ij} dR_j = [\boldsymbol{\phi} \times d\boldsymbol{R}]_i$$

with $\phi_1 = \omega_{32}$, $\phi_2 = \omega_{13}$, $\phi_3 = \omega_{21}$. From mechanics (rigid body) we know that this corresponds to a rotation of $d\boldsymbol{R}$ with angle $|\boldsymbol{\phi}|$ around the $\boldsymbol{\phi}$ axis (passing through $\boldsymbol{R}$).

The symmetric part ϵ_{ij} may be diagonalized by changing to a rotated coordinate system with basis $\boldsymbol{e}_\alpha$, $\alpha = 1, 2, 3$, which are the principle axes of the tensor. In this system $d\boldsymbol{R} = \hat{R}_\alpha \boldsymbol{e}_\alpha$ and the action of ϵ on $\hat{R}_i$ is particularly simple; it just corresponds to the multiplication with the corresponding eigenvalue $\epsilon^{(\alpha)}$ as $\epsilon_{\alpha\beta} = \epsilon^{(\alpha)} \delta_{\alpha\beta}$.

Such a simple rescaling of length in 3 directions corresponds to a *deformation* of infinitesimal volume elements around $\boldsymbol{R}$. This type of motion transforms a cube oriented parallel to the principle axis into a general rectangular box without changing the direction of the axes of the cube. A deformation will in general change the volume of a volume element. For a pure deformation ($\omega_{ij} = 0$) we have from $d\boldsymbol{u} = d\boldsymbol{x} - d\boldsymbol{R}$:

$$dx_\alpha = (1 + \epsilon^{(\alpha)}) dR_\alpha$$

(no summation over α here!) and thus for an infinitesimal volume element

$$dV_{deformed} = dx_1 dx_2 dx_3 = (1 + \epsilon^{(1)})(1 + \epsilon^{(2)})(1 + \epsilon^{(3)}) dR_1 dR_2 dR_3$$

This relation can only hold to first order in ϵ as we only considered this order in defining the distortion tensor. Thus

$$dV_{deformed} = (1 + \epsilon^{(1)} + \epsilon^{(2)} + \epsilon^{(3)}) dV$$

Remember that the trace of a second rank tensor is invariant under rotations, so that the sum over eigenvalues may also be replace by $\mathrm{Tr}(\epsilon) = \epsilon_{ii}$ and we may write in coordinate-free notation

$$dV_{deformed} = [1 + \mathrm{Tr}(\epsilon)] dV \quad .$$

Deformations, which do not change volume elements are called *shear deformations*. Every deformation can be decomposed into a shear and an *isotropic compression* (characterized by a diagonal deformation tensor $\epsilon_{ij} = \epsilon\delta_{ij}$). The decomposition looks as follows

$$\epsilon_{ij} = \left(\epsilon_{ij} - \frac{\mathrm{Tr}(\epsilon)}{3}\delta_{ij}\right) + \frac{\mathrm{Tr}(\epsilon)}{3}\delta_{ij}$$

To remember:

> An arbitrary infinitesimal displacement field of a deformable medium can be locally decomposed into
>
> - a translation
> - a rotation
> - a deformation
>
> This decomposition is unique.

3.1.2 The Strain Tensor

Interactions between material points (atoms!) depend upon distances between these points. The change in distance between neighboring points may be written as

$$dx^2 - dR^2 = 2(\partial_j u_i)dR_i dR_j + (\partial_i u_k)(\partial_j u_k)dR_i dR_j$$

Note that in the second term on the right hand side, it is only the *symmetric part* of the distortion tensor that actually enters as

$$(\partial_i u_j)dR_i dR_j = (\partial_j u_i)dR_i dR_j$$

by renaming of summation indices. Therefore

$$dx^2 - dR^2 = 2E_{ij}dR_i dR_j$$

with

$$E_{ij} = \frac{1}{2}[\partial_i u_j + \partial_j u_i + (\partial_i u_k)(\partial_j u_k)]$$

E_{ij} is called the (Lagrangian) *strain tensor*.
In many cases of practical importance (the overwhelming majority, in fact), deformations are small enough to safely neglect the term quadratic in distortions. A "small deformation" is one, for which all elements of the distortion tensor are small compared to 1 everywhere ($|\partial_i u_j| \ll 1$) In these case

$$E_{ij} \approx \epsilon_{ij}$$

We will use this approximation in the rest of this chapter.
For situations with flow, the choice of *Eulerian coordinates* has many advantages. In an Eulerian description, we try to express everything in terms of locations $\boldsymbol{x}$ of the distorted body in a fixed laboratory frame. The connection to Lagrangian coordinates is given by the relation

$$\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{R}(\boldsymbol{x}))$$

$\boldsymbol{R}$ is the Lagrangian coordinate indexing a material point in the undistorted body. If we consider a point $\boldsymbol{x}$ in our laboratory frame, the material point sitting there is just the one which started out from $\boldsymbol{x} - \boldsymbol{u}$ in the undistorted body. Note that the relation implies that

$$dR_i = dx_i - \frac{\partial u_i}{\partial x_j} dx_j$$

and therefore the change of local length scales are given by

$$dx^2 - dR^2 = [\partial_j u_i + \partial_i u_j - (\partial_i u_k)(\partial_j u_k)] dx_i dx_j$$

Thus we may also introduce an *Eulerian strain tensor* in analogy to the Lagrangian strain tensor:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial R_j} + \frac{\partial u_j}{\partial R_i} + \frac{\partial u_k}{\partial R_i}\frac{\partial u_k}{\partial R_j}\right) \qquad \text{Lagrangian}$$

$$E_{ij}^{Euler} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \qquad \text{Eulerian}$$

For small deformations, we get

$$E_{ij} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial R_j} + \frac{\partial u_j}{\partial R_i}\right)$$

as well as

$$E_{ij}^{Euler} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

which shows that we do not have to bother about the differences of Lagrangian and Eulerian coordinates in this approximation.

3.2 The Stress Tensor

If a material body is undistorted, its molecules or atoms are in thermodynamic equilibrium, which implies that they are in mechanical equilibrium. Thus there are no forces on an arbitrary volume element (finite or infinitesimal) within the body. On the other hand, in a distorted body, there will be

forces acting between volume elements within the body. If the interactions between the molecules or atoms do only depend on relative distances, these forces will only appear for deformations, not for rotations or translations. Such internal forces due to deformations are called *stresses*.

> Here we first encounter an important concept in field theory, which is inherent in nearly all field theories of practical importance, relativistic or not, classical or quantum. It is the *principle of locality*. In the field theories of deformable media, it appears as the statement: *All microscopic forces between the material points are short ranged, their action extends over microscopic distances only*

Consider the resulting force on a certain volume V within the material. It is the linear superposition of all the forces acting on all the "material points" or infinitesimal volume elements, which may be expressed as an integral over a *force density field* according to our general remarks on classical fields:

$$\boldsymbol{F}^V = \int_V d^d r \boldsymbol{f}(\boldsymbol{r})$$

Note that forces between material points *within* a given volume have to balance to zero because of Newton's *actio=reactio*. Thus a resulting nonzero force has to emerge from interactions between material points *outside* the considered volume with material points *inside* this volume. Due to the short (microscopic) range of molecular forces, this means that non-zero forces on a volume element result entirely from forces located within a microscopic distance of its surface. Thus, every component F_i^V must be representable as an integral over the boundary (surface) ∂V of the volume element,

$$F_i^V = \int_V d^d r f_i = \int_{\partial V} d\boldsymbol{a} \cdot \boldsymbol{\sigma}_i \ .$$

Here $d\boldsymbol{a}$ denotes the surface element (parallel to the outward normal vector) and we introduced $\boldsymbol{\sigma}_i$, which has the physical dimension of a *force per area*. $\sigma_{ik} da_k$ is the i-th component of the force acting on the surface element $d\boldsymbol{a}$. For Cartesian coordinates, the surface elements are small parts of the xy, xz and yz planes. Thus, for example σ_{xx} denotes the x-component of force (per area) acting on the plane, which is orthogonal to the x-axis. Thus, $\sigma_{xx} dydz$ is the force acting normal to the y-z plane element. The components $\sigma_{yx} dydz$ and $\sigma_{zx} dydz$ are components of the *tangential force*.
From Gauss law we obtain

$$\int_{\partial V} \boldsymbol{\sigma_i} d\boldsymbol{a} = \int_V d^d r \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_i = \int_V d^d r \partial_k \sigma_{ik} \ .$$

Thus the force density $\boldsymbol{f}$ must be representable as

$$f_i = \partial_k \sigma_{ik} \quad .$$

The tensor σ_{ij}, which contains all the information about stress in a material is called the (Lagrangian) *stress tensor*.
Finally, we can include *external force fields* acting on the material. These *volume forces* act via a given force density $\boldsymbol{f}^{ext}$ and equilibrium requires that $\boldsymbol{f} + \boldsymbol{f}^{ext} = 0$ everywhere inside the material. Thus

$$\partial_k \sigma_{ik} + f_i^{ext} = 0$$

3.2.1 Torque and the Symmetry of the Stress Tensor

Let us now consider the torque acting on a volume V inside a material body

$$M_{ik} = \int_V d^d r (f_i x_k - f_k x_i)$$

which may be written using the stress tensor as

$$\begin{aligned} M_{ik} &= \int_V d^d r [(\partial_j \sigma_{ij}) x_k - (\partial_j \sigma_{kj}) x_i] \\ &= \int_V d^d r \partial_j (\sigma_{ij} x_k - \sigma_{kj} x_i) - \int_V d^d r (\sigma_{ij} \partial_j x_k - \sigma_{kj} \partial_j x_i) \quad . \end{aligned}$$

In the second line, we have produced a divergence term in the first integral, which can be transformed into a surface integral by Gauss law. Using $\partial_j x_i = \delta_{ji}$ the torque may be written as the sum of a surface and a volume term

$$M_{ik} = \int_{\partial V} da_j (\sigma_{ij} x_k - \sigma_{kj} x_i) + \int d^d r (\sigma_{ik} - \sigma_{ki})$$

At this point, we distinguish between 2 types of material:

- *material without internal structure* for which the torque on each volume is produced from contributions at the boundary. In other words, there are no intrinsic properties of the material, which could produce volume densities of torque.
- *material with internal structure*, where torque may also have volume density contributions. We will not consider those materials in our lecture.

For materials without internal structure, the second integral has to vanish. This is obviously the case, if the stress tensor is symmetric

$$\sigma_{ik} = \sigma_{ki}$$

However, this sufficient condition is not necessary. In fact, if the antisymmetric part is itself a divergence, the second integral may also be converted into a surface integral. Thus we only have to require

$$\sigma_{ik} - \sigma_{ki} = 2\partial_j b_{ikj}$$

(Note that $b_{ikj} = -b_{kij}$ by definition.)
One can even go one step further and show that the stress tensor can be given in a symmetric form for every material without internal structure (Martin, Parodi and Pershan 1972). This observation relies o the fact, that the defining relation $f_i = \partial_k \sigma_{ik}$ does not fix the stress tensor completely. The tensor

$$\tilde{\sigma}_{ik} = \sigma_{ik} + \partial_j \chi_{ikj}$$

leads to the same forces, if the additional term $\partial_k \partial_j \chi_{ikj}$ vanishes, which is always fulfilled if $\chi_{ikj} = -\chi_{ijk}$. Martin et. al. showed that if the antisymmetric part of a tensor is a divergence, then the tensor can be made symmetric by a transformation of the above type. Explicitly

$$\tilde{\sigma}_{ik} = \sigma_{ik} + \partial_j (b_{kji} + b_{ijk} - b_{ikj})$$

First let us check that $\tilde{\sigma}$ is symmetric. This can be seen by slightly rewriting the right-hand side. The last term $-\partial_j b_{ikj} = (\sigma_{ki} - \sigma_{ik})/2$ and thus

$$\tilde{\sigma}_{ik} = \sigma_{ik} + \frac{\sigma_{ki} - \sigma_{ik}}{2} + \partial_j (b_{kji} + b_{ijk})$$

In this form, symmetry is obvious.
Now we check that the additional term has the required symmetry

$$\chi_{ikj} = (b_{kji} + b_{ijk} - b_{ikj}) = -\chi_{ijk}$$

by direct inspection.
To remember:

> The stress tensor of a material without internal structure can always be made symmetric, although it may not appear symmetric in the form it is obtained from some calculation.

To finish this discussion of fundamental notions let me present the three most important special cases of stresses: *isotropic pressure*, *shear stress* and *tension*. The corresponding forms of the stress tensor are:

$\sigma_{ij} = -p\delta_{ij}$	for pressure
$\sigma_{ij} = \sigma_0(\delta_{iy}\delta_{jz} + \delta_{iz}\delta_{jy})$	for a shear stress
$\sigma_{ij} = T\delta_{iy}\delta_{jy}$	for tension

It is important to note that the pressure p means the force per unit area exterted by the environment on the medium (see also Eq. 5.6 in section 5.1.2).

3.3 The Stress-Strain Relation

The relation between an applied stress and the resulting strain (or applied strain and resulting stress) characterizes a particular material. In the next two chapters we will study two simple but important types of material in detail: *linear elastic solids* and *simple fluids*.

3.3.1 Elastic Solids

A solid is characterized by the fact that all static stresses cause finite static strains. In a *fluid*, on the other hand, the application of a static shear stress causes the fluid to flow indefinitely and no finite static strain emerges[3]. In an *elastic* solid, the application of external forces (stresses) leads to a static strain field of the body, such that *the undistorted state is reestablished* after removing the external forces. In contrast to elastic materials, a material is called *plastic*, if there remain deformations after removal of the external forces.
If we perform *quasistatic* (i.e. the system always stays in thermal equilibrium), infinitesimally small deformations (located in a finite volume), characterized by the displacement field $\delta u_i(\boldsymbol{r})$, we can easily calculate the work done during the the deformation:

$$\delta W = \int_V d^d r f_i \delta u_i = \int_V d^d r (\partial_j \sigma_{ij}) \delta u_i$$

We perform a partial integration

$$\delta W = \int_{\partial V} d\boldsymbol{a} \cdot \boldsymbol{e}_j \sigma_{ij} \delta u_i - \int_V d^d r (\partial_j \delta u_i) \sigma_{ij}$$

and discard the boundary term by considering an integration volume with $\delta u_i = 0$ on its boundary. As the stress tensor is symmetric, we can rewrite the volume term as

$$\delta W = -\frac{1}{2} \int_V d^d r \sigma_{ij} [\partial_j \delta u_i + \partial_i \delta u_j] = -\int_V d^d r \sigma_{ij} \delta \epsilon_{ij}$$

So the *density of work* $\delta w(\boldsymbol{r})$ for this process is

$$\delta w(\boldsymbol{r}) = -\sigma_{ij} \delta \epsilon_{ij}$$

Thermodynamically, the internal energy of a *simple elastic material* depends only upon strain and entropy. More precisely, the (infinitesimal) change of the density of internal energy $e(\boldsymbol{r})$ at point $\boldsymbol{r}$ due to quasistatic processes changing work and heat is given by

$$de(\boldsymbol{r}) = Tds - \delta w = Tds(\boldsymbol{r}) + \sigma_{ij}(\boldsymbol{r}) d\epsilon_{ij}(\boldsymbol{r})$$

[3] Note that the application of a static pressure causes finite strains in both fluids and solids.

The total internal energy of the body is given by

$$E = \int_V d^d r e(\boldsymbol{r})$$

Let us show that for *homogeneous, hydrostatic pressure* $\sigma_{ij} = -p\delta_{ij}$, this relation reduces to the well known form of internal energy for simple fluids or gases $dE = TdS - pdV$, which you have learned in the introductory thermodynamics course. For a pure pressure term, $\delta W = -pd\epsilon_{ii}$. Remember that $\epsilon_{ii} = \mathrm{Tr}\epsilon$ is the relative change of volume elements $dV_{deformed}(\boldsymbol{r}) = (1 + \mathrm{Tr}\epsilon(\boldsymbol{r})))dV(\boldsymbol{r})$. Integrating over the entire body gives[4] $V_{deformed} - V = \int_V d^d r \mathrm{Tr}\epsilon = dV$.
Since $E(S, \epsilon)$ constitutes a thermodynamic potential for simple elastic solids, we may change to any other thermodynamic potential by Legendre transformations. For example,

$$df(\boldsymbol{r}) = -s(\boldsymbol{r})dT + \sigma_{ij}(\boldsymbol{r})d\epsilon_{ij}(\boldsymbol{r})$$

is the differential of the *free energy density* and

$$dg(\boldsymbol{r}) = -s(\boldsymbol{r})dT - \epsilon_{ij}(\boldsymbol{r})d\sigma_{ij}(\boldsymbol{r})$$

is the *Gibbs free enthalpy density* of simple elastic solids. Stress and strain are related to thermodynamic potentials via

$$\sigma_{ij}(\boldsymbol{r}) = \left(\frac{\partial f(\boldsymbol{r})}{\partial \epsilon_{ij}(\boldsymbol{r})}\right)_T$$

and

$$\epsilon_{ij}(\boldsymbol{r}) = -\left(\frac{\partial g(\boldsymbol{r})}{\partial \sigma_{ij}(\boldsymbol{r})}\right)_T$$

so that *a thermodynamic potential of an elastic material also fixes the stress-strain relation*.

3.3.2 Fluids

For fluids, the action of hydrostatic pressure produces isotropic compression, which is reversible. So the situation is the same as for elastic solids and this part of the stress strain relation is not qualitatively different from solids, $\mathrm{Tr}(\epsilon)\delta_{ij}(p)$ is a function determined by thermostatics. Static shear stresses, however, cause flow and drives the fluid from its rest state forever. To describe flow kinematically, we have to introduce the *velocity* of a material point, which is given by the time derivative of the displacement,

$$\frac{d\boldsymbol{x}(t, \boldsymbol{R})}{dt} = \frac{d\boldsymbol{u}(t, \boldsymbol{R})}{dt} \quad .$$

[4] Do not confuse $dV(\boldsymbol{r})$ (the volume element at point $\boldsymbol{r}$) with dV (the infinitesimal change in the total volume of the body

Note that the *velocity field* of a flow is given somewhat implicitly by

$$\boldsymbol{v}(\boldsymbol{R}+\boldsymbol{u},t)=\frac{d\boldsymbol{u}(t,\boldsymbol{R})}{dt}$$

in Lagrangian coordinates (for which $\boldsymbol{R}$ indicates the initial position of a flowing material point). If the velocities of the material points change with time, the *acceleration* of a material point is given by

$$\frac{d\boldsymbol{v}(t,\boldsymbol{x}=\boldsymbol{R}+\boldsymbol{u}(t,\boldsymbol{R}))}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+\frac{\partial\boldsymbol{v}}{\partial x_i}v_i=\left[\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\right]\boldsymbol{v} \tag{3.1}$$

The operator in brackets is known as the *comoving time derivative* or *substantial time derivative*.

Chapter 4

Elasticity

4.1 Linear Elasticity

4.1.1 Hooke's Law

For small deformations, we may expand the internal energy or the free energy[1] in terms of ϵ. The first terms in this expansion appear at order ϵ^2:

$$f = f_0 + \frac{1}{2}\epsilon_{ij} K_{ij,kl} \epsilon_{kl}$$

The 4th order tensor K is called the *tensor of elastic modules*. As ϵ_{ij} is symmetric, the tensor has the following symmetry properties

$$K_{ij,kl} = K_{ji,kl} = K_{ji,lk} = K_{lk,ji}$$

You can figure out that from the 81 components of K, only 21 are independent due to these symmetries. If the material has no further symmetries (as is the case for crystals with *triclinic symmetry*) you need to know all the 21 modules to describe the linear elastic behavior of the material. In the following, we will consider *isotropic* materials, which are described by only 2 independent modules.
In general, the stresses of linear elastic bodies are connected to the strains by

$$\sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}} = K_{ij,kl}\epsilon_{kl}$$

This stress-strain relation is the generalization of Hooke's law you all know from elementary mechanics.
Let us turn to isotropic materials. As f is a scalar (density) it must be a linear combination of all terms of second order in ϵ, which are invariant under rotations. The coefficients of these terms must be scalars, because they only depend on intrinsic properties of the undeformed medium, where all directions are equivalent. We can form 2 independent invariants of 2nd order:

$$\epsilon_{ii}^2, \qquad \epsilon_{ij}\epsilon_{ij} = \epsilon_{ij}\epsilon_{ji}$$

Note that it is very easy to construct invariants under rotation from products of tensor elements: You just have to "pair" all the indices, so that no "free" indices remain, which would turn the thing into a tensor again. For example, for 3rd order you have to "pair" indices in $\epsilon_{ij}\epsilon_{kl}\epsilon_{mn}$ in all possible ways.

[1] We only consider situations with a homogeneous temperature T. The undeformed state $\epsilon = 0$ is the thermodynamic equilibrium, if external forces are absent and the temperature remains unchanged. The last condition should not be overlooked (think of the effects of thermal expansion)

Thus, the most general form of f becomes

$$f = f_0 + \frac{\lambda}{2}\epsilon_{ii}^2 + \mu\epsilon_{ij}\epsilon_{ij}$$

These elastic modules of an isotropic medium are known as *Lame coefficients*.
As we already saw, it is very convenient for many purposes to decompose ϵ_{ij} into *pure shear deformations*, which do not change the volume, and an *isotropic* or *hydrostatic compression* $\propto \delta_{ij}$. For the decomposition we obtained

$$\epsilon_{ij} = \left(\epsilon_{ij} - \frac{1}{3}\mathrm{Tr}(\epsilon)\delta_{ij}\right) + \frac{1}{3}\mathrm{Tr}(\epsilon)\delta_{ij} = \hat{\epsilon}_{ij} + \frac{1}{3}\mathrm{Tr}(\epsilon)\delta_{ij} \tag{4.1}$$

By definition $\mathrm{Tr}(\hat{\epsilon}) = 0$.
Using this decomposition in f leads to

$$f = f_0 + \mu\hat{\epsilon}_{ij}\hat{\epsilon_{ij}} + \frac{K}{2}\mathrm{Tr}(\epsilon)^2$$

The quantity

$$K = \lambda + \frac{2}{3}\mu$$

is called the *compression module*, μ is referred to as the *torsion module*.
For an isotropic linear elastic medium, the stress-strain relation becomes

$$\sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}} = K \cdot \mathrm{Tr}(\epsilon)\delta_{ij} + 2\mu\hat{\epsilon}_{ij} \tag{4.2}$$

We can also express the deformations by the stresses, because taking the trace of the above relation leads to

$$\mathrm{Tr}(\epsilon) = \frac{1}{3K}\mathrm{Tr}(\sigma)$$

Thus we see that relative volume changes in an isotropic linear elastic medium can only be caused by hydrostatic pressure. If we now express all the $\mathrm{Tr}(\epsilon)$ terms by $\mathrm{Tr}(\sigma)$ in the stress-strain relation (4.2) and use (4.1) (with $\mathrm{Tr}(\epsilon)$ also replaced by $\mathrm{Tr}(\sigma)$) to express $\hat{\epsilon}$, it is easy to invert (4.2) into a strain-stress relation

$$\epsilon_{ij} = \frac{1}{9K}(\mathrm{Tr}\sigma)\delta_{ij} + \frac{1}{2\mu}\hat{\sigma}_{ij}$$

with

$$\hat{\sigma} = \sigma - \frac{1}{3}\mathrm{Tr}(\sigma) \quad .$$

4.1.2 Static Field Equation for Displacements

Now we want to calculate the displacement field $\boldsymbol{u}(\boldsymbol{r})$ for linear elastic media in the presence of external forces. Obviously this is a question of much technical relevance (Think of deformations of buildings!) as well as of principle concern in the theory of condensed matter (Think of the deformations caused by a defect in a material). We start from the balance of forces

$$\partial_j \sigma_{ij} + f_i^{ext} = 0$$

and insert the stress-strain relation

$$\partial_j [K(\mathrm{Tr}\epsilon)\delta_{ij} + 2\mu\hat{\epsilon}_{ij}] + f_i^{ext} = 0$$

Now we express the strain tensor by the displacement field, $2\epsilon_{ij} = \partial_i u_j + \partial_j u_i$, to find the following partial differential equation for the displacements fields:

$$\left(K - \frac{2}{3}\mu\right) \partial_j(\partial_l u_l) + \mu\partial_j(\partial_i u_j + \partial_j u_i) + f_i^{ext} = 0 \quad . \tag{4.3}$$

This equation may also be written in vector form, where it is even a little bit more compact:

$$(K + \frac{1}{3}\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} + \boldsymbol{f}^{ext} = 0 \tag{4.4}$$

In many important cases, all the forces acting on the body are located at its boundary, so that $\boldsymbol{f}^{ext}$ can be expressed via boundary conditions and does not have to appear explicitly in the balance equation. Then we can get two simpler balance equations. Taking the divergence of 4.3 gives

$$\nabla^2(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0$$

and applying the Laplace operator to 4.3, using the above relation for the first two terms, gives

$$\nabla^2(\nabla^2\boldsymbol{u}) = 0$$

Functions obeying this equation are called *biharmonic*.

4.1.3 Dynamic Field Equation and Elastic Waves

If the parts of a linear elastic body are not in mechanical equilibrium, the volume elements will move. The equation of motion for a volume element is just Newton's equation, which we now want to set up for a linear elastic medium. Let us suppose, that we have a medium with a homogeneous distribution of identical material points everywhere. If you want to apply this concept to

real solids, like for example, crystals, you have to be careful. It applies to a monoatomic ideal crystal lattice (no vacancies, no interstitials). Then the displacement vector $\boldsymbol{u}(\boldsymbol{r})$ may be identified with the displacements of all the material points at $\boldsymbol{r}$ and $\partial_t \boldsymbol{u}(\boldsymbol{r}, t)$ is the velocity of these material points. Let the mass density of the points be ρ, then $\rho \partial_t \boldsymbol{u}$ is the *momentum density* and Newton' law of motion can be written as

$$\rho \partial_t^2 u_i = f_i = \partial_j \sigma_{ij} + f_i^{ext} \tag{4.5}$$

If, however, you have more microscopic structure like, for example, vacancies and interstitials, you may have mass and momentum transfer, which is *not* described by $\rho \partial_t \boldsymbol{u}$. Luckily, for most experimental conditions, the density of the point defects is too small or their motion is too slow to modify the main results obtained from the simplified theory presented here. Keep in mind, however, that additional microscopic structures, which may carry momentum will lead to additional field degrees of freedom.
Let us insert the stress-strain relation to obtain a differential equation for the displacement field:

$$\rho \partial_t^2 \boldsymbol{u} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) \ ,$$

where we have used $K + 2\mu/3 = \lambda + \mu$.
The physical implications of this equation become much more transparent, if we decompose the displacement field into a *divergence free* or *transverse* part $\boldsymbol{u}^t$ and a *rotation free* or *longitudinal part* $\boldsymbol{u}^l$. It is a general theorem of vector analysis that the decomposition

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^l + \boldsymbol{u}^t \quad \text{with} \\ 0 &= \boldsymbol{\nabla} \times \boldsymbol{u}^l \\ 0 &= \boldsymbol{\nabla} \cdot \boldsymbol{u}^t \end{aligned}$$

is unique [10]. Let us insert this decomposition into the field equation

$$\partial_t^2 (\boldsymbol{u}^l + \boldsymbol{u}^t) = \frac{\mu}{\rho} \nabla^2 (\boldsymbol{u}^l + \boldsymbol{u}^t) + \frac{\lambda + \mu}{\rho} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{u}^l)$$

and then take the divergence and the rotation of the equation, respectively. Applying the divergence leads to

$$\boldsymbol{\nabla} \cdot (\partial_t^2 \boldsymbol{u}^l - c_l^2 \nabla^2 \boldsymbol{u}^l) = 0$$

with $c_l^2 = (\lambda + 2\mu)/\rho$, whereas taking the rotation gives

$$\boldsymbol{\nabla} \times (\partial_t^2 \boldsymbol{u}^t - c_t^2 \nabla^2 \boldsymbol{u}^t) = 0$$

with $c_t^2 = \mu/\rho$. Note that that not only the divergence but also the rotation of the first expression in brackets vanishes (due to $\boldsymbol{\nabla} \times \boldsymbol{u}^l = 0$). A vector field with vanishing rotation and divergence has to vanish identically due to the above mentioned decomposition theorem. Applying an analogous argument to the rotation part of the field equation, we conclude that the field equation is equivalent to the following set of two wave equations:

$$\begin{aligned} \partial_t^2 \boldsymbol{u}^l - c_l^2 \nabla^2 \boldsymbol{u}^l &= 0 \\ \partial_t^2 \boldsymbol{u}^t - c_t^2 \nabla^2 \boldsymbol{u}^t &= 0 \end{aligned}$$

If we consider *monochromatic plane wave solutions*

$$\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{u}(\boldsymbol{k}) \exp(i\boldsymbol{k}\boldsymbol{r} - i\omega t)$$

we find that

$$\begin{aligned} \omega &= \pm c_l |\boldsymbol{k}| \qquad \text{for } \boldsymbol{u}^l(\boldsymbol{k}) \times \boldsymbol{k} = 0 \\ \omega &= \pm c_t |\boldsymbol{k}| \qquad \text{for } \boldsymbol{u}^t(\boldsymbol{k}) \cdot \boldsymbol{k} = 0 \end{aligned}$$

From these relations you see that the vector $\boldsymbol{u}(\boldsymbol{k})$, called the *polarization vector of the monochromatic plane wave* is directed parallel to $\boldsymbol{k}$ for a longitudinal mode and orthogonal to $\boldsymbol{k}$ for a transverse mode.
Remember that relative changes in volume (dilations) are described by $\mathrm{Tr}(\epsilon) = \partial_i u_i = \boldsymbol{\nabla} \cdot \boldsymbol{u}$. Therefore longitudinal wave modes imply volume changes, whereas transverse modes do not.
Note that the polarization vector of a monochromatic plane wave solution for a crystal with a tensor of elastic modules $K_{ij,kl}$ has to obey

$$[\rho\omega^2 \delta_{im} - K_{ij,lm} k_j k_l] u_m(\boldsymbol{k}) = 0$$

by a simple generalization (see exercise). This is a 3×3 homogeneous linear system. The solvability condition

$$\det[\rho\omega^2 \delta_{im} - K_{ij,lm} k_j k_l] = 0$$

leads to the *dispersion relations* $\omega^2 = \omega_\alpha^2(\boldsymbol{k})$ for the three eigenmodes (which are no longer longitudinal and transverse). However, the direction of polarizations of the three eigenmodes are still orthogonal, because they are the eigenvectors of a real symmetric matrix.
Physically, elastic waves correspond to *sound waves* with *sound velocities* c_t and c_l. Note that the velocity of longitudinal sound waves is lager than those of transverse sound waves from the definitions.

Chapter 5

Hydrodynamics

5.1 Hydrodynamics

A simple fluid consists of a single, homogeneous substance (think of a liquid or gas), which may be in macroscopic motion[1]. Our task now is to set up a closed system of field equations for simple fluids.

5.1.1 Balance Equation

Let $\varrho^M(\boldsymbol{r},t)$ denote the *mass density* of the fluid. How does ϱ^M change with time? We define the vector field *mass current density* $\boldsymbol{j}^M(\boldsymbol{r},t)$ as the mass flowing across a plane with normal vector $\boldsymbol{n}$ parallel to $\boldsymbol{j}^M$ at $\boldsymbol{r}$ per time and per area.
More precisely, let $M_A(t_2,t_1)$ be the total mass, which flows through an arbitrary surface A from time t_1 to time t_2. It is given by

$$M_A(t_2,t_1)=\int_{t_1}^{t_2}\mu_A(t)dt$$

with μ_A denoting the mass flow across A (in direction of the surface normal). In terms of the mass current density, we can express the mass flow as the sum (or rather integral) of all mass flows crossing all the area elements $d\boldsymbol{a}$ of the arbitrary surface A:

$$\mu_A(t)=\int_A d\boldsymbol{a}\cdot\boldsymbol{j}^M(\boldsymbol{r},t)$$

Consider now a volume V bounded by the surface ∂V. The normal vectors of the closed surface ∂V are chosen to point *outwards*.
The important law of **conservation of mass** implies that the mass inside of V only changes due to mass flow across the boundary of V. Thus

$$\frac{dM_V(t)}{dt}=\int_V d^3r\frac{\partial\varrho^M(\boldsymbol{r},t)}{\partial t}=-\int_{\partial V}d\boldsymbol{a}\cdot\boldsymbol{j}^M(\boldsymbol{r},t)$$

Here we apply **Gauss's Law**

$$\int_{\partial V}d\boldsymbol{a}\cdot\mathbf{j}^M(\boldsymbol{r},t)=\int_V d^3r\nabla\cdot\boldsymbol{j}^M(\boldsymbol{r},t)$$

As the relations between the integrals hold for arbitrary volumes, they must hold for the integrands. This leads to the **continuity equation**

$$\frac{\partial\varrho^M}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}^M=0\ . \tag{5.1}$$

[1] We use the term macroscopic to distinguish this type of motion from the random thermal motion of the material points making up the substance on microscopic scales.

Obviously, this is a field equation connecting the mass density and the mass current density. But the same reasoning applies to every conserved quantity. This is an important and general concept, which helps to set up field equations.

> Continuity equations hold for the densities of all the physical quantities, which are *additive* and *conserved* (for example, mass, charge, momentum, energy,...).

If a quantity is not only transported by fluxes, but may also be created in sources or destroyed in sinks, the field equation takes on the more general form

$$\frac{\partial \varrho(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{j}(\boldsymbol{r},t) = q(\boldsymbol{r},t) \tag{5.2}$$

where q is the term describing sources and sinks. It has to be specified from other principles. Field equations of this type are called *balance equations*.
Although we have found a field equation, it is not closed. It just connects two fields. So we have to continue by either expressing one field by the other, or by finding another field equation connecting the 2 fields.
First note that from simple physical reasoning, mass is transported via a current, which is simply given by

$$\boldsymbol{j}^M(\boldsymbol{r},t) = \varrho^M(\boldsymbol{r},t)\boldsymbol{v}(\boldsymbol{r},t) \quad . \tag{5.3}$$

For other additive and conserved quantities, the connection between its current density $\boldsymbol{j}$, its density ϱ and the velocity of the fluid $\boldsymbol{v}$ may be much more complicated because transport can be accomplished in two very different ways:

- via *convection*, which just means that the quantity in each volume element is transported passively with the motion of the volume element, so that $\boldsymbol{j} = \varrho\boldsymbol{v}$
- via *conduction*, which is a form of transport, which may be present even if either $\boldsymbol{v}$ or ϱ (or both) vanishes. As two prominent examples, consider electric currents in regions, where the electric charge density vanishes, and heat conduction in a fluid, which is macroscopically at rest.

5.1.2 Momentum Balance and Angular Momentum Balance

A very interesting and illuminating example of a balance equation and of a transport involving both convection and conduction is that of *momentum*. In a simple fluid, the momentum density is given by

$$\boldsymbol{\rho}^P = \varrho^M\boldsymbol{v}$$

Thus momentum density is exactly equal to the mass current density, $\boldsymbol{\rho}^P = \boldsymbol{j}^M$! Therefore, the balance equation of momentum may lead to a closed set of equations for the simple fluid.
Generally, momentum is a vector quantity and therefore we have three component densities, which make up momentum density:

$$\rho_i^P \qquad \text{i=1,2 or 3}$$

The continuity equations of momentum density are closely related to *Newton's* equations of motion, because both express the time change of momentum. In the language of continuity equations, we would write

$$\frac{\partial \rho_i^P}{\partial t} + \partial_k j_{ik}^P = 0 \tag{5.4}$$

for a closed system. The *momentum current densities* j_{ik}^P form the components of a second rank tensor.
In the language of Newton's equation, we would read this equation analogous to the equation (4.5), as we may identify $\rho_i^p = \varrho^M \boldsymbol{v} = \varrho^M \partial_t \boldsymbol{u}$ in a simple fluid. Thus the momentum current density is the exact analogue of the object we called the stress tensor in a deformable medium. This makes sense, since force is momentum per time and

$$\boldsymbol{j}_{i\,.}^P \cdot d\boldsymbol{a}$$

is the i-th component[2] of *momentum per time flowing across* $d\boldsymbol{a}$. This is nothing but the i-th component of *force, exerted by the fluid* on the surface element $d\boldsymbol{a}$. Thus you should keep in mind that

> *momentum current density* and *stress tensor* are just two names for the same physical concept

In hydrodynamics, one likes to split the convective part of the momentum current density from the *stress tensor* and only calls the conductive part *hydrodynamic stress tensor* σ

$$j_{ik}^P = \rho_i^P v_k + \sigma_{ik} = \varrho^M v_i v_k + \sigma_{ik} \tag{5.5}$$

This is a very useful convention, because we have to remember that the convective velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ depends on the choice of a Galilean frame of reference. A fluid at rest in one frame may become a streaming fluid with homogeneous velocity in another frame. Splitting of the convection thus makes the

[2] Note that $\boldsymbol{j}_{i\,.}^P = (j_{i1}, j_{i2}, j_{i3})$ is the current density of the i-th momentum component, whereas $\boldsymbol{j}_{.\,k}^P = (j_{1k}, j_{2k}, j_{3k})$ is the current density of the momentum vector, transported in the direction of the k-th Cartesian unit vector $\boldsymbol{e}_k$. At first sight, these two vectors seem to be unrelated, but see below!

hydrodynamic stress tensor an *intrinsic property of the material*. We will come back to the usefulness of the decomposition in the next subsections.
Traditionally one splits off one further term of the stress tensor, which is present in any simple fluid at rest, *the hydrostatic pressure* p

$$\sigma_{ik} = p\delta_{ik} + \hat{\sigma}_{ik} \quad . \tag{5.6}$$

Note that, in contrast to elasticity, the hydrostatic pressure here is defined as the force per unit area a volume element of the fluid exerts *on its environment*! We will analyze the stress tensor in more detail in the next subsection.
Finally, we may extend the balance equation of momentum density to situations with external forces by simply replacing the right hand side of Eq. (5.4) by f_i^{ext}.
Let us now consider the balance equation of *angular momentum density*. At first sight the situation looks very similar to the case of momentum density: we have a three component vector density $\boldsymbol{\rho}^L$, the balance equations take the form

$$\frac{\partial \rho_i^L}{\partial t} + \partial_k j_{ik}^L = d_i \tag{5.7}$$

and the sources of angular momentum are described by a *torque density* $\boldsymbol{d}$.

> *However, for simple fluids there is neither internal angular momentum nor internal torque! It is a material without internal structure as we encountered in section 3.2.1.*

Consequently, the angular momentum is simply given by $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and the torque exerted on a small fluid element is determined by the force acting on it as $\boldsymbol{D} = \boldsymbol{r} \times \boldsymbol{F}$. Therefore

- $\boldsymbol{\rho}^L(\boldsymbol{r}) = \boldsymbol{r} \times \boldsymbol{\rho}^P(\boldsymbol{r})$
- $\boldsymbol{j}_{\cdot k}^L(\boldsymbol{r}) = \boldsymbol{r} \times \boldsymbol{j}_{\cdot k}^P(\boldsymbol{r})$
- $\boldsymbol{d}(\boldsymbol{r}) = \boldsymbol{r} \times \boldsymbol{f}$

The first and third of these relations are pretty obvious. The meaning of the second relation is: the angular momentum current in any direction equals the cross product of position and of momentum current in the same direction!
Given these connections it is clear that there is no independent balance equation of angular momentum for simple fluids. Instead, there is an additional constraint on the tensor of momentum current density, which is easily derived, if we insert the special forms of angular momentum density and current density into the balance equation Eq. (5.7)

$$\boldsymbol{r} \times \frac{\partial \boldsymbol{\rho}^P}{\partial t} + \partial_k (\boldsymbol{r} \times \boldsymbol{j}_{\cdot k}^P) = \boldsymbol{r} \times \boldsymbol{f}$$

The spatial derivatives give us two terms, which we regroup as follows

$$\boldsymbol{r} \times \left(\frac{\partial \boldsymbol{\rho}^P}{\partial t} + \partial_k \boldsymbol{j}^P_{\cdot k} - \boldsymbol{f} \right) + \frac{\partial \boldsymbol{r}}{\partial x_k} \times \boldsymbol{j}^P_{\cdot k} = 0$$

The balance equation of momentum implies that the terms in brackets vanish. Let us write down explicitly the first component of the remaining term (the other two are completely analogous)

$$\sum_{k=1}^{3} \left(\frac{\partial x_2}{\partial x_k} j^P_{3k} - \frac{\partial x_3}{\partial x_k} j^P_{2k} \right) = j^P_{32} - j^P_{23} = 0 \quad .$$

Thus conservation of angular momentum in a simple fluid implies that the tensor of momentum current density is symmetric,

$$j^P_{ik} = j^P_{ki} \quad . \tag{5.8}$$

We will make use of this property in the next subsection.

5.1.3 Ideal Fluids, Viscosity and the Navier-Stokes Equation

We now return to our main task: to derive a closed set of field equations for a simple fluid. We have seen already that the momentum current density contains a convective part and a stress term, which is again split into pressure and a rest $\hat{\sigma}$. What we need to find for a closed set of field equations is a *stress-strain relation*. As such a relation depends on the material, we have to specify material properties.
Let us start with a very simple case, the so called *ideal fluid*, where there are no internal forces apart from hydrostatic pressure.

A simple fluid with $\hat{\sigma} = 0$ is called an ideal fluid.

For an ideal fluid, the balance equation for momentum becomes

$$\frac{\partial \varrho}{\partial t} v_i + \varrho \frac{\partial v_i}{\partial t} + \partial_k (\varrho v_i v_k + p \delta_{ik}) = f_i$$

which may be rewritten as

$$\frac{\partial \varrho}{\partial t} v_i + \varrho \frac{\partial v_i}{\partial t} + v_i \partial_k (\varrho v_k) + \varrho v_k \partial_k v_i + \partial_i p = f_i$$

The first and the third term add up to zero due to the continuity equation of mass. Using this simplification, the balance equation of momentum takes on a form called *Euler's equation*

$$\varrho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) \right) + \boldsymbol{\nabla} p = \boldsymbol{f} \tag{5.9}$$

The term in brackets and the whole equation have a very intuitive interpretation. Note that for every field defined on the moving fluid,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}(F(t+\Delta t,\boldsymbol{r}+\boldsymbol{v}(t,\boldsymbol{r})\Delta t)-F(t,\boldsymbol{r}))=\left(\frac{\partial F}{\partial t}+(\boldsymbol{v}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla})F\right)\quad .$$

As already mentioned in connection with Eq. (3.1), the left hand side is called the *comoving time derivative* or *substantial time derivative*. It contains the temporal changes of a quantity in a frame of reference, which moves along with the fluid. Sometimes an extra symbol for such a time derivative is introduced, for example DF/Dt. In particular we see that the bracketed term in equation (5.9) corresponds to the comoving derivative of the velocity field itself. Euler's equation takes on the very simple form

$$\varrho\frac{D\boldsymbol{v}}{Dt}=\boldsymbol{f}-\boldsymbol{\nabla}p$$

which is just a form of Newton's second law.
Let us rediscover some elementary results about fluids, which you should already know. First suppose that the forces are conservative $\boldsymbol{f}=-\boldsymbol{\nabla}\phi$ and the velocity field is stationary $\partial_t\boldsymbol{v}=0$. Euler's equation reduces to

$$\varrho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=-\boldsymbol{\nabla}(\phi+p)$$

Furthermore, assume that the fluid is *incompressible*, i.e.

$$\varrho(\boldsymbol{r},t)=\varrho_0$$

If you make use of the identity from vector analysis

$$\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}))=\frac{1}{2}\boldsymbol{\nabla}(v^2)-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$$

you see that Euler's equation may be written in the form

$$\boldsymbol{\nabla}\left(\frac{\varrho_0}{2}v^2+p+\phi\right)=\varrho_0\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v})$$

If the velocity field does not contain circulation, i.e. $\boldsymbol{\nabla}\times\boldsymbol{v}=0$, we recover *Bernoulli's law*

$$\frac{\varrho_0}{2}v^2+p+\phi=\text{const}$$

which expresses energy conservation. If the fluid is at rest, Euler's equation (5.9) reduces to the well known equilibrium condition of *hydrostatics*

$$\boldsymbol{\nabla}p=\boldsymbol{f}$$

At this point it is a good exercise, to try to derive *Archimedes law of buoyancy* and admire its simplicity!

We proceed now and consider another class of materials, called *Newtonian fluids*. They are characterized by the fact that the stresses $\sigma(\boldsymbol{r}, t)$ depend only on the strain rates at the same point and at the same time:

$$\hat{\sigma}(\boldsymbol{r}, t) = \hat{F}\left(\frac{\partial \epsilon(\boldsymbol{r}, t)}{\partial t}\right)$$

Let us explain some physics behind this type of stress-strain relations, by introducing the most important special case of a Newtonian fluid, the *simple viscous fluid*. First note, that $\hat{\sigma}$ must depend on *spatial derivatives* , $\partial_i v_k$, of the streaming velocity, because it has to vanish for $\boldsymbol{v} = \boldsymbol{const}$ (a fluid, in which all parts move with constant velocity is physically equivalent to a fluid at rest, - by Galilean transformation). So the stress is a function $\hat{\sigma}(\partial_i v_k, \partial_i \partial_j v_k, \cdots)$. For *smooth velocity profiles* we neglect higher order derivatives and for small velocity gradients we expand $\hat{\sigma}$ to leading order,

$$\hat{\sigma}_{kl} = \eta_{ij,kl} \partial_i v_j + \cdots$$

These are the approximations leading to a linear stress-strain relation, which is the *Navier-Stokes fluid*. The tensor $\partial_i v_j$ may be decomposed into a symmetric and an antisymmetric part $\partial_i v_j \pm \partial_j v_i$. If the whole fluid is rotating, there is no relative motion of fluid elements, i.e. the stress $\hat{\sigma}$ vanishes. On the other hand, the velocity profile corresponding to a uniform rotation is given by $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. For this profile, the antisymmetric part of the tensor $\partial_i v_j$ does not vanish (whereas the symmetric part does!). Thus we conclude that $\hat{\sigma}$ can only depend on the symmetric combinations of $\partial_i v_k$. Now we can proceed in close analogy to our arguments we used to fix the stress-strain relation of an isotropic, linear elastic solid. The most general linear relation between stress and strain-rate for an isotropic fluid thus becomes

$$\hat{\sigma}_{ij} = a(\partial_i v_j + \partial_j v_i) + b(\partial_k v_k)\delta_{ik}$$

In analogy to linear elasticity, it is very convenient to write this expression in a slightly different form

$$\hat{\sigma} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}(\partial_k v_k)\delta_{ij}\right) + \zeta(\partial_k v_k)\delta_{ij} \tag{5.10}$$

In this decomposition, the first term is traceless and the second term is proportional to the *local volume change* $\boldsymbol{\nabla} \cdot \boldsymbol{v}$.
η and ζ are called *viscosities*. For most simple fluids, they are constants (independent on pressure and temperature) to a good approximation. The stress enters the balance equation of momentum in the form (see Eqs. (5.4) and (5.5))

$$\partial_k \hat{\sigma}_{ik} = \eta(\partial_k \partial_k v_i) + \left(\zeta + \frac{\eta}{3}\partial_i(\partial_j v_j)\right)$$

Using this result we can finally write the balance equation of momentum as

$$\varrho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right]=-\boldsymbol{\nabla}p+\eta\nabla^2\boldsymbol{v}+(\zeta+\eta/3)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})+\boldsymbol{f} \tag{5.11}$$

This equation is known as the *Navier-Stokes Equation* of a compressible fluid. Together with the continuity equation we thus have four equations for the five fields $(\boldsymbol{v},\varrho,p)$ to be determined. We still need one more equation. The missing equation comes from thermodynamics as an *equation of state* in the form

$$p=p(\varrho,T)$$

Note that this equation depends on the particular material under consideration and is no longer universal for all simple fluids. Furthermore, it will close our system of equations only if we can consider the temperature as constant throughout the fluid. If this is not the case, we need another equation, which tells us, how heat is transported within the fluid.
Many simple fluid are (to a good approximation) incompressible, i.e.

$$\varrho(\boldsymbol{r},t)=\varrho_0.$$

Incompressibility implies that all terms $\propto \boldsymbol{\nabla}\cdot\boldsymbol{v}$ will vanish, because they correspond to compressions. This is an immediate consequence of the continuity equation of mass. Obviously, the Navier-Stokes equation simplifies to

$$\varrho_0\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right]=-\boldsymbol{\nabla}p+\eta\nabla^2\boldsymbol{v}+\boldsymbol{f}$$

which has to be supplemented by the condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$$

5.1.4 Similarity: The Power of Dimensional Analysis

The Navier-Stokes equation is not easy to solve, even for the simplest physical problems. Here we like to show that dimensional analysis can be become a very powerful, quantitative tool in field theories, even if you cannot find exact solutions. Let us demonstrate this for the Navier-Stokes Equation of an incompressible fluid, which we write in the standard form

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=\frac{-\boldsymbol{\nabla}p}{\varrho_0}+\frac{\eta}{\varrho_0}\nabla^2\boldsymbol{v}$$

This equation has to be supplemented by $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$ and the functions $\boldsymbol{v}(\boldsymbol{r},t)$ and $p(\boldsymbol{r},t)$ have to be determined from these equations. Note that the only material

property entering the equations is $\eta/\varrho_0 = \nu$, which is also called *kinematic viscosity*.
A typical problem of hydrodynamics is to determine the stationary flow pattern, which emerges from a rigid body moving relative to the fluid (or a resting rigid body in a fluid streaming with constant velocity far away from the body). Let the relative velocity (in x-direction) be u. Now consider a collection of rigid bodies, which are generated from a prototype by homogeneous dilations. This implies, that any linear dimension l of the bodies are scaled as λl.
Now we turn to dimensional analysis. There are three model parameters with physical dimension:

a) the velocity u has dimension [length/time]

b) the kinematic viscosity ν has dimension [length^2/time]

c) the linear dimension (any) of the rigid body has dimension [length]. This quantity does not appear explicitly in the Navier-Stokes equation, but enters the problem via the boundary conditions.

From these three quantities we can form a single dimensionless ratio:

$$\text{Reynolds number} \qquad Re = \frac{ul}{\nu}$$

Every other dimensionless parameter can be written as a function of Re. Now let us make lengths and times in the Navier-Stokes equation dimensionless by measuring lengths in terms of l and velocities in terms of u, $\boldsymbol{R} = \boldsymbol{r}/l$ and $\boldsymbol{V} = \boldsymbol{v}/u$. As $\boldsymbol{V}$ is dimensionless, it must by a (dimensionless) function of dimensionless quantities, i.e. for stationary flows

$$\boldsymbol{v} = u\boldsymbol{f}(\boldsymbol{r}/l, Re)$$

Although nothing beyond dimensional analysis was used to derived this result, it is quite powerful. It states, that flow patterns of fluids with different viscosities flowing around bodies of different length scales are related by *similarity*. This is the origin of the powerful method of *experimental hydrodynamic models*, which allow to study inaccessible conditions by scaled models (scaling either the linear dimensions to turn large structures small or by scaling viscosities).
In the context of field theories, the Reynolds number is called a *dimensionless coupling*. The name becomes obvious if we write the stationary Navier Stokes equation using the dimensionless rescaled variables $\boldsymbol{V}$ and $\boldsymbol{R}$

$$-\left(\frac{\nu u}{l^2}\right)\nabla^2\boldsymbol{V} + \frac{u^2}{l}(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = -\frac{\boldsymbol{\nabla}p}{l\varrho_0}$$

(Here $\boldsymbol{\nabla}$ means differentiations with respect to $\boldsymbol{R}$). Multiplying by $\frac{l^2}{\nu u}$ we get

$$-\nabla^2 \boldsymbol{V} + Re(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = -\left(\frac{l}{\varrho_0 \nu u}\right) \boldsymbol{\nabla} p$$

For small Reynold's number, the second term on the left hand side can be neglected and the Navier Stokes equations reduce to

$$\nabla^2 \boldsymbol{V} = \frac{Re}{\varrho_0 u^2} \boldsymbol{\nabla} p$$

Since furthermore $\varrho_0 u^2$ has the dimension [force/length^2] it can be used to transform the pressure p into a dimensionless rescaled variable P, too, to obtain

$$\nabla^2 \boldsymbol{V} = Re \boldsymbol{\nabla} P$$

Thus, the inhomogeneities of in the system, described by the local changes of the pressure, couple to the velocity field of the fluid with a coupling strength described by Reynold's number Re.

Part II

Relativistic Field Theories

Chapter 6

Special Relativity

The examples for field theories like Ginzburg-Landau theory or the theory of deformable media in part I were built on the observation, that the actual "granular" structure of the systems on microscopic scales (individual spins or atoms) becomes unimportant for phenomena which are characterized by length scales much larger than the microscopic scales (but still much smaller than the macroscopic dimensions of the specism). Therefore, the introduction of continuously varying objects, called "fields", related to the macroscopic phenomena which were obtained by averaging over on a microscopic scale large regions seems to be a possibly convenient, but by no means fundamental physical concept. As already pointed out in the introduction, the actually first theory, which *had* to introduce fields as fundamental physical objects *without* any underlying microscopic structure is Maxwell's theory of the electromagnetic phenomena. At the same time, Maxwell's theory is, as we will learn in the following chapters, also the prototype of a manifestly covariant or Lorentz invariant field theory, i.e. it is consistent with the theory of of relativity as introduced by Einstein.
Thus, I will start this part with an introduction to the concepts of this fundamental theory before turning to the theory of electromagnetic phenomena.

6.1 Relativity Principles

An *observer* $\mathcal{B}$ introduces a *frame of reference* by introducing a *coordinate system*, which consists of standards of length together with a system of coordinate lines resting with respect to $\mathcal{B}$ and a system of *synchronized clocks*, so that a point located in space and time (*event*) may be represented by a quadruple of real numbers. In an *inertial frame of reference* one can introduce coordinates (the so called *natural coordinates*) that all mass points, which are not affected by forces always move on a straight lines with constant velocities. A frame of reference, which is at rest with respect to the fixed stars is a good approximation of an inertial frame of reference, if one chooses three orthonormal unit vectors as references of length and "appropriately synchronized" clocks to measure time (we will come back to the question of synchronization).
The existence of inertial frames is a very fundamental property of space and time. Its importance was discovered by *Galilei* and the following two statements are sometimes called *Galilean Principle of Relativity*. These statements are valid both in Newtonian and in Einsteinian mechanics (special relativity).

(GI) *Space and time are homogeneous and isotropic.*Stated in mathematical language: Space is an affine (Euclidean) three dimensional point space and time is a one-dimensional affine point space. Thus space-time has the structure of a four-dimensional affine point space $\mathcal{A}^4$.

(GII) *All frames of reference, which move with constant velocity vector relative to an inertial frame of reference are also inertial frames. All inertial frames of reference are physically equivalent.* This means that an observer, which can perform any kind of experiment *inside* an inertial reference system (not looking outside!) cannot determine, in which frame he is located. This requires that all observable physical processes obey the same laws of nature in all inertial frames of reference.

Important consequences of the Galilean relativity are

a) Transformation $T_{I\to I'}(\boldsymbol{r},t) = (\boldsymbol{r}',t')$ of events from one inertial frame I to another inertial frame I' can be composed $T_{I\to J} = T_{K\to J}\circ T_{I\to K}$ and $T^{-1}_{I\to J} = T_{J\to I}$.

b) The transformations can only depend on *relative* motion of I with respect to I' and not on properties of the motion of I or I' separately. If you are sitting in a frame J there is always a frame J', which moves relative to J exactly the same way that I' moves relative to I. Otherwise I and J were not physically equivalent as is required by GII. Therefore we may

state that for any three inertial frames I, I', J there is exactly one fourth inertial frame J' such that $T_{I\to I'} = T_{J\to J'}$.

From a) and b) we can conclude that the transformations have to form a group. Obviously, there is an identity $T_{I\to I}$ and each element has an inverse. From b) we get the extension of the composition law to arbitrary transformations $T_{I\to J}$ and $T_{L\to M}$. According to b) there is an N such that $T_{L\to M} = T_{J\to N}$, so that -according to a)- we can compose any two transformations: $T_{L\to M} \circ T_{I\to J} = T_{J\to N} \circ T_{I\to J} = T_{I\to N}$.
There are further restrictions on the structure of the group of transformations. They have to take all motions with constant velocity vectors $\boldsymbol{v} = (\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1))/(t_2 - t_1)$ into motions with constant velocity vectors $\boldsymbol{v}' = (\boldsymbol{r}'(t_2') - \boldsymbol{r}'(t_1'))/(t_2' - t_1')$. Such transformations (called *affine*) have the general form

$$\boldsymbol{r}' = \boldsymbol{A}\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{r}_0 \tag{6.1a}$$

$$t' = \gamma t + \boldsymbol{u}\cdot\boldsymbol{r} + t_0 \tag{6.1b}$$

Here $\boldsymbol{A}$ is a 3×3 matrix.
Newtonian mechanics completes the two Galilean principles (GI) and (GII) with the assumption

(NIII) *Space and Time are absolute*, i.e. spatial and temporal distances are independent of the frame of reference.

This fixes $\gamma = 1,\ \boldsymbol{u} = 0$ (obvious) and $\boldsymbol{A}$ to be a rotation matrix (corresponding to fixed rotation of one system relative to another). This group of transformations is called *Galilei group*. You should know it from elementary mechanics.
Einstein chose a different completion of (GI) and (GII)

(EIII) The velocity of light in vacuum c is a universal constant of nature.

Obviously, (NIII) and (EIII) are incompatible. With (EIII), synchronization of distant clocks is no longer a trivial process (as it is with (NIII)) and we have to give an operative definition of "synchronous" by specifying a synchronization procedure. This is Einstein's version:

> Consider a clock C resting at the origin of an inertial frame I and another clock $C(\boldsymbol{r})$ at rest in point $\boldsymbol{r}$. As C shows time t_1, a light source at the origin emits a pulse, which is reflected at $\boldsymbol{r}$ and returns to the origin at time t_3 (shown on clock C). We call C and $C(\boldsymbol{r})$ synchronized, if the light pulse reaches $\boldsymbol{r}$ at time $t_2 = t_1 + 1/2(t_3 - t_1) = t_1 + |\boldsymbol{r}|/c$.

An inertial frame of reference with natural coordinates and a set of Einstein-synchronized clocks, which allow to measure time at every point in space is called a *Lorentz system*.

6.2 Lorentz Transformations

As it is usual in special relativity we use a coordinate $x^0 = ct$ for time. In the $\mathbb{R}^4$ of quadruples characterizing events we choose the canonical basis. The basis vectors will be denoted as follows: $\boldsymbol{e}_\mu$ with $\mu = 0, 1, 2, 3$ (Indices 1,2 and 3 correspond to space, index 0 to time).The coordinates of an event are denoted as x^μ. Objects with all lower indeces are called *covariant*, if they carry all upper indeces they are denoted as *contravariant*.
We will use *Einstein's index convention* in the following. This means:

- Greek indices (small Greek letters) run from 0 to 3, Latin indices from 1 to 3.
- indices, which appear both in raised and in lowered position have to be summed, so for example $x^\mu y_\mu = \sum_{\mu=0}^{3} x^\mu y_\mu$.

The quadruple, characterizing an event can be written as a vector in $\mathbb{R}^4$, $x = x^\mu \boldsymbol{e}_\mu$.
Now we are looking for the explicit form of transformations $T_{I\to K}(x^\mu)$ between two Lorentz systems . The form Eq. (6.1a, 6.1b) can be written as follows

$$(x')^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu \tag{6.2}$$

The a^μ correspond to shifts of the origin in space and time. In the following we will concentrate on the $\Lambda^\mu{}_\nu$. We will call these transformations *general Lorentz transformations* and the transformations including a^μ are called *general Poincaré transformations*. Shortly, we will explain the term general. But the reader should be warned that there is no unique denomination of these transformations in the literature (whereas Lorentz and Poincaré transformations are common, see below) The set of general Lorentz (Poincaré) transformations will be called $\mathcal{G}_L$ ($\mathcal{G}_P$).
Now we turn to the explicit characterization of the general Lorentz transformations. Consider a light pulse, emitted at time x^0/c at location $\boldsymbol{r}$ in the direction of the 3-vector $\boldsymbol{y} - \boldsymbol{x}$. At time $y^0/c > x^0/c$ the pulse will arrive at $\boldsymbol{y}$ iff

$$|\boldsymbol{y} - \boldsymbol{x}| = y^0 - x^0 \quad .$$

Such vectors $y - x$ are called *light-like*. As $(\boldsymbol{y} - \boldsymbol{x})^2$ is the (squared) Euclidean distance in space, light-like vectors obey

$$(y^0 - x^0)^2 - (\boldsymbol{y} - \boldsymbol{x})^2 = 0 \ .$$

It is convenient to introduce a quadratic form in the $\mathbb{R}^4$ of events,

$$g(x, y) := g_{\mu\nu} x^\mu y^\nu \tag{6.3}$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that the form (6.3) is *not positive definite*! A vector space equipped with such a quadratic form is called a *pseudo-Euclidean* vector space and in particular $\mathbb{R}^4$ with this quadratic form is called *Minkowski space*.
The Einstein relativity principle (EIII) implies, that vectors, which are *light-like* in one inertial frame stay light-like in every other inertial frame. Thus the homogeneous Lorentz transformations Λ have to obey

$$g(x, x) = 0 \Rightarrow g(\Lambda x, \Lambda x) = 0 \tag{6.4}$$

The important fact to remember is, that this quite lucid physical fact completely fixes the explicit matrix form of admissible Λ's (in fact, *it uniquely determines the whole symmetry group of special relativity*). The mathematical result is contained in the following Lemma (called a Lemma, because it involves mathematics, not physics):
Lemma: From the condition 6.4 it follows that

$$g(\Lambda x, \Lambda y) = b(\Lambda) g(x, y)$$

with $b(\Lambda) > 0$. The proof of this lemma is rather technical and will be omitted here.
In index notation, the lemma takes on the form

$$\Lambda_\mu{}^\rho g_{\rho\lambda} \Lambda^\lambda{}_\nu = b(\Lambda) g_{\mu\nu}$$

The factor $b(\Lambda)$ simply corresponds to a *scaling transformation* of all natural coordinates,

$$T_b(x) = bx \ .$$

The scaling transformations form an obvious subgroup of all general Lorentz transformations. Every general Lorentz transformation can be composed of a scaling and a *Lorentz transformation* obeying

$$\Lambda_\mu{}^\rho g_{\rho\lambda}\Lambda^\lambda{}_\nu = g_{\mu\nu} \tag{6.5}$$

So a general Lorentz transformation is composed of a scaling and a Lorentz transformation:

$$\mathcal{G}_L = \mathcal{R}^+ \times L$$

Now we sketch the structure of the group of transformations, which are determined by this condition.

a) as $\det(g) = \det(\Lambda^\dagger g \Lambda) = \det(\Lambda)^2 \det(g)$ the Λ matrices must have determinants

$$\det(\Lambda) = \pm 1 \ .$$

b) As

$$g_{00} = 1 = \Lambda_0{}^\mu g_{\mu\nu}\Lambda^\nu{}_0 = \Lambda_0{}^0\Lambda^0{}_0 - \sum_{i=1}^{3}\Lambda_0{}^i\Lambda^i{}_0 = |\Lambda_0{}^0|^2 - \sum_{i=1}^{3}|\Lambda_0^i|^2$$

it follows that

$$|\Lambda^0{}_0| \geq 1$$

We define 4 subsets of Lorentz transformations according to

$$\begin{aligned}
L_+^\uparrow &= \{\Lambda \in L |\ det(\Lambda) = +1,\ \Lambda^0{}_0 \geq +1\} \\
L_+^\downarrow &= \{\Lambda \in L |\ det(\Lambda) = +1,\ \Lambda^0{}_0 \leq -1\} \\
L_-^\uparrow &= \{\Lambda \in L |\ det(\Lambda) = -1,\ \Lambda^0{}_0 \geq +1\} \\
L_-^\downarrow &= \{\Lambda \in L |\ det(\Lambda) = -1,\ \Lambda^0{}_0 \leq -1\}
\end{aligned}$$

It is easy to find all transformations just from $L_+^\uparrow$ (also called *restricted orthochronous Lorentz transformations*) by applying to the $\Lambda \in L_+^\uparrow$ the special transformations *parity*

$$P\begin{pmatrix} x^0 \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} x^0 \\ -\boldsymbol{r} \end{pmatrix}$$

and *time reversal*

$$T\begin{pmatrix} x^0 \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} -x^0 \\ \boldsymbol{r} \end{pmatrix}$$

c) The transformations, which only act on space coordinates are the same as in the Galilei group: *rotations* and the *parity operator*. A space rotation (around the x^1 axis), represented as a matrix operation in Minkowski space for examples looks as follows:

$$\hat{R}(\varphi, \boldsymbol{e}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix}$$

As rotations keep all scalar products in Euclidean 3-space fixed, they do not change the pseudo-Euclidean product, either.

Let us now consider the really interesting transformations, which involve both spatial and temporal coordinates. Let us start with transformations, which keep x^2 and x^3 fixed. They must be of the form

$$\begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & 0 & 0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix coefficients have to obey the relations following directly from Eq. (6.5)

$$\begin{aligned} +1 = g(\Lambda e_0, \Lambda e_0) &= |\Lambda^0{}_0|^2 - |\Lambda^1{}_0|^2 \\ -1 = g(\Lambda e_1, \Lambda e_1) &= |\Lambda^0{}_1|^2 - |\Lambda^1{}_1|^2 \\ 0 = g(\Lambda e_0, \Lambda e_1) &= \Lambda_0{}^0 \Lambda^0{}_1 - \Lambda_0{}^1 \Lambda^1{}_1 \end{aligned}$$

A fourth equation comes from the condition $\det(\Lambda) = 1$

$$1 = \det(\Lambda) = \Lambda^0{}_0 \Lambda^1{}_1 - \Lambda^1{}_0 \Lambda^0{}_1$$

A little contemplation shows that all solutions of all these four equations (together with the condition $\Lambda^0{}_0 \geq 1$) can be written as

$$\begin{aligned} \Lambda^0{}_0 &= \Lambda^1{}_1 = \cosh(\theta) \\ \Lambda^1{}_0 &= \Lambda^0{}_1 = -\sinh(\theta) \end{aligned} \tag{6.6}$$

If you have problems with this parametrization, just remember that the formal replacement[1]

$$t \to i\tau$$

[1] This replacement is also called *Wick rotation*.

transforms the pseudo Euclidean form into a Euclidean form

$$-g(x,x) = -(ct)^2 + \boldsymbol{r}^2 \rightarrow (c\tau)^2 + \boldsymbol{r}^2$$

and the Lorentz transformations become 4-dimensional rotations!
The above special Lorentz transformations ($\Lambda_1(\theta)$) are called *boosts* in 1-direction and the parameter θ is called *rapidity*. This parameter has a simple, physical interpretation, which becomes obvious if we apply $\Lambda_1(\theta)$ to a vector x, $x' = \Lambda x$

$$\begin{aligned} x'^0 &= x^0\cosh(\theta) - x^1\sinh(\theta) \\ x'^1 &= x^1\cosh(\theta) - x^0\sinh(\theta) \end{aligned}$$

($x'^2 = x^2$ and $x'^3 = x^3$). Thus if $x'^1 = 0$, $x^1 = x^0\tanh(\theta) = ct\tanh(\theta)$. So the relative velocity of origin the system I' relative to the origin of I is

$$v = c\tanh(\theta)$$

So the rapidity is just a reparametrization of the relative velocity. It is a simple task to reexpress the transformation in terms of the relative velocity. In this form it is usually presented in elementary texts on special relativity

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \end{aligned}$$

with

$$\beta = v/c \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

In analogy to rotations, boosts are characterized by a direction $\boldsymbol{n}$ and a rapidity θ ($\Lambda_1(\theta)$ can alternatively be referred to as $\Lambda(\boldsymbol{n} = \boldsymbol{e}_1, \theta)$). The action of a general boost $\Lambda(\boldsymbol{n}, \theta)$ on a 4-vector x can be read-off from the special case Eq. (6.6).

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta\boldsymbol{n}\boldsymbol{x}) \\ \boldsymbol{x}'_{||} &= \gamma(\boldsymbol{x}_{||} - \beta\boldsymbol{n}x^0) \end{aligned}$$

where $\boldsymbol{x}_{||} = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{x})$ is the projection of $\boldsymbol{x}$ onto the direction of the boost. The components of $\boldsymbol{x}$ perpendicular to $\boldsymbol{n}$ do not change under the boost.

6.3 Time Dilation and Length Contraction

Here we discuss two immediate, albeit much discussed consequences of Lorentz transformations between inertial frames: time dilation and length contraction.
Time dilation

Consider a periodic clock with period Δt resting at $\boldsymbol{r}_c$ in the frame I. Consider the two 4-vectors $x = (ct, \boldsymbol{r}_c)$ and $y = (ct + c\Delta t, \boldsymbol{r}_c)$ corresponding to identical states of the clock. Let us introduce the difference 4-vector between these events $\Delta x = y - x = (c\Delta t, \mathbf{0})$, which encodes that the intervals in time and space between identical states of the clock are Δt and $\mathbf{0}$, respectively. Now we analyze the clock and its readings from an inertial frame I', which is obtained from I by a boost with velocity $-\boldsymbol{v} = -v\hat{\boldsymbol{n}}$. The velocity of the clock in I' is $\boldsymbol{v} = \beta c\hat{\boldsymbol{n}}$. We transform Δx to find out the distances in space and time between identical states of the clock:

$$\begin{aligned} c(\Delta t)' &= \gamma c(\Delta t) \\ \Delta \boldsymbol{r}'_{||} &= -\gamma\beta c\Delta t\hat{\boldsymbol{n}} \end{aligned}$$

The first of these relation tells us that the time interval between identical states of the clock has become

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} > \Delta t$$

An immediate consequence is that that the period of every clock is shortest in the frame of reference, in which the clock is at rest. This is called *Eigenzeit*.

Length Contraction

Consider a bar at rest in frame I. The two events $x = (ct, \boldsymbol{x})$ and $y = (ct, \boldsymbol{y})$ mark the *simultaneously* measured positions of the ends of the bar. It is important that we define the *physical length* of the bar as the spatial distance of *simultaneous events at its ends*. The effect of length contraction does only apply if this definition is used. One can, of course, imagine other definitions. For example, suppose you make a photograph of the bar. Then the light pulses emerging from its ends will travel different distances until they reach the photographic plate *at the same time*. Therefore, the photograph does not show simultaneous events at both ends! The difference is very important and a source of misunderstandings. As your eyes are based on the principles of photography, you will not "see" the Lorentz contraction but rather a bar rotated by $\phi = \arctan(v/c)$ (exercise).

According to our definition, we consider the difference vector $\Delta x = (0, \boldsymbol{y} - \boldsymbol{x}) = \Delta l \cdot (0, \hat{\boldsymbol{e}})$, with Δl denoting the length of the bar in I.

Now perform a boost with velocity $-\boldsymbol{v}$ to system I', in which the bar moves with velocity $\boldsymbol{v} = \beta c\hat{\boldsymbol{n}}$. Note that the transformed events, which appeared simultaneously in I are no longer simultaneous in I', but rather are separated by the time interval

$$c(\Delta t)' = \beta \frac{\Delta l}{\sqrt{1 - v^2/c^2}} (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{e}})$$

The length intervals calculated from applying the boost transformation to Δx are therefore not appropriate to calculate the length of the bar in I'. Rather we have to find simultaneous events in I' and therefore we first transform two *non-simultaneous events* in I, located at the ends of the bar and then require that they have been chosen such that two simultaneous events in I' will result:

$$\begin{aligned} (x^0)' &= \gamma(x^0 + \beta\hat{\boldsymbol{n}}\cdot\boldsymbol{x}) & (y^0)' &= \gamma(y^0 + \beta\hat{\boldsymbol{n}}\cdot\boldsymbol{y}) \\ \boldsymbol{x}'_{||} &= \gamma(\boldsymbol{x}_{||} + \beta\hat{\boldsymbol{n}}x^0) & \boldsymbol{y}'_{||} &= \gamma(\boldsymbol{y}_{||} + \beta\hat{\boldsymbol{n}}y^0) \\ \boldsymbol{x}'_{\perp} &= \boldsymbol{x}_{\perp} & \boldsymbol{y}'_{\perp} &= \boldsymbol{y}_{\perp} \end{aligned}$$

The requirement of simultaneity in I' , $(x^0)' = (y^0)'$ leads to

$$y^0 - x^0 = -\beta\hat{\boldsymbol{n}}\cdot(\boldsymbol{y}-\boldsymbol{x}) = -\beta\Delta l(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{e}})$$

so that

$$\begin{aligned} \boldsymbol{y}'_{||} - \boldsymbol{x}'_{||} &= \gamma[(\boldsymbol{y}_{||} - \boldsymbol{x}_{||}) + \beta(y^0 - x^0)\hat{\boldsymbol{n}}] \\ &= \gamma[\Delta l(\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} - \beta^2\Delta l(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{e}})] \\ &= \gamma(1-\beta^2)\Delta l(\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \end{aligned}$$

The spatial distance between simultaneous events at the ends of the bar in I' is

$$(\Delta l)' = \sqrt{(\boldsymbol{y}'_{||} - \boldsymbol{x}'_{||})^2 + (\boldsymbol{y}'_{\perp} - \boldsymbol{x}'_{\perp})^2}$$

Note that only the length scale *parallel* to the boost velocity is affected. The quantitative expression becomes particularly simple, if the boost velocity is parallel to the bar: $\boldsymbol{v}||\hat{\boldsymbol{e}}$:

$$\Delta l' = \Delta l\gamma(1-\beta^2) = \Delta l\sqrt{1 - v^2/c^2}$$

As the length scales perpendicular to $\boldsymbol{v}$ remain unaffected, a volume element will change with the same scale factor as a bar:

$$(\Delta V)' = \Delta V\sqrt{1 - v^2/c^2}$$

Consequently, the length of bars and the volumes are largest in the frame, in which they are at rest.

Addition of velocities

In non-relativistic mechanics, velocities add linearly. Consider a point moving with constant velocity $\boldsymbol{u}$ in frame I, now perform a $-\boldsymbol{v}$-boost to I'. For simplicity, let us first study the case $\boldsymbol{u}||\boldsymbol{v} = -c\beta\hat{\boldsymbol{n}}$.The Lorentz transformation

from I to I' affects both the length and the time intervals traveled by the point and the velocity of the point in I' is given by

$$\boldsymbol{u}' = c\frac{\Delta \boldsymbol{r}'}{\Delta (x^0)'} = c\frac{\Delta \boldsymbol{r} - \beta c \Delta t \hat{\boldsymbol{n}}}{c\Delta t - \beta \Delta \boldsymbol{r} \cdot \hat{\boldsymbol{n}}}$$

Inserting $\Delta \boldsymbol{r} = \boldsymbol{u}\Delta t$ we get

$$\boldsymbol{u}' = \frac{\boldsymbol{u} + \boldsymbol{v}}{1 + (\boldsymbol{u} \cdot \boldsymbol{v})/c^2}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, we can split $\boldsymbol{u} = \boldsymbol{u}_{||} + \boldsymbol{u}_{\perp}$ in analogy to the arguments on length contraction. As $\Delta \boldsymbol{r}_{\perp}$ remains unaffected by the Lorentz boost, we find the general law of transformation of velocities

$$\begin{aligned} \boldsymbol{u}'_{||} &= \frac{\boldsymbol{u}_{||} + \boldsymbol{v}}{1 + (\boldsymbol{u} \cdot \boldsymbol{v})/c^2} \\ \boldsymbol{u}'_{\perp} &= \sqrt{1 - v^2/c^2}\boldsymbol{u}_{\perp} \quad . \end{aligned}$$

Note that this law implies that velocities can never exceed the velocity of light.

6.4 A second look at vectors, tensors and summation conventions

Before we proceed with physics, we will briefly consider some geometrical concepts. We have learned that events are elements (points) of a Minkowski space, which is a $\mathbb{R}^4$ vector space equipped with a pseudo-Euclidean bilinear form

$$g(x, y) = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \quad .$$

Let us draw some analogies to Euclidean geometry. In an Euclidean vector space (dimension n), you may define a *metric tensor* after you have chosen a particular set of base vectors $\boldsymbol{b}_i, i = 1, 2, \ldots n$ by making use of the Euclidean scalar product

$$g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$$

Note that the metric tensor is symmetric by definition. Here, we will *not* assume, that the base vectors are orthogonal, so they define a oblique, rectilinear coordinate system. For such situations, the Einstein-type summation convention becomes a very elegant and helpful tool in calculus (originally due to Ricci). A vector is written as

$$\boldsymbol{x} = x^i \boldsymbol{b}_i$$

and the scalar product between 2 vectors may be written using the metric tensor:

$$\boldsymbol{x} \cdot \boldsymbol{y} = x^i y^j g_{ij}$$

To the chosen set of base vectors (called *covariant base* in this context) we may construct a second set (called *contravariant base*) and denoted by $\boldsymbol{b}^i, i = 1 \ldots n$ if we require

$$\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i{}^j$$

with $\delta_i{}^j$ denoting the Kronecker symbol. A vector may be expanded in either of the two bases

$$\boldsymbol{x} = x^i \boldsymbol{b}_i = x_i \boldsymbol{b}^i$$

and the components x^i (x_i) are called *contravariant components* (*covariant components*). You may have guessed by now that the metric tensor g_{ij} is called *covariant metric tensor* and that

$$g^{ij} = \boldsymbol{b}^i \boldsymbol{b}^j$$

is the *contravariant metric tensor*. Of course, we may also expand covariant base vectors in terms of contravariant base vectors. Such an expansion looks as follows: $\boldsymbol{b}_i = A_{ij} \boldsymbol{b}^j$. Multiplying by $\boldsymbol{b}_k$ we easily see that $(\boldsymbol{b}_k \cdot \boldsymbol{b}_i) = A_{ij} \boldsymbol{b}_k \cdot \boldsymbol{b}^j = A_{ij} \delta^j{}_k = A_{ik}$ So we get

$$\boldsymbol{b}_i = g_{ij} \boldsymbol{b}^j$$

Analogously, we obtain

$$\boldsymbol{b}^i = g^{ij} \boldsymbol{b}_j$$

Inserting such an expansion into the representation $\boldsymbol{x} = x^i \boldsymbol{b}_i = x^i g_{ij} \boldsymbol{b}^j$ we see that the metric tensor may be used to *raise* and *lower* indices

$$x_j = g_{ji} x^i$$

($g_{ij} = g_{ji}$ has been used here). Analogously

$$x^j = g^{ji} x_i$$

Finally, the covariant and the contravariant metric tensors are inverses of each other, as you may easily check from the fact, that first expanding the $\boldsymbol{b}_i$ in terms of $\boldsymbol{b}^j$ and afterwards expanding the $\boldsymbol{b}^j$ in terms of $\boldsymbol{b}_k$ must lead you back to the original expression. This leads to

$$g_{ij} g^{jk} = \delta_i{}^k$$

What we have briefly presented here is the algebraic part of *Ricci's calculus*. Note that all differences between co- and contravariant coordinates vanish for an orthonormal system, where the covariant and the contravariant base vectors become identical.

Let us now switch back to the pseudo-Euclidean Minkowski space and note, that we may easily transfer Ricci's calculus. We define a set of base vectors $\boldsymbol{e}_\mu, \mu = 0, 1, 2, 3$ such that the pseudo-Euclidean products give [2]

$$g(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = g_{\mu\nu}$$

with

$$g_{00} = 1 \qquad g_{ii} = -1$$

and all other elements vanishing. We may call $g_{\mu\nu}$ the covariant, pseudo-Euclidean metric tensor. Although it is diagonal, it is not proportional to the unit tensor, and therefore, a distinction between co- and contravariant coordinate systems remains. The pseudo-Euclidean product between two vectors may be written as

$$g(\boldsymbol{x}, \boldsymbol{y}) = x^\mu x^\nu g_{\mu\nu} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

The contravariant set of base vectors $\boldsymbol{e}^\mu$ have to obey

$$g(\boldsymbol{e}_\mu, \boldsymbol{e}^\nu) = \delta_\mu{}^\nu$$

and from the orthogonality of co- and contravariant metric tensors , $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu{}^\lambda$ you can trivially read off

$$g^{00} = 1 \qquad g^{ii} = -1$$

i.e the elements of co- and contravariant metric tensor are identical.
Now we may use the g tensors to raise and lower indices:

$$\begin{aligned} x_\mu &= g_{\mu\nu} x^\nu \\ x^\mu &= g^{\mu\nu} x_\nu \end{aligned}$$

Thus contravariant 4-vector

$$(x^0, x^1, x^2, x^3)$$

has the covariant components

$$(x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3)$$

Finally, let us discuss, how partial derivatives of co- and contravariant coordinates transform. Consider the total differential of a scalar function $f(\boldsymbol{x})$:

$$df(\boldsymbol{x}) = \sum_{\mu=0}^{3} \frac{\partial f}{\partial x^\mu} dx^\mu$$

[2] Here we will temporarily denote 4 vectors by bold face symbols, because we need both base vectors, which have an index and components, which have an index. Shortly, we can return to our leaner notation used outside of this subsection

As the total differential of a scalar function is again a scalar we conclude that the partial differentials with respect to contravariant coordinates are covariant components (and vice versa). Therefore we define

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &= \partial_\mu \\ \frac{\partial}{\partial x_\mu} &= \partial^\mu \end{aligned}$$

Note that

$$\partial_0 = \partial^0 \quad \text{and} \quad \partial_i = -\partial^i$$

so that

$$\Box := \partial_\mu \partial^\mu = \frac{1}{c^2}\partial_t^2 - \nabla^2$$

becomes the *wave operator* (D'Alembert-Operator). Let us sum up the relations between vectors and their co- or contravariant components:

$$\begin{aligned} \partial_\mu = \partial/\partial x^\mu \to (c^{-1}\partial_t, +\boldsymbol{\nabla}) \qquad x^\mu = (ct, +\boldsymbol{r}) \\ \partial^\mu = \partial/\partial x_\mu \to (c^{-1}\partial_t, -\boldsymbol{\nabla}) \qquad x_\mu = (ct, -\boldsymbol{r}) \end{aligned} \tag{6.7}$$

6.5 World Lines, 4-Velocity, 4-Acceleration

A point particle in non-relativistic mechanics moves along a trajectory $\boldsymbol{r}(t)$. Obviously, this is not a Lorentz invariant concept. In special relativity, we replace the trajectory by the sequence of events $(x^0(\sigma), \boldsymbol{x}(\sigma))$, which correspond to the presence of the particle at a point $\boldsymbol{x}$ in space at time x^0. The sequence of these events is a curve in Minkowski space, called a *world line*, which can be parametrized by an arbitrary parameter σ (Note that σ has no particular physical significance). For convenience, let us only consider parametrizations, for which σ increases with increasing time

$$\frac{dx^0}{d\sigma} > 0$$

Such parameters are also called *topological time*. As the velocity of the particle can never exceed the velocity of light, all world lines have to obey an important constraint:

$$\left|\frac{d\boldsymbol{x}}{dt}\right| \leq c$$

Making use of the parametrization, this condition is equal to

$$\left|\frac{d\boldsymbol{x}}{d\sigma}\right| \leq \frac{dx^0}{d\sigma}$$

or in an equivalent, manifestly Lorentz invariant form

$$\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma} \geq 0$$

In Minkowski space, world lines have to stay within the so called *light cone* (see Fig. 6.1), which can be constructed around any arbitrary event (chosen as the origin without loss of generality) and is bounded by $|\boldsymbol{x}| = ct = x^0$. Points with $x^0 > 0$ lie in the future of the chosen event and points with $x^0 < 0$ in its

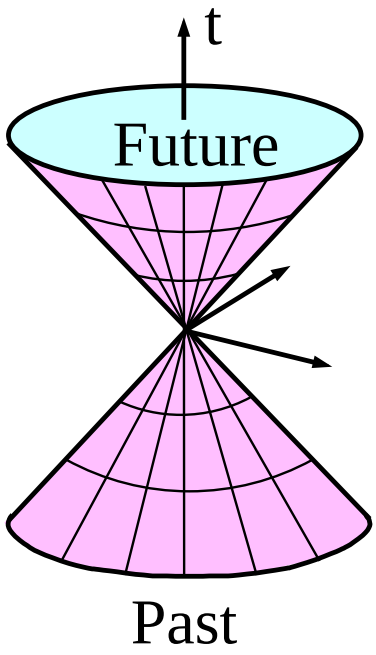

Figure 6.1: The light-cone of an event in Minkowski space.

past.

> The chosen event can only be a consequence of events, which occurred in its past light cone and it can only influence events in its future light cone. This is a Lorentz invariant formulation of *causality*.

Events x, y with $g(x, y) > 0$ (i.e. x is in the light-cone of y and vice versa) are called *time-like*. Events with $g(x, y) = 0$ (on the boundary of the light-cone) are called *light-like* and events with $g(x, y) < 0$ are called *space-like*. Space-like pairs of events are not causally connected.
The arclength of a world line

$$\tau_{12} = \frac{1}{c}\int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}$$

is obviously Lorentz invariant (a scalar under Lorentz transformations). It has a simple and important physical meaning. As

$$\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma} = c^2(\frac{dt}{d\sigma})^2 - (\frac{d\boldsymbol{x}}{d\sigma})^2$$

we may factor $dt/d\sigma$, use $d\sigma = dt(d\sigma/dt)$ and get

$$\tau_{12} = \int_{t_1}^{t_2} dt\sqrt{1 - \frac{\boldsymbol{v}(t)^2}{c^2}} = \int_{t_1}^{t_2} dt\gamma^{-1}(v(t))$$

This corresponds to adding up small time intervals seen on clocks in inertial frames, which have the same velocity as the point particle at the time t of measurement (*local rest frame* or *tangential frame*). To give a more explicit explanation of this important physical interpretation, consider an observer sitting in an inertial frame I and a clock moving around. At each moment of time, we can introduce the local rest frame I', which is an inertial frame rigidly tight to the moving clock for an infinitesimal time interval.
In our (global) inertial frame I, the moving clock proceeds a distance $\sqrt{(d\boldsymbol{r})^2}$ during an infinitesimal time interval dt (as measured in I), whereas in I', the clock is momentarily at rest, so that $d\boldsymbol{r}' = 0$. As space-time intervals are invariant under a Lorentz transformation (connecting I and I'), we have

$$ds^2 = c^2dt^2 - d\boldsymbol{r}^2 = c^2(dt')^2$$

Note that $(dt)'$ corresponds to an infinitesimal interval of the Eigenzeit of the moving clock. It may equivalently be written as

$$dt' = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}}$$

with $\boldsymbol{v} = d\boldsymbol{r}/dt$ denoting the velocity of the moving clock observed in I.
The Eigenzeit may be read off from moving clocks, which are not influenced by accelerations[3]. Note that

$$\tau_{12} \leq \int_{t_1}^{t_2} dt$$

so the Eigenzeit can never be longer than the time observed in any global inertial frame ("moving clocks go more slowly"). This leads to the famous *Zwillingsparadoxon*. Note the local relation between time and Eigenzeit reads

$$\frac{d\tau}{dt} = \frac{1}{\gamma(v(t))} \quad .$$

Given a world line

$$x(\sigma) = \begin{pmatrix} x^0(\sigma) \\ \boldsymbol{x}(\sigma) \end{pmatrix}$$

[3]Biological clocks of people moving in space ships may be a reasonable approximation.

how can we characterize the kinematic elements of motion in a Lorentz invariant way? The worldline contains useful kinematic information, if we parametrize it by the Eigenzeit τ, which is a scalar under Lorentz transformations. The quantity

$$u = \frac{dx}{d\tau}$$

is called *4-velocity* and it is obviously a 4-vector. Its components are

$$u = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma(v(t)) \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix} \tag{6.8}$$

In the *non-relativistic limit* $|\boldsymbol{v} \ll c|$ the spatial components of the 4-velocity are the usual velocity of a particle. Note that the the square of the 4-velocity is *constant* for all trajectories

$$u^\mu u_\mu = \gamma^2(c^2 - \boldsymbol{v}^2) = c^2$$

We may also define a *4-acceleration*

$$a = \frac{du}{d\tau} = \frac{du}{dt}\frac{dt}{d\tau} = \gamma(v)\frac{du}{dt} \tag{6.9}$$

The time derivative may be performed straightforwardly and the result may be written using the 3-acceleration $\boldsymbol{a} = d\boldsymbol{v}/dt$

$$a^\mu = \gamma(v)^2 \begin{pmatrix} 0 \\ \dot{\boldsymbol{v}} \end{pmatrix} + \frac{\gamma(v)^4 \boldsymbol{v}\cdot\dot{\boldsymbol{v}}}{c^2} \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

Note that in a local rest frame

$$a(t) = \begin{pmatrix} 0 \\ \boldsymbol{a}(t) \end{pmatrix} \ , \ \ \boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t)$$

so that the 4-acceleration coincides with the "usual" acceleration. Note furthermore, that

$$u^2 = c^2 \quad \rightarrow \quad \frac{d\,(u^\mu u_\mu)}{d\tau} = 2u^\mu \frac{du_\mu}{d\tau} = 2g(u,a) = 0$$

We should note that in order to construct the 4-velocity and the 4-acceleration, we only need the usual velocity and the usual acceleration. The additional component of the 4-vectors is not free due to the constraints $u^\mu u_\mu = c^2$ and $u^\mu a_\mu = 0$. We have just constructed a manifestly Lorentz invariant kinematics without adding any further ingredients.

Now that we have found kinematic quantities as 4-vectors it is tempting to set up a generalization of Newtonian mechanics to the regime of special relativity. We might, for example, simply define a 4-vector of momentum

$$p^\mu = m u^\mu$$

and a generalized law of motion

$$ma^\mu = F^\mu$$

introducing a 4-vector of force[4]. Obviously, this generalization is in accordance with the requirements of Lorentz invariance. Furthermore, the law of motion implies that Newton's second law remains unchanged in a local rest frame (see above). It is possible to show that the above relations contain the usual definition of momentum and Newton's second law as limiting cases in the non-relativistic regime. However, we will construct relativistic mechanics from the point of view of Lagrangian mechanics, for reasons we explain on the way.

6.6 The Principle of Least Action

In the Theoretical Mechanics course you have learned that classical mechanics with conservative forces can be formulated as the solution of a variational problem. Let us very briefly recall this approach.
The physical trajectory $\boldsymbol{r}(t)$ of a point particle is the one, which is the stationary point of the action functional

$$S = \int_{t_1}^{t_2} dt L(\boldsymbol{r}, d\boldsymbol{r}/dt)$$

where the Lagrange function $L = T - V$ is the difference between kinetic and potential energy. In Cartesian coordinates

$$L = \frac{m}{2}(\frac{d\boldsymbol{r}}{dt})^2 - V(\boldsymbol{r})$$

The stationary condition

$$\frac{\delta S}{\delta \boldsymbol{r}(t)} = 0$$

leads to the Euler-Lagrange equation,

$$\frac{\partial L}{\partial \boldsymbol{r}} - \frac{d}{dt}\frac{\partial L}{\partial (d\boldsymbol{r}/dt)} = 0$$

which exactly reproduces Newtons law of motion.
The value of this approach is twofold:

- it is coordinate free, so it is comparatively easy to treat complicated coordinate systems, as arise, for example, in problems with holonomic constraints

[4]Note that $p^\mu F_\mu = 0$ due to the constraint $u^\mu a_\mu = 0$.

- the connection between symmetries and conservation laws is particularly transparent

Both properties make it an ideal tool in the study of relativistic theories, both particle theories and field theories. Let us briefly introduce relativistic particle mechanics using the principle of least action.
Consider a single material point in free space without any forces. The action S is a functional of the world line of the point and it must be independent of the frame of reference, so it must be a Lorentz scalar.

$$S = \int_{\sigma_1}^{\sigma_2} d\sigma L(x(\sigma), dx/d\sigma)$$

The Lagrange function cannot depend on higher order derivatives, because we require that the physical state is fixed by x and first order derivatives. The only scalar of this type is the arclength of the world line connecting to events x_1 and x_2.

$$S = -\alpha \int_{x_1}^{x_2} ds = -\alpha \int_{\tau_1}^{\tau_2} \sqrt{u_\mu u^\mu} d\tau \ \ ,$$

where we used $ds = \sqrt{u_\mu u^\mu} d\tau$ in the last step. The constant of proportionality $-\alpha$ remains undetermined at present. We can express the Eigenzeit integral in the middle as a time integral (see above) using $u_\mu u^\mu = c^2$ and get

$$S = -\alpha c \int_{t_1}^{t_2} dt \sqrt{1 - v^2/c^2}$$

where $\boldsymbol{v} = d\boldsymbol{r}/dt$ is the particle velocity. Thus we read off the following form of the Lagrange function for a free relativistic particle

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = -\alpha c \sqrt{1 - v^2/c^2}$$

In the nonrelativistic regime, this should become the Lagrange function of a free, non-relativistic particle. Expanding in powers of v/c leads to

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = -\alpha c + \frac{\alpha v^2}{2c}$$

The first term is just an unimportant constant. From the second term we identify

$$\alpha = mc$$

so that

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = -mc^2 \sqrt{1 - v^2/c^2}$$

or

$$L(x, u) = -mc \sqrt{u_\mu u^\mu} \ (= -mc^2) \ \ . \tag{6.10}$$

Now we approach the question of definitions of momentum and energy from a point of view, which can easily be transferred to the more complex situation of field theories. We define momentum and energy as conserved quantities emerging from symmetries, in particular:

- Momentum is the quantity, which is conserved due to translational symmetry.
- Energy is the quantity, which is conserved due to time translational symmetry

The technique to determine a conserved quantity from a (continuous) symmetry group is called *Noether's Theorem*. You should know it from the Theoretical Mechanics course, but we will rephrase what is needed for our purposes. We do this to convince you that there is nothing in Noether's theorem, which reduces its applicability to non-relativistic mechanics.
So consider infinitesimal *space-time translations*

$$(x^\mu)' = x^\mu + \delta x^\mu$$

Such transformations are obviously symmetry transformation for a free particle, as they transform every world line solution for the particle into another world line solution for this particle.
Let us first calculate the variation of the action functional under an infinitesimal variation δx^μ of the wordline under the constraint $\delta x^\mu = 0$ at the end points of the world line:

$$\begin{aligned}
\delta S &= -mc \int_{x_1}^{x_2} (\sqrt{d(x^\mu + \delta x^\mu) d(x_\mu + \delta x_\mu)} - ds) \\
&= -m \int_{x_1}^{x_2} u^\mu d\delta x_\mu \\
&= -m \int_{s_1}^{s_2} u^\mu \frac{d\delta x_\mu}{ds} ds
\end{aligned}$$

where $u^\mu = c dx^\mu/ds = dx^\mu/d\tau$ is the 4-velocity we introduced before. Now we perform a partial integration and use that $\delta x_\mu = 0$ at the end points to obtain

$$\delta S = \int_{s_1}^{s_2} \frac{d}{ds} (m u^\mu) \, \delta x_\mu ds$$

From the stationary condition $\delta S = 0$ and the arbitrariness of δx^μ the equation of motion

$$m a^\mu = m \frac{du^\mu}{d\tau} = 0 \tag{6.11}$$

for a free particle follows. This is exactly what we would have expected, because it implies that the 4-velocity (and thus also the usual velocity) is constant.

On the other hand, the Lagrange function (6.10) is surely invariant under *homogeneous* translations

$$x'^{\mu} = x^{\mu} + \xi^{\mu} =: h^{\mu}(x, \xi)$$

in space-time. Note that these translations comprise spatial as well as temporal translations, i.e. if we find a conservation law it will give us combined energy-momentum conservation!
Going through exactly the same algebra as in classical mechanics, one arrives at the conclusion that if L is invariant under such a transformation, the quantity

$$I_{\mu}(x, u) = \frac{\partial L(x, u)}{\partial u^{\nu}} \left. \frac{\partial h^{\nu}(x, \xi)}{\partial \xi^{\mu}} \right|_{\xi=0} = -\frac{mcu_{\mu}}{\sqrt{u^{\mu}u_{\mu}}} = -mu_{\mu}$$

is a constant of motion with respect to τ. In the last step, we again used $u^{\mu}u_{\mu} = c^2$. The quantity mu^{μ} appearing in the equation of motion (6.11) thus indeed constitutes the *energy-momentum 4-vector*

$$p^{\mu} = mu^{\mu} = \begin{pmatrix} E/c \\ \boldsymbol{p} \end{pmatrix} = m\gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix} \tag{6.12}$$

of relativistic mechanics. The non-relativistic limit for

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \to m\boldsymbol{v}$$

and for

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \to mc^2 + \frac{mv^2}{2}$$

correspond to the Newtonian definitions as is required[5]. The additional term in the energy looks like an uninteresting shift of the zero of energy, but in special relativity it acquires important physical significance, which we will discuss in the next subsection.
From the constraint $u_{\mu}u^{\mu} = c^2$ we get

$$p^{\mu}p_{\mu} = m^2c^2$$

or

$$E^2 = c^2\boldsymbol{p}^2 + m^2c^4$$

This *energy-momentum relation* replaces the formula from Newtonian mechanics, which reads $E = p^2/(2m)$.
You should keep in mind that the definitions of generalized momenta and of energy and Hamiltonian you learned in the Theoretical mechanics course need

[5]The first component of the 4-vector is E/c, because the we considered shifts of $x^0 = ct$

not be replaced. We presented the above arguments in a way, which makes the connection to space-time symmetries obvious. But you will easily check that the definition of momentum

$$\boldsymbol{p} = \frac{\partial L}{\partial v}$$

still holds if you use $L = -mc^2\sqrt{1 - v^2/c^2}$. Furthermore, the energy may still be obtained by

$$E = \boldsymbol{p} \cdot \boldsymbol{v} - L$$

as you have learned. This is the beauty of a coordinate free formulation.

6.7 Physical Relevance of Rest Energy: Mass Defect

Let us consider a body consisting of many (interacting) particles. We may study the motion of the body as a whole. If it is at rest, its energy must be $E_0 = mc^2$ Thus applying special relativity to composite bodies leads us to a *new operative definition* of mass.

Mass is the energy of a body at rest, divided by c^2

Note that this energy will in general contain kinetic energies of the constituent particles and the interaction energies between these particles. This concept of mass differs completely from the Newtonian picture, where the mass of a composite body is just the sum of masses of its constituents $m = \sum_\alpha m_\alpha$ and mass is always conserved.

In Einsteinian mechanics, mass is not a conserved quantity.

The difference between the mass of the composite body and the sum of masses of its constituents is called the *mass defect*.

$$\Delta m = m - \sum_\alpha m_\alpha$$

Let us suppose that a composite body at rest spontaneously decomposes into two parts with masses m_1, m_2 and velocities v_1, v_2 (as an example, think of nuclear fission). Conservation of energy requires

$$mc^2 = \frac{m_1 c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - v_2^2/c^2}}$$

Note that this equation can only be fulfilled if $m > m_1 + m_2$, so that the mass defect must be positive $\Delta M > 0$. This is a necessary condition for fission.

6.8 Particles in Fields

In Newtonian mechanics, interactions between particles can be described by force fields, which are just a mode of description. We may imagine that a mass creates a gravitational field, which is experienced by other masses, but we can also eliminate this field from our description and stick to the *action at a distance* force law of Newton. In special relativity, this concept has to be revised completely. A change in position of one mass can be sensed by other bodies only after a lapse of time, sufficient to transport the disturbance with velocity of light. We can no longer speak of direct interaction between particles at a distance. The field, as a medium of carrying the disturbances emerging from one body and being sensed by another one, acquires physical reality. An interaction process is decomposed into local interactions between particles and fields and the dynamics of the fields themselves.

Therefore, relativistic mechanics requires fields and you will not see a lot of examples with point particles moving around under the influence of static forces. Here we extend the principle of least action to situations, where charged particles interact with an electromagnetic field. We will study the dynamics of the field in the next chapter.

Without much ado, let us just give the action functional and then show that it produces the results you all know. The coupling between point particles of charge q and the electromagnetic field is expressed via a 4-vector field $A^\mu = (\phi, \boldsymbol{A})$, known as the 4-vector potential

$$S = -\int_{x_1}^{x_2} \left(mcds + \frac{q}{c} A^\mu dx_\mu \right)$$

Note that this a manifestly Lorentz invariant action. If we parametrize the world lines by time, the same action looks as follows

$$S = \int_{t_1}^{t_2} \left(-mc^2 \sqrt{1 - v^2/c^2} + \frac{q}{c} \boldsymbol{A} \cdot \boldsymbol{v} - q\phi \right) dt$$

In the non-relativistic limit, this reduces to the form of the action functional known from the Theoretical Mechanics course.

However, from the above discussion you should be aware now, that this action is not complete. A charged particle does not only *experience* electromagnetic fields, it also *creates* such fields. So we cannot, in general, derive valid equations of motion from this action. Nevertheless, in some situations it is a reasonable approximation to consider given electromagnetic fields and neglect the back-action of the charge. Obviously, the charge should be small enough. We will not give precise conditions here, but rather take the approximation for

granted. Then it is straightforward to derive the equation of motion for the point particle. It is just the Euler equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} = 0$$

Let us consider the terms separately. The term $\partial L/\partial \boldsymbol{v}$ gives the generalized momentum

$$\boldsymbol{P} = \gamma m \boldsymbol{v} + \frac{q}{c}\boldsymbol{A} = \boldsymbol{p} + \frac{q}{c}\boldsymbol{A}$$

Now consider $\partial L/\partial \boldsymbol{r} = \boldsymbol{\nabla} L$:

$$\boldsymbol{\nabla} L = \boldsymbol{\nabla}(\boldsymbol{A}(\boldsymbol{r}, t) \cdot \boldsymbol{v}) - q\boldsymbol{\nabla}\phi$$

These terms are reshuffled a bit. First we use the identity from vector analysis

$$\boldsymbol{\nabla}(\boldsymbol{A}(\boldsymbol{r}) \cdot \boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

so that the Euler Lagrange equation takes on the form

$$\frac{d}{dt}\left(\boldsymbol{p} + \frac{q}{c}\boldsymbol{A}\right) = \frac{q}{c}(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \frac{q}{c}\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) - q\boldsymbol{\nabla}\phi$$

Now we insert

$$\frac{d\boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{A}$$

and find a form of the equation of motion, you may already know

$$\frac{d\boldsymbol{p}}{dt} = -\frac{q}{c}\frac{\partial \boldsymbol{A}}{\partial t} - q\boldsymbol{\nabla}\phi + \frac{q}{c}\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

You easily identify two parts of the electromagnetic force, one which is independent of velocity. This force (per unit charge) is called *electric field*

$$\boldsymbol{E} = \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi$$

The factor of $\boldsymbol{v}/c$ (per unit charge) is called *magnetic field*

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

This leads to the well known *Lorentz force*

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v} \times \boldsymbol{B}$$

Although this looks exactly like the formula you might remember from Newtonian mechanics, do not forget that $\boldsymbol{p} = \gamma m \boldsymbol{v}$ and that this equation of motion holds for all velocities, not just for small ones.

Chapter 7

Maxwell‘s Field Theory

7.1 Static Fields

Electric charge is a fundamental property of matter (like mass). Particles with electric charge interact, i.e. exert *forces* onto each other (in the sense of mechanics, i.e. they appear in the equations of motion). Electric charges in relative motion to each other correspond to **electric currents**. Electric currents exert forces onto each other, which cannot be reduced to forces between charges. These additional forces are called *magnetic*. Electromagnetic theory (**Maxwell Theory**) provides exact and quantitative relations between *sources* (i.e. charges and currents) and *fields*, i.e. forces. I will discuss these relations in three steps:

a.) *Forces between static charges. Electric fields*

b.) *Forces between currents. Magnetic fields*

c.) *Generalizations*

Furthermore, I will assume that you are familiar with the fundamental experiments on classical electrodynamics as well as the basic concepts of electro- and magnetostatic as presented in the base level courses in physics. Last, but not least, I shall in general study electromagnetic phenomena in vacuum, i.e. I shall *not distinguish* between electric field and dielectric displacement or magnetic field and magnetic induction. This distinction becomes important in the presence of polarizable media, and will be touched briefly in section 7.3.2.

7.1.1 Coulomb's Law and the electric field

Consider two point particles in vacuum, located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, with charges q_1 and q_2. Experiment tells us that the force $\boldsymbol{F}_{1\to 2}$, which particle 2 exerts on particle 1, is given by

$$\boldsymbol{F}_{2\to 1}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = k_{el}\; q_1\; q_2\; \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \tag{7.1}$$

This force, called the *Coulomb force*, has the same functional form as the Newton's gravitational force.
The constant k_{el} is traditionally and conveniently put to

$$k_{el} = 1/(4\pi\epsilon_0).$$

The *dielectric constant of the vacuum*, ϵ_0, depends upon the system of units. In *cgs units (Gaussian system)* $k_{el} = 1$, i.e. $\epsilon_0 = 1/4\pi$. There is no independent unit of charge, rather

$$[q] = 1\; cm\; dyn^{1/2}$$

as derived from the force law. (see standard texts on electrodynamics for MKSA units). We will reconsider the question of units in electromagnetic theory later.
The Coulomb force obeys two most important "axioms" of Newtonian forces:

a) Newton's third law: $\boldsymbol{F}_{1\to 2} = -\boldsymbol{F}_{2\to 1}$

b) the superposition law, i.e. the force exerted by fixed point charges $q_2, \cdots, q_N$ onto point charge q_1 is given by

$$\boldsymbol{F}_{2\ldots N\to 1} = \frac{q_1}{4\pi\epsilon_0} \sum_{i=2}^{N} q_i \frac{\boldsymbol{r}_1 - \boldsymbol{r}_i}{|\boldsymbol{r}_1 - \boldsymbol{r}_i|^3} \tag{7.2}$$

Note that the forces on q_1 are always proportional to q_1. Therefore it is convenient to introduce the *force on a unit test charge*. This force (per charge) is called **electric field**.

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{F}_{test}/q_{test} \tag{7.3}$$

The electric field may be considered as being *created* by *sources of electric charge*, because it only depends on these charges. A test charge then experiences this field by coupling to it via its own electric charge. This view of sources creating fields and coupling to these fields by some physical property is a very general one in all sorts of field theories, both classical and quantum.
Following the strategy described in section 1.1, we can also introduce *charge densities*. The electric charge in an infinitesimal volume element is $dq = \rho(\boldsymbol{r})d^3r$ and by superposition we get

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r_1 \, \rho(\boldsymbol{r}_1) \frac{\boldsymbol{r} - \boldsymbol{r}_1}{|\boldsymbol{r} - \boldsymbol{r}_1|^3} \tag{7.4}$$

From the force law (7.2) it furthermore follows that

electrostatic forces are conservative.

This important property becomes obvious by noting that

$$\frac{\boldsymbol{r} - \boldsymbol{r}_1}{|\boldsymbol{r} - \boldsymbol{r}_1|^3} = -\nabla_{\boldsymbol{r}} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} \quad .$$

Inserting this relation into Eq. (7.4), we see that $\boldsymbol{E}$ may be written as the gradient of a scalar potential

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}) \tag{7.5}$$

with

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r_1 \frac{\rho(\boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|} \tag{7.6}$$

7.1.2 Field Equations of Electrostatics

Now we can easily set up differential equations, which determine the electric field of static charge distributions. First, as $\boldsymbol{E}$ is a gradient, it must obey

$$\nabla \times \boldsymbol{E} = 0 \qquad \textbf{(Homogeneous Maxwell E)} \tag{7.7}$$

A second equation, which connects the electric field to its sources is obtained, if we use the relation[1]

$$\Delta_r \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} = -4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r}_1) \ , \tag{7.8}$$

which when applied to Eq. (7.6) leads to

$$\begin{aligned} \Delta_r \phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3 r_1 \rho(\boldsymbol{r}_1) \Delta_r \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} \\ &= -\frac{1}{\epsilon_0}\rho(\boldsymbol{r}) \end{aligned} \tag{7.9}$$

The equation

$$\Delta_r \phi(\boldsymbol{r}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{r}) \tag{7.10}$$

is called the **Poisson equation**; for vanishing charge density $\rho(\boldsymbol{r}) = 0$ it is called **Laplace equation**. Using Eq. (7.5), we may rewrite it as a field equation for $\boldsymbol{E}$:

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r}) \qquad \textbf{Inhomogeneous Maxwell E} \tag{7.11}$$

The Poisson (or Laplace) equation (7.10) has to be supplemented by *boundary conditions*. As these problems have been treated extensively in the undergraduate courses, I will not discuss them here but ask those interested to study the detailed presentation in the standard textbooks on electrodynamics.

7.1.3 The Law of Biot-Savart

We have built a theory of electrostatics starting from point charges. Now we want to study *time independent* electric current densities $\boldsymbol{j}(\boldsymbol{r})$ and the forces they exert on charges and on each other (*magnetostatics*), using an analogous idealization for currents, the *infinitesimal current filament*. From experiment we know that the electric charge is a strictly conserved quantity, i.e. the charge density has to obey a continuity equation of the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \ .$$

[1] see e.g. Jackson "*Electrodynamics*".

In static situations $\partial\rho/\partial t$ has to vanish and thus the admissible current densities have to obey

$$\boldsymbol{\nabla}\cdot\boldsymbol{j}=0\ .$$

Note that $\boldsymbol{j}$-field lines of magnetostatics have to be closed: Every field line starting in a volume element dV transports current out of this volume element, so it has to return to dV in order to cancel the net flux through the surface of the volume as is required by $\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$ in conjunction with Gauss' law.

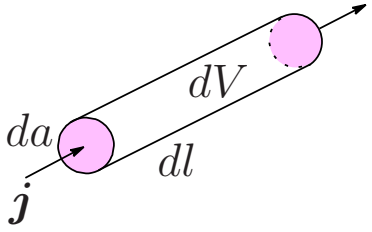

Figure 7.1: The law of Biot-Savart: An infintesimal current filament.

Let us imagine a current distribution to be composed of filaments $\boldsymbol{j}(\boldsymbol{r}) = j(\boldsymbol{r})\boldsymbol{t}(\boldsymbol{r})$, where $\boldsymbol{t}(\boldsymbol{r})$ denotes the unit vector in the direction of $\boldsymbol{j}(\boldsymbol{r})$. An infinitesimal part of such a filament is the analogue of a point charge in electrostatics. The current flowing through an area element $d\boldsymbol{a}$ is $I=\boldsymbol{j}\cdot d\boldsymbol{a}=jda$ and this current has to be transported within the current filament, see Fig. 7.1. Thus we have to put

$$\boldsymbol{j}dV(\boldsymbol{r})=j\boldsymbol{t}(\boldsymbol{r})dlda=jdad\boldsymbol{l}=Id\boldsymbol{l}\ . \tag{7.12}$$

If I is finite, the current density has to be singular (just like a charge density representing a point charge has to be singular). On the other hand, this formula allows us to consider every current density as a collection of (infinitesimal) parts of current filaments with infinitesimal currents (just like a continuous charge distribution can be represented by as a collection of infinitely many infinitesimal point charges).
The analogue of Coulomb's law is the force exerted by a line element $d\boldsymbol{l}_2$ (at $\boldsymbol{r}_2$) of a current filament with current I_2 onto a line element $d\boldsymbol{l}_1$ (at $\boldsymbol{r}_1$) of a current filament with current I_1. Experiment tells us that it is given by

$$\boldsymbol{F}_{2\to1}=k_{magn}\ I_1I_2\ d\boldsymbol{l}_1\times\left(d\boldsymbol{l}_2\times\frac{(\boldsymbol{r}_1-\boldsymbol{r}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2)|^3}\right)\ . \tag{7.13}$$

The constant k_{magn} is conveniently written as

$$k_{magn}=\frac{\kappa^2\mu_0}{4\pi}\ .$$

The reason for introducing an extra constant κ will become clear in section 7.2.3.
We will call this law the force law of Biot-Savart[2], since, like Coulomb's law, it contains the complete information about forces due to currents and can be generalized to distribution of currents similar to (7.2).
We can separate the force into a factor, which depends on the source ($d\boldsymbol{l}_2$) and one, which depends on the "test" current filament ($d\boldsymbol{l}_1$) and in this way introduce the *magnetic field*

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi}\, I_2 \frac{d\boldsymbol{l}_2 \times (\boldsymbol{r} - \boldsymbol{r}_2)}{|\boldsymbol{r} - \boldsymbol{r}_2|^3} \quad . \tag{7.14}$$

Thus the force onto an element of a current filament $I_1 d\boldsymbol{l}_1$ is

$$\boldsymbol{F}_{2\to 1} = \kappa I_1 d\boldsymbol{l}_1 \times \boldsymbol{B}(\boldsymbol{r}) \quad .$$

Let us now generalize this result to arbitrary *current densities* by using Eq. (7.12) and the important property that *the magnetic forces obey the superposition law*:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi} \int d^3 r_2\, \boldsymbol{j}(\boldsymbol{r}_2) \times \frac{\boldsymbol{r} - \boldsymbol{r}_2}{|\boldsymbol{r} - \boldsymbol{r}_2|^3} \quad . \tag{7.15}$$

Finally, we may also combine many current filament elements $I_1 dl_1 = \boldsymbol{j}(\boldsymbol{r}) d^3\boldsymbol{r}$ to obtain the force of a magnetic field $\boldsymbol{B}$ on an arbitrary current density :

$$\boldsymbol{F} = \kappa \int d^3\boldsymbol{r} \boldsymbol{j}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}) \quad . \tag{7.16}$$

On the other hand, if we specialize to a moving point charge (charge density $\rho(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{r}(t))$ and current density $\boldsymbol{j}(\boldsymbol{r}) = q\boldsymbol{v}\delta(\boldsymbol{r} - \boldsymbol{r}(t))$) we arrive at the *Lorentz force*

$$\boldsymbol{F} = \kappa q \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}(t)) \quad . \tag{7.17}$$

From Eq. (7.15), it is easy to see that $\boldsymbol{B}$ may be written as the rotation of a vector field (analogous to $\boldsymbol{E}$ being represented as a gradient of a scalar field). Just use

$$\boldsymbol{j}(\boldsymbol{r}_2) \times \frac{\boldsymbol{r} - \boldsymbol{r}_2}{|\boldsymbol{r} - \boldsymbol{r}_2|^3} = \nabla_{\boldsymbol{r}} \times \left(\frac{\boldsymbol{j}(\boldsymbol{r}_2)}{|\boldsymbol{r} - \boldsymbol{r}_2|} \right)$$

and you get

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{7.18}$$

with

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi} \int d^3 r_2 \frac{\boldsymbol{j}(\boldsymbol{r}_2)}{|\boldsymbol{r} - \boldsymbol{r}_2|} \quad . \tag{7.19}$$

$\boldsymbol{A}$ is called the *vector potential* (of magnetostatics).

[2]Conventionally, integrated forms of Eq. (7.13) are called the Biot-Savart law.

7.1.4 Field Equations for Magnetostatics

We can now derive the field equations for the magnetic field. $\boldsymbol{B}$ can be expressed as the rotation of a vector field, so its divergence has to vanish

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \textbf{Homogeneous Maxwell B} \ . \tag{7.20}$$

Second, take the rotation of Eq. (7.15) and use

$$\nabla \times [\nabla \times \boldsymbol{c}(\boldsymbol{r})] = \nabla[\nabla \cdot \boldsymbol{c}(\boldsymbol{r})] - \Delta \boldsymbol{c}(\boldsymbol{r})$$

with $\boldsymbol{c}(\boldsymbol{r}) = \boldsymbol{j}(\boldsymbol{r}_2)/|\boldsymbol{r} - \boldsymbol{r}_2|$.
The first term vanishes in magnetostatics. This can be seen, if we use

$$\boldsymbol{j}(\boldsymbol{r}_2) \cdot \boldsymbol{\nabla} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_2|} = -\boldsymbol{j}(\boldsymbol{r}_2) \cdot \boldsymbol{\nabla}_2 \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_2|}$$

and then perform a partial integration to let the partial derivatives act on $\boldsymbol{j}(\boldsymbol{r}_2)$. This produces a $\boldsymbol{\nabla}_2 \cdot \boldsymbol{j}(\boldsymbol{r}_2)$, which vanishes in magnetostatics.
In the second part the Laplacian is applied to $|\boldsymbol{r} - \boldsymbol{r}_2|^{-1}$, which leads to a delta function (see Eq. (7.8)) and thus produces

$$-\int d^3 r_2 \Delta \frac{\boldsymbol{j}(\boldsymbol{r}_2)}{|\boldsymbol{r} - \boldsymbol{r}_2|} = 4\pi \boldsymbol{j}(\boldsymbol{r}) \ ,$$

so that we find

$$\nabla \times \boldsymbol{B} = \kappa \mu_0 \, \boldsymbol{j} \qquad \textbf{Inhomogeneous Maxwell B} \ . \tag{7.21}$$

7.2 Dynamics

All static electric and magnetic phenomena are contained in the equations

$$\begin{array}{ll} \nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \rho(\boldsymbol{r})/\epsilon_0 & \nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \\ \nabla \times \boldsymbol{B}(\boldsymbol{r}) = \kappa \mu_0 \boldsymbol{j}(\boldsymbol{r}) & \nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \ . \end{array} \tag{7.22}$$

To generalize these equations to dynamic phenomena, we have to take into account one more basic experimental finding.

7.2.1 Faraday's Law of Induction

Faraday's law of induction can be stated as follows:

> If there is a time dependent magnetic flux $\Phi(t) = \int_{\mathcal{A}(t)} d\boldsymbol{a} \cdot \boldsymbol{B}(t)$ through a surface $\mathcal{A}(t)$ bounded by a (possibly time dependent)

conducting loop $\gamma(t) = \partial\mathcal{A}(t)$, there is an *electromotive force* or *induced voltage*, U_{ind}, for charges moving in γ, which is proportional to the time derivative of the magnetic flux:

$$U_{ind} = -k\frac{d\Phi}{dt} \quad .$$

The minus sign is *not* a convention but is again dictated by experiment, which tells us that the reaction of the charges is *opposite* to the field change (Lentz's rule). The voltage U_{ind} is defined as the (virtual) work per charge necessary to move a unit charge once around γ, i.e. if $\boldsymbol{F}$ denotes the electromagnetic force acting on the charge q

$$qU_{ind} = \oint\limits_{\gamma} d\boldsymbol{x} \cdot \boldsymbol{F} \quad .$$

First, a remark to avoid confusion. Although it may seem, that Faraday's law involves *surfaces*, bounded by conducting loops, it is only about loops! Note that there are infinitely many surfaces bounded by a given loop. If we consider two of them, say $\mathcal{A}_1$ and $\mathcal{A}_2$, the union $\mathcal{A}_1 \cup \mathcal{A}_2$ encloses a finite volume. Using Gauss' law and the Maxwell equation (7.21), we see that the flux across $\mathcal{A}_1$ must be equal to the flux across $\mathcal{A}_2$. So the magnetic flux is only a property of the loop and you can choose any surface you like (as long as it is bounded by the loop) to compute the flux integral.

To turn Faraday's law into a field equation, let us first consider the situation of a fixed, time-independent loop, which is penetrated by a time dependent magnetic field. The time derivative of the flux then acts only on the $\boldsymbol{B}$-field. The force per unit charge fixed to the loop is given by an electric field (per definition $\boldsymbol{F} = q\boldsymbol{E}$) so that

$$\oint\limits_{\partial\mathcal{A}} d\boldsymbol{l} \cdot \boldsymbol{E} = -k \int\limits_{\mathcal{A}} d\boldsymbol{a} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \quad .$$

Here we use Stokes theorem to transform the line integral into a surface integral, so that

$$\int\limits_{\mathcal{A}} d\boldsymbol{a} \cdot \nabla \times \boldsymbol{E} = -k \int\limits_{\mathcal{A}} d\boldsymbol{a} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

and as this has to hold for arbitrary loops, the integrands must be equal. This leads to the field equation

$$\nabla \times \boldsymbol{E} = -k\frac{\partial \boldsymbol{B}}{\partial t} \tag{7.23}$$

which appears as a generalization of Eq. (7.7).

It seems that this equation is still not complete, because we assumed that the conducting loop was at rest. We will now show that this is *not* the case. In fact, analyzing the situation of a time dependent loop $\gamma(t)$ only fixes the constant of proportionality in the field equation, which still remained undetermined so

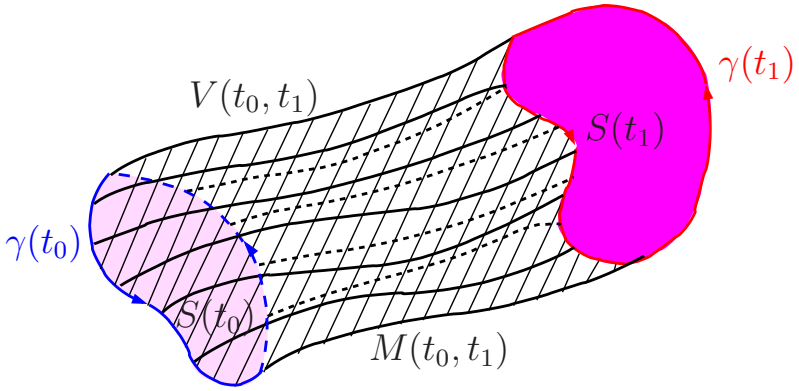

Figure 7.2: Sketch of a time-dependent loop.

far. Let us analyze this situation in detail. Consider a surface $\mathcal{S}(t_0)$, bounded by the loop $\gamma(t_0) = \partial\mathcal{S}(t_0)$. As time proceeds from t_0 to t_1, the loop spans a surface $\mathcal{M}(t_0, t_1)$. This surface corresponds to the mantle of a deformed cylinder with volume $V(t_0, t_1)$, which is completed by $\mathcal{S}(t_0)$ and $\mathcal{S}(t_1)$. The situation is shown in Fig. 7.2. For the following argument, keep t_0 and t_1 fixed. Then we may introduce parametrizations

a) of the loops $\gamma(t) = \{\boldsymbol{r}(t,\tau), 0 \le \tau \le \tau_m\}$ with a curve parameter τ.

b) of the surfaces $\mathcal{S}(t) = \{\boldsymbol{r}(t,\tau,\sigma),\ 0 \le \tau \le \tau_m,\ 0 \le \sigma \le \sigma_m\}$

c) of the mantle surface $M(t_0,t_1) = \{\boldsymbol{r}(t,\tau),\ 0 \le \tau \le \tau_m,\ t_0 \le t \le t_1\}$

so that the locations within the volume of the deformed cylinder are within the parametrized set $\{\boldsymbol{r}(t,\tau,\sigma)|0 \le \tau \le \tau_m,\ 0 \le \sigma \le \sigma_m,\ t_0 \le t \le t_1\}$.
Consider the magnetic flux through $\mathcal{S}(t)$. Its time derivative is given by

$$\frac{d}{dt}\int\limits_{\mathcal{S}(t)} d\boldsymbol{a}\cdot\boldsymbol{B}(t) = \int\limits_{\mathcal{S}(t)} d\boldsymbol{a}\cdot\frac{\partial\boldsymbol{B}}{\partial t} + \frac{d}{dt'}\int\limits_{\mathcal{S}(t')} d\boldsymbol{a}\cdot\boldsymbol{B}(t)\Bigg|_{t'=t} .$$

In the second term on the right hand side, the time differentiation only acts on the time dependence of the integration surface.

Let us first consider a time independent magnetic field. We apply Gauss theorem to the deformed cylinder:

$$\int\limits_{V(t_0,t_1)} d^3r\ \nabla\cdot\boldsymbol{B} = \int\limits_{\mathcal{S}(t_1)} d\boldsymbol{a}\cdot\boldsymbol{B} + \int\limits_{\mathcal{M}(t_0,t_1)} d\boldsymbol{a}\cdot\boldsymbol{B} - \int\limits_{\mathcal{S}(t_0)} d\boldsymbol{a}\cdot\boldsymbol{B}\ .$$

Since no magnetic charges exist[3], the left hand side has to vanish and we get

$$\frac{d}{dt_1}\int\limits_{\mathcal{S}(t_1)} d\boldsymbol{a}\cdot\boldsymbol{B} = -\frac{d}{dt_1}\int\limits_{\mathcal{M}(t_0,t_1)} d\boldsymbol{a}\cdot\boldsymbol{B}\ .$$

Now we use the introduced parametrization

$$\int\limits_{\mathcal{M}(t_0,t_1)} d\boldsymbol{a}\cdot\boldsymbol{B} = \int\limits_{t_0}^{t_1} dt \int\limits_0^{\tau_m} d\tau \left(\frac{\partial\boldsymbol{r}(t,\tau)}{\partial\tau}\times\frac{\partial\boldsymbol{r}(t,\tau)}{\partial t}\right)\cdot\boldsymbol{B}$$

and $\boldsymbol{X}\cdot(\boldsymbol{Y}\times\boldsymbol{Z}) = (\boldsymbol{Z}\times\boldsymbol{X})\cdot\boldsymbol{Y}$ to get

$$\frac{d}{dt_1}\int\limits_{\mathcal{S}(t_1)} d\boldsymbol{a}\cdot\boldsymbol{B} = -\int\limits_{\gamma(t_1)} d\boldsymbol{l}\cdot(\boldsymbol{v}\times\boldsymbol{B})$$

where $\boldsymbol{v} = \partial\boldsymbol{r}(t_1,\tau)/\partial t_1$ is the velocity of a point of the loop $\gamma(t_1)$ and $d\boldsymbol{l} = d\tau(\partial\boldsymbol{r}/\partial\tau)$ is an element of the curve $\gamma(t_1)$. Note that for a time independent magnetic field and a moving loop, the force on a test charge in this loop comes from the motion of the loop and is in accordance with the force a magnetic field exerts on a moving point charge (Lorentz force) if we choose the constant of proportionality appropriately.

Finally, we can generalize our argument to time dependent $\boldsymbol{B}$ fields. We make use the field equation $-k\partial\boldsymbol{B}/\partial t = \nabla\times\boldsymbol{E}$ to treat the term arising from differentiating the magnetic field with respect to time.

The complete time derivative of the flux through $\mathcal{S}(t)$ takes on the form

$$\begin{aligned} -k\frac{d}{dt}\int\limits_{\mathcal{S}(t)} d\boldsymbol{a}\cdot\boldsymbol{B}(t) &= k\int\limits_{\gamma(t_1)} d\boldsymbol{l}\cdot(\boldsymbol{v}\times\boldsymbol{B}) + k\int\limits_{\mathcal{S}(t)} d\boldsymbol{a}\cdot\frac{\partial\boldsymbol{B}}{\partial t} \\ &= \oint\limits_{\gamma(t_1)} d\boldsymbol{l}\cdot(k\,\boldsymbol{v}\times\boldsymbol{B}+\boldsymbol{E})\ . \end{aligned} \tag{7.24}$$

In the last step we used Stokes' theorem. Note that we obtained Faraday's law of induction in the form

$$-\kappa\frac{d\Phi}{dt} = U_{ind} = \oint\limits_{\gamma(t)} d\boldsymbol{l}\cdot\boldsymbol{F}\ .$$

[3]Dirac has shown this experimental fact to be a direct consequence of the quantization of electric charges. See e.g. Jackson, "*Electrodynamics*".

Thus we conclude that the field equation Eq. (7.23) implies Faraday's Law for arbitrarily moving loops if we use the electromagnetic force exerted by electric and magnetic fields on a point charge and fix

$$k = \kappa.$$

7.2.2 Maxwell's displacement current

We have established the field equation required by Faraday's law of induction, and the complete set of field equations of electromagnetism now reads

$$\begin{array}{ll} \nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r}) & \nabla \times \boldsymbol{E}(\boldsymbol{r}) = -\kappa \frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{B}(\boldsymbol{r}) = \kappa\mu_0 \boldsymbol{j}(\boldsymbol{r}) & \nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \ . \end{array} \tag{7.25}$$

It is easy to see that this set of equations is not compatible with the fundamental law of charge conservation. Just take the divergence of the inhomogeneous equation for $\boldsymbol{B}$. This gives $0 = \nabla \cdot (\nabla \times \boldsymbol{B}) = \kappa\mu_0 \nabla \cdot \boldsymbol{j}$, which is in conflict with the continuity equation $\partial\rho/\partial t = -\nabla \cdot \boldsymbol{j}$. It was Maxwell's original idea to add an additional current density like term: $\nabla \times \boldsymbol{B} = \kappa\mu_0(\boldsymbol{j} + \boldsymbol{j}_M)$. To reconcile this equation with the equation of continuity, the additional term has to obey

$$\nabla \cdot \boldsymbol{j}_M = \frac{\partial \rho}{\partial t} = \epsilon_0 \frac{\partial(\boldsymbol{\nabla} \cdot \boldsymbol{E})}{\partial t}$$

and so we finally may *postulate* the complete form of Maxwell's equations of electromagnetism

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \tag{7.26a}$$

$$\nabla \times \boldsymbol{E} = -\kappa \frac{\partial \boldsymbol{B}}{\partial t} \tag{7.26b}$$

$$\nabla \times \boldsymbol{B} = \kappa\mu_0 \boldsymbol{j} + \kappa\mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{7.26c}$$

$$\nabla \cdot \boldsymbol{B} = 0 \ . \tag{7.26d}$$

7.2.3 Units

We have found that Maxwell's theory contains 3 constants with physical dimensions:

$$\epsilon_0, \ \mu_0 \text{ and } \kappa \ .$$

From dimensional analysis it follows that $[\kappa^2\mu_0\epsilon_0] = time^2/length^2$. Let us therefore introduce a velocity via

$$\kappa^2\epsilon_0\mu_0 =: 1/c^2 \tag{7.27}$$

which will turn out to be the velocity of light later on.
The constant $\kappa, \epsilon_0, \mu_0$ will change with the system of units. We distinguish two classes of unit systems:

$$\text{asymmetric units}: \ \kappa = 1 \quad \epsilon_0\mu_0 = \frac{1}{c^2}$$

and

$$\text{symmetric units}: \ \kappa = 1/c \quad \epsilon_0\mu_0 = 1$$

The most prominent example of the first class is the international MKSA system (*SI units*). Remember that in this system

$$\epsilon_0 = \frac{10^7}{4\pi}\frac{1}{c^2}\left[\frac{A^2}{N}\right], \ \mu_0 = 4\pi \cdot 10^{-7}\left[\frac{N}{A^2}\right]$$

Well known examples of the second class are

$$\text{Gaussian units}: \ \epsilon_0 = \frac{1}{4\pi}, \ \mu_0 = 4\pi$$

and

$$\text{Heaviside units}: \ \epsilon_0 = \mu_0 = 1$$

Some of the later sections will use a particular system of units, because it is used most frequently in the contexts under consideration and the expressions look particularly simple in a special system of units.

7.3 Electrodynamics and Matter

Maxwell's equations are linear, inhomogeneous partial differential equations, *provided the charge and current densities are given*. However, if we consider charged matter, the fields created by this matter will lead to forces acting on the very same matter and thus Maxwell's equations cannot be considered a complete description of charged matter. As soon as we take the back reaction of the fields into account, we have to provide additional equations describing the action of the electro-magnetic forces on charged matter. These equations will in general be non-linear and will lead to very complicated theories. We will consider here only simple situations, which can be treated with modest effort. For more complicated situations, see textbooks like Landau Lifshitz Vol. 8.

7.3.1 Conductors

Conductors (metals) are characterized on a macroscopic scale as a form of condensed matter, which includes charge that can flow around. Thus, in *static equilibrium situations*[4] the charge inside metals will redistribute itself until all the forces acting on the flowing charged particles will vanish. This implies that the electric field inside a conductor has to vanish in static equilibrium:

$$\boldsymbol{E}_{in} = 0 \ ,$$

and the Maxwell equation $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0$ implies that the charge density inside the conductor also has to vanish

$$\rho_{in} = 0 \ .$$

Charges appearing as the reaction upon externally applied fields can only appear directly on the surface of the conductor. Such charge distributions, which are constrained to a surface will be described by a *charge suface density* λ with physical dimension $[\lambda] = charge/length^2$. Note that a surface charge density corresponds to a singular volume charge density, which contains a delta function in the direction normal to the surface[5]. The surface charge is not given, but has to be calculated, which seems a complicated problem. But the problem can be stated in a much simpler form. Surface charges can float freely on the surface of the conductor, and thus the electric field vector at the surface

[4] Conduction in a metal is not such a situation!

[5] So, if the surface is given in the form $z = z(x, y)$, the volume charge density $\rho(x, y, z)$ corresponding to a surface charge density can be written in the form $\rho(x, y, z) = \lambda(x, y)\delta(z - z(x, y))$.

must have vanishing components in directions tangential to the surface. As $\boldsymbol{\nabla}\phi = -\boldsymbol{E}$, this implies that

$$\boldsymbol{t}(\boldsymbol{r}) \cdot \boldsymbol{\nabla}\phi(\boldsymbol{r}) = 0$$

for every vector $\boldsymbol{t}(\boldsymbol{r})$ tangential to the surface at $\boldsymbol{r}$ or, stated otherwise

$$\phi(\boldsymbol{r}) = \text{const} \quad \text{for all surface point } \boldsymbol{r} \ .$$

If the conductor fills a volume V we denote the (outwardly oriented) surface by ∂V and state the boundary condition in the brief form

$$\phi|_{\partial V} = 0 \quad \text{(Dirichlet boundary condition)} \ . \tag{7.28}$$

The standard task of electrostatics is then to find the solution to charge distributions and fields in a given arrangment of conductors under the boundary condition (7.28), which leads to the concept of *capacitance* and *capacitors* (for more details see for example [4] or the lecture *Physics II*).

7.3.2 Polarizable Media

The situation is different for insulators. In these materials, charges cannot move freely, but an applied external electric field will lead to a redistribution of charges on a microscopic scale which will result in an induced polarization of the medium modifying the electric field. Again, a true microscopic calculation of this effect is far beyond the scope of this lecture and in fact a research frontier in modern solid state physics. Let us therefore invoke the concepts already used in the theory of elastic media and try to set up a *macroscopic model* for the effects of polarizable media, so-called *dielectrics*.
To this end we again assume that our external fields are slowly varying with respect to the atomic length scales[6]. We can then, as for elastic media, divide our sample into portions large compared to microscopic scales but still small compared to the macroscopic dimensions of the sample and in particular small enough so that we can assume that the fields are *constant* over the dimension of the volume elements. An applied electric field will then *polarize* the volume elements such that positive charges will accumulate on the boundary in the direction of the field and negative charges on the opposite boundary (see Fig. 7.3). Obvioulsy, the field $\boldsymbol{E}_{ind}$ induced by this charge imbalance will act against the applied external field $\boldsymbol{E}$.

[6] This is a reasonable assumption, even for electromagnetic waves up to frequencies of visible light.

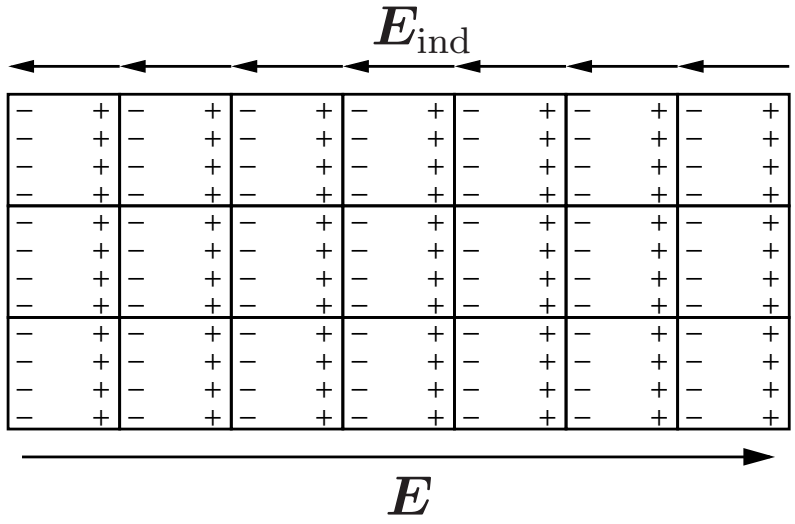

Figure 7.3: Response of a polarizable medium to an external electric field $\boldsymbol{E}(\boldsymbol{r})$.

If we denote with $\boldsymbol{R}_i$ the position of the i-the volume element and with $\langle \boldsymbol{d} \rangle_i(\boldsymbol{r})$ its (average) contribution to the dipole moment at a point $\boldsymbol{r}$, the total *polarization*, i.e. the dipole moment per unit volume, is given as

$$\boldsymbol{P}(\boldsymbol{r}) = \frac{1}{\text{Vol.}} \sum_i \langle \boldsymbol{d} \rangle_i(\boldsymbol{r}) \quad .$$

On the other hand, a dipole $\boldsymbol{d}(\boldsymbol{r}')$ at $\boldsymbol{r}'$ produces a contribution[7]

$$\delta\phi_{\text{dipol}}(\boldsymbol{r}) = \boldsymbol{d}(\boldsymbol{r}') \cdot \boldsymbol{\nabla}_{\boldsymbol{r}'} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

to the potential $\phi(\boldsymbol{r})$ at $\boldsymbol{r}$. Together with the potential produced by the true charges we obtain

$$\begin{aligned} \phi(\boldsymbol{r}) &= \int d^3r' \left\{ \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} + \boldsymbol{P}(\boldsymbol{r}') \cdot \boldsymbol{\nabla}_{\boldsymbol{r}'} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right\} \\ &\overset{\text{P.I.}}{=} \int d^3r' \frac{\rho(\boldsymbol{r}') - \boldsymbol{\nabla}_{\boldsymbol{r}'} \cdot \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \quad . \end{aligned}$$

Finally, with $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla}^2 |\boldsymbol{r}|^{-1} = -4\pi\delta(\boldsymbol{r})$ we arrive at

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \left[\rho - \boldsymbol{\nabla} \cdot \boldsymbol{P} \right] \quad .$$

Comparing this result with (7.26a), we can set up a similar equation for the effective field $\boldsymbol{D} := \boldsymbol{E} + 4\pi\boldsymbol{P}$ called *dielectric displacement*. With this definition,

[7] For convenience I choose Gaussian units in the following.

the equation (7.26a) valid in vacuum has to be replaced by

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho \ , \tag{7.26a'}$$

where $\rho(\boldsymbol{r})$ denotes the *external charge density*, i.e. excluding charges possibly induced by the electric field.
With similar arguments, one finds that in the presence of a medium the equation (7.26c) must be replaced by the expression

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} \ , \tag{7.26c'}$$

where $\boldsymbol{H} := \boldsymbol{B} - 4\pi\boldsymbol{M}$ is called *magnetic field*, $\boldsymbol{M}(\boldsymbol{r})$ denotes the magnetization induced in the medium by the *magnetic induction* $\boldsymbol{B}(\boldsymbol{r})$ and $\boldsymbol{j}(\boldsymbol{r})$ the *external current density* excluding currents induced by the varying magnetic induction.
As already mentioned it is in general an extremely complicated task to calculate $\boldsymbol{P}(\boldsymbol{r})$ and $\boldsymbol{M}(\boldsymbol{r})$ for a given distribution of electric field, magnetic induction, external charges and currents from a microscopic theory. However, in case of weak fields and isotropic and homogenous media[8], one can show that $\boldsymbol{D}(\boldsymbol{r}) = \epsilon\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r}) = \mu\boldsymbol{H}(\boldsymbol{r})$, where $\epsilon > 1$ and $\mu > 0$ are called *dielectric constant* and *magnetic permeability*, respectively. Under these conditions, the general equations (7.26a') and (7.26c') can be rewritten in terms of electric field and magnetic induction as

$$\nabla \cdot \boldsymbol{E} = \frac{4\pi}{\epsilon}\rho \tag{7.26a''}$$

and

$$\nabla \times \boldsymbol{B} = \frac{4\pi\mu}{c}\boldsymbol{j} + \frac{\epsilon\mu}{c}\frac{\partial \boldsymbol{E}}{\partial t} \ . \tag{7.26c''}$$

If we specialize to *harmonic* temporal and spatial variations of the fields

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) &= \boldsymbol{E}_0 e^{i\omega t - \boldsymbol{q}\cdot\boldsymbol{r}} \\ \boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{B}_0 e^{i\omega t - \boldsymbol{q}\cdot\boldsymbol{r}} \ , \end{aligned}$$

the above relations can be generalized to

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{r},t) &= \epsilon(\boldsymbol{q},\omega)\boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) &= \mu(\boldsymbol{q},\omega)\boldsymbol{H}(\boldsymbol{r},t) \ . \end{aligned}$$

The *dielectric function* $\epsilon(\boldsymbol{q},\omega)$ is particularly interesting, because it completely determines the optical properties of a medium.

[8] And in the absence of magnetic or electric ordering.

7.4 Initial Value Problem of Maxwell's Equations

Maxwell's equations constitute a system of 8 partial differential equations for 2 (times 3) fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Is the system of field equations overdetermined? We expect that for given external sources ρ and $\boldsymbol{j}$, the equations will uniquely determine the time evolution of a given initial field configuration $\boldsymbol{E}(\boldsymbol{r}, t_0)$ and $\boldsymbol{B}(\boldsymbol{r}, t_0)$. Mathematical theorems guarantee the existence and uniqueness of solutions of the *initial value problem*, if we only consider the two partial differential equations (7.26b) and (7.26c), which are first order in time. The other two equations, which do not contain any time derivatives, pose constraints, which have to be automatically fulfilled. Otherwise, Maxwell's equations would suffer from severe consistency problems. Luckily, consistency can easily be shown as a direct consequence of charge conservation. The fields have to obey the constraints at initial time. Let us therefore consider the time evolution

$$\begin{aligned} \partial_t(\boldsymbol{\nabla}\cdot\boldsymbol{E} - \rho/\epsilon_0) &= \boldsymbol{\nabla}\cdot(\partial_t\boldsymbol{E}) - \frac{1}{\epsilon_0}\partial_t\rho \\ &= \boldsymbol{\nabla}\cdot(\frac{1}{\epsilon_0}\boldsymbol{j} + \partial_t\boldsymbol{E}) \end{aligned}$$

In the last step, we used the continuity equation, which reflects charge conservation. Now we may use the Maxwell equations again and replace the term in brackets by $\boldsymbol{\nabla}\times\boldsymbol{B}/(\kappa\epsilon_0\mu_0)$. As the divergence of the rotation term vanishes, we have shown that $\boldsymbol{\nabla}\cdot\boldsymbol{E} - \rho/\epsilon_0$ is time independent and therefore stays zero if it was zero initially. With the same strategy, we can show that for the other constraint

$$\partial_t(\boldsymbol{\nabla}\cdot\boldsymbol{B}) = \boldsymbol{\nabla}\cdot(-\boldsymbol{\nabla}\times\boldsymbol{E}/\kappa) = 0 \ ,$$

too, where we again made use of Maxwell's equations in the last step.

7.4.1 Potentials and Gauge Transformations

We have found that the homogeneous Maxwell equations are constraints. These constraints can be eliminated by the introduction of *potentials*. From $\nabla\cdot\boldsymbol{B} = 0$ we conclude $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. $\boldsymbol{A}$ is called *vector potential*. With this definition, the other homogeneous Maxwell equation can be rewritten as

$$\nabla\times\left(\boldsymbol{E} + \kappa\frac{\partial\boldsymbol{A}}{\partial t}\right) = 0$$

which implies that

$$\boldsymbol{E} = -\nabla\phi - \kappa\frac{\partial\boldsymbol{A}}{\partial t} \ .$$

The scalar field ϕ is called the *scalar potential*. These relations generalize the relations between potentials and fields we found in electro- and magnetostatics.

It is important to note that $\boldsymbol{E}$ and $\boldsymbol{B}$ do not uniquely determine $\boldsymbol{A}$ and ϕ. Stated otherwise, there exist *gauge transformations*

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi \tag{7.29a}$$

$$\phi' = \phi - \kappa\frac{\partial\chi}{\partial t} \tag{7.29b}$$

which leave the physically measurable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ unchanged.
Some mathematical relations involving potentials become particularly simple, if special restrictions are posed on the otherwise arbitrary function χ. This is called *choosing or fixing a gauge*, and the restrictions are called *gauge conditions*. All gauge transformations, which are in accordance with a gauge condition are called *residual gauge transformations*. Let us give the most prominent examples:

$$\text{gauge condition :} \quad \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \quad \text{(Coulomb gauge)} \tag{7.30}$$

with residual gauge transformations corresponding to functions χ which have to obey

$$\nabla^2\chi = 0$$

and

$$\text{gauge condition :} \quad (\kappa\epsilon_0\mu_0)\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \quad \text{(Lorentz gauge)} \tag{7.31}$$

with residual gauge transformations corresponding to functions χ which have to obey

$$-\frac{1}{c^2}\partial_t^2\chi + \nabla^2\chi = 0$$

($c^{-2} = \kappa^2\epsilon_0\mu_0$).
Introducing potentials has the advantage that the four Maxwell equations reduce to two. However, the two remaining equations do not look particularly simple if we express them in terms of potentials:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = -\nabla^2\phi - \kappa\boldsymbol{\nabla}\cdot\partial_t\boldsymbol{A} = \rho/\epsilon_0$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \kappa\epsilon_0\mu_0\partial_t\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) - \kappa\epsilon_0\mu_0(-\partial_t\boldsymbol{\nabla}\phi - \kappa\partial_t^2\boldsymbol{A}) = \kappa\mu_0\boldsymbol{j} \ .$$

We can replace

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = -\nabla^2\boldsymbol{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})$$

although this does not simplify the equations immediately. In Coulomb and Lorentz gauge, however, remarkable simplifications occur.

Coulomb Gauge

The equations may be written in the following form:

$$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho$$

which looks exactly like in electrostatics and

$$\frac{1}{c^2}\partial_t^2\boldsymbol{A} - \nabla^2\boldsymbol{A} = \kappa\mu_0\boldsymbol{j}_T$$

with

$$\boldsymbol{j}_T = \boldsymbol{j} - \epsilon_0\boldsymbol{\nabla}\partial_t\phi \ .$$

The physical meaning of $\boldsymbol{j}_T$ becomes obvious if we consider

$$\boldsymbol{\nabla}\cdot\boldsymbol{j}_T = \boldsymbol{\nabla}\cdot\boldsymbol{j} - \epsilon_0\partial_t\nabla^2\phi = \boldsymbol{\nabla}\cdot\boldsymbol{j} + \partial_t\rho = 0 \ ,$$

where we have used the continuity equation in the last step. Thus $\boldsymbol{j}_T$ is the *transverse*, i.e. divergence free part of the current density.

Lorentz gauge

In this gauge the remaining Maxwell equations take on the most simple and transparent form

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\phi = \frac{1}{\epsilon_0}\rho \tag{7.32}$$

and

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\boldsymbol{A} = \kappa\mu_0\boldsymbol{j} \ . \tag{7.33}$$

These are just 4 decoupled, linear wave equations with inhomogeneities. Obviously this form of the Maxwell equations is particularly well suited to find the general solution.

7.5 Solution of Maxwell's Equations and Electromagnetic Waves

7.5.1 Waves in Vacuum

In vacuum, i.e. in regions of space without charges or currents, Maxwell's equations read

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0 \quad , \quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\kappa\frac{\partial\boldsymbol{B}}{\partial t} \quad , \tag{7.34a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \quad , \quad \boldsymbol{\nabla}\times\boldsymbol{B}=\frac{1}{\kappa c^2}\frac{\partial\boldsymbol{E}}{\partial t} \quad , \tag{7.34b}$$

where we used the definition (7.27) to replace ϵ_0 and μ_0.
Instead of using the equations for the potentials in the Lorentz gauge, we can here proceed directly from Maxwell's equations. For example, applying $\boldsymbol{\nabla}\times$ on both sides of the second equation in (7.34a) yields

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=-\kappa\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{B}}{\partial t}=-\kappa\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{B}\overset{(7.34b)}{=}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} \quad .$$

Employing further $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E})-\boldsymbol{\nabla}^2\boldsymbol{E}$, we end up with

$$\Box\boldsymbol{E}=\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\boldsymbol{E}=0 \quad . \tag{7.35}$$

Likewise, we obtain

$$\Box\boldsymbol{B}=\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\boldsymbol{B}=0 \tag{7.36}$$

for the magnetic field. Note that the equations (7.32) and (7.33) for the potentials have precisely the same form in the absence of external sources $\rho=0$ and $\boldsymbol{j}=0$.
The above equations are three-dimensional wave equations with plane waves

$$\boldsymbol{E}(\boldsymbol{r},t) \;=\; \boldsymbol{E}_0 e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})} \tag{7.37a}$$

$$\boldsymbol{B}(\boldsymbol{r},t) \;=\; \boldsymbol{B}_0 e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})} \tag{7.37b}$$

with $\omega^2=c^2|\boldsymbol{k}|^2$ as solutions for $\boldsymbol{E}$ and $\boldsymbol{B}$. More precisely, these solutions describe plane waves propagating with velocity c in direction $\boldsymbol{k}/k$ with angular frequency $\omega=ck$, where $k:=|\boldsymbol{k}|$. Note that for electromagnetic waves in vacuum, both *phase* and *group* velocity are independent of $\boldsymbol{k}$ and equal to c:

$$v_{\mathrm{Ph}}:=\frac{\omega(\boldsymbol{k})}{k}=c \quad , \quad v_{\mathrm{Gr}}:=|\boldsymbol{\nabla}\omega(\boldsymbol{k})|=c \quad . \tag{7.38}$$

According to the discussion in section 7.4, not all solutions to (7.35) and (7.36) can be also solutions to Maxwell's equations. Inserting our solutions into (7.34a) and (7.34b), one obtains the following additional conditions:

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0 \quad , \quad \boldsymbol{k}\cdot\boldsymbol{B}_0 = 0 \ , \tag{7.39a}$$

$$\boldsymbol{k}\times\boldsymbol{E}_0 = \kappa\omega\boldsymbol{B}_0 \quad , \quad \boldsymbol{k}\times\boldsymbol{B}_0 = -\frac{\omega}{\kappa c^2}\boldsymbol{E}_0 \ . \tag{7.39b}$$

Equation (7.39a) means that electromagnetic waves in vacuum are *transverse*, i.e. $\boldsymbol{E}$ and $\boldsymbol{B}$ oscillate perpendicular to the direction of propagation $\boldsymbol{k}/k$. Furthermore, equation (7.39a) follows from (7.39b) which allows to express $\boldsymbol{B}_0$ with $\boldsymbol{E}_0$ and vice versa, where for consistency reasons again $\omega^2 = c^2k^2$ has to hold. Taking all these observations together we can state:

> Plane wave solutions of Maxwell's equations in vacuum have the form (7.37a) and (7.37b) with *dispersion relation* $\omega = c|\boldsymbol{k}|$ and $\kappa c|\boldsymbol{B}| = |\boldsymbol{E}|$. The vectors $\boldsymbol{k}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ form a positively oriented orthogonal set.

The *physical* solutions of Maxwell's equations of course have to be real. However, since Maxwell's equations are real and linear, real solutions are constructed simply as the real part of complex solutions.
To keep the rest of the discussion simple, let us choose $\boldsymbol{k} = k\boldsymbol{e}_3$. Then we have from (7.39b)

$$\boldsymbol{k} = k\boldsymbol{e}_3 \ , \quad \boldsymbol{E}_0 = a\boldsymbol{e}_1 + b\boldsymbol{e}_2 \ , \quad \boldsymbol{B}_0 = \frac{1}{\kappa c}\left(a\boldsymbol{e}_2 - b\boldsymbol{e}_1\right)$$

with $a, b \in \mathbb{C}$. If a and b have (apart from a sign) the same phase, i.e. $a = |a|e^{i\alpha}$ and $b = \pm|b|e^{i\alpha}$, we say that the wave has *linear polarization*, because then

$$E_1(\boldsymbol{r},t) = |a|e^{-i(\omega t - kx_3 - \alpha)}$$
$$E_2(\boldsymbol{r},t) = \pm|b|e^{-i(\omega t - kx_3 - \alpha)}$$

and analogous for $\boldsymbol{B}$. This means, that both $\boldsymbol{E}$ and $\boldsymbol{B}$ oscillate along a fixed line in the $x_1 - x_2$ plane, whose orientation with respect to the x_1 and x_2 axes is given by the angle $\Theta = \arctan(\pm|b|/|a|)$.
In general, a and b will however be complex numbers with independent phases, say $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$. In this case one says that the wave has *elliptical polarization*, because going through a little algebra one can show that the tips of $\boldsymbol{E}$ and $\boldsymbol{B}$ move on an ellipse in the x_1-x_2-plane, oriented with respect to the x_1 and x_2 axes by the angle

$$\Theta = \frac{1}{2}\arctan\left(\frac{2|a||b|}{|a|^2 - |b|^2}\cos(\alpha - \beta)\right)$$

and axes

$$a_\pm = \frac{2|a|^2|b|^2\sin^2(\alpha-\beta)}{|a|^2+|b|^2 \mp \sqrt{(|a|^2-|b|^2)^2+4|a|^2|b|^2\cos^2(\alpha-\beta)}} \ .$$

A particularly important special case is $|b| = |a|$ and $\beta = \alpha \pm \pi/2$, i.e. $b = \pm ia$. In this case we speak of *circular polarization*, because the tips of both $\boldsymbol{E}$ and $\boldsymbol{B}$ move on a circle in the x_1-x_2-plane. For the direction of the rotation the following convention is used: If one looks opposite to the direction of propagation, one speaks in the case of mathematically positive rotation (i.e. counter-clock wise) of left-circular polarization or *right-handedness* or *positive helicity*. In the other case we speak of *left-handedness* or *right-circular polarization* or *negative helicity*.

The concepts of handedness or *chirality* are especially useful in relativistic quantum mechanics (see also the lecture Quantum Mechanics II), where the helicity of a massless particle is either parallel to its propagation direction (helicity$= +1$) or anti-parallel (helicity$= -1$). We have discussed this property in Quantum Mechanics II for the phonons.

7.5.2 Wave guides

In applications of electrodynamics one typically encounters problems like transporting electromagnetic signals (such as TV or radio signals) or filtering or creating waves within a given narrow frequency band. The first task can be accomplished with so-called *wave guides*, the second set of tasks with the help of *(resonance) cavities*. The latter are also employed in accelerators, because the *standing waves* in cavities allow to tailor oscillating electrical fields that can be used to accelerate charged particles.

Here we want to treat wave guides only. The calculation for cavities is very similar, just introducing two additional boundaries, and will be given as an exercise. For our wave guide we assume a hollow cylinder (although not necessarily with a circular cross-section) extending infinitely in the x_3-direction. Therefore, the electromagnetic fields can be written as

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0(x_1,x_2)e^{-i(\omega t - k_3 x_3)} \quad , \quad \boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}_0(x_1,x_2)e^{-i(\omega t - k_3 x_3)} \tag{7.40}$$

and the remaining task is to determine the functions $\boldsymbol{E}_0$, $\boldsymbol{B}_0$ and $\omega(k_3)$ from Maxwell's equations and the boundary conditions due to the wave guide. For simplicity we further assume that the mantle of our wave guide consists of a perfect conductor, i.e. the tangential components of $\boldsymbol{E}$ on the boundary have to vanish to avoid the induction of infinite currents. From $\boldsymbol{\nabla}\times\boldsymbol{E} = -\kappa\partial\boldsymbol{B}/\partial t$

we obtain for the *normal* component B_n of the magnetic field

$$i\kappa\omega B_n = (\boldsymbol{\nabla} \times \boldsymbol{E})_n = (\boldsymbol{\nabla} \times \boldsymbol{E}_t) \cdot \boldsymbol{e}_n = 0 \ ,$$

because $\boldsymbol{E}_t = 0$. Thus it follows that $B_n = 0$. For convenience of notation we employ in the following the Heaviside units, i.e. $\kappa = 1/c$ and $\epsilon_0 = \mu_0 = 1$.
Inside the wave guide Maxwell's equations become

$$(\boldsymbol{\nabla} \times \boldsymbol{E})_2 = -\frac{1}{c}\frac{\partial B_2}{\partial t} \quad \rightarrow \quad ik_3 E_{0,1} - \frac{\partial E_{0,3}}{\partial x_1} = i\frac{\omega}{c} B_{0,2} \ , \tag{7.41a}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{B})_1 = \frac{1}{c}\frac{\partial E_1}{\partial t} \quad \rightarrow \quad \frac{\partial B_{0,3}}{\partial x_2} - ik_3 B_{0,2} = -i\frac{\omega}{c} E_{0,1} \tag{7.41b}$$

and a corresponding set of equations for $E_{0,2}$ and $B_{0,1}$. Let us define as usual $k := \omega/c$ and $k_\perp^2 := k^2 - k_3^2$. Then, multiplying (7.41a) with $-ik_3$ and (7.41b) with ik and adding both equations, one obtains

$$k_\perp^2 E_{0,1} = ik_3 \frac{\partial E_{0,3}}{\partial x_1} + ik\frac{\partial B_{0,3}}{\partial x_2} \ , \tag{7.42}$$

while repeating the procedure with $-ik$ for (7.41a) and ik_3 for (7.41b) results in

$$k_\perp^2 B_{0,2} = ik\frac{\partial E_{0,3}}{\partial x_1} + ik_3 \frac{\partial B_{0,3}}{\partial x_2} \ . \tag{7.43}$$

Again, corresponding equations hold for $E_{0,2}$ and $B_{0,1}$. The longitudinal components are determined from the wave equation, resulting in

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k_\perp^2\right) E_{0,3}(x_1, x_2) = 0 \ , \tag{7.44a}$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k_\perp^2\right) B_{0,3}(x_1, x_2) = 0 \ . \tag{7.44b}$$

For $k_\perp \neq 0$ one can readily show (exercise), that solutions of (7.44a) and (7.44b) also fulfill the remaining Maxwell equations.
Evidently, $E_{0,3}$ and $B_{0,3}$ are independent of each other. Note that, at least for $k_\perp \neq 0$, one of them has to be non-zero to allow for non-trivial solutions. Consequently one distinguishes between so-called TE-modes (transverse electric modes) with $E_{0,3} = 0$ and $B_{0,3} \neq 0$ and TM-modes (transverse magnetic modes) where $B_{0,3} = 0$ and $E_{0,3} \neq 0$.
Let us now return to the boundary conditions. From (7.42), (7.43) and the corresponding equations for the remaining components one can form

$$k_\perp^2 (E_{0,1}\boldsymbol{e}_1 + E_{0,2}\boldsymbol{e}_2) = ik_3 \boldsymbol{\nabla} E_{0,3} - ik\boldsymbol{e}_3 \times (\boldsymbol{\nabla} B_{0,3}) \ , \tag{7.45a}$$

$$k_\perp^2 (B_{0,1}\boldsymbol{e}_1 + B_{0,2}\boldsymbol{e}_2) = ik_3 \boldsymbol{\nabla} B_{0,3} + ik\boldsymbol{e}_3 \times (\boldsymbol{\nabla} E_{0,3}) \ . \tag{7.45b}$$

Besides the normal $\boldsymbol{e}_n$ to the boundary and $\boldsymbol{e}_3$, we can introduce a third unit vector $\boldsymbol{e}_c := \boldsymbol{e}_3 \times \boldsymbol{e}_n$. The tangential plane of the surface is then spanned by $\boldsymbol{e}_3$ and $\boldsymbol{e}_c$. Since $\boldsymbol{e}_n$ and $\boldsymbol{e}_c$ both lie in the x_1-x_2-plane, we may rewrite the previous equations as

$$\begin{aligned} k_\perp^2 (E_{0,n}\boldsymbol{e}_n + E_{0,c}\boldsymbol{e}_c) &= ik_3 (\boldsymbol{e}_n \partial_n E_{0,3} + \boldsymbol{e}_c \partial_c E_{0,3}) - \\ &\quad ik (\boldsymbol{e}_c \partial_n B_{0,3} - \boldsymbol{e}_n \partial_c B_{0,3}) \end{aligned} \tag{7.46a}$$

$$\begin{aligned} k_\perp^2 (B_{0,n}\boldsymbol{e}_n + B_{0,c}\boldsymbol{e}_c) &= ik_3 (\boldsymbol{e}_n \partial_n B_{0,3} + \boldsymbol{e}_c \partial_c B_{0,3}) + \\ &\quad ik (\boldsymbol{e}_c \partial_n E_{0,3} - \boldsymbol{e}_n \partial_c E_{0,3}) \quad . \end{aligned} \tag{7.46b}$$

On the surface, the boundary conditions require $E_{0,3} = E_{0,c} = B_{0,n} = 0$, which then leads to

$$k_\perp^2 E_{0,c} = ik_3 \partial_c E_{0,3} - ik \partial_n B_{0,3} \ , \tag{7.47a}$$

$$k_\perp^2 B_{0,n} = ik_3 \partial_n B_{0,3} - ik \partial_c E_{0,3} \ . \tag{7.47b}$$

Since $E_{0,3} = 0$, these conditions require $\partial_n B_{0,3} = 0$. Thus the two fundamental modes TM and TE are determined by the following eigenvalue problems

$$\text{TM mode:} \quad \left(\partial_1^2 + \partial_2^2 + k_\perp^2\right) E_{0,3} = 0 \ , \quad E_{0,3} = 0 \quad \text{at the surface} \tag{7.48a}$$

$$\text{TE mode:} \quad \left(\partial_1^2 + \partial_2^2 + k_\perp^2\right) B_{0,3} = 0 \ , \quad \partial_n B_{0,3} = 0 \quad \text{at the surface} \tag{7.48b}$$

fixing $k_\perp$. Finally, the dispersion law reads

$$\omega(k_3) = c\sqrt{k_3^2 + k_\perp^2} \ . \tag{7.49}$$

Before turning to the example of a wave guide with rectangular cross section let us discuss the possibility of having both $E_{0,3} = B_{0,3} = 0$ (TEM modes). Obviously, this is possible only if $k_\perp = 0$ or $k_3 = \pm k$. Furthermore, from (7.41a) and (7.41b) one finds $B_{0,2} = \pm E_{0,1}$ and $B_{0,1} = \mp E_{0,2}$. For $B_{0,3} = 0$ we moreover conclude $(\boldsymbol{\nabla} \times \boldsymbol{E})_3 = 0$, so that we can represent $\boldsymbol{E}_0$ as gradient of a scalar potential

$$\boldsymbol{E}_0 = -\boldsymbol{\nabla}\Phi(x_1, x_2) \ ,$$

which due to $\boldsymbol{\nabla} \cdot \boldsymbol{E}_0 = 0$ has to fulfill Laplace's equation

$$(\partial_1^2 + \partial_2^2)\Phi = 0 \ .$$

The boundary condition $\boldsymbol{E}_t = 0$ on the mantle translates into Φ =const., i.e. nontrivial solutions of this equation exist only in multiply connected regions such as *outside* a hollow cylinder or for a coaxial wire or outside two wires.
As specific example I want to discuss the modes of a rectangular wire with dimensions a in x_1 and b in x_2 directions. For the TM modes we obtain from (7.48a) and a product ansatz $E_{0,3}(x_1, x_2) = f(x_1)g(x_2)$ the equation

$$f''g + g''f + k_\perp^2 fg = 0 \ ,$$

or equivalently

$$\frac{f''}{f} + \frac{g''}{g} = -k_\perp^2$$

meaning that both f''/f and g''/g have to be constants. Together with the boundary condition $E_{0,3} = 0$ on the mantle the solutions are

$$E_{0,3}(x_1, x_2) = E_0 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{b}\right) \ , \quad n, m \in \mathbb{N} \tag{7.50}$$

and

$$k_\perp^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \ . \tag{7.51}$$

For the TE modes one obtains with a corresponding ansatz and the boundary condition $\partial_n B_{0,3} = 0$

$$B_{0,3}(x_1, x_2) = B_0 \cos\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{b}\right) \ , \quad n, m \in \mathbb{N}_0 \ , \quad m + n \geq 1 \tag{7.52}$$

and expression (7.51) for $k_\perp^2$.

7.5.3 Solution of Maxwell's Equations

In Lorentz gauge, Maxwell's equations become inhomogeneous wave equations for the potentials

$$\begin{aligned} (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\boldsymbol{A} &= \kappa\mu_0 \boldsymbol{j} \\ (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\phi &= \frac{\rho}{\epsilon_0} \ . \end{aligned}$$

Note that solutions have to obey the constraint posed by the gauge condition

$$\nabla \cdot \boldsymbol{A} + \frac{1}{\kappa c^2}\frac{\partial \phi}{\partial t} = 0 \ .$$

Now we consider the general solution of an inhomogeneous wave equation

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\psi = g \ .$$

The general strategy to solve such types of differential equations is to construct a so-called *Green's function* which obeys the differential equation

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)G(\boldsymbol{r} - \boldsymbol{r}', t - t') = 4\pi\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t') \ . \tag{7.53}$$

Then the solution for a general inhomogeneity is given by

$$\psi(\boldsymbol{r}, t) = \frac{1}{4\pi}\int d\boldsymbol{r}' dt' G(\boldsymbol{r} - \boldsymbol{r}', t - t')g(\boldsymbol{r}', t')$$

plus an arbitrary solution of the homogeneous wave equation.
An important condition on the solution is imposed by the fundamental law of causality. A perturbation $g(t')$ can only contribute to the (physical) wave field $\psi(t)$ at later times $t > t'$. The wave equation is symmetric in time direction ($t \to -t$) and propagates disturbances both forward and backward in time. Thus we have to look for solutions with the special (time) boundary condition

$$G_{ret}(\boldsymbol{r}, t) = 0 \text{ for } \quad t < 0 \ .$$

This Green's function is called the *retarded Green's function* of the wave equation.
In general, the construction of $G_{ret}(\boldsymbol{r}, 0)$ is a tedious task. For problems with spatial and temporal translational invariance (as we have here) a good strategy usually is to consider the Fourier transform of Eq. (7.53) with respect to both space and time, which for the wave equation becomes (exercise)

$$(c^2k^2 - \omega^2)\hat{G}(\boldsymbol{k}, \omega) = 4\pi c^2 \ . \tag{7.54}$$

The Green's function $G(\boldsymbol{r}, t)$ is then obtained from the inverse Fourier transformation, i.e.

$$G(\boldsymbol{r}, t) = 4\pi c^2 \int \frac{d\omega}{2\pi} \frac{d^3k}{(2\pi)^3} \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t)}{(ck)^2 - \omega^2} \ ,$$

which will pick the proper solution. Note that the integrand of the ω-integration has two poles at $\omega = \pm ck$ and is thus ill-defined. This is not a surprise, because we have not yet introduced boundary conditions,
How to handle these types of integrals we have learned in the lecture "*Mathematische Methoden der Physik*": We extend the ω-integration into the complex plane and note that for $t < 0$ the integrand decays exponentially for $|\omega| \to \infty$ in the upper half plane ($\Im m\omega > 0$), whereas it grows exponentially in the lower half plane; for $t > 0$ the situation is reversed. Thus if we shift the integration path slightly into the upper half-plane[9], we can close the path by adding $\mathcal{C}_+$ for $t < 0$ or $\mathcal{C}_-$ for $t > 0$ as shown in Fig. 7.4 without changing the value of the integral. Since the region bounded by $C = \mathcal{C}_{ret} \cup \mathcal{C}_+$ does not contain any singularities of the integrand the integral vanishes for $t < 0$ by Cauchy's Theorem. In this way, we have implemented retarded boundary conditions. For $t > 0$, the integral may be evaluated using the theorem of residues. Let us concentrate on the ω dependent parts

$$I = \int_C d\omega \frac{e^{-i\omega t}}{(ck)^2 - \omega^2}$$

[9]This is equivalent to the procedure used in "*Mathematische Methoden der Physik*", where we avoided the poles on the real axis by introducing small semicircles in the upper half plane.

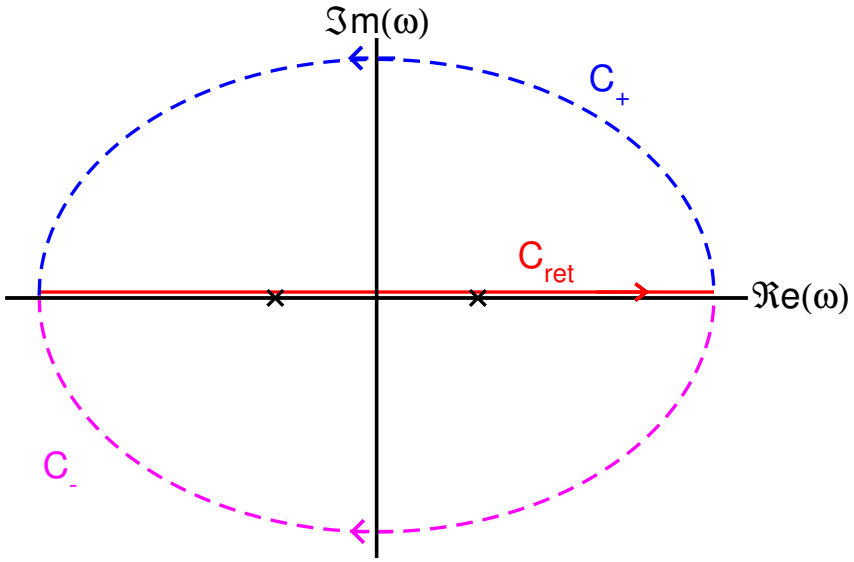

Figure 7.4: Complex integration paths for the ω-integral in $G_{ret}(\boldsymbol{r},t)$. The crosses denote the positions of the poles on the real axis.

and use

$$\frac{1}{(ck)^2-\omega^2}=\frac{1}{2ck}\left(\frac{1}{\omega+ck}-\frac{1}{\omega-ck}\right)$$

Thus the integrand possesses two poles of first order. Remember the formula to calculate the residue at such a pole

$$\mathrm{Res}(f(z))=\lim_{z\to z_0}(z-z_0)f(z)$$

which leads to

$$I=-\frac{2\pi i}{2ck}(e^{-ickt}-e^{ickt}) \quad .$$

The minus sign takes care of the orientation of the path $C=C_{ret}\cup C_-$; it is traversed in mathematically negative direction (clockwise). and thus

$$G_{ret}(\boldsymbol{r},t)=\Theta(t)\frac{ic}{4\pi^2}\int d^3k\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k}(e^{ickt}-e^{-ickt}) \quad .$$

The remaining $\boldsymbol{k}$-integration can be easily performed by introducing polar coordinates and remembering that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dxe^{ikx}=\delta(k)$$

to yield

$$G_{ret}(\boldsymbol{r},t)=\frac{c}{r}\Theta(t)\delta(r-ct)=\frac{1}{r}\Theta(t)\delta(t-r/c) \quad . \tag{7.55}$$

A solution of the inhomogeneous wave equation may now be obtained from

$$\psi(\boldsymbol{r},t) = \frac{1}{4\pi}\int d^3r'dt'\; G_{ret}(\boldsymbol{r}-\boldsymbol{r'},t-t')g(\boldsymbol{r'},t') \quad .$$

Obviously, the time integration may be performed by inspection. If we introduce the so-called *retarded time*

$$t_{ret} = t - |\boldsymbol{r}-\boldsymbol{r'}|/c$$

the solution may be written as

$$\psi(\boldsymbol{r},t) = \frac{1}{4\pi}\int d^3r'\;\frac{g(\boldsymbol{r'},t_{ret})}{|\boldsymbol{r}-\boldsymbol{r'}|}$$

We now specialize to Maxwell's equations to obtain

$$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\;\frac{\rho(\boldsymbol{r'},t_{ret})}{|\boldsymbol{r}-\boldsymbol{r'}|} \tag{7.56}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\kappa\mu_0}{4\pi}\int d^3r'\;\frac{\boldsymbol{j}(\boldsymbol{r'},t_{ret})}{|\boldsymbol{r}-\boldsymbol{r'}|} \tag{7.57}$$

for the potentials. This is a beautiful formula. The connection between sources and potentials looks completely analogous to the corresponding formulas of electro- and magnetostatics! However, the time argument is *not simply* t, but $t_{ret}(t) = t - |\boldsymbol{r}-\boldsymbol{r'}|/c$. This actually means that changes in the source distributions at a certain point are not transported to another point with infinite velocity! Rather they propagate with the velocity of light! Finally, we have to show that our solution obeys the Lorentz gauge condition. This is a straightforward task and left as an exercise.

7.5.4 The Quasistatic Approximation

If we have a setup which is small enough that we may neglect the difference between t and t_{ret}, we may approximate the solution to

$$\phi(\boldsymbol{r},t) \approx \frac{1}{4\pi\epsilon_0}\int d^3r'\;\frac{\rho(\boldsymbol{r'},t)}{|\boldsymbol{r}-\boldsymbol{r'}|}$$

$$\boldsymbol{A}(\boldsymbol{r},t) \approx \frac{\kappa\mu_0}{4\pi}\int d^3r'\;\frac{\boldsymbol{j}(\boldsymbol{r'},t)}{|\boldsymbol{r}-\boldsymbol{r'}|}$$

This is the *quasistatic approximation of Maxwell's equations*, which neglects retardation effects. It is the realm for most applications in electrical engineering and the basis which allows to describe passive electric circuits as combinations of capacitors and inductors. We already discussed capacitors in the context of electrostatics. In the quasistatic approximation, this discussion is still valid.

7.5.5 Generation of Waves

Let us return to the full solution and study consequences of the retardation. For simplicity (and because it is of great practical importance) let us consider oscillating sources, i.e.

$$\begin{aligned} \rho(\boldsymbol{r},t) &= \rho(\boldsymbol{r})e^{-i\omega t} \\ \boldsymbol{j}(\boldsymbol{r},t) &= \boldsymbol{j}(\boldsymbol{r})e^{-i\omega t} \end{aligned}$$

located in space (i.e. $\rho = 0,\ \boldsymbol{j} = 0$ outside a bounded region). The restriction to just one frequency is not severe, because we may use the linearity of the problem to solve more complicated time dependencies by superposition. Note that the time dependence of potentials is then also given by

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r},t) &= \boldsymbol{A}(\boldsymbol{r})e^{-i\omega t} \\ \boldsymbol{\phi}(\boldsymbol{r},t) &= \phi(\boldsymbol{r})e^{-i\omega t} \end{aligned}$$

while the spatial variation is described by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi}\int d^3r' \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}e^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c} \tag{7.58}$$

$$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}e^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c}\ . \tag{7.59}$$

If we are interested in positions outside the charge- and current distribution, we need only calculate the vector potential. If $\boldsymbol{j}(\boldsymbol{r},t) = 0$, the inhomogeneous Maxwell equation for $\boldsymbol{B}$-fields takes on the form

$$\nabla \times \boldsymbol{B} = \frac{1}{\kappa c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

and thus

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{i}{\omega}\kappa c^2 \nabla \times (\nabla \times \boldsymbol{A}(\boldsymbol{r}))$$

How do we evaluate $\boldsymbol{A}$? Further progress is possible, if we have a clear *separation of the following length scales*:

- d: the linear dimension of the region containing the sources
- r: the distance of an observer from the sources
- $\lambda = 2\pi c/\omega$, a length scale resulting from retardation, which turns out to be the wavelength of emitted radiation

Here we will mainly be interested in the regime

$$d \ll \lambda\ ,\quad d \ll r$$

i.e. the sources are *well localized*. This situation is the one with the broadest range of applicability (radiating atoms). This assumptions allows us to repeat the technique, which lead to the multipole expansion in static problems. We use expansions in the small parameter r'/r, for example

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} + \cdots \quad .$$

Furthermore

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{r^2 + r'^2 - 2\boldsymbol{r}\cdot\boldsymbol{r}'} = r\sqrt{1 - \frac{r'^2}{r^2} - 2\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2}}$$

leads to the expansion

$$|\boldsymbol{r}-\boldsymbol{r}'| = r\left(1 - \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2} + \cdots\right)$$

which we insert in the exponential to get

$$e^{i(\omega/c)|\boldsymbol{r}-\boldsymbol{r}'|} = e^{i(\omega/c)r}\, e^{-i(\omega/c)(\boldsymbol{r}\cdot\boldsymbol{r}'/r)} \cdots \quad .$$

The exponent of the second exponential function is of the order $d/\lambda \ll 1$, which is small due to the first condition for localized sources. Thus we may expand the second exponential function. Combining all the expansions we get

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{e^{ikr}}{r}\left[1 + \left(\frac{1}{r} - ik\right)\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r} + \cdots\right] \quad , \quad k := \frac{\omega}{c}$$

This may be inserted into the integral, which determines the vector potential:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\boldsymbol{j}(\boldsymbol{r}') + \frac{\kappa\mu_0}{4\pi}\left(\frac{1}{r} - ik\right)\frac{e^{ikr}}{r}\int \boldsymbol{j}(\boldsymbol{r}')\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r} + \cdots \quad .$$

Further progress is still possible, if we introduce an additional separation of length scales. Two regimes emerge:

- the *near field* with $d \ll r \ll \lambda$
- the *far field* with $d \ll \lambda \ll r$

In the near field zone $(\omega/c)|\boldsymbol{r}-\boldsymbol{r}'| \ll 1$ and thus we may replace the exponential in the integral by 1. Then we are back to the same expressions we already evaluated in statics! This is entirely consistent, because the near field regime is actually where the quasistatic approximation holds, i.e. retardation effects can be neglected.
The leading term in the far field regime is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\boldsymbol{j}(\boldsymbol{r}')$$

which does only require well localized sources. We will now consider it in more detail. First note, that the integral over the current density would vanish in a static situation, as you have found in magnetostatic multipole expansion in *Physik II*. This was a consequence of the continuity equation, which reduced to $\nabla \cdot \boldsymbol{j} = 0$ in static situations. In dynamics, this is no longer true. Still we may use the same trick we employed in statics to express the volume integral of the current density in terms of divergences. First note that

$$\partial_k(x_i j_k) = x_i(\nabla \cdot \boldsymbol{j}) + j_i \ ,$$

so that analogous to the discussion in magnetostatics (see *Physik II*)

$$\int d^3r' \boldsymbol{j}(\boldsymbol{r}') = -\int d^3r' \boldsymbol{r}' \nabla \cdot \boldsymbol{j}(\boldsymbol{r}') \ .$$

The left hand side is transformed using the continuity equation, which takes on a simple form for oscillating sources:

$$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho(\boldsymbol{r}, t)}{\partial t} = i\omega\rho(\boldsymbol{r}) \ .$$

Therefore the volume integral over $\boldsymbol{j}$ may be connected to the *electrical dipole moment* $\boldsymbol{p}$:

$$\int d^3r' \boldsymbol{j}(\boldsymbol{r}') = -i\omega \int d^3r' \boldsymbol{r}' \rho(\boldsymbol{r}') = -i\omega \boldsymbol{p} \ ,$$

and the vector potential in the far field takes on the form

$$\boldsymbol{A}(\boldsymbol{r}) = -i\frac{\kappa\mu_0}{4\pi}\omega \boldsymbol{p} \frac{e^{i(\omega/c)r}}{r} \ .$$

This type of radiation is called *electrical dipole radiation*. It will dominate the fields in the far field regime, unless the electrical dipole moment of the sources vanishes. Then magnetic dipoles and higher order multipole moments will generate the far field radiation.

Finally, we can calculate the electric and magnetic field for electric dipole radiation. Note that the expression for the vector potential did only require $d \ll r, \lambda$ and thus is valid in a larger regime. Therefore, we may also find terms in $\boldsymbol{E}$ and $\boldsymbol{B}$, which are not of leading order in the far field. We leave the explicit calculations as an exercise. For $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ one finds

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi} c \left(\frac{\omega}{c}\right)^2 \frac{e^{i(\omega/c)r}}{r} \left(1 + i\frac{c}{\omega r}\right) \frac{\boldsymbol{r}}{r} \times \boldsymbol{p} \ .$$

In the far field the leading term is

$$\boldsymbol{B}_d(\boldsymbol{r}) = \frac{\kappa\mu_0}{4\pi} c \left(\frac{\omega}{c}\right)^2 \frac{e^{i(\omega/c)r}}{r} \frac{\boldsymbol{r}}{r} \times \boldsymbol{p} \ . \tag{7.60}$$

The calculation of $\boldsymbol{E}$ is a bit more clumsy but still straightforward. The result is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{e^{i(\omega/c)r}}{r}\left\{\left(\frac{\omega}{cr}\right)^2[(\boldsymbol{r}\times\boldsymbol{p})\times\boldsymbol{r}] + \frac{1}{r}\left(\frac{1}{r} - i\frac{\omega}{c}\right)\left[\frac{1}{r^2}3\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{p}) - \boldsymbol{p}\right]\right\} .$$

However, the leading term in the far field is just the first term in curly brackets. It is obviously connected to the leading $\boldsymbol{B}$ term by

$$\boldsymbol{E}_d(\boldsymbol{r}) = \kappa c\left(\boldsymbol{B}_d(\boldsymbol{r})\times\frac{\boldsymbol{r}}{r}\right) \quad . \tag{7.61}$$

As an interesting application let us discuss the scattering of light from a small object with dimension $d \ll \lambda$ (such as atoms or molecules) located at $\boldsymbol{r}_0 = 0$. We assume the incoming wave to be a plane wave with polarization $\boldsymbol{\varepsilon}$, i.e.

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{\varepsilon}E_0 e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \quad .$$

The incoming wave will lead to a periodic polarization of the object. Under the above assumption it is justified to consider only the induction of an electric dipolmoment

$$\boldsymbol{p}(t) = 4\pi\epsilon_0\chi(\omega)E_0 e^{-i\omega t}\boldsymbol{\varepsilon}$$

where the factor $\chi(\omega)$ characterizes the (linear) response of the microscopic object to the external field and is called *dielectric susceptibility*. The electromagnetic radiation emitted from our object in the far-field region is then given by Eqs. (7.60) and (7.61). In particular, we can readily read off these results that linearly polarized incoming radiation will also produce linearly polarized radiation, because with $\boldsymbol{\varepsilon} \in \mathbb{R}^3$ we also have $\boldsymbol{\varepsilon}_t := (\boldsymbol{e}_r \times \boldsymbol{\varepsilon}) \times \boldsymbol{e}_r \in \mathbb{R}^3$ and $\boldsymbol{E}_d$ oscillates in direction of $\boldsymbol{\varepsilon}_t$.

Typically, we are interested in the differential cross-section

$$\frac{d\sigma}{d\Omega}(\omega;\boldsymbol{n},\boldsymbol{\varepsilon},\boldsymbol{n}',\boldsymbol{\varepsilon}')$$

for the scattering of an incoming plane wave with frequency ω and polarization $\boldsymbol{\varepsilon}$ propagating along $\boldsymbol{n}$ into a solid angle element $d\Omega$ in direction $\boldsymbol{n}'$ with polarization $\boldsymbol{\varepsilon}'$, which we can assume to be $\perp \boldsymbol{n}'$ without loss of generality. As usual, the differential cross section is defined as the ratio of the scattered intensity to the incoming intensity times r^2. These intensities are proportional to $|\boldsymbol{E}_d \cdot (\boldsymbol{\varepsilon}')^*|^2$ and $|\boldsymbol{E}|^2$, respectively. With

$$\left[(\boldsymbol{n}'\times\boldsymbol{\varepsilon})\times\boldsymbol{n}'\right]\cdot(\boldsymbol{\varepsilon}')^* = -\left[(\boldsymbol{n}'\cdot\boldsymbol{\varepsilon})\boldsymbol{n}' - (\boldsymbol{n}'\cdot\boldsymbol{n}')\boldsymbol{\varepsilon}\right]\cdot(\boldsymbol{\varepsilon}')^* = \boldsymbol{\varepsilon}\cdot(\boldsymbol{\varepsilon}')^*$$

we obtain the result

$$\frac{d\sigma}{d\Omega}(\omega;\boldsymbol{n},\boldsymbol{\varepsilon},\boldsymbol{n}',\boldsymbol{\varepsilon}') = \frac{\omega^4|\chi(\omega)|^2}{c^4}\left|\boldsymbol{\varepsilon}\cdot(\boldsymbol{\varepsilon}')^*\right|^2 \tag{7.62}$$

for the differential cross section. Note, that the intensity of the incoming wave $\propto E_0^2$ cancels, as expected.
The further discussion we restrict to the case of linearly polarized incoming light, because all other situations can be constructed from that by suitable linear combinations of different polarization directions. Let us define the *scattering angle* $\cos\Theta = \boldsymbol{n}\cdot\boldsymbol{n}'$ and the three orthonormal vectors

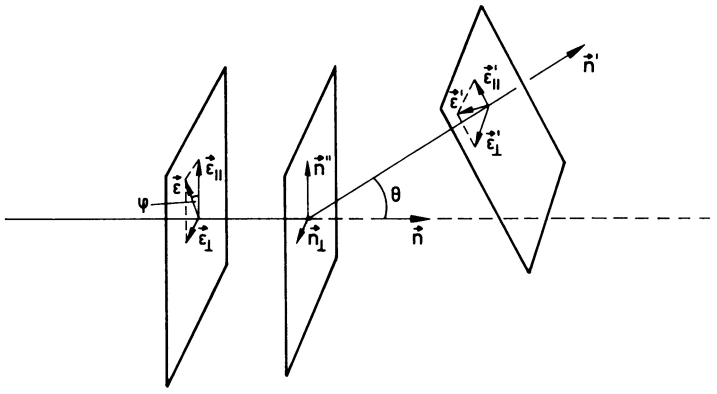

Figure 7.5: Geometry for the scattering of light.

$$\boldsymbol{n}\ ,\ \ \boldsymbol{n}'' := \frac{\boldsymbol{n}' - \cos\Theta\boldsymbol{n}}{\sin\Theta}\ ,\ \ \boldsymbol{n}_\perp := \frac{\boldsymbol{n}\times\boldsymbol{n}'}{\sin\Theta}\ \ ,$$

Then, we can write

$$\boldsymbol{\varepsilon} = \cos\varphi\,\boldsymbol{n}'' + \sin\varphi\,\boldsymbol{n}_\perp$$

and (since $\boldsymbol{\varepsilon}' \in \mathbb{R}^3$ for linear polarization)

$$\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' = \pm\cos\Theta\cos\varphi \pm \sin\varphi\ \ ,$$

where we made use of $\boldsymbol{\varepsilon}' \perp \boldsymbol{n}'$, i.e. $\boldsymbol{\varepsilon}'\cdot\boldsymbol{n} = \pm\sin\Theta$, and hence $\boldsymbol{\varepsilon}'\cdot\boldsymbol{n}'' = \pm\cos\Theta$ and $\boldsymbol{\varepsilon}'\cdot\boldsymbol{n}_\perp = \pm 1$. The differential cross section thus reads

$$\frac{d\sigma}{d\Omega}(\omega;\boldsymbol{n},\boldsymbol{\varepsilon},\boldsymbol{n}',\boldsymbol{\varepsilon}') = \frac{\omega^4|\chi(\omega)|^2}{c^4}\left(\cos\Theta\cos\varphi \pm \sin\varphi\right)^2\ \ . \tag{7.63}$$

If the incoming radiation is unpolarized, i.e. can be viewed as superposition of all possible polarization angles φ, the resulting differential cross section is the average over all angles φ and can be written as the sum of two contributions

$$\frac{d\sigma_{||}}{d\Omega} := \frac{\omega^4|\chi(\omega)|^2}{2c^4}\cos^2\Theta$$

and

$$\frac{d\sigma_{\perp}}{d\Omega} := \frac{\omega^4|\chi(\omega)|^2}{2c^4}\ \ .$$

which can be interpreted as scattering with final polarizations *parallel* to the scattering plane defined by $\boldsymbol{n}''$ and $\boldsymbol{n}_\perp$, and scattering with final polarizations *perpendicular* to that plane. The total differential cross section thus is

$$\frac{d\sigma}{d\Omega} = \frac{\omega^4 |\chi(\omega)|^2}{2c^4}(1 + \cos^2\Theta) \quad , \tag{7.64}$$

while the *degree of polarization* is given by

$$P(\Theta) := \frac{d\sigma_\perp - d\sigma_{||}}{d\sigma_\perp + d\sigma_{||}} = \frac{\sin^2\Theta}{1 + \cos^2\Theta} \quad . \tag{7.65}$$

Thus, for $\Theta = \pi/2$ (scattering into a direction perpendicular to the propagation of the incoming wave) we find $P(\pi/2) = 1$, i.e. the scattered light is completely polarized, while for forward scattering or backscattering ($\Theta = 0$ or $\Theta = \pi$) we find $P = 0$, i.e. completely unpolarized light.
Finally, one can calculate the *total cross section* by integrating over all scattering angles Θ, yielding

$$\sigma_{tot} = \frac{8\pi\omega^4 |\chi(\omega)|^2}{3c^4} \quad . \tag{7.66}$$

A perfectly conducting sphere is a very primitive, but for certain purposes nevertheless usable model for an air molecule. In this case, $\chi(\omega)$ =const. and hence

$$\sigma_{tot} \propto \frac{8\pi\omega^4}{3c^4} \quad .$$

This expression together with Eq. (7.65) is sufficient to explain polarization and color of the sun light scattered by the air molecules: One finds maximal polarization if one looks into a direction perpendicular to the axis sun-earth, while due to $\sigma_{tot} \propto \omega^4$ light with short wave-lengths (i.e. large $\omega = ck = 2\pi c/\lambda$) will be scattered more strongly, thus explaining the blue color of the sky.

7.6 Maxwell meets Einstein

In the previous sections we have learned that Maxwell's theory of electromagnetism describes among others a variety of phenomena connected with the propagation of electromagnetic waves. These waves propagate with a speed we denoted as c, determined by the ϵ_0 and μ_0. If we use standard SI units (i.e. $\kappa = 1$) and insert numbers obtained from precision measurements, we obtain as value of c precisely the value for the speed of light, which is according to Einstein's postulate also the bound for any transport of information. This immediately raises the question, how Maxwell's equations fit into the framework of Einstein's theory of relativity.
Originally, the problem was actually seen the other way round. Before the dawn of special relativity, it was believed that the true "microscopic" theory was Newton's mechanics and that Maxwell's equations, which turned out to violate invariance against *Galilei transformations* were assumed to be derived in a very special frame of reference. It was believed that a special "substance", called *aether*, which carries the electromagnetic phenomena was at rest in this frame of reference. As you all know, this point of view was wrong, all efforts to find a relative motion with respect to the aether failed and the correct story is told by the theory of special relativity. Let us now see if Maxwell's equation are consistent with this fundamental physical principle.

7.6.1 Maxwell's Theory in Lorentz Covariant Form

We start from the Lorentz force acting on a particle moving with velocity $\boldsymbol{v}$, which in the frame of reference of a spectator at rest reads

$$\frac{d\boldsymbol{p}}{dt} = q\left(\boldsymbol{E}(\boldsymbol{r},t) + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r},t)\right) \quad .$$

Note that we used $\boldsymbol{p}$ here, because we already know from section 6.6 on relativistic kinematics that the momentum of a particle moving with respect to our rest frame is given by (6.12), i.e. $\boldsymbol{p} = \gamma m\boldsymbol{v}$, where $\gamma = 1/\sqrt{1 - v(t)^2/c^2}$ is now time dependent. We can change the differentiation on the left hand side to the Eigenzeit τ , remembering that $\gamma\frac{d\tau}{dt} = 1$, to obtain

$$\frac{d\boldsymbol{p}}{d\tau} = q\gamma\left(\boldsymbol{E}(\boldsymbol{r},t) + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r},t)\right) = \frac{q}{c}\left(u^0\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{u}\times\boldsymbol{B}(\boldsymbol{r},t)\right) \quad .$$

In the last step, we used the definition $u^\mu = (\gamma c, \gamma\boldsymbol{v})$ for the 4-velocity.
The left hand side of this equation is, by definition, the space part of a 4-vector. We can even state the corresponding time component by noting, that dp^0/dt is

the rate of change of energy, which for a moving charge is given as

$$\frac{dE}{dt} = \boldsymbol{v} \cdot \boldsymbol{F}_{Lorentz} = q\boldsymbol{v} \cdot \boldsymbol{E} = c\frac{dp^0}{dt}$$

since $p^0 = E/c$. Again, we may replace dt by $d\tau$ to obtain

$$\frac{dp^0}{d\tau} = \frac{q}{c}\boldsymbol{u} \cdot \boldsymbol{E}(\boldsymbol{r}, t) \ .$$

Thus, the Lorentz force can be written in 4-vector notation as

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu \ , \tag{7.67}$$

if we *define*[10]

$$F^{\mu\nu}(x^\tau) := \begin{pmatrix} 0 & -E^1(x^\tau) & -E^2(x^\tau) & -E^3(x^\tau) \\ +E^1(x^\tau) & 0 & -B^3(x^\tau) & +B^2(x^\tau) \\ +E^2(x^\tau) & +B^3(x^\tau) & 0 & -B^1(x^\tau) \\ +E^3(x^\tau) & -B^2(x^\tau) & +B^1(x^\tau) & 0 \end{pmatrix} \tag{7.68}$$

To prove that (7.67) is indeed covariant, we have to prove two things: (i) the quantity q is a Lorentz scalar and (ii) the tensor $F^{\mu\nu}$ properly transforms under the Poincaré group. The first problem is in fact the deepest. There is no mathematical proof, because we cannot reduce the concept of "charge" to a more fundamental one that would allow to show its Lorentz invariance. However, all experiments done so far show with extremely high precision (10^{-21} or better), that

the charge q is independent of the frame of reference, i.e. a Lorentz scalar.

For a physicist, this is sufficient to use "conservation of charge" as a well-founded working hypothesis until some culprit will prove the contrary and thus cause a landslide like the discovery of the constancy of c did in the beginning of the 20th century.

The (observational) fact, that q is a Lorentz scalar can be used to set up a further 4-vector: Let us assume that, in some Lorentz frame, we have a *charge density* ϱ at rest. The charge $dq = \varrho d^3r$ contained in a small volume element is Lorentz invariant, as is the volume element $d^4x = dx^0 d^3r$ of the Minkowski space. Consequently, the charge density must transform like the time component of a 4-vector. Next, we know (also from experiment) that the continuity equation must hold, i.e.

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \ .$$

[10] Remember that up to now we work in a fixed reference frame, i.e. using the 4-vector notation is possibly suggestive but does not necessarily mean that we have a covariant formulation. We still have to prove the latter!

Obviously, the right hand side is a Lorentz scalar, i.e. the left hand side *must* be a Lorentz scalar, too, which means that charge and current density form a 4-vector, the *4-current density*

$$j^\mu := (c\rho, \boldsymbol{j}) \tag{7.69}$$

so that continuity equation becomes

$$\partial_\mu j^\mu = 0 \ . \tag{7.70}$$

In Lorentz gauge, Maxwell's equations can be transformed into wave equations for the potentials ϕ and $\boldsymbol{A}$ as[11]

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}^2 \boldsymbol{A} &= \frac{4\pi}{c}\boldsymbol{j} \\ \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \boldsymbol{\nabla}^2 \phi &= 4\pi\varrho \end{aligned}$$

plus a gauge condition

$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \ .$$

Apparently, the right hand sides of the wave equations form the 4-current density, while the differential operator on the left hand side is $\Box = \partial^\mu\partial_\mu$, i.e. a Lorentz scalar. Consequently, the potentials must form a 4-vector, the *4-vector potential*

$$A^\mu := (\phi, \boldsymbol{A}) \tag{7.71}$$

and the wave equations and gauge condition can be written in the compact form

$$\Box A^\mu = \frac{4\pi}{c} j^\mu \ , \tag{7.72}$$

$$\partial_\mu A^\mu = 0 \ . \tag{7.73}$$

Did you wonder where this all leads to? Well, I will use the back door to prove that $F^{\mu\nu}$ is indeed a proper 4-tensor and consistent with Maxwell' equations. This can now be done in two steps. First note, that the fields are related to the potentials by

$$\begin{aligned} \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi \\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A} \ . \end{aligned}$$

[11] I choose Gaussian units form now on.

For example, for E_1 and B_1 this leads to the explicit formulas

$$\begin{aligned} E_1 &= -\frac{1}{c}\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{dx_1} = -(\partial^0 A^1 - \partial^1 A^0) \ , \\ B_1 &= \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = -(\partial^2 A^3 - \partial^3 A^2) \ , \end{aligned}$$

where we used $\partial^\mu = (\partial/\partial x_0, -\boldsymbol{\nabla})$. Thus, the fields can be represented as a second rank 4-tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{7.74}$$

which fulfills the required transformation properties under the Poincaré group because ∂^μ and A^μ do! That $F^{\mu\nu}$ is indeed the tensor (7.68) is left as an exercise. With $F^{\mu\nu}$, the homogeneous Maxwell equations read (exercise)

$$\partial^\kappa F^{\mu\nu} + \partial^\mu F^{\nu\kappa} + \partial^\nu F^{\kappa\mu} = 0 \ , \quad , \kappa \neq \mu \neq \nu \in \{0,1,2,3\} \ .$$

To complete this discussion we merely need to write down Maxwell's equations in a covariant form. For this purpose we need the so-called *dual field-strength tensor* defined by

$$\mathcal{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ +B_1 & 0 & +E_3 & -E_2 \\ +B_2 & -E_3 & 0 & +E_1 \\ +B_3 & +E_2 & -E_1 & 0 \end{pmatrix} \tag{7.75}$$

where we have defined the *totally antisymmetric fourth rank tensor*

$$\epsilon^{\alpha\beta\gamma\delta} := \begin{cases} +1 & \text{for } \{\alpha,\beta,\gamma,\delta\} \text{ an even permutation of } \{0,1,2,3\} \\ -1 & \text{for } \{\alpha,\beta,\gamma,\delta\} \text{ an odd permutation of } \{0,1,2,3\} \\ 0 & \text{if any two indices are equal} \end{cases} \tag{7.76}$$

which you know in $\mathbb{R}^3$ as Levi-Civita symbol. Maxwell's equations then reduce to the beautiful compact form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu \tag{7.77a}$$

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{7.77b}$$

One of the immediate results we get from the introduction of the tensor F is the transformation behavior of the electric and magnetic field under Lorentz transformation. This behavior was not at all clear from the outset, but now it is obvious. Given a transformation $\Lambda^\nu{}_\mu$, you get the transformed field strength tensor

$$(F')^{\mu\nu}(x') = \Lambda^\mu{}_\rho \Lambda^\nu{}_\kappa F^{\rho\kappa}(x)$$

from which you can read off the fields (exercise).

Chapter 8

Lagrangian Field Theory

8.1 Lagrangian Formulation of Field Theories

First, we illustrate how to formulate a field theory of continuum mechanics by starting from a system of mass points. To give a useful but simple example, we consider an elastic string, which we treat as a limit of a linear chain of mass points coupled by ideal springs.

8.1.1 Lagrangian Mechanics

Consider a classical mechanical system with N degrees of freedom. The N generalized coordinates will be denoted as $\phi_1, \phi_2, \ldots, \phi_N$. In the Lagrangian formulation of mechanics, we start from the *action functional*, which is determined by the *Lagrange function*

$$S = \int_{t_1}^{t_2} L(\phi_i, \dot{\phi}_i) \tag{8.1}$$

The equations of motion follow by finding an extremum of S (the so-called *principle of least action*) subject to the boundary conditions

$$\delta\phi_i(t_1) = \delta\phi_i(t_2) = 0 \tag{8.2}$$

From $\delta S = 0$ you will find the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_i} - \frac{\partial L}{\partial \phi_i} = 0. \tag{8.3}$$

8.1.2 Linear Elastic Chain

Let us apply this formalism to a one-dimensional chain of point masses of mass m coupled by elastic springs (spring constant k) (see Fig. 8.1), which are con-

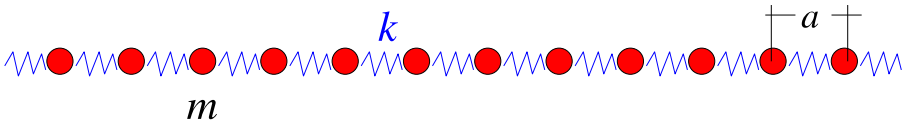

Figure 8.1: Linear elastic chain.

nected to form a ring (*periodic boundary conditions*). As generalized coordinates we use the *displacements* of the masses from their resting positions. Then, the *periodic boundary conditions* can be expressed as

$$\phi_{i+N} = \phi_i \tag{8.4}$$

The Lagrangian of such a system is given by $L = T - V$ with kinetic energy T and potential energy V. Thus

$$L = \sum_{i=1}^{N} \left[\frac{m}{2} \dot{\phi_i}^2 - \frac{k}{2} (\phi_{i+1} - \phi_i)^2 \right] \tag{8.5}$$

The system of Euler-Lagrange equations is straightforwardly derived to be

$$m \frac{d^2 \phi_i}{dt^2} = k \left[(\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1}) \right] \tag{8.6}$$

In fact, you can also derive these equations from Newton's second law directly. What we will show now is, that the limit from the chain to a string with constant mass density can be very easily obtained within the Lagrangian formulation.

8.1.3 From an Elastically Coupled Chain to an Elastic String

Let a be the equilibrium distance between mass points. The limit to a string is obtained by letting

$$a \to 0,$$

while at the same time keeping the equilibrium length $l = Na$ fixed, so that the number of mass points has to go to infinity. What we want to keep fixed is the *mass per length*

$$\frac{m}{a} \to \rho$$

Do we have to scale the spring constant, too?
The limit from chain to spring is most easily controlled within the Lagrangian, which we write in the form

$$L = a \sum_{i=1}^{N} \left(\frac{1}{2} \frac{m}{a} \dot{\phi_i}^2 - \frac{1}{2} ka (\frac{\phi_{i+1} - \phi_i}{a})^2 \right) = a \sum_i L_i \tag{8.7}$$

You see that this expression can be considered as a *Riemann sum*, which approaches the integral

$$L = \int_0^l dx \mathcal{L}(\phi, \partial\phi/\partial x) \tag{8.8}$$

over a *Lagrangian density*

$$\mathcal{L} = \frac{1}{2} \left(\rho (\frac{\partial \phi}{\partial t})^2 - Y (\frac{\partial \phi}{\partial x})^2 \right) \tag{8.9}$$

provided that not only $m/a \to \rho$ but also

$$ka \to Y$$

Y is called *Young module* of the elastic string. Note that the discrete index i counting the mass points has become a continuous variable x, which is the coordinate along the string. $\phi(x,t)$ has become a classical field, called the *longitudinal displacement field* of the string (we did not consider transverse displacement of the mass points, which would point in directions perpendicular to the x-axis.) The physical meaning of $\phi(x)$ is the (longitudinal) displacement of a point on the string from its equilibrium position x.
The *field equation* may now be obtained from the principle of least action with boundary conditions

$$\begin{aligned} \delta\phi(x,t_1) &= \delta\phi(x,t_2) = 0 \\ \delta\phi(0,t) &= \delta\phi(l,t) \end{aligned} \tag{8.10}$$

The resulting *Euler-Lagrange equation* is easily obtained

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\phi)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x\phi)} - \frac{\partial \mathcal{L}}{\partial\phi} = 0 \tag{8.11}$$

This leads to the field equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = 0 \tag{8.12}$$

with $c^2 = Y/\rho$.
The field equation is a one-dimensional *wave equation*.

8.1.4 Hamiltonian

The Hamiltonian is obtained from the Lagrangian by a *Legendre Transformation*. For the chain, this means that

$$H(\{p_i,\phi_i\}) = \sum_i p_i\dot{\phi}_i - L \tag{8.13}$$

with

$$p_i = \frac{\partial L}{\partial\dot{\phi}_i} \tag{8.14}$$

This leads to the very obvious result

$$H = \sum_i \left(\frac{p_i^2}{2m} + \frac{k}{2}(\phi_{i+1}-\phi_i)^2\right) \tag{8.15}$$

To determine the Hamiltonian of the elastic string, note that

$$p_i = a\frac{\partial L_i}{\partial \dot{\phi}_i} = m\dot{\phi}_i$$

so that in the continuum limit

$$\pi = \lim_{a\to 0}\frac{p_i}{a} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \rho\dot{\phi} \tag{8.16}$$

and

$$H = \lim_{a\to 0} a\sum_i \left(\dot{\phi}_i\frac{\partial L_i}{\partial \dot{\phi}_i} - L_i\right) = \int_0^l (\pi\dot{\phi} - \mathcal{L}) \tag{8.17}$$

8.1.5 Lagrangian Field Theories

Some of the above features can be generalized to the Lagrangian formulation of arbitrary field theories for a collection of fields $\phi^{(1)}, \ldots, \phi^{(n)}$. The (non-quantum) field theory is completely characterized by its Lagrange density

$$\mathcal{L}(\{\phi^{(k)}, \partial_\mu \phi^{(k)}\})$$

The Lagrange function of this theory is given by

$$L = \int d^d r \mathcal{L}$$

and the action functional has the usual form

$$S = \int_{t_1}^{t_2} dt L$$

Relativistically covariant field equations require relativistically invariant Lagrange densities (more precisely: $\mathcal{L}$ has to be a 4-scalar density). The derivation of the Euler-Lagrange equation is analogous to the derivation for particles. We consider variations $\delta\phi^{(k)}$, which vanish on the hypersurfaces $t = t_1$ and $t = t_2$. At this point, one may be worried that the variational problem might break the Lorentz invariance because it treats the time coordinate different from the space coordinates. We can, however, easily reformulate the problem in a manifestly covariant way. Instead of the two hypersurfaces $t = t_1,\ t = t_2$ we consider a closed hypersurface $\partial\Sigma$, which contains the 2 hypersurfaces as parts and is closed by additional parts at spatial infinity. Then we define the action functional as

$$S = \int_\Sigma d^{d+1}x\mathcal{L}$$

and consider variations, which vanish on $\partial\Sigma$.
Let us demonstrate the typical steps towards the Euler-Lagrange equation: First we insert $\phi^{(k)} + \delta\phi^{(k)}$ and expand in $\delta\phi^{(k)}$. As all the $\delta\phi^{(k)}$ are independent for different k, let us pick variations for a particular field component with $k = k_0$ and for simplicity denote $\phi^{(k_0)} = \phi$. Then we get

$$\delta S = \int\limits_{\Sigma} \sum_k \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi \right\}$$

We have used that the variation of $\partial_\mu \phi$ is equal to the derivative of $\delta\phi$, which is true if the variations are smooth. The second term is transformed by performing a partial integration. Boundary terms on $\partial\Sigma$ vanish due to our choice of $\delta\phi$ and we get

$$\delta S = \int\limits_{\Sigma} \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\} \delta\phi$$

Due to the arbitrariness of the $\delta\phi$ we conclude that the term in curly brackets has to vanish, which gives us the form of the Euler-Lagrange equations for field theories (summation convention!)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

To every field we define a canonically conjugate momentum as

$$\pi^{(k)}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^{(k)})}$$

and the *Hamilton density* of the field theory is given by

$$\mathcal{H} = \sum_k \pi^{(k)} \partial_0 \phi^{(k)} - \mathcal{L}$$

Note that the definition of the canonically conjugate momentum is in accordance with the definition: "the quantity, which is conserved as a consequence of translational symmetry in space". We will give a more systematic discussion of such conservation laws in the section 8.3.

8.2 Relativistic Field Theories

In the previous section, we have considered the mechanics of an elastic string. The resulting field equation turned out to be the wave equation, which, as we know by now, is a Lorentz invariant equation. Obviously, the underlying theory we used was Newtonian mechanics and from this point of view, the Lorentz invariance is a completely unphysical artifact. Our derivations are only valid in a very special frame of reference, where the string is at rest and only the longitudinal density modulations are time dependent.

Maxwell's equations were considered from this point of view by the time they were found. It was believed that the true "microscopic" theory was Newton's mechanics and that Maxwell's equations, which turned out to violate invariance against Galilei transformations were assumed to be derived in a very special frame of reference. It was believed that a special "substance" called *aether*, which carries the electromagnetic phenomena, was at rest in this of reference. As you all know, this point of view was wrong, all efforts to find a relative motion with respect to the aether failed and the correct story is told by the theory of special relativity.

Now, if we forget about its derivation, we can take the Lagrangian density Eq. (8.9) as a simple example of a manifestly Lorentz invariant field theory. Let us write its action in covariant notation

$$S = \frac{1}{2} \int d^4x \, (\partial_\mu \phi)(\partial^\mu \phi)$$

This theory is also called the *massless scalar field* or *massless Klein-Gordon field*. Consider a slight generalization of the elastic chain, where every mass point is not only coupled to its neighboring mass points but also to its resting position by ideal springs. The additional potential energy $V = m^2/2 \sum \phi_i^2$ gives rise to an additional term in the Lagrangian density of the string. Adding this term to our action in covariant form, we get the so called *massive scalar field* or *Klein-Gordon field with mass m*

$$S = \frac{1}{2} \int d^4x \, [(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2] \tag{8.18}$$

The field equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

is considered as a Lorentz invariant generalization of Schrödinger's equation for a free particle (which is definitely not a Lorentz invariant equation). Indeed, we did encounter this theory already in *Quantum Mechanics II* as possible extension of Schrödinger's non-relativistic wave equation to scalar, massive, free,relativistic particles.

8.2.1 The Lagrangian of Maxwell's Theory

The Lagrange density should reproduce the Maxwell equations as the stationary point of the action functional. First we have to decide which fields to chose as dynamical variables. It may seem that the field strengths are the most sensible choice. However, there are two strong arguments, which favor the 4-potential:

1) If you choose the $F^{\mu\nu}$, you have to reproduce both the homogeneous and the inhomogeneous Maxwell equations as the stationary point. The homogeneous Maxwell equations pose constraints on the 6 independent fields contained in $F^{\mu\nu}$. Remember that we have introduced the 4-potential exactly to resolve these constraints.

2) Remember the Lagrangian of (relativistic) particles in a given electromagnetic field. In Sec. 6.8 we learned that the Lagrangian of interaction between particles and fields is expressed by means of the 4-potential

$$L_{int} = -\frac{1}{c} A^\mu j_\mu$$

This term reproduces the Lorentz force and is in accordance with Maxwell' equations, but it is not given by the field strengths.

So let us choose the 4-potential A_μ as the dynamical field variables.
Obviously, the Lagrange density must be a Lorentz scalar (density). Furthermore, gauge transformations $A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$ should leave the equations of motions untouched. This is indeed the case, because in $\int d^4x (A^\mu + \partial^\mu \Lambda) j_\mu$ the additional term vanishes after a partial integration $\int d^4x j_\mu \partial^\mu \Lambda = -\int d^4x \Lambda \partial^\mu j_\mu$ as a consequence of the continuity equation $\partial_\mu j^\mu = 0$. As the equations of motion are second order partial differential equations and only contain derivatives of the A^μ (not the A^μ themselves) the Lagrangian should only depend on $\partial_\mu A_\nu$. So it is quite suggestive to try to construct a part of the Lagrangian from the field strength tensor. It should have the form

$$a F_{\mu\nu} F^{\mu\nu}$$

to be a Lorentz invariant. So much for plausibility. There are some more arguments to fix some aspects of the Lagrangian, but we just prefer to give it straight away and show it reproduces the Maxwell equations:

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A^\mu j_\mu \tag{8.19}$$

Now we consider the terms of the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} - \partial_\rho \frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\lambda)} = 0$$

The first term is straightforward to compute, but the second term needs a little contemplation. First note that the Lagrangian should be expressed by the dynamical field variables before taking the derivatives. Thus we have to use

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)g^{\mu\kappa}g^{\nu\sigma}(\partial_\kappa A_\sigma - \partial_\sigma A_\kappa)$$

A simple way to perform the partial differentiation with respect to $\partial_\rho A_\lambda$ is to use the chain rule

$$\frac{\partial}{\partial(\partial_\rho A_\lambda)} = \frac{\partial F_{\mu\nu}}{\partial(\partial_\rho A_\lambda)}\frac{\partial}{\partial F_{\mu\nu}} = (\delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\rho})\frac{\partial}{\partial F_{\mu\nu}}$$

Using this rule and $F^{\nu\mu} = -F^{\mu\nu}$ we get

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\lambda)} = -2\frac{1}{16\pi}\partial_\rho(\delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\rho})F^{\mu\nu} = -\frac{1}{4\pi}\partial_\rho F^{\rho\lambda}$$

Now we can combine the elements of the Euler-Lagrange Equations and recover the inhomogeneous Maxwell equations in covariant form

$$\partial_\rho F^{\rho\lambda} = \frac{4\pi}{c}j^\lambda$$

Finally, we give the Lagrangian and the corresponding Hamiltonian of Maxwell's theory in terms of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. Multiplying the two 4×4 matrices $F_{\mu\nu}$ and $F^{\mu\nu}$ and taking the trace of the resulting matrix $F_{\mu\nu}F^{\nu\mu}$ gives

$$\mathcal{L} = \frac{1}{8\pi}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - \frac{1}{c}j^\mu A_\mu$$

To compute the Hamiltonian, we first have to find the canonically conjugate momenta

$$\begin{aligned}
\pi^{(0)} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0 \\
\pi^{(i)} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = -\frac{1}{4\pi}F^{0i} = \frac{1}{4\pi}E^i
\end{aligned}$$

Surprisingly, the conjugate momentum of A_0 vanishes! The Hamiltonian density

$$\mathcal{H} = \sum_{i=1}^{3}\pi^{(i)}\partial_0 A_i - \mathcal{L} = \frac{1}{4\pi}\sum_{i=1}^{3}E^i\partial_0 A_i - \mathcal{L}$$

is rewritten using $E^i = -\partial^i A_0 - \partial_0 A^i$ and $A^i = -A_i$:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{4\pi}\boldsymbol{E}\cdot(\boldsymbol{E} + \boldsymbol{\nabla}A_0) - \frac{1}{8\pi}(\boldsymbol{E}^2 - \boldsymbol{B}^2) + \frac{1}{c}j^\mu A_\mu \\
&= \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \frac{1}{4\pi}\boldsymbol{E}\cdot\boldsymbol{\nabla}A_0 + \frac{1}{c}j^\mu A_\mu
\end{aligned}$$

The first term we already know from the undergraduate courses as energy density of the electromagnetic field.
If we compute the Hamiltonian from this density, we may use a partial integration to get a much more transparent form of

$$\begin{aligned}\frac{1}{4\pi}\int d^3r\ \boldsymbol{E}\cdot\boldsymbol{\nabla}A_0 &= -\frac{1}{4\pi}\int d^3r\ (\boldsymbol{\nabla}\cdot\boldsymbol{E})A_0\\ &= -\int d^3r\ \rho A_0\end{aligned}$$

As $j^0 = c\rho$ we may combine the term with the other current dependent term $j^\mu A_\mu - j^0 A_0 = -\boldsymbol{j}\cdot\boldsymbol{A}$. Thus we finally find for the Hamiltonian H:

$$\begin{aligned}H &= \int d^3r\ \mathcal{H}\\ &= \int d^3r\ \left[\frac{1}{8\pi}(\boldsymbol{E}^2+\boldsymbol{B}^2) - \frac{1}{c}\boldsymbol{j}\cdot\boldsymbol{A}\right]\end{aligned}$$

In the absence of external sources ($\boldsymbol{j} = 0$) it is the first term, which gives the energy (density) of a free Maxwell field.
When using the expressions for the Lagrangian and the Hamiltonian you must always keep in mind that they apply to situations, where the external sources are *given and fixed*. In particular, you should not apply it to situations, where the sources are themselves dependent on the electromagnetic fields (for example, currents in a conductor). If you want to treat these more complicated situations, you have to add a dynamical theory of the degrees of freedom, which generate the currents.

8.3 Noether's Theorem

8.3.1 Internal Symmetry: Simple Example

In classical mechanics you learn that space-time symmetries are closely connected to conservation laws. For example, the translation symmetry implies the conservation of momentum, the time translation symmetry implies conservation of energy and the rotation symmetry implies conservation of angular momentum. Emmy Noether has shown how to generalize this and how to connect the symmetry with respect to some continuous group of transformations to conservation laws. This deep theorem has its most remarkable applications in field theory. There is even a "straightforward" quantum field theory version of Noether´s Theorem, called Ward identities. To understand the power of Noether´s Theorem, let us consider a relativistic field theory for a system consisting of two scalar fields ϕ_1, ϕ_2 with Lagrange density

$$\mathcal{L} = \sum_{a=1,2} (\partial_\mu \phi_a \partial^\mu \phi_a \; - \; m^2 \phi_a^2)$$

Such field theories do in fact arise in physical problems, but for the moment, we don´t give any more interpretation. Note that the Lagrangian does not change under transformations

$$\begin{aligned} \phi_1' &= \cos\alpha \; \phi_1 + \sin\alpha \; \phi_2 \\ \phi_2' &= -\sin\alpha \; \phi_1 + \cos\alpha \; \phi_2 \end{aligned}$$

corresponding to rotations in some *internal* two-dimensional space. Now we follow the strategy of Noether´s theorem from classical mechanics.
Let us consider an infinitesimal angle $\delta\alpha$ and put $\sin\delta\alpha \approx \delta\alpha$ and $\cos\delta\alpha \approx 1$ so that $\delta\phi_a = \phi_a' - \phi_a$ becomes:

$$\begin{aligned} \delta\phi_1 &= \delta\alpha \; \phi_2 \\ \delta\phi_2 &= -\delta\alpha \; \phi_1 \end{aligned}$$

We insert $\phi_a(x) + \delta\phi_a(x)$ in the Lagrangian and calculate $\delta\mathcal{L}$ as

$$\begin{aligned} \delta\mathcal{L} &= \sum_{a=1,2} (\partial_\mu \delta\phi_a)(\partial^\mu \phi_a) + (\partial_\mu \phi_a)(\partial^\mu \delta\phi_a) - 2m^2 \delta\phi_a \phi_a \\ &= 2 \sum_{a=1,2} \left[(\partial_\mu \delta\phi_a)(\partial^\mu \phi_a) - m^2 \delta\phi_a \phi_a \right] \end{aligned}$$

Now let us add and subtract $(\partial_\mu \partial^\mu \phi_a)\delta\phi_a$ to obtain

$$\delta\mathcal{L} = 2 \sum_{a=1,2} \partial_\mu \left[\delta\phi_a \partial^\mu \phi_a \right] - 2 \sum_{a=1,2} \delta\phi_a \left[\partial_\mu \partial^\mu + m^2 \right] \phi_a \quad .$$

The last term contains the field equations and thus vanishes. Since by assumption we must have $\delta\mathcal{L} = 0$, it follows that

$$\sum_{a=1,2} \partial_\mu \left[\delta\phi_a \partial^\mu \phi_a\right] = 0$$

or, after inserting the variations $\delta\phi_a$,

$$\delta\alpha \; \partial_\mu[\phi_2 \partial^\mu \phi_1 - \phi_1 \partial^\mu \phi_2] = 0 \quad .$$

Due to the arbitrariness of $\delta\alpha$, the 4-divergence has to vanish. Thus we have found a continuity equation

$$\partial_\mu j^\mu = 0$$

with a 4-current density

$$j^\mu = (c\rho, \boldsymbol{j}) = \phi_2 \partial^\mu \phi_1 - \phi_1 \partial^\mu \phi_2$$

Note that the 0-component is the density of a conserved "generalized charge"

$$Q = \int d^3 r j^0$$

8.3.2 General One-Parameter Symmetry Group

The generalization of the above, simple example is straightforward. We have a field theory with a collection $\{\psi^{(s)}\},\ s = 1 \cdots n$ of fields as dynamical variables, and consider a general, one-parameter group of transformations with parameter λ

$$\overline{\psi}^{(r)}(x) = G^{(r)}_\lambda(\{\psi^s(x)\})$$

such that $G^{(r)}_{\lambda=0} = id$. It is important to note that these transformations act on the fields only, but not on the space-time argument x. The general strategy for transformations which modify x, too, will be discussed in the next section. Transformations near the identity result in small changes in the fields

$$\delta\psi^{(r)} = \overline{\psi}^{(r)} - \psi^{(r)} = \left.\frac{dG^{(r)}}{d\lambda}\right|_{\lambda=0} \cdot \lambda + O(\lambda^2)$$

and the corresponding change in the Lagrange density is

$$\delta\mathcal{L} = \sum_{r=1}^{n} \left[\frac{\partial\mathcal{L}}{\partial\psi^{(r)}}\delta\psi^{(r)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{(r)})}\partial_\mu\delta\psi^{(r)}\right]$$

To proceed we simply add and subtract the 4-divergence

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{(r)})}\right)\delta\psi^{(r)}$$

so that the change in Lagrange density becomes

$$\delta\mathcal{L} = \sum_{r=1}^{n} \left[\frac{\partial\mathcal{L}}{\partial\psi^{(r)}} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{(r)})} \right) \right] \delta\psi^{(r)} + \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{(r)})} \delta\psi^{(r)} \right)$$

If the Lagrange density is invariant under the transformations G_λ (simple symmetry transformations), we can conclude that for every trajectory (where the term in angular brackets vanishes), the 4-current density

$$j^\mu = \sum_{r=1}^{n} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{(r)})} \left. \frac{dG^{(r)}}{d\lambda} \right|_{\lambda=0} \tag{8.20}$$

obeys a continuity equation

$$\partial_\mu j^\mu = 0.$$

Obviously, j^μ is only defined up to a multiplicative constant.

8.3.3 Energy-Momentum Tensor

The transformations discussed in the previous section are of a type that change the field configurations *locally*. In the "classical" formulation of Noether's theorem they correspond to transformations acting on the coordinates q_i only but do not involve time. As you may remember, for a general type of transformation including also time, Noether's theorem looks slightly different.
In the following, I want to discuss the consequences of space-time symmetries, which corresponds to the classical application in mechanics. The simplest transformation of this type is the *space-time* translation

$$x^\mu \to x^\mu + a^\mu \quad .$$

These translations combine two symmetry operations of classical mechanics: space translations (leading to momentum conservation) and time-translation (leading to energy conservation). So we expect to find the field version of the conservation of the energy-momentum 4-vector. The form of Noether's theorem we must use here then reads

> If $\delta S = 0$ for a transformation compatible with the Euler-Lagrange equations, this transformation is connected to a conserved current.

Let us now apply this "Noether strategy" to a field ϕ with Lagrange density $\mathcal{L}(\phi, \partial^\mu\phi)$. As usual we assume infinitesimal transformations, i.e. we can expand the field at the translated position $x' = x + \delta a(x)$ according to

$$\phi(x') = \phi(x + \delta a(x)) \approx \phi(x) + \delta a^\mu \partial_\mu \phi(x)$$

and insert this expansion into the action functional to calculate the linear variation by standard steps.

$$\delta S_a = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \left(\delta a^\mu \partial_\mu \phi\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \, \partial_\nu (\delta a^\mu \partial_\mu \phi) \right\}$$

From $\partial_\nu(\delta a^\mu \partial_\mu \phi) = \delta a^\mu \partial_\nu \partial_\mu \phi \; + \; (\partial_\nu \delta a^\mu)(\partial_\mu \phi)$ the first term is combined with the other term proportional to δa^μ. The other term is partially integrated. So we get

$$\int d^4x \delta a^\mu \left\{ \partial_\mu \mathcal{L} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right) \right\}$$

The integral may be written in the form

$$\int d^x \delta a^\mu \partial_\nu T^\nu{}_\mu \quad . \tag{8.21}$$

with

$$T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu{}_\mu \mathcal{L}$$

This quantity is called the *canonical energy-momentum tensor* and has a form reminding us of the result for time-translations in classical mechanics, were the conserved quantity is the Hamilton function

$$H = \frac{dL}{d\dot{q}^i} \dot{q}^i - L \quad .$$

The expression for $T^0{}_0$ is indeed exactly what you would expect for a Hamiltonian density

$$T^0{}_0 = \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \partial_t \phi - \mathcal{L}$$

(compare with the "elastic string"). Quite generally, the 0-component $T^0{}_\mu$ corresponds to a 4-vector of densities of conserved quantities, i.e.

$T^0{}_0$	energy density
$T^0{}_i$	momentum density

For every relativistic field theory, we can furthermore read off now (Note that $T^{\nu\mu} = g^{\mu\kappa} T^\nu{}_\kappa$))

- the *energy current density* or *energy flux*: $T^i{}_0$
- the *momentum density* $\rho^i_P = T^{0i} = -T^0{}_i$
- the *momentum current density tensor* or *stress tensor* $\sigma^{ij} = T^{ij}$.

In this way, you find that the concepts of balance equations we encountered in hydrodynamics show up naturally for every relativistic field theory.

8.3.4 Energy-Momentum Tensor of Electromagnetic Fields

It is straightforward to generalize the energy-momentum tensor to the case of several fields $\phi^{(k)}$:

$$T^{\nu}{}_{\mu} = \left\{ \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^{(k)})} \partial_\mu \phi^{(k)} \right\} - \delta^{\nu}{}_{\mu} \mathcal{L}$$

Let us now specialize this result to the case of an electromagnetic field. The Lagrange density is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

External sources have to be put equal to zero, because otherwise we would not have energy-momentum conservation at all (translational symmetry is explicitly broken).
In case of gauge potentials as fields, it is necessary to consider the variations of fields a little bit more carefully. From

$$A^\nu(x^\mu + \delta a^\mu) \approx A^\nu(x) + \partial_\mu A^\nu(x) \delta a^\mu$$

we get variations of the vector potential, which are *not gauge invariant*. Therefore, we cannot guarantee that small variations in one gauge will stay small after application of a gauge transformation. We can construct manifestly gauge invariant small variations, if we combine the space time translation with a special gauge transformation (which is still a symmetry transformation, of course!). We choose as gauge field $\chi = \delta a_\mu A_\mu$ and obtain

$$\delta A^\nu = \delta a^\mu \left(\partial_\mu A^\nu - \partial^\nu A_\mu \right) \quad .$$

Using the previously obtained result

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi} F^{\mu\nu}$$

we get

$$T^{\mu\nu} = \frac{1}{4\pi} \left[F^{\mu\sigma} F_\sigma{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$

with

$$\partial_\mu T^{\mu\nu} = 0$$

as long as $j^\mu = 0$. Of course the definition of $T^{\mu\nu}$ is valid also without this assumption. In that case we can calculate the divergence of the tensor with the help of Maxwell's equations as (exercise)

$$\partial_\mu T^{\mu\nu} + \frac{1}{c} F^{\nu\alpha} j_\alpha = 0 \quad .$$

The individual components of the energy-momentum tensor can be evaluated straightforwardly (exercise) and read

$$T^{00} = \frac{1}{8\pi}\left\{\boldsymbol{E}^2 + \boldsymbol{B}^2\right\} =: u(\boldsymbol{r},t) \tag{8.22a}$$

$$T^{0i} = \frac{1}{4\pi}\left[\boldsymbol{E}\times\boldsymbol{B}\right]_i =: cP_i(\boldsymbol{r},t) \tag{8.22b}$$

$$T^{i0} = -\frac{1}{4\pi}\left[\boldsymbol{E}\times\boldsymbol{B}\right]_i =: -\frac{1}{c}S_i(\boldsymbol{r},t) \tag{8.22c}$$

$$T^{ik} = -\frac{1}{4\pi}\left[E_kE_i + B_kB_i - \frac{1}{2}\delta_{ik}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\right] \quad . \tag{8.22d}$$

The notations on the right hand sides show the physical meaning of the components: The first is the *energy density* $u(\boldsymbol{r},t)$ of the fields, the quantity $\boldsymbol{P} = (P_1, P_2, P_3)$ is interpreted as *momentum density*. The vector $\boldsymbol{S}$ is the energy flux density and called *Poynting vector*. The space-space components build Maxwell's stress tensor.

Chapter 9

Gravitation

The second fundamental classical field theory besides Maxwell's theory of electromagnetism is Einstein's theory of gravitation or general relativity. In this theory, the fields are related to the geometric structure of space and time. We will see, that gravitation is a consequence of a deviation from the flat space-time structure of special relativity or Minkowski metric. However, before we can discuss the basic principles of general relativity and go through simple applications of the theory, we have to introduce some mathematical concepts and notations from *differential geometry*, that allow us to deal with curved manifolds.

9.1 Exterior Forms on $\mathbb{R}^n$

This chapter is meant to give you a glimpse "behind the curtains" of the mathematical basis of field theories, especial their coordinate-free formulation. As usual it is not meant to be complete in any sense, in particular not in any mathematical.

Let us consider a smooth manifold M. This can, on the one hand, be a simple, flat $\mathbb{R}^n$; on the other hand, we can also think of the surface of a n-dimensional sphere. For the former, it is obviously possible to introduce one *global* coordinate system or *chart* and express all physical quantities in terms of this coordinate frame. The latter, however, does not permit such a global coordinate system. As we move along the surface of the sphere, the directions of the coordinate vectors will have to change, too, and we will have to deal with *local charts* and the description of physical objects like the components of fields will have to be adapted to the changed coordinate frames. On the other hand, physics and the quantities necessary to describe them should not depend on a particular choice of a coordinate system. It is the aim of differential geometry to introduce concepts that allow to express physical quantities in curved manifolds in a coordinate-free notation.

At a given point $x \in M$, we can define the tangential space T_xM to M in x and a mapping

$$\overset{1}{\omega}: T_xM \longrightarrow \mathbb{R}\ , \quad v \mapsto \overset{1}{\omega}(v)$$

We will call $\overset{1}{\omega}$ an exterior one-form. A particularly important example is the total differential *df* of a smooth function f, which acts on a vector $v \in T_xM$ as

$$df(v) = \sum_{i=1}^{n} v^i \partial_i f \equiv v(f)$$

and simply yields the derivative of f in x along the direction of v. Note that we used upper indices for the *components* of the field v, but lower indices for the

partial derivatives. In the last step we made use of the fact, that the tangential space at a point $x \in M$ can be represented via the directional derivatives (think of the Taylor expansion). One can thus write in a short-hand way

$$v = \sum_{i=1}^{n} v^i \partial_i$$

and consider $\{\partial_i\}$ as *basis of the tangential space in point* x. Thus, any smooth *vector field* V on M can be decomposed as

$$V = \sum_{i=1}^{n} v^i(x) \partial_i$$

with smooth *component functions* $v^i(x)$.
Obviously, the coordinates x^i are smooth functions themselves and the differentials dx^i are exterior one-forms one calls basis one-forms. The set $\{dx^i\}$ is *dual* to the basis $\{\partial_i\}$ and

$$dx^i(\partial_k) = \partial_k(x^i) = \delta^i{}_k$$

and one can expand every exterior one-form with respect to this basis, i.e.

$$\overset{1}{\omega} = \sum_{i=1}^{n} \omega_i dx^i \ , \ \ \omega_i(x) = \overset{1}{\omega}(\partial_i) \ \ .$$

For an arbitrary, smooth vector field we then have

$$\overset{1}{\omega}(V) = \sum_{i=1}^{n} V^i(x) \omega_i(x) \ \ .$$

Let us define the following *wedge product* for exterior forms by stating it for the basis forms:

$$\left(dx^i \wedge dx^j\right)(v, w) := v^i w^j - v^j w^i$$

where $v, w \in T_x M$ arbitrary. The wedge product is antisymmetric, i.e. $dx^i \wedge dx^j = -dx^j \wedge dx^i$, and obviously a generalization of the cross-product for $\mathbb{R}^3$ (exercise). It can be generalized to an arbitrary number of factors. For example for a 3-dimensional manifold we obtain

$$(dx^i \wedge dx^j \wedge dx^k)(u, v, w) = \det \begin{pmatrix} u^i & v^i & w^i \\ u^j & v^j & w^j \\ u^k & v^k & w^k \end{pmatrix}$$

The products $dx^i \wedge dx^j$ form a basis for smooth *exterior two-forms*

$$\overset{2}{\omega} = \sum_{i<j} \omega_{ij}(x) dx^i \wedge dx^j \ \ .$$

and so on. For some $k > 0$ one can thus generate $\binom{n}{k}$ basis k-forms $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ and define corresponding k-forms

$$\overset{k}{\omega} = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k}(x) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \quad .$$

These k-forms are also called covariant tensor fields of rank k, and the associated space is labeled $\Lambda^k(M)$. The dimensions are

$$\dim \Lambda^k(M) = \binom{n}{k} \quad .$$

Let us now define a mapping

$$d : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M) \ , \quad \overset{k}{\omega} \rightarrow d \overset{k}{\omega} \quad .$$

For a smooth function, d is simply the total differential

$$f \rightarrow df = \sum_i \partial_i f dx^i$$

Applying d to the wedge product of two forms can be evaluated via the *Leibniz rule*:

$$d(\overset{r}{\omega} \wedge \overset{s}{\omega}) = (d \overset{r}{\omega}) \wedge \overset{s}{\omega} + (-1)^r \overset{r}{\omega} \wedge (d \overset{s}{\omega}) \quad .$$

Finally, an explicit representation of $d \overset{k}{\omega}$ is given by

$$d \overset{k}{\omega} = \sum_j \sum_{i_1 < \ldots < i_k} \frac{\partial \omega_{i_1 \ldots i_k}(x)}{\partial dx^j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \quad .$$

We call d the *exterior differential*. An interesting property is that $d \circ d = 0$. Obviously, the spaces Λ^k and Λ^{n-k} have the same dimension and are thus isomorphic. We can now define a map

$$\star : \Lambda^k \longrightarrow \Lambda^{n-k} \ , \quad \omega \mapsto \star\omega$$

(Hodge's star operator). If we have an **oriented** basis $\{e_i\}$ it follows

$$(\star\omega)(e_{i_{k+1}}, \ldots, e_{i_n}) = \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} \omega(e_{i_1}, \ldots, e_{i_k})$$

where $\epsilon_{\ldots}$ is Levi-Civita symbol extended to n indices. In particular for $n = 3$ we have:

$$\star dx^i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} dx^j \wedge dx^k \ ,$$

$$\star \left(dx^i \wedge dx^j \right) = \epsilon_{ijk} dx^k \ ,$$

$$\star \left(dx^i \wedge dx^j \wedge dx^k \right) = 1 \quad .$$

For general n the relation

$$\star(\star\omega) = (-1)^{k(n-k)}\omega$$

holds.
Finally, we may combine the exterior differential with the star operation into

$$\delta := (-1)^{n(k+1)+1} \star d\star \ , \quad \Delta_{\mathrm{LdR}} := d\circ\delta + \delta\circ d \ \ .$$

The left operator, which is called *codifferential*, is to some extent the opposite to d, because it is a mapping $\delta : \Lambda^k \longrightarrow \Lambda^{k-1}$. In particular, its application to a function $f \in \Lambda^0(M)$ yields $\delta f = 0$. The second operator is called *Laplace-de-Rahm operator* and does not change k. Applied to a function $f \in \Lambda^0(M)$ it reduces to the usual Laplace operator (exercise).

9.2 Maxwell's equation in coordinate free notation

As a first example let us now consider the application of the formalism developed above to Maxwell's theory.

9.2.1 Field-strength tensor and Lorentz force

If we now denote with dx^μ the basis one-forms in Minkowski space, our field-strength tensor $F_{\mu\nu}(x)$ becomes a 2-form

$$\overset{2}{\omega}_F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

where we had to introduce the factor $1/2$ to allow the use of the sum convention (taking into account that both $F_{\mu\nu}$ and $dx^\mu \wedge dx^\nu$ are antisymmetric). If we take into account that $F_{\mu\nu}$ differs from $F^{\mu\nu}$ in (7.68) only by the sign in the time-space components (i.e. $E_i \to -E_i$), we may write this as

$$\begin{aligned}\overset{2}{\omega}_F = \ & dx^0 \wedge \left[E_1 dx^1 + E_2 dx^2 + E_3 dx^3\right] - \left[B_3 dx^1 \wedge dx^2 + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1\right] \\ & = dx^0 \wedge \overset{1}{\omega}_{\boldsymbol{E}} - \overset{2}{\omega}_{\boldsymbol{B}} \ \ .\end{aligned}$$

In Minkowski space, both electric and magnetic field become 2-forms, conveniently written as

$$\overset{2}{\omega}_{\boldsymbol{E}} := \sum_{i=1}^{3} E_i(\boldsymbol{r},t) dx^0 \wedge dx^i \tag{9.1a}$$

$$\overset{2}{\omega}_{\boldsymbol{B}} := \frac{1}{2}\sum_{ijk} \epsilon_{ijk} B_i(\boldsymbol{r},t) dx^j \wedge dx^k \tag{9.1b}$$

If we apply $\overset{2}{\omega}_F$ only to one vector field, say u^μ, we find

$$\overset{2}{\omega}_F\,(u,\cdot) = K_\mu(x)dx^\mu\ ,\quad K_\mu(x) = \frac{q}{c}F_{\mu\nu}(x)u^\nu\quad ,$$

i.e. the Lorentz force (appearing as one-form here).

9.2.2 Maxwell's equations

Before we can write down Maxwell's equation, we have to find the properties of Hodge's $\star$ operator. This is somewhat subtle, because in contrast to normal $\mathbb{R}^4$, Minkowski space has the metric $g_{\mu\nu}$. The result is as follows (summation convention!):

$$\begin{aligned}
\star\, dx^\mu &= \frac{1}{3!}g^{\mu\lambda}\epsilon_{\lambda\nu\sigma\tau}dx^\nu\wedge dx^\sigma\wedge dx^\tau\ , && \text{(9.2a)}\\
\star\,(dx^\mu\wedge dx^\nu) &= \frac{1}{2!}g^{\mu\lambda}g^{\nu\rho}\epsilon_{\lambda\rho\sigma\tau}dx^\sigma\wedge dx^\tau\ , && \text{(9.2b)}\\
\star\,(dx^\mu\wedge d^\nu\wedge dx^\sigma) &= g^{\mu\lambda}g^{\nu\rho}g^{\sigma\eta}\epsilon_{\lambda\nu\eta\tau}dx^\tau\ , && \text{(9.2c)}\\
\star\left(dx^0\wedge d^1\wedge dx^2\wedge dx^3\right) &= \det(g) = -1\ . && \text{(9.2d)}
\end{aligned}$$

After these considerations we can write down Maxwell's equations using the exterior 2-form $\overset{2}{\omega}_F$, and the 1-form $\overset{1}{\omega}_j = j_\mu(x)dx^\mu$ as

$$\begin{aligned}
d\,\overset{2}{\omega}_F &= 0 \qquad \text{(homogeneous equations)} && \text{(9.3a)}\\
\delta\,\overset{2}{\omega}_F &= \frac{4\pi}{c}\,\overset{1}{\omega}_j \qquad \text{(inhomogeneous equations)} && \text{(9.3b)}
\end{aligned}$$

Like $d\circ d = 0$, one also has $\delta\circ\delta = 0$. Thus applying δ to (9.3b) a second time yields

$$\delta\,\overset{1}{\omega}_j = 0\ , \qquad (9.4)$$

which, when translated into functions, is the continuity equation.
Although it is obviously a nice mental gymnastic to invent new nomenclatures to make formulas look even more compact (and unreadable to the uninitialized), the obvious question is: Does it do any good? The answer is, of course, yes. Note that Maxwell's equation in the form (9.3a) and (9.3b) are written without any reference to a particular coordinate system, but in a calculus respecting the Minkowski metric. Thus, they are *manifestly Lorentz covariant*. Second, the above calculus is still valid if one leaves the realm of a flat metric and enters general relativity with curved time-space manifolds. The representation of the forms, basis etc. by actual functions becomes in fact much more cumbersome, the structure of the equations, however, remains.

9.2.3 Vector potential and covariant derivative

With the vector potential $A_\nu(x)$ we can build a 1-form

$$\overset{1}{\omega}_A := A_\mu(x)dx^\mu$$

with derivative

$$d\,\overset{1}{\omega}_A = dA_\nu(x) \wedge dx^\nu = \partial_\mu A_\nu(x) dx^\mu \wedge dx^\nu \quad .$$

The right hand side tells us that

$$\overset{2}{\omega}_F = d\,\overset{1}{\omega}_A \quad .$$

Note that the above relation implies that the homogeneous Maxwell equations are trivially fulfilled, because $d\ \overset{2}{\omega}_F = d \circ d\ \overset{1}{\omega}_A = 0$ by virtue of $d \circ d = 0$. Furthermore, we can change

$$\overset{1}{\omega}_A \to \overset{1}{\omega}_A + d\chi$$

without changing $\overset{1}{\omega}_F$, i.e. we can implement the gauge transformation in a natural way in this language. Finally let us now apply the Laplace-de-Rahm operator to $\overset{1}{\omega}_A$:

$$\Delta_{\mathrm{LdR}}\ \overset{1}{\omega}_A = (d \circ \delta + \delta \circ d)\ \overset{1}{\omega}_A = -(d * d * + * d * d)\ \overset{1}{\omega}_A = (\Box A_\mu(x) dx^\mu) \quad .$$

Finally, the particularly important Lorentz gauge reads in this formalism

$$\delta A = 0$$

Since gauge transformations seem to play a particularly important role in field theories, we may elaborate on them a little further. First, let us define the operator (*covariant derivative*)

$$D_A := d + i\frac{q}{\hbar c}\ \overset{1}{\omega}_A$$

The appearance of $\hbar$ is pure convention here, necessary only to obtain the correct units, but already suggests that this will become especially interesting in quantum mechanics. On an arbitrary exterior form, this operator acts as

$$D_A\omega = d\omega + i\frac{q}{\hbar c}\ \overset{1}{\omega}_A \wedge \omega$$

In particular for an arbitrary smooth function we have

$$D_A f = \left(\partial_\mu f + i\frac{q}{\hbar c} A_\mu f\right) dx^\mu$$

Applied twice, we have (the calculations are straightforward and left as exercise)

$$D_A \circ D_A = i\frac{q}{\hbar c}\overset{2}{\omega}_F$$

This result becomes even clearer when we rename $A := i\frac{q}{\hbar c}\overset{1}{\omega}_A$ and $F := i\frac{q}{\hbar c}\overset{2}{\omega}_F$ to find

$$D_A = d + A\ ,\ \ D_A^2 = dA + A \wedge A = dA = F \tag{9.5}$$

These type of equations are well known in differential geometry: A is called *connection (form)*, $D_A = d + A$ is called *covariant derivative* and F the *curvature (form)*. It can be shown that $F = D_A^2$ can be interpreted as "round trip" through a small closed path in M. Such an equation is also called *structure relation*.

9.3 Elements of general relativity

9.3.1 Principle of Equivalence

One particular property of gravitation or gravitational fields is, that all bodies move in them in the same manner, irrespective of their mass, charge, ..., provided the initial conditions are the same. For example, the free fall in the gravitational field of the earth is the same for all bodies; whatever their mass, the acceleration is always the same.
This special property of gravitational fields can be used to establish an analogy of the motion of a body in such a field and the motion without field, but in a noninertial reference system: A freely moving body in a uniformly accelerated reference system has relative to that noninertial system a constant acceleration equal and opposite to the acceleration of the system. The same applies to a body moving in a uniform gravitational field. As a Gedankenexperiment let us consider the situation depicted in Fig. 9.1:

- Case 1: The rocket is placed in a part of the universe far removed from gravitating bodies. The rocket is accelerated forward with a constant acceleration g relative to an inertial observer. The observer releases a body from rest and sees it fall to the floor with acceleration g.

- Case 2: The rocket motor is switched off so that the ship undergoes uniform motion relative to the inertial observer. A released body is found to remain at rest relative to the observer in the ship.

- Case 3: The rocket is next placed on the surface of the earth, whose rotational and orbital motions are ignored. A released body is found to fall to the floor with acceleration g.

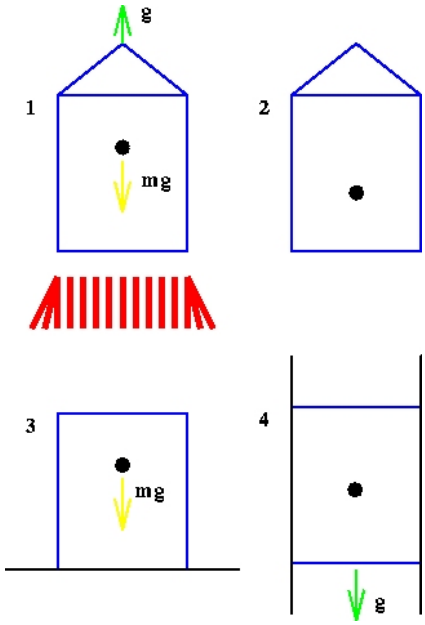

Figure 9.1: The lift experiments

- Case 4: Finally, the rocket is allowed to fall freely towards the center of the earth. A released body is found to remain at rest relative to the observer.

Thus, motion in an uniform gravitational field is equivalent to free motion in an uniformly accelerated noninertial reference system. This is the *principle of equivalence*.

There is, however, a subtle difference. The "fields", to which noninertial systems are equivalent, will vanish if we transform to an inertial system. This is not possible for a "true" gravitational field produced by a mass and which has to vanish at infinity. Fields produced by noninertial systems, on the other hand, will stay uniform or even increase like the centrifugal acceleration as one approaches infinity. The best one can do is to find a certain region in space where one can consider the gravitational field as uniform and transform *locally* to a suitable noninertial system where the field vanishes *locally*.

In an inertial reference system, the space time interval ds^2 is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad .$$

If we now transform e.g. into a uniformly rotating reference frame with

$$x = x' \cos\omega t - y' \sin\omega t \ , \ y = x' \sin\omega t + y' \cos\omega t \ , \ z = z'$$

the corresponding expression for ds^2 is

$$ds^2 = \left[c^2 - \omega(x'^2 + y'^2)\right] dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\omega y' dx' dt - 2\omega x' dy' dt$$

Obviously it is impossible to write such an expression as sum of squares of coordinate differentials. Thus, in a noninertial system we will have to use the general form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where the *metric tensor* $g_{\mu\nu}$ is a function of the coordinates now and uniquely determines the geometric properties of the space-time manifold.
Consequently, we can now interpret gravitational fields as changes of the space-time metric leading to a metric tensor that is different from the Minkowski form $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The geometric structure of space and time is thus determined by physical phenomena and not an a priory given property of space and time. Also, since gravitational fields cannot be "gauged" away globally, there does not exist a reference frame where the metric tensor takes on the Minkowski form; this can be achieved at most locally for a (infinitesimally small) region about a given point p, where the gravitational field can be considered as uniform. Globally, space-time will have to be considered as a curved manifold.

9.3.2 Curves, Torsion and Curvature

To work in a curved space time manifold M, we need the concept of differentiation of fields on that manifold. The definition in a flat space is to take the difference of the field evaluated at a certain point p and its neighboring point $p + \delta p$ in a certain direction and divide by the coordinate difference δp. In $\mathbb{R}^3$ we then arrive at the conventional result

$$\boldsymbol{\nabla V} = \begin{pmatrix} \partial_1 V_1 & \partial_1 V_2 & \partial_1 V_3 \\ \partial_2 V_1 & \partial_2 V_2 & \partial_2 V_3 \\ \partial_3 V_1 & \partial_3 V_2 & \partial_3 V_3 \end{pmatrix}$$

which is a tensor field over $\mathbb{R}^3$.
If we try to adopt this definition to a curved manifold $\mathbb{M}$, we have to account for the change of coordinate system when picking a point $p + \delta p$ in the neighborhood of p. Simply setting $p \to p + \delta p$ will transport a vector parallel with respect to the *local coordinate system at* p. In Fig. 9.2 this situation has been visualized with the dashed arrow. This is, however, not what we want, because in this case a certain component μ of our vector in the coordinate system at p will in general become a different component (or a linear combination of components) relative to the coordinate system at $p + dp$. We rather need a definition

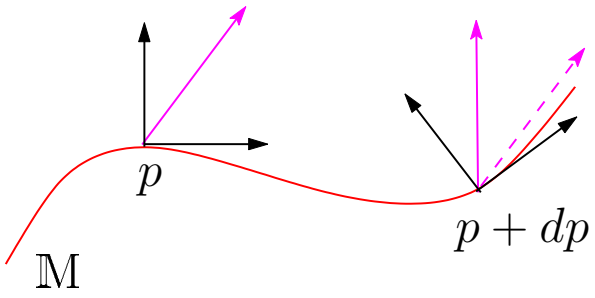

Figure 9.2: Parallel transport in a curved manifold.

of *parallel transport* such that the vector at $p + dp$ is oriented with respect to *the local coordinate system at* $p + dp$ as it was at p, corresponding to the full arrow in Fig. 9.2.
As usual, we use as definition of the derivative the linear change in the vector field V, i.e. $V(p + dp) = V(p) + (DV)(p) \cdot dp$, where we denote the operation of differentiation with D. The corresponding expression for a vector field $V = v^\mu \partial_\mu$ becomes, according to our rules developed in the section on differential geometry, a mixed tensor field

$$DV = (\partial_\nu v^\mu + \delta_\nu v^\mu)\, \partial_\mu \otimes dx^\nu \quad .$$

The field $\delta_\nu v^\mu$, which takes into account the change of coordinate system, has to depend on the field V in a linear way to ensure that D is a linear operation. We can thus make the ansatz

$$\delta_\nu v^\mu = \Gamma^\mu_{\nu\kappa} v^\kappa \quad .$$

The 4^3 functions $\Gamma^\kappa_{\mu\nu} : M \longrightarrow \mathbb{R}$ are called *Christoffel symbols* or *connection coefficients*. The operation

$$DV = (\partial_\nu v^\mu + \Gamma^\mu_{\nu\kappa} v^\kappa)\, \partial_\mu \otimes dx^\nu \tag{9.6}$$

thus defined is called *covariant derivative* or *absolute derivative* of V. For a certain *direction* we will write[1]

$$D_{\partial_\nu} V = (\partial_\nu v^\mu + \Gamma^\mu_{\nu\kappa} v^\kappa)\, \partial_\mu \quad .$$

Note, that the Christoffel symbols cannot form a tensor field. In fact, if they were a tensor field, then, because we can always assume in an infinitesimally

[1] In literature, the short hand notations ${v^\mu}_{,\nu}$ for the partial derivative $\partial_\nu v^\mu$ and ${v^\mu}_{;\nu}$ or ${v^\mu}_{||\nu}$ for the covariant derivative are frequently used.

small region about a point $p \in M$ a flat manifold where surely $\Gamma = 0$, then due to the properties of a tensor under coordinate transformations we must have $\Gamma = 0$ in M and consequently M must be globally flat.
The formula (9.6) can easily be extended to tensor fields taking into account that a tensor field $T^{\mu\nu}$ has to transform like the tensor product $A^\mu B^\nu$ of two vector fields. One then arrives at

$$DT = \left(\partial_\kappa T^{\mu\nu} + \Gamma^\mu_{\lambda\kappa} T^{\lambda\nu} + \Gamma^\nu_{\lambda\kappa} T^{\mu\lambda}\right) \partial_\mu \otimes \partial_\nu \otimes dx^\kappa$$

as expression for the covariant derivative of a contravariant tensor field.
The most important tensor field is the metric tensor, because it directly determines the geometric properties of space-time. Let us therefore try to find out, what we can say about Dg. To this end let us note that, as for any 4-vector,

$$A^\mu = g^{\mu\lambda} A_\lambda \ .$$

Obviously, this is also true for the μ-th component of DA, i.e.

$$(DA)^\mu = g^{\mu\lambda} (DA)_\lambda \ .$$

However, from the first relation we obtain

$$(DA)^\mu = g^{\mu\lambda} (DA)_\lambda + (Dg)^{\mu\lambda} A_\lambda \ .$$

Thus, the metric tensor $g_{\mu\nu}$ has to satisfy[2]

$$Dg = 0 \ , \tag{9.7}$$

which leads to the explicit expression

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\nu g_{\kappa\mu} + \partial_\mu g_{\kappa\nu} - \partial_\kappa g_{\mu\nu}\right) \tag{9.8}$$

for the Christoffel symbols (exercise).
Next, we must define what motion in a curved space-time manifold means. To that end, let us denote with

$$\gamma : \mathbb{R} \longrightarrow M \ , \ \ \tau \mapsto \gamma(\tau)$$

a curve in M with a curve parameter τ (for example the proper time) and with V a smooth vector field (defined at least in an open neighborhood of γ). One calls the vectorfield $D_{\dot\gamma} V$ the *covariant derivative along* γ or *total derivative*,

[2] This is also called *Riemannian or Levi-Civita connection*.

denoted also as $DV/d\tau$. With $V = v^\mu \partial_\mu$ and $\dot\gamma = \frac{dx^\mu}{d\tau}\partial_\mu$ the representation of $D_{\dot\gamma}V$ in terms of coordinates is

$$D_{\dot\gamma}V = \left(\frac{dv^\kappa}{d\tau} + \Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}v^\nu\right)\partial_\kappa \ .$$

One calls V to be *autoparallel* along γ if

$$D_{\dot\gamma}V = 0$$

holds. A special application involves the field $\dot\gamma$ (which belongs to the tangent space at each point $x(\tau)$ and is thus a valid vector field). If we have

$$D_{\dot\gamma}\dot\gamma = 0 \ \Leftrightarrow \ \ddot x^\kappa + \Gamma^\kappa_{\mu\nu}\dot x^\mu \dot x^\nu = 0 \tag{9.9}$$

we may say that along γ the "velocity" $\dot\gamma$ is constant. Therefore such a curve γ is called a *geodesic*.

Let us discuss the formula (9.9) from a different point of view. Remember, that in special relativity the equation of motion for a free particle was given by

$$\frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = 0 \ .$$

Obviously, for a covariant description in a curved space-time we have to use the replacement

$$\frac{du^\kappa}{d\tau} \to \frac{Du^\kappa}{d\tau} = \frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

and thus the geodesic can be viewed to describe the free motion of a body. Furthermore, denoting by $m\ddot x^\mu$ the 4-acceleration, we can interpret $-m\Gamma^\kappa_{\mu\nu}\dot x^\mu \dot x^\nu$ as "force field" arising from the curvature of space-time, the *gravitational field*. Note that we now can identify the metric tensor as "potential of the gravitational field" and the Christoffel symbols as "field intensities" determined by the derivatives of the "potentials". Thus, the equation $Du^\kappa/d\tau = 0$ of a geodesic can be regarded as the generalization of the law of inertia in the presence of a curvature in space-time, which we will call *gravitational field* in the following. It has also become unnecessary to introduce two different "masses" (heavy and inert ones), eliminating a certain element of "magic" from Newton's theory, where one has to postulate their equivalence a posteriori.

How can we quantify the deviation of our manifold $\mathbb{M}$ from a flat metric? A possible way is to consider parallel transport from a starting point x to a final point $x + dx + d\bar x$ along different, infinitesimal paths. Such a construction for a general vector V is shown in Fig. 9.3. Going through the algebra, the change of $v^\mu(x)$ along the two different paths is found to be

$$\Delta v^\mu = -R^\mu_{\rho\sigma\nu}v^\rho(x)d\bar x^\nu dx^\sigma \ .$$

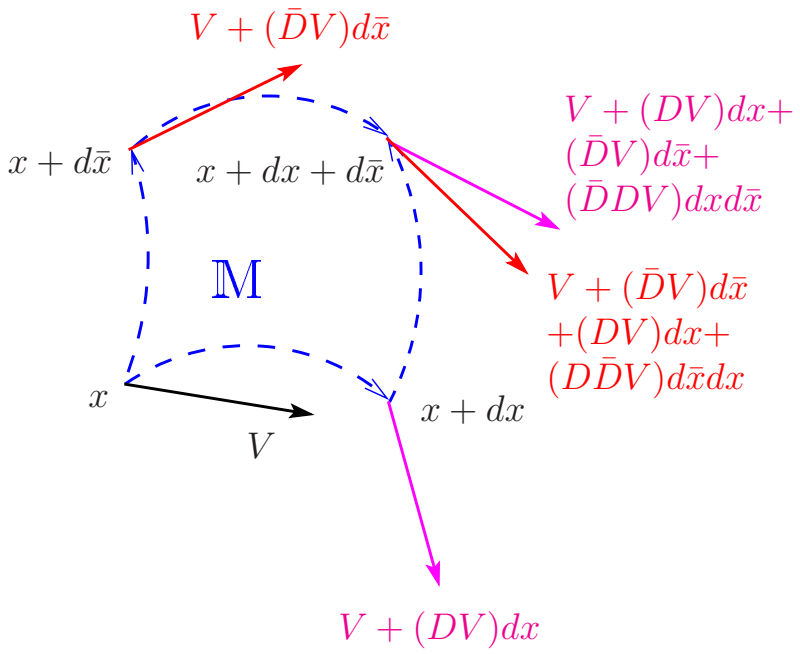

Figure 9.3: Parallel transport along a closed, infinitesimal path.

The quantity R appearing on the right hand side directly determines the deviation of $\mathbb{M}$ from a flat manifold and is called *curvature tensor*

$$R = \frac{1}{3!} R^{\mu}_{\nu\kappa\lambda} dx^{\nu} \wedge dx^{\kappa} \wedge dx^{\lambda} \otimes \partial_{\mu}$$

$$R^{\mu}_{\nu\kappa\lambda} = \partial_{\kappa}\Gamma^{\mu}_{\lambda\nu} - \partial_{\lambda}\Gamma^{\mu}_{\kappa\nu} + \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\mu}_{\mu\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\mu}_{\lambda\sigma} \ .$$

In terms of the *curvature form*

$$\Omega^{\mu}_{\nu} := \frac{1}{2} R^{\mu}_{\nu\kappa\lambda} dx^{\kappa} \wedge dx^{\lambda} \tag{9.10}$$

and the *connection form*

$$\omega^{\mu}_{\nu} := \Gamma^{\mu}_{\kappa\nu} dx^{\kappa} \ . \tag{9.11}$$

it can be represented as

$$\Omega^{\mu}_{\nu} = d\omega^{\mu}_{\nu} + \omega^{\mu}_{\kappa} \wedge \omega^{\kappa}_{\nu} \ .$$

Comparing this equation with expression (9.5), we see that the curvature form Ω^{μ}_{ν} plays a role similar to the field tensor in Maxwell's theory. Furthermore, the potentials $A^{\mu}(x)$ there are replaced by the Christoffel symbols. Note that there exists a very important difference: The potential in Maxwell's theory is a scalar function, i.e. $\omega_A \wedge \omega_A = 0$. In general relativity, however, the potentials are matrices which in general do not commute, hence $\omega_\Gamma \wedge \omega_\Gamma \neq 0$.

That curvature and curvature tensor deserve these names can be seen from the theorems

> *A pseudo-Riemannian space is locally flat iff the curvature*[3] *vanishes.*

and

> *Parallel transport is independent of the path iff the curvature tensor vanishes.*

Note that both theorems state equivalences. In particular, one can always achieve that for a space locally flat in a given point[4] p the Christoffel symbols vanish ($\Gamma^{\mu}_{\nu\kappa}(p) = 0$). These systems we will call *local inertial systems*.
There exist a lot more interesting theorems and relations on space-time structure (or differential forms, if you like the mathematical perspective more), which can fill a lecture on its own. Those concepts presented up to now are however sufficient to grasp the basic ideas behind general theory of relativity.

9.3.3 The Newtonian Limit

Let us now consider the limit of a slowly moving particle in a weak gravitational field. The latter means, that we may introduce a coordinate system which is nearly Lorentzian, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \ \ |h_{\mu\nu}| \ll 1 \ , \ \ \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \ \ .$$

For a slowly moving particle we furthermore have $dx^0/d\tau \approx c$, i.e. $|dx^i/d\tau| \ll 1$. One then obtains

$$\frac{d^2x^i}{dt^2} \approx \frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \approx -c^2\Gamma^i_{00} \tag{9.12}$$

and

$$\Gamma^i_{00} \approx \frac{1}{2}(\partial_i h_{00} - \partial_0 h_{0i}) \ \ .$$

We now make the further assumption that the gravitational field is stationary, i.e. ignore the second term, and arrive at

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{c^2}{2}\boldsymbol{\nabla} h_{00} \ \ ,$$

which coincides with the Newtonian equation of motion if $h_{00} = 2\Phi/c^2$ or equivalently

$$g_{00} \approx 1 + \frac{2}{c^2}\Phi \ \ .$$

[3] Since there exist other connections one must in fact state here: "of the Riemannian connection".

[4] plus within an open region, of course

This looks very appealing, but what is Φ? Obviously, we are still lacking an important ingredient to our theory, viz a relation that actually determines the metric tensor.

9.3.4 Einstein's field equations

In electrodynamics, the physical fields (or gauge fields) were determined by Maxwell's equations, which represent the *field equations of that theory*. These equations were determined from experiment. A similar approach will be used here. If our theory shall make sense at all, we have to obey certain conservation laws and limiting cases. One limiting case obviously is Newton's law of gravity, which means that the field Φ has to fulfill

$$\Delta\Phi = 4\pi G\varrho$$

for a given mass density ϱ. Since we assume static mass distributions (i.e. work in the non-relativistic limit), the mass density is equivalent to the energy density, which in turn is related to T_{00} of the energy-momentum tensor

$$T_{00} \approx c^2\varrho \quad .$$

Hence,

$$\Delta g_{00} \approx \frac{2}{c^2}\Delta\Phi \approx \frac{8\pi}{c^2}G\varrho \;\Leftrightarrow\; \Delta g_{00} \approx \frac{8\pi G}{c^4}T_{00} \quad .$$

Now consider the following *contraction*

$$R_{\mu\nu} := R^{\kappa}_{\mu\kappa\nu} = \partial_\kappa\Gamma^{\kappa}_{\mu\nu} - \partial_\nu\Gamma^{\kappa}_{\kappa\mu} + \Gamma^{\sigma}_{\nu\mu}\Gamma^{\kappa}_{\kappa\sigma} - \Gamma^{\sigma}_{\kappa\mu}\Gamma^{\kappa}_{\nu\sigma}$$

of the curvature tensor, called *Ricci tensor*. Within the Newtonian limit, $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ time independent. The Christoffel symbols are related to $g_{\mu\nu}$ and thus $h_{\mu\nu}$ via (9.8). Ignoring terms quadratic in h, we may write

$$R_{\mu\nu} \approx \partial_\kappa\Gamma^{\kappa}_{\mu\nu} - \partial_\nu\Gamma^{\kappa}_{\kappa\mu} \quad .$$

Since the fields are static, we especially have

$$R_{00} \approx \partial_l\Gamma^{l}_{00}$$

and, since for the same reasons $\Gamma^{l}_{00} \approx \partial_l g_{00}/2$ we find

$$R_{00} \approx \Delta g_{00} \approx \frac{4\pi G}{c^4}T_{00} \quad .$$

This relation suggests that a general ansatz for the field equations might be

$$R_{\mu\nu} = \frac{4\pi G}{c^4}T_{\mu\nu} \quad ,$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the field theory. However, this ansatz would violate the conservation law

$$(DT)^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\nu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\nu\lambda} T^{\mu\lambda} = 0$$

for energy, momentum and angular momentum density. Note, that in this equation the simple partial derivative had to be replaced by the covariant derivative to make the equation manifestly covariant. The correct form can be however obtained by *requiring* these conversation laws, leading to

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\kappa{}_\kappa = \frac{8\pi G}{c^4} T^{\mu\nu} \quad . \tag{9.13}$$

These are *Einstein's field equations*. Note that these equations are *nonlinear, partial differential equations* even in vacuum, because the curvature tensor depends quadratically on the fields (Christoffel symbols). Evidently, this feature of general relativity means that solving these equations, even numerically, is extremely cumbersome.
A remarkable fact is that for four dimensional space-time one can prove the following statement:

> *The most general tensor $\mathcal{D}^{\mu\nu}(g)$ satisfying (i) $\mathcal{D}^{\mu\nu}(g) = T^{\mu\nu}$ and $(D\mathcal{D})^{\mu\nu} = 0$ (required by conservation laws) is given by*
>
> $$\mathcal{D}^{\mu\nu}(g) = \frac{1}{\kappa} G^{\mu\nu} + \frac{\Lambda}{\kappa} g^{\mu\nu}$$
>
> *with*
>
> $$G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\lambda{}_\lambda$$
>
> *the so-called* Einstein tensor.

Note that the whole theory leaves only two additional constants κ and Λ open! By comparing this expression with the Newtonian limit, we see that $\kappa = 8\pi G/c^4$. The quantity Λ is called *cosmological constant* and would imply the existence of a homogeneous mass distribution in addition to the "physical" masses. While conventionally one assumed $\Lambda = 0$, recent discoveries of a deceleration of the expansion of our universe raised the question, whether one indeed has to consider $\Lambda \neq 0$ to account for this effect ("dark energy").

9.3.5 Linearized Theory of Gravity

As before, we consider the case, that our system is – at least in a certain space-time region – nearly flat. Thus, a coordinate system exists for which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \quad |h_{\mu\nu}| \ll 1 \quad .$$

For example, for our solar system we have $|h_{\mu\nu}| \sim |\Phi|/c^2 \sim GM_\odot/c^2R_\odot \sim 10^{-6}$. However, the gravitational fields may vary with time now.
For such fields, we expand the field equations in powers of $h_{\mu\nu}$ and keep only the linear terms. The Ricci tensor is then given by

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu}$$

with

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(\partial_\nu h^\alpha{}_\mu + \partial_\mu h^\alpha{}_\nu - \partial^\alpha h_{\mu\nu}\right)$$

where we used the convention that indices are raised and lowered with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. For the Ricci tensor we then obtain

$$R_{\mu\nu} = \frac{1}{2}\left[\partial_\nu\partial_\lambda h^\lambda{}_\mu \Box h_{\mu\nu} - \partial_\mu\partial_\nu h^\lambda{}_\lambda + \partial\mu\partial_\lambda h^\lambda{}_\nu\right] \quad .$$

Inserting this into the field equations[5] $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ one arrives at

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h^\lambda{}_\lambda - \partial_\mu\partial_\lambda h^\lambda{}_\mu - \partial_\nu\partial_\lambda h^\lambda{}_\nu - \eta_{\mu\nu}\Box h^\lambda{}_\lambda + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} = -16\pi G T_{\mu\nu} \quad .$$

As one can show by direct computation, these field equations require $\partial_\nu T^{\mu\nu} = 0$, which means that the sources that produce the field do not feel a back reaction of this field.
It is now useful to introduce the quantity

$$\gamma_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad ,$$

where $h := h^\lambda{}_\lambda$. In terms of $\gamma_{\mu\nu}$, the field equations read

$$-\Box\gamma_{\mu\nu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta\gamma_{\alpha\beta} + \partial^\alpha\partial_\nu\gamma_{\mu\alpha} + \partial^\alpha\partial_\mu\gamma_{\nu\alpha} = 16\pi G T_{\mu\nu} \quad .$$

An important observation now is that, as in Maxwell's theory, there exist gauge transformations which leave the linearized Einstein tensor invariant. These transformations are of the form

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\nu\xi_\mu + \partial_\mu\xi_\nu$$

with an arbitrary vector field ξ^μ. The gauge invariance means, that we can always find a gauge, such that (*Hilbert gauge*)

$$\partial_\beta\gamma^{\alpha\beta} = 0 \tag{9.14}$$

(proof as exercise) and the field equations reduce to the simple form

$$\Box\gamma_{\mu\nu} = -16\pi G T_{\mu\nu} \quad .$$

[5] In the following we set $c = 1$ for convenience.

As we have learned in section 7.5.3, the solution of these equations can be found by using the Green function. Since we again have to deal with the D'Alembert operator, we already know the solution, which is given by Eq. (7.55). Thus, we may write

$$\gamma_{\mu\nu}(x) = -4G \int \frac{T_{\mu\nu}(x^0 - |\boldsymbol{r} - \boldsymbol{r}'|, \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r' + \text{solution of the homogeneous equation.} \tag{9.15}$$

Within our linearized theory, we thus have found that the field consists of two terms, the first being created by the sources (i.e. our sun), while the second represents *gravitational waves* traveling with the speed of light.
To be able to proceed further, let us now consider nearly Newtonian sources, for which

$$T_{00} \gg |T_{0j}|,\ |T_{ij}|$$

holds. If we further assume small velocities, we may also neglect retardation effects in (9.15) to obtain

$$\gamma_{00} = 4\Phi\ ,\quad \gamma_{0j} = \gamma_{ij} = 0$$

with

$$\Phi = -G \int \frac{T_{00}(ct, \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r' \quad . \tag{9.16}$$

For the metric we obtain with $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma^{\lambda}{}_{\lambda}$

$$g_{00} = 1 + 2\Phi\ ,\quad g_{0i} = 0\ ,\quad g_{ij} = -(1 - 2\Phi)\delta_{ij} \quad .$$

At large distances $r \gg r'$ from the sources, we can perform a multipole expansion for (9.16) and keep only the *monopole* contribution, observe that $T_{00} \approx \varrho$, use a static mass density and finally obtain

$$g_{00} = 1 - 2\frac{GM}{r}\ ,\quad g_{0i} = 0\ ,\quad g_{ij} = -(1 + 2\frac{GM}{r})\delta_{ij} \quad . \tag{9.17}$$

The first term is what we have already found in the beginning of our tour through general relativity. The additional changes of the space components of the metric tensor give already rise to observable effects. Let us consider as an example the behavior of a massless particle (e.g. light) passing a mass (e.g. the sun). Since massless particles travel with the speed of light, i.e. their world lines are light-like, all vectors have to fulfill $a^\mu a_\mu = 0$. In particular, the energy-momentum relation now reads $E = p$ ($c = 1$) or $E^2 = p^2$. Although the world lines cannot be parametrized by the proper time any more (since $ds^2 = 0$), one can still find a curve parameter, say λ, and from $E^2 = p^2$ we infer the covariant relation

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \quad .$$

For a free particle, the Lagrangian is given by the covariant expression[6]

$$L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

which here implies $L = 0$.
For the further discussion it is helpful to use spherical coordinates. For that purpose we can write the metric tensor (9.17) as exterior form as

$$g = (1 - 2\frac{GM}{r})dx^0 \otimes dx^0 - (1 + 2\frac{GM}{r})(dx \otimes dx + dy \otimes dy + dz \otimes dz) \quad .$$

Using the basis vectors dr, $rd\theta$ and $r\sin\theta d\phi$ the metric tensor for spherical coordinates becomes

$$g = (1-2\frac{GM}{r})dx^0\otimes dx^0-(1+2\frac{GM}{r})\left(dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)\right) \quad .$$

With this result we find for the Lagrangian ($\alpha := GM$)

$$L = (1 - 2\frac{\alpha}{r})(\dot{x}^0)^2 - (1 + 2\frac{\alpha}{r})\left(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right) \quad .$$

Note that as in Newtonian mechanics ϕ is cyclic, i.e.

$$\dot{\phi} = \frac{l}{r^2\sin^2\theta(1 + 2\frac{\alpha}{r})}$$

is a constant of motion. Furthermore, for θ we obtain the equation $(r^2\dot{\theta})^{\cdot} = r^2\sin\theta\cos\theta\dot{\phi}^2$ which means that if we choose $\theta = \pi/2$ initially, $\dot{\theta} = 0$ holds and the motion is confined to the equitorial plane. The second cyclic coordinate, x^0, leads to the conservation of the quantity $\dot{x}^0(1 - 2\alpha/r) =: E$, which we may identify as energy. Finally, we are left with the equation

$$(1 - 2\frac{\alpha}{r})^{-1}E^2 - (1 + 2\frac{\alpha}{r})\dot{r}^2 - \frac{l^2}{r^2(1 + 2\frac{\alpha}{r})} = 0$$

to determine $r(\phi)$, for example. With $r' = \frac{dr}{d\phi} = \dot{r}/\dot{\phi}$ and the new variable $u = 1/r$ we have

$$\dot{r}^2 = (r')^2\frac{l^2}{r^4(1 + 2\frac{\alpha}{r})^2} = (u')^2\frac{l^2}{(1 + 2\alpha u)^2}$$

and finally

$$\frac{E^2}{l^2}\frac{1 + 2\alpha u}{1 - 2\alpha u} - u'^2 - u^2 = 0 \quad .$$

[6]We actually use L^2 here, which however does not make any difference as long as we are interested in the Euler-Lagrange equations only.

Within our linearized theory we must, for consistency, neglect terms α^2/r^2 and obtain as equation for the orbit

$$u'^2 + u^2 = \frac{E^2}{l^2}\left(1 + 4\alpha u\right) \quad . \tag{9.18}$$

Since αu is typically very small, we can solve this equation by iteration. Ignoring the second term on the right hand side, we have with $R = l/E$

$$u'^2 + u^2 = \frac{1}{R^2} \quad ,$$

with the solution

$$u(\phi) = \frac{\sin\phi}{R} \quad .$$

Inserting $x = r\cos\phi$, $y = r\sin\phi$ one obtains ($u = 1/r$)

$$R = r\sin\phi = y \quad ,$$

i.e. the curve is a straight line parallel to the x-axis at distance R.
For the full solution we make the ansatz $u = R^{-1}\sin\phi + a$ and obtain from (9.18)

$$\frac{\cos\phi}{R^2} + \left(\frac{\sin\phi}{R} + a\right)^2 = \frac{1}{R^2} + \frac{4\alpha}{R^2}\left(\frac{\sin\phi}{R} + a\right)$$

or

$$2\left(a - \frac{2\alpha}{R^2}\right)\sin\phi + a\left(aR - \frac{4\alpha}{R}\right) = 0 \quad .$$

This latter equation can be solved for a by iteration with the result

$$a = \frac{2\alpha}{R^2} + O\left(\frac{\alpha^2}{R^4}\right) \quad ,$$

and hence

$$u(\phi) \approx \frac{\sin\phi}{R} + \frac{2\alpha}{R^2} \quad .$$

Let me now replace $x = r\cos\phi$, $y = r\sin\phi$ as before. We then find

$$R = y + \frac{2\alpha}{R}\sqrt{x^2 + y^2} \quad ,$$

which describes a hyperbola. The asymptotes are obtained for $x \to \infty$ as (see Fig. 9.4)

$$y = R \pm \frac{2\alpha}{R}x$$

and the angle between these asymptotes and the line $y = R$, ϕ_∞, becomes

$$\left|\frac{dy}{dx}\right| = |\tan\phi_\infty| \approx |\phi_\infty| = \frac{2\alpha}{R} \quad .$$

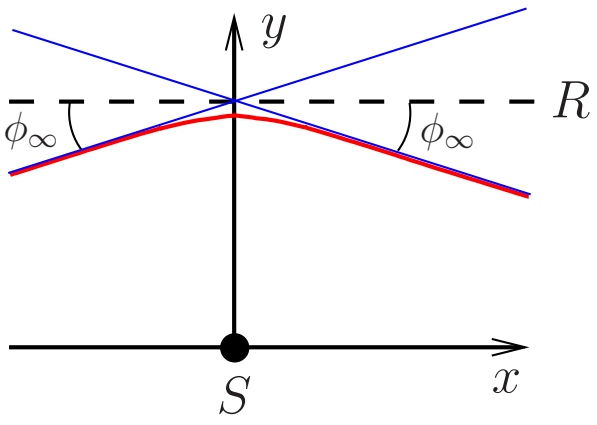

Figure 9.4: Schematic representation of the solution $u(\phi)$.

Thus the total *angle of deflection* for light by a mass M is given by

$$\delta = 2|\phi_\infty| \approx \frac{4MG}{Rc^2} \approx 1.75'' \frac{R_\odot}{R}$$

in case of the sun. This prediction by general relativity has been confirmed by observation pretty soon after it has been made during a total eclipse of the sun in March 1919.

A similar calculation can be done for a massive particle, where the Lagrangian is

$$L = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1$$

with proper time τ. Within this theory, one can for example discuss the advance of the perihelion of a bound orbit. It turns however out, that the result obtained is slightly too large in comparison with a more refined theory due to the neglect of nonlinear contributions to the metric.

Without external sources the linearized theory in Hilbert gauge leads to the field equations

$$\Box \gamma_{\mu\nu} = 0 \ . \tag{9.19}$$

Even within the Hilbert gauge, one still has the freedom to choose the gauge such that in addition $\gamma^\lambda{}_\lambda = 0$ (exercise). As in Maxwell's theory, the most general solutions can be represented as plane waves

$$h_{\mu\nu} = \Re e \left\{ \epsilon_{\mu\nu} e^{-ik_\alpha x^\alpha} \right\}$$

with $k^2 := k_\alpha k^\alpha = 0$ from the wave equation (9.19), $k_\mu \epsilon^\mu{}_\nu = 0$ from the Hilbert gauge (9.14) and $\epsilon^\mu{}_\mu = 0$ from the additional gauge condition $\gamma^\lambda{}_\lambda = 0$. Further analyzing the properties of the *polarization tensor* $\epsilon_{\mu\nu}$, one can show, that – as

for vacuum light waves – there exist only two independent polarization states. One finds that for the gravitational waves under consideration here the *helicity* is ± 2 instead of ± 1 for electromagnetic waves. In a quantized theory, one calls the normal modes of gravitational waves *gravitons* and from the helicity can conclude that they carry a spin $S = 2$.
Treating the generation of waves by sources as in Maxwell's theory, one finds again similar structures. In contrast to electrodynamics, were the dominant contribution is dipole radiation, gravitational radiation is generated from the quadrupolar terms in the source distribution. Thus, it is in general weaker and the decay will be much faster.

9.3.6 Beyond the linear approximation

The linearized theory of the previous section already showed the power of the theory of general relativity even within the linear approximation. To go beyond this linear theory, one typically has to "guess" a certain form of the metric and then determine the parameters by solving Einstein's field equations (9.13). The best known among these solutions is the Schwarzschild metric, which is given by

$$g = (1 - 2\frac{GM}{c^2 r})dx^0 \otimes dx^0 - \frac{1}{1 - 2\frac{GM}{c^2 r}} dr \otimes dr - r^2 \left(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi \right) \quad , \tag{9.20}$$

which describes the gravitational field *outside* of a *nonrotating* and *radial symmetric* mass distribution. The Schwarzschild metric (9.20) seems to have a serious defect, i.e. a divergence at the *Schwarzschild radius*

$$R_S := 2\frac{GM}{c^2} \quad .$$

The problem comes about by the assumption of a static metric. It turns out, that this cannot be true any longer for $r < R_S$. In fact, the radius R_S marks the distance from the mass distribution where a light pulse takes infinitely long to reach an observer at $r > R_S$ (*event horizon*). Light and particles at $r < R_S$ are captured inside the event horizon. Note that for this to become possible it is required that the mass is concentrated within R_S. In such a case one speaks of a *black hole*. A very nice discussion of this issue can be found in the book by H. Goenner [12].
Let us now briefly discuss the geodesics of the Schwarzschild metric. These can be obtained from the Lagrangian

$$\tilde{L} = \left(1 - \frac{R_S}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{R_S}{r}\right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right) = \epsilon \quad ,$$

where $\epsilon = c^2$ for massive particles and $\epsilon = 0$ for light. The dot means differentiation with respect to the curve parameter (for example the proper time in case of massive particles).
As before, we have two cyclic variables, t and ϕ with integrals of motion

$$\begin{aligned} \left(1 - \frac{R_S}{r}\right)\dot{t} &= \frac{e}{c} \\ \sin^2\theta\, r^2\dot{\phi} &= l \ . \end{aligned}$$

Moreover, from the Euler-Lagrange equation

$$-r^2 \sin\theta\cos\theta\,\dot{\phi}^2 + (r^2\dot{\theta}\dot{)} = 0$$

follows for the initial condition $\theta(0) = \pi/2$ that $\dot{\theta} = 0$, i.e. the motion is restricted to the equitorial plane as before. Finally, for r we obtain the equation

$$\dot{r}^2 = e^2 - \left(1 - \frac{R_S}{r}\right)\left(\epsilon + \frac{l^2}{r^2}\right) \ . \tag{9.21}$$

Setting $\epsilon = 0$ in (9.21), we can again deduce the result for the deflection of light; with $\epsilon = c^2$ we could, for example calculate the advance of the perihelion of mercury.
A different way to discuss the possible trajectories without calculations can be obtained by rewriting (9.21) as

$$\dot{r}^2 + V_{eff}(r) = e^2 \ , \tag{9.22}$$

where I introduced an effective potential

$$V_{eff} = c^2\left(1 - \frac{R_S}{r}\right)\left(1 + \frac{l^2}{c^2 R_S^2}\frac{R_S^2}{r^2}\right) \ . \tag{9.23}$$

The effective potential has extrema at the points

$$\rho_\pm = a^2\left(1 \pm \sqrt{1 - \frac{3}{a^2}}\right) \ ,$$

where $\rho = r/R_S$ and $a^2 = l^2/(c^2 R_S^2)$. Note that real extremal points exist only for $a \geq \sqrt{3}$, i.e. a minimal value of l. Further analysis shows that ρ_- is a maximum and ρ_+ a minimum. The form of V_{eff} is shown in Fig. 9.5. In contrast to Newton's law of gravitation, one has in total four different types of trajectories: Trajectory a directly leads into the center of the source of gravitation; this type of trajectory does *not* exist in Newtonian theory of gravity. Similarly, trajectory b hits the (instable) extremal point and corresponds to a trajectory, where the perihelion is never reached but circles infinitely many times. Trajectories c and d correspond to the "classical" hyperbolas and elliptic trajectories

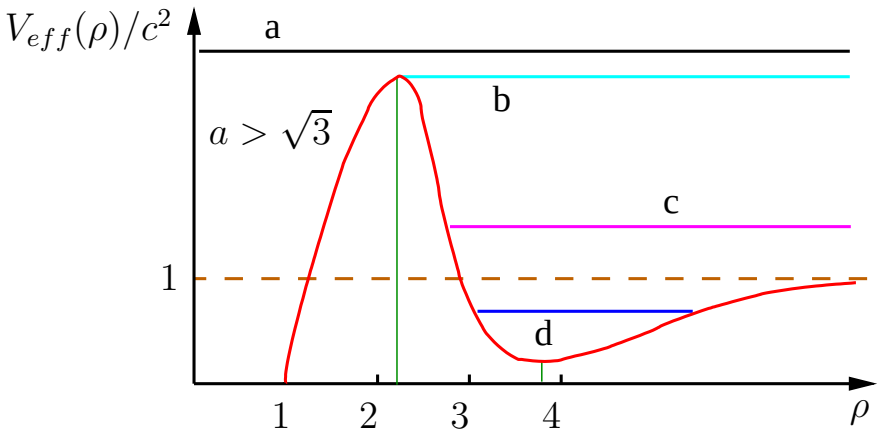

Figure 9.5: Schematic representation of V_{eff} in Eq. 9.23 for a massive particle.

also present in Newtonian theory. However, here the perihelion (or aphelia) is not fixed but moves in time, as observed e.g. for mercury.
A nice Java applet showing these features can be found at

http://www.fourmilab.ch/gravitation/orbits.

There are a lot more interesting further effects emerging in general relativity. These things go unfortunately far beyond the scope of this lecture, but can for example be found in Ref. [12].

Chapter 10

Gauge Fields and Fundamental Interactions

10.1 Quantum Particles in Classical Electromagnetic Fields: Gauge Invariance

We have stressed repeatedly that the Maxwell theory is only part of a consistent theory of electromagnetic fields and charged matter. The charge density and current density, which enter Maxwell's equations have to be given, before the electric and magnetic fields can be calculated. To set up a consistent and self-contained theory of charged matter we also need a Lagrangian describing the dynamics of matter and the coupling of matter to electromagnetic fields.

We start with a non-relativistic example, which will help us to grasp a beautiful idea about the connection of *symmetry* and *interactions*, which is the basis of all modern theories of fundamental interactions. We consider (non-relativistic) *quantum particles* and treat the electromagnetic fields as non-quantized. The example is not entirely academic. There are important practical applications, where the electromagnetic fields are "macroscopic" and need not be quantized, for example in "accelerator physics". For us, this example is a perfect setting for extending the Maxwell Lagrangian by a matter field, which gives rise to electric charges and currents.

Schrödinger's equation for a free particle is the stationary condition of an action with Lagrangian density

$$\mathcal{L}_S = \frac{i\hbar}{2}[\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi] - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2$$

This Lagrange density has an obvious symmetry, which you already know from quantum mechanics: the wave function ψ can be multiplied by a constant phase

$$\overline{\psi}(\boldsymbol{r},t) = e^{i\alpha}\psi(\boldsymbol{r},t)$$

Note that these transformations form a *continuous group* (under multiplication). The group of all *unitary* $n \times n$ *matrices* $\boldsymbol{U}$ is called $U(n)$ and the phase factor multiplication corresponds to the special case $n = 1$. Therefore we state that 1-particle non-relativistic quantum mechanics has a $U(1)$ symmetry. In the following section we will introduce some important concepts on continuous groups.

Noether's theorem tells us, that there is a conserved 4-current given by Eq. (8.20), which for the present example leads to

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}(-i\psi^*) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(i\psi)$$

as a conserved quantity as a consequence of $U(1)$ symmetry. Using

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} = \frac{i\hbar}{2}\psi^* \tag{10.1a}$$

$$\frac{\partial \mathcal{L}}{\boldsymbol{\nabla}\psi} = \frac{-\hbar^2}{2m}\boldsymbol{\nabla}\psi^* \tag{10.1b}$$

and the corresponding relations for the complex conjugate field ψ^* we get a 4-vector[1]

$$j^\mu = -\hbar\left(c|\psi|^2, \frac{\hbar}{2mi}(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)\right)$$

consisting of the well known probability density and current density of elementary quantum mechanics and obeying the continuity equation

$$\partial_\mu j^\mu = -\hbar\left(\partial_t|\psi|^2 + \boldsymbol{\nabla}\left[\frac{\hbar}{2mi}(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)\right]\right) = 0$$

If the quantum particle carries a charge q and moves in an electromagnetic field, described by the 4-potential $(\phi, \boldsymbol{A})$ you know that Schrödinger's equation is changed to

$$i\hbar\partial_t\psi = \frac{1}{2m}[-i\hbar\boldsymbol{\nabla} - \frac{q}{c}\boldsymbol{A}]^2\psi + q\phi\psi$$

and the corresponding Lagrange density is

$$\mathcal{L}_S = \frac{i\hbar}{2}[\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi] - \frac{1}{2m}|(-i\hbar\boldsymbol{\nabla} - \frac{q}{c}\boldsymbol{A})\psi|^2 - q\phi|\psi|^2$$

Adding the Maxwell Lagrange density

$$\mathcal{L}_M = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

we have a complete semi-classical theory $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_M$ of charged quantum particles in classical electromagnetic fields. A gauge transformation

$$\begin{aligned}\phi' &= \phi - \frac{1}{c}\partial_t\chi \\ \boldsymbol{A}' &= \boldsymbol{A} + \boldsymbol{\nabla}\chi\end{aligned}$$

leads to changes in $\mathcal{L}$, which may be written in form of a 4-divergence of the current.

Now we come to a remarkable point. The combined Lagrange density has a much higher symmetry than its parts, if we combine gauge transformations of

[1] For the derivation remember that $\partial_\mu = (\frac{1}{c}\partial_t, \boldsymbol{\nabla})$ (Eq. (6.7))!

the 4-potential with *smooth, space-time dependent phase factors*, multiplying the wave function

$$\psi'(\boldsymbol{r},t) = e^{i\alpha(\boldsymbol{r},t)}\psi(\boldsymbol{r},t)$$

Under such transformations the derivatives of the wave function transform as follows:

$$\begin{aligned}\partial_t\psi' &= e^{i\alpha(\boldsymbol{r},t)}\big\{\partial_t + i(\partial_t\alpha)\big\}\psi \\ \boldsymbol{\nabla}\psi' &= e^{i\alpha(\boldsymbol{r},t)}\big\{\boldsymbol{\nabla} + i(\boldsymbol{\nabla}\boldsymbol{\alpha})\big\}\psi\end{aligned}$$

Thus $\mathcal{L}_S$ is changed under the space-time dependent phase factors, which are not symmetry transformations by themselves. However, it easy to check, that these changes may be compensated by an appropriate gauge transformation of the 4-potential. The time derivatives in $\mathcal{L}_S$ produce an extra term

$$\delta_1\mathcal{L} = -\hbar(\partial_t\alpha)|\psi|^2$$

whereas a gauge transformation of the scalar potential produces an extra term

$$\delta_2\mathcal{L} = \frac{q}{c}(\partial_t\chi)|\psi|^2$$

Thus, choosing

$$\alpha = \frac{q}{\hbar c}\chi$$

leads to $\delta_1\mathcal{L} + \delta_2\mathcal{L} = 0$. In the terms with spatial derivatives we get additional terms

$$[-i\hbar\boldsymbol{\nabla} - (q/c)\boldsymbol{A}']\psi' = e^{i\alpha}[-i\hbar\boldsymbol{\nabla} - (q/c)\boldsymbol{A} + \hbar(\boldsymbol{\nabla}\boldsymbol{\alpha}) - (q/c)(\boldsymbol{\nabla}\chi)]\psi$$

which also compensate each other exactly. So, surprisingly, the complete theory is invariant under combined gauge transformations and space-dependent phase factor multiplications of the wave function.

At this point, an interesting (although at first sight rather bizarre) possibility emerges: Is it possible that electromagnetic fields have to exist, because the Lagrange density of the "matter fields" (the wave function in our example) must be invariant under *local U(1) transformations*? In this perspective the 4-potential appears as a *compensating field*, because it compensates (via its gauge transformations) the changes in the Lagrange density, which appear from local phase transformations. This has turned out to be an extremely fruitful hypothesis, *because it relates interactions to symmetry principles*. The requirement of invariance under local phase transformations forces the existence of gauge fields and fixes the form of the coupling between gauge fields and matter fields.

10.2 Geometric Interpretation of Gauge Fields and the Construction of Non-Abelian Gauge Theories

The idea of a compensating field may seem a bit artificial to you. To convince you that this is indeed a very "natural" construction, we show you that the underlying idea is a geometric one. Thus the construction of interactions is reduced to a geometric problem – just like in Einstein's theory of gravitation!

10.2.1 Compendium: Maxwell's theory

To start with, let us briefly recapitulate the properties of Maxwell's theory with respect to gauge transformations:

- It is invariant under *global gauge transformations*

$$G_{\text{em}} = U(1) := \left\{ e^{i\alpha} | \alpha \in \mathbb{R} \ (\text{mod}\, 2\pi) \right\} \tag{10.2a}$$

  as well as *local gauge transformations*

$$\mathcal{G}_{\text{em}} := \left\{ e^{i\alpha(x)} | \alpha \in C^\infty(\mathbb{M}) \ (\text{mod}\, 2\pi) \right\} \tag{10.2b}$$

  where $\mathbb{M}$ denotes Minkowski space. The group (10.2a) we shall call *structure group*; the infinite dimensional group (10.2b) derived from it is called *gauge group* and fixes the form of the gauge transformation.

  For Maxwell's theory the structure group thus is $U(1)$. This Abelian group has only one *generator* which we may denote as $\mathbf{1}$.

- A gauge transformation is given by $A'_\mu = A_\mu - \partial_\mu \alpha(x)$. With the help of exterior forms this can be written as[2] $A' = A - d\alpha$. For $g \in \mathcal{G}_{\text{em}}$ an obviously equivalent way of writing this is

$$iA' = igAg^{-1} + g(dg^{-1}) \ , \tag{10.3}$$

  where the parentheses mean that the derivative acts only on g^{-1}. Obviously, it is convenient to include the factor i into the definition of A (see also section 9.2.3), hence we will use the replacement $A \to iA$ in the following.

  In section 9.2.3 we have shown, that the electromagnetic fields are the curvature form $F = D^2$ obtained from the covariant derivative $D = d + A$. In the present case we have $D^2 = (dA) + A \wedge A = (dA)$. We

[2] Unless stated otherwise we shall use $\hbar = c = 1$ from now on.

furthermore note in passing that the behavior of D and F under gauge transformations is

$$\begin{aligned} D' &= d + A' = gg^{-1}d + gAg^{-1} + g(dg^{-1}) = gdg^{-1} + gAg^{-1} \\ &= gDg^{-1} \qquad (10.4) \\ F' &= D'^2 = gDg^{-1}gDg^{-1} = gFg^{-1} = F \ , \qquad (10.5) \end{aligned}$$

i.e. a *conjugation* or *equivalence relation* with respect to the gauge group.

- The coupling of the electromagnetic field to a charged particle field can be expressed via the covariant derivative (principle *of minimal coupling*) through the "kinetic energy" as[3] $\partial_\mu \psi^* \partial^\mu \psi \to (D_\mu \psi)^\dagger D^\mu \psi$. Its behavior under gauge transformations is given by

$$\begin{aligned} \left(D'_\mu \psi'\right)^\dagger D'^\mu \psi' &= \left(gD_\mu g^{-1} g\psi\right)^\dagger gD^\mu g^{-1} g\psi \\ &= \left(gD_\mu \psi\right)^\dagger gD^\mu \psi \\ &= \left(D_\mu \psi\right)^\dagger g^\dagger gD^\mu \psi \\ &= \left(D_\mu \psi\right)^\dagger D^\mu \psi \ , \qquad (10.6) \end{aligned}$$

  i.e. it is invariant provided we apply the same gauge transformation $g \in \mathcal{G}_{\rm em}$ to both the electromagnetic field and the particle field.

We already learned in the theory of gravitation, that the structure equation $D^2 = \Omega$ together with field equations (=Einstein's equations) for the potentials (=Christoffel symbols) determine the geometric structure of the space (=space-time) the physical objects (=masses) live in. The deviation from a flat space-time manifold was interpreted as forces acting between the objects living in this space, i.e. gravitational forces between masses.
In the present case we have found a very similar structure. Our physical objects are now charges, which however live in an abstract space spanned by the particle fields ψ. The geometric structure of this space is determined by the structure equation $D^2 = F$ and the field equations (=Maxwell's equations) for the potentials. Finally, the electromagnetic forces acting between the charges can be interpreted as resulting from a deviation from a flat ψ-space.
This similarity between the two completely unrelated phenomena of electromagnetism and gravitation lets the initially rather bizarre idea of giving the potentials and gauge fields a fundamental physical meaning appear in a completely new light. It thus seems to be a quite general principle that

[3] We replace $D_\mu \psi^*$ by $(D_\mu \psi)^\dagger$ for reasons apparent from the further discussions.

fundamental interactions can be understood as geometric properties of a general field-space (describing generalized charges) determined by a suitably chosen gauge group.

10.2.2 Non-Abelian gauge groups

Obviously, the gauge group $\mathcal{G}_{\rm em}$ of electromagnetism is very special in that it is Abelian. There is in fact nothing to prevent us from having a system with a non-Abelian gauge group; in fact, the modern understanding of the fundamental interactions would be impossible without this possibility. However, in contrast to Abelian groups, which only have one-dimensional irreducible representations, non-Abelian groups have in general a series of irreducible representations of different dimensionality, i.e. the physics will critically depend on the actual representation induced by the physical particle fields. Furthermore, for non-Abelian groups the group elements do not commute, and one has to find out what modifications to the previous formulas become necessary. Last but not least, not all groups are possible candidates for gauge groups. It must be possible to define a gauge-invariant, positive-semidefinite kinetic energy of the form $(D_\mu\psi, D^\mu\psi)$, where $(\ldots,\ldots)$ denotes a bilinear form invariant under the structure group.

Under these conditions, the possible structure groups are so-called *compact Lie groups*. A (finite dimensional) Lie group is a smooth manifold G of transformations fulfilling the group axioms and where products $g_1 \circ g_2$ and g^{-1} are at least $\in C^1$ with respect to the group parameters. A well-known example is the $SO(3)$, i.e. the group of all orthogonal 3×3 matrices R over $\mathbb{R}$ with $\det R = 1$. Every element $R \in SO(3)$ can for example be characterized by the Euler angles and the matrices are in fact $\in C^\infty$ with respect to these parameters, which are all taken from a *compact* interval[4].

A further necessary prerequisite is that the structure group must contain the identity (=no gauge transformation), which means that it must be the so-called *identity component*. The identity component contains all group elements, which can be obtained from the identity by smooth variations of the group parameters. For example, the *orthogonal group* $O(3)$ contains also those matrices R for which $\det R = -1$. Obviously, these elements are (in $\mathbb{R}$!!) not smoothly connected to the identity (but they are in $\mathbb{C}$).

[4]The Lorentz group is an example of a non-compact Lie group which is important in physics. The Lorentz transformations depend on the rapidity $\theta \in [0, \infty)$

Quite generally, the elements of the identity component can be represented as

$$g = \exp\left\{ i \sum_{k=1}^{N} \alpha_k \mathcal{T}_k \right\} \tag{10.7}$$

and are obviously smooth functions of the real variables α_k. The *generators* $\mathcal{T}_k$ span the *Lie algebra* $\mathfrak{g} := \mathrm{Lie}(G)$, i.e. they form a basis of $\mathfrak{g}$. The algebra is characterized by the commutators

$$[\mathcal{T}_i, \mathcal{T}_j] = i C_{ij}^k \mathcal{T}_k \ , \quad i,j,k = 1,2,\ldots,\dim \mathfrak{g} \tag{10.8}$$

with *structure constants* $C_{ij}^k \in \mathbb{R}$. By a suitable choice of generators, the structure constants can always be made totally antisymmetric in all three indices.
As you already know from group theory, the theory of representations is of central importance. In the case of Lie groups one is especially interested in a mapping

$$\begin{aligned} \varrho : \mathfrak{g} &\rightarrow \mathbb{C}^n \times \mathbb{C}^n \\ \mathcal{T}_k &\mapsto T(\mathcal{T}_k) \ , \end{aligned}$$

where the complex $n \times n$ matrices $T(\mathcal{T}_k)$ fulfill the same commutation relations as $\mathcal{T}_k$. One can show, that the matrices T can be chosen to be hermitian, $T = T^\dagger$. There are two particularly important representations, the *fundamental or defining* representation and the *adjoint* representation. The former is the representation – different from the trivial representation – with the lowest dimension, the latter is defined through the structure constants themselves, i.e.

$$T_{nm}^{(\mathrm{ad})}(\mathcal{T}_k) = -i C_{kn}^m \ .$$

Let me make these concepts transparent by discussing an important example, the $SU(2) \subset U(2)$. The elements of $SU(2)$ have to fulfill the additional constraint $\det g = 1$. If we write such an element $g \in SU(2)$ – which is a 2×2 matrix – as $g = \exp\{i\mathcal{A}\}$, we can immediately read off $g^\dagger g = 1 \Leftrightarrow \mathcal{A}^\dagger = \mathcal{A}$ and $\det g = 1 \Leftrightarrow \mathrm{Tr}\mathcal{A} = 0$. Now, any traceless 2×2 matrix can be represented as sum over the Pauli matrices, i.e.

$$\mathcal{A} = \sum_{i=1}^{3} a_i \sigma_i \ , \quad a_i \in \mathbb{R}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The Pauli matrices – multiplied with a factor $1/2$ – are thus an obvious choice for the generators[5] of $SU(2)$. The commutators are

$$\left[\frac{\sigma_k}{2}, \frac{\sigma_l}{2}\right] = i\epsilon_{klm}\frac{\sigma_m}{2}$$

i.e. the structure constants are given by[6] $C_{ij}^k = \epsilon_{ijk}$. For the fundamental representation we obtain the spinor representation (2-dimensional) with $T^{(\mathrm{f})}(\mathcal{T}_i) = \sigma_i/2$, and the adjoint representation is given by

$$T^{(\mathrm{ad})}(\mathcal{T}_1) = -i\epsilon_{1lm} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{10.9a}$$

$$T^{(\mathrm{ad})}(\mathcal{T}_2) = -i\epsilon_{2lm} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{10.9b}$$

$$T^{(\mathrm{ad})}(\mathcal{T}_3) = -i\epsilon_{3lm} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10.9c}$$

An important property of simple[7] Lie groups like $SU(2)$ is that one can always choose the representation matrices such that

$$\mathrm{Tr}\left(T(\mathcal{T}_i)T(\mathcal{T}_j)\right) = \kappa\delta_{ij} \quad , \tag{10.10}$$

where $\kappa > 0$ depends on the representation but not on the indices. For example, for $SU(2)$ one finds

$$\mathrm{Tr}\left(T^{(\mathrm{f})}(\mathcal{T}_i)T^{(\mathrm{f})}(\mathcal{T}_j)\right) = \frac{1}{4}\mathrm{Tr}\left(\sigma_i\sigma_j\right) = \frac{1}{2}\delta_{ij}$$

and

$$\mathrm{Tr}\left(T^{(\mathrm{ad})}(\mathcal{T}_i)T^{(\mathrm{ad})}(\mathcal{T}_j)\right) = \ldots = 2\delta_{ij} \tag{10.11}$$

(exercise).

We now have defined the structure group of our system. The next step would be to set up a Lagrange density for the scalar[8] fields $\Phi = (\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(m)})$ of our theory. The fields span a representation of the structure group, too, which may be reducible or irreducible. To assure that the Lagrange density is a scalar with respect to the Lorentz and structure group we have to define

[5] Does that ring a bell in your head?

[6] The bell must be pretty loud by now!

[7] A group G is called simple if it is (i) non-Abelian and (ii) does not contain an invariant subgroup $H \subset G$, i.e. a subgroup with $gHg^{-1} \subseteq H$ for all $g \in G$.

[8] The concept can be easily extended to multicomponent fields.

a scalar product with respect to the structure group. Such a scalar product will in general depend on the group and the representation spanned by the fields. We will denote it as (X, Y) and the terms entering the Lagrange density will be of the form (Φ, Φ) and $(\partial_\mu \Phi, \partial^\mu \Phi)$. For example, for the $L = 1$ triplet representation of $SO(3)$ such a scalar product would be the Clebsch-Gordan coupling to total angular momentum $L = 0$[9] and read

$$(X, Y) = \sum_{i=-1}^{1} (-1)^{1-i} X_i^* Y_{-i} \quad .$$

Finally, we have to construct the gauge group by replacing the parameters α_k in (10.7) by smooth functions $\alpha_k(x)$ of space-time. With this operation we allow gauge transformations to be restricted to a finite space-time volume, i.e. remote fields will remain unaffected. This means that we attach a local copy of the structure group to each point $x \in \mathbb{M}$, which can be viewed as an internal additional space at this point[10]. For two points $x \neq y$ the copies of $G(x)$ and $G(y)$ are disjoint and consequently the representations induced by $\Phi(x)$ and $\Phi(y)$ live in disjoint vector spaces. This means, that for $\Phi(x)$ and $\Phi(x+dx)$ one cannot simply identify a particular component $\Phi^{(i)}(x)$ with $\Phi^{(i)}(x + dx)$. If we ask the question, which transformation will lead from $\Phi^{(i)}(x)$ to $\Phi^{(i)}(x + dx)$, we are again at a point where we have to define what parallel transport means.

10.2.3 Potentials and covariant derivative

By now we have learned how one can construct such a transformation. Let us denote with $N = \dim \mathfrak{g}$ the dimension of our Lie algebra. One can then define the generalized potential

$$A := iq \sum_{k=1}^{N} A^{(k)} \mathcal{T}_k \quad . \tag{10.12}$$

In this definition we included a constant q, which is a generalized "charge", and a factor i for the reasons encountered in (10.3). There are as many $A^{(k)}$ as the Lie group has generators and they are one forms over $\mathbb{M}$

$$A^{(k)} = A^{(k)}_\mu dx^\mu \quad ,$$

with four smooth real fields $A^{(k)}_\mu$. The potential defined in that way has a dual nature. On the one hand it is a one form over Minkowski space, on the other hand it has its values in the Lie algebra.

[9] A quantity with $L = 0$ transforms as a scalar under $SO(3)$.

[10] One calls this construction a *principal fiber bundle* with $\mathbb{M}$ as base manifold and G as *typical fiber*.

Guided by our experience from Maxwell's and Einstein's theories we define the parallel transport between to points x and $x + dx$ as

$$\Phi(x + dx) = \left(\mathbb{1} - T^{(\Phi)}(A)\right) \Phi(x) \ , \tag{10.13}$$

where $T^{(\Phi)}(\ldots)$ denotes the representation induced by the field Φ. If the definition (10.13) correctly describes parallel transport, it must have the property, that a gauge transformation $g(x + dx)$ applied to (10.13) commutes with $\mathbb{1} - T^{(\Phi)}(A)$ in the sense that

$$\begin{aligned} T^{(\Phi)}(g(x + dx))\Phi(x + dx) &= T^{(\Phi)}(g(x + dx)) \left(\mathbb{1} - T^{(\Phi)}(A)\right) \Phi(x) \\ &= \left(\mathbb{1} - T^{(\Phi)}(A')\right) T^{(\Phi)}(g(x))\Phi(x) \ . \end{aligned}$$

In other words

$$g(x + dx)\,(\mathbb{1} - A) = \left(\mathbb{1} - A'\right) g(x) \ ,$$

where we used the fact, that this property must be independent of the actual field or the representation $T^{(\Phi)}$. Taylor expansion of $g(x + dx) \approx g(x) + dg$ and using $0 = d(gg^{-1}) = (dg)g^{-1} + gdg^{-1}$ we arrive at the gauge condition

$$A' = gAg^{-1} + gdg^{-1} \ , \tag{10.14}$$

which has precisely the same form as in Maxwell's theory. Note, however, that here the order of the products is crucial, because the gauge group is non-commutative!

We are now in the position to write down the covariant derivative

$$D_A := d + A$$

for our theory which is tantamount to the replacement

$$\partial_\mu \Phi(x) \to \left\{ \partial_\mu \mathbb{1} + iq \sum_{k=1}^{N} A_\mu^{(k)}(x) T^{(\Phi)}(\mathcal{T}_k) \right\} \Phi(x) \ .$$

The gauge condition (10.14) then again leads to

$$D_{A'} = g D_A g^{-1} \ ,$$

i.e. the transformation behavior of the covariant derivative is much simpler than that of the potentials and given by a conjugation.

10.2.4 Field tensor and curvature

The next step is the construction of the curvature form, which in close analogy to electromagnetism is given by

$$F := D_A^2 = (dA) + A \wedge A \ .$$

In contrast to Maxwell's theory, the fact that G is non-Abelian now leads to

$$\begin{aligned} A \wedge A &= -q^2 \sum_{k,l=1}^{N} \mathcal{T}_k \mathcal{T}_l \sum_{\mu,\nu=0}^{3} A_\mu^{(k)}(x) A_\nu^{(l)}(x) dx^\mu \wedge dx^\nu \\ &= -q^2 \sum_{k,l=1}^{N} [\mathcal{T}_k, \mathcal{T}_l] \sum_{\mu<\nu=0}^{3} A_\mu^{(k)}(x) A_\nu^{(l)}(x) dx^\mu \wedge dx^\nu \\ &= -iq^2 \sum_{k,l,m=1}^{N} C_{kl}^m \mathcal{T}_m \sum_{\mu<\nu=0}^{3} A_\mu^{(k)}(x) A_\nu^{(l)}(x) dx^\mu \wedge dx^\nu \ , \end{aligned} \tag{10.15}$$

where we used the antisymmetry of $dx^\mu \wedge dx^\nu$ when going from the first to the second line. Looking again at Maxwell's theory, we may define

$$F =: iq \sum_{k=1}^{N} \mathcal{T}_k \sum_{\mu<\nu} F_{\mu\nu}^{(k)}(x) dx^\mu \wedge dx^\nu \ , \tag{10.16}$$

with

$$F_{\mu\nu}^{(k)}(x) := \partial_\mu A_\nu^{(k)}(x) - \partial_\nu A_\mu^{(k)}(x) - q \sum_{n,m=1}^{N} C_{nm}^k A_\mu^{(n)}(x) A_\nu^{(m)}(x) \ . \tag{10.17}$$

The $N = \dim \mathfrak{g}$ tensor fields $F_{\mu\nu}^{(k)}$ are the direct generalization of the field tensor in electrodynamics.
Let me finally note that the behavior $D_{A'} = g D_A g^{-1}$ again leads to the transformation behavior

$$F \mapsto F' = g F g^{-1}$$

for the field form.

10.2.5 Gauge invariant Lagrange densities

With the prerequisites from the previous sections we are now in the position to write down Lagrange densities which are Lorentz invariant and invariant under transformations from a given gauge group $\mathcal{G}$. The necessary ingredients are

- A compact Lie group G and corresponding gauge group $\mathcal{G}$,

- a potential A as defined by (10.12),
- a set of fields $\Phi = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x))$ which span a (reducible or irreducible) representation of G. For simplicity we assume scalar fields here, but the generalization to fields with more complex transformation properties under the Lorentz group (e.g. spinors etc.) is straightforward.

Let us start with the part containing the gauge fields, i.e. the curvature tensor $F_{\mu\nu}$. Since this quantity has values in the Lie algebra, we have to find a way to construct a scalar under the group transformations. The answer is given by group theory: For a Lie algebra valued quantity X the functional invariant under all group elements has the form $\mathrm{Tr}\left(T^{(ad)}(X)\right)$. Guided by the Lagrangian of electromagnetism, we thus will have a Lagrange density containing $\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$, where the trace has to be evaluated over the adjoint representation.
If we abbreviate the first term in the field tensor (10.17) as $f^{(k)}_{\mu\nu}(x) := \partial_\mu A^{(k)}_\nu(x) - \partial_\nu A^{(k)}_\mu(x)$, the contraction will contain the term

$$\sum_{k,l=1}^{N} \mathrm{Tr}\left(T^{(ad)}(\mathcal{T}_k)T^{(ad)}(\mathcal{T}_l)\right) f^{(k)}_{\mu\nu}(x) f^{(l)\mu\nu}(x) \quad .$$

According to (10.10) the trace does not depend on k and l, but on the representation via the constant $\kappa^{(ad)}$. Furthermore, this term has the same form as in the Lagrangian of Maxwell's theory (see (8.19)), i.e. the normalization constant is fixed by the same arguments here and given by $-(16\pi q^2\kappa^{(ad)})^{-1}$. With this we can write down the Lagrange density for the gauge fields as

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{16\pi q^2\kappa^{(ad)}}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \quad . \tag{10.18}$$

The index YM stands for "Yang-Mills" in honor of the physicists who first introduced the concept of local gauge theories into physics[11].
Finally, the particle fields will be described by a Lagrange density

$$\mathcal{L}_\Phi = \frac{1}{2}\left[(\partial_\mu\Phi, \partial^\mu\Phi) - m^2\left(\Phi, \Phi\right)\right] - W(\Phi(x))$$

where m is the mass of the particles (assumed to be the same for all particle fields) and $W(\Phi(x))$ an additional self-interaction term, which of course has to be invariant under the structure group. This Lagrange density is globally, but not locally gauge invariant. To obtain a locally gauge invariant theory

[11] C.N. Yang, and R.L. Mills, Phys. Rev. **96**, 191(1954).

we must (i) replace the derivative by the covariant derivative and (ii) add the Yang-Mills part, leading to

$$\mathcal{L} = -\frac{1}{16\pi q^2 \kappa^{(ad)}} \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2}\left[(D_\mu\Phi, D^\mu\Phi) - m^2\left(\Phi,\Phi\right)\right] - W(\Phi(x)) \tag{10.19}$$

10.2.6 Where is the physics?

Up to now the discussion remained, apart from some loans from Maxwell's theory, rather mathematical. The obvious question is what kind of physical treasures are buried in the concepts introduced. To find an answer let us first note that from the Lagrange density (10.19) one can deduce equations with a structure similar to Maxwell's equations in electrodynamics. In particular, one obtains a "radiative" and a "material" part. The latter describes the generation of the G-fields from sources provided by the particle fields.

In quantum mechanics we have learned that the quantization of the electromagnetic field leads to the notion of "massless particles", called photons, which mediate the electromagnetic interaction. The algebra of fields describing these particles shows that they are bosons, and because they emerge from the quantization of the gauge fields one calls them *gauge bosons*. Since photons are massless and do not interact with themselves they mediate a long ranged interaction.

A similar line of action can be taken for non-Abelian gauge theories. Here, too, one can quantize the free fields to obtain a set of N gauge bosons mediating the interaction described by the gauge group G. Again, these particles are necessarily massless, because an explicit mass term in the Lagrange density would have to be of the form

$$\frac{\lambda^{(k)\,2}}{8\pi} A^{(k)}_\mu(x) A^{(k)\mu}(x)$$

and violate the local gauge invariance incurably.

Due to the non-Abelian structure group, however, the Lagrange density (10.18) contains terms which are third and fourth order in the gauge fields. When one sets up the Euler-Lagrange equations, these terms will lead to non-linear terms in the equation of motion, which can (and must) be interpreted as *genuine interactions among the gauge bosons* in the quantized theory. This particular property, viz the self-interaction of the bosons responsible for mediating the interaction, is a well-known feature in particle physics; thus the concept of non-Abelian gauge theories seems to be not completely academical. On the other hand, experiment tells us that most of these gauge bosons have finite masses, which at

first sight seems to contradict the present theoretical approach. A way out of this dilemma was proposed by Higgs[12] and others. Namely, if we add to $\mathcal{L}_{\rm YM}$ true scalar particle fields (so-called *Higgs fields*) which have a potential energy $W(\Phi(x))$ such that at $\Phi_0 \neq 0$ there exists an absolute minimum, the Lagrange density remains gauge invariant, but the gauge bosons can become massive. This phenomenon is similar to the spontaneous generation of a magnetic order in a rotationally invariant solid and also runs under the notion of *spontaneous symmetry breaking*: The physical system has, to lower its energy, chosen a state which seems to have a lower symmetry than the Langrange density. In this approach the Lagrange density still has the full gauge invariance, but it is hidden. One calls this kind of symmetry therefore commonly *hidden symmetry*.
Does that concept appear rather outrageous and artificial to you? Well, in fact you already encountered an example in solid state physics, where precisely this mechanism is at work: Superconductivity. As you know from solid state lectures, the superconducting state is characterized by a *condensation of Cooper pairs*. The existence of particle pairs always means, that the wave function must contain pairs of creation operators, which are not compensated by annihilation operators. Such a combination obviously breaks the global gauge invariance of Schrödinger's equation. Since, on the other hand, electrons are charged, they couple to the electromagnetic field as gauge field. Within Ginzburg-Landau theory (or any mean-field theory), the condensation of the Cooper pairs immediately leads to a generation of a "mass term" for the photons, i.e. *in a superconducting solid the photons become massive*[13]. Experimentally, this manifests itself in a finite penetration depth of electromagnetic fields, known as *London penetration depth* and the expelling of magnetic fields from the interior of a superconductor. Thus, Higgs' mechanism is something well-known and well-understood in solid state physics!
Note that this concept requires the existence of a massive particle with non-vanishing vacuum expectation value. These days, this *Higgs particle* is the most wanted particle searched for in high-energy experiments. Obviously, if it will *not* be found, the standard theory, which is so very successful in describing the properties of elementary particles and their interactions, will be in deep peril.

10.3 The $U(2)$ theory of electroweak interaction

Let us now discuss the theory of the combined electromagnetic and weak interaction as a specific example for all the concepts introduced before. The

[12] P.W. Higgs, Phys. Lett. **12**, 132 (1964); Phys. Rev. **145**, 1156 (1966).
[13] P.W. Anderson, Phys. Rev. **130**, 439 (1963)

weak interaction is for example responsible for the β- and μ-decay and its short range of $10^{-16} \ldots 10^{-15}$cm requires that the gauge boson(s) mediating this interaction have to be massive. On the other hand, the photon as the gauge boson of the electromagnetic interaction is always massless. This poses the first challenge. If we furthermore intend to combine both types of interactions into one unified theory, we will have to use a structure group with at least two generators and a possible $U(1)$ subgroup. The simplest structure group which fulfills this latter requirement is the non-Abelian group[14] $U(2) = U(1) \times SU(2)$. Historically, the necessity to describe the weak interaction by the non-Abelian group $U(2)$ was suggested by experiments, which could – at least for low energies – be explained by the existence of charged and neutral currents. From the observed parity violation in processes involving the weak interaction one could also conclude that the neutral current must contain a certain contribution from electromagnetic currents. The amount can be quantified by a factor $\sin\Theta_W$, where Θ_W is called *Weinberg angle* and has a value $\sin^2\Theta_W \approx 0.228$. The charged current and the non-electromagnetic part of the neutral current can be understood as a representation of the $SU(2)$ algebra, hence Sheldon Glashow suggested in 1961 the description of the electroweak interaction as a $U(1) \times SU(2)$ non-Abelian gauge theory. As already mentioned, the problem in this theory was that due to the short-ranged nature of the weak interaction some of the gauge bosons had to be massive, while at that time a gauge theory produced only massless gauge fields. After Higgs' proposal, Steven Weinberg and Abdus Salam put forward the present standard model for the electroweak interaction in 1967, known as Glashow-Salam-Weinberg (GSW) model. To be precise, the GSW model describes the electroweak interaction among the six fermionic leptons, i.e. the particle fields should contain, in addition to the Higgs field, also a corresponding set of bispinor fields. Here, I however do not want to discuss the full theory but just how the concepts introduced in the previous sections lead to the experimentally observed properties of the electroweak "fields". Thus, in the following I will not consider particle fields apart from the Higgs field.

The group $U(2)$ has four generators, one for the $U(1)$ and three for the $SU(2)$ factor. We thus have actually got more than we bargained for, namely four gauge bosons instead of the two originally wanted. This is in fact *a theoretical prediction* we just arrived at: The unified description of electroweak interactions *requires* the existence of four gauge bosons, one of which is the massless photon. This theoretical prediction has been experimentally verified long after its proposal and is the first overwhelming success of the concept of local gauge

[14] Something like $U(1) \times U(1) \cong U(1)$ would not do the job, because it has only one generator!

theories.
The general structure of the elements of the group $U(2)$ is

$$U(2) \ni g = e^{i\alpha} \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \ , \ \text{with } \alpha \in \mathbb{R},\, u, v \in \mathbb{C} \text{ and } |u|^2 + |v|^2 = 1 \ .$$

The generators of the corresponding Lie algebra are

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 \ , \tag{10.20a}$$

$$\mathcal{T}_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1 \ , \tag{10.20b}$$

$$\mathcal{T}_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_2 \ , \tag{10.20c}$$

$$\mathcal{T}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_3 \tag{10.20d}$$

with commutators

$$[\mathcal{T}_0, \mathcal{T}_i] = 0 \ , \tag{10.20e}$$

$$[\mathcal{T}_k, \mathcal{T}_l] = i\epsilon_{klm}\mathcal{T}_m \ . \tag{10.20f}$$

Note that $U(2)$ is not simple, because it contains $U(1)$ as invariant subgroup.
The gauge group, the potentials and the covariant derivative are thus determined. From (10.11) we also know the factor $\kappa^{(ad)} = 2$ entering (10.18) if the structure group were $SU(2)$ only. Since the factor group $U(1)$ of $U(2)$ is Abelian, the normalization of its adjoint representation $T^{(ad)}(\mathcal{T}_0)$ can be chosen freely. In order to have a common normalization in (10.18) the proper choice is $\sqrt{\kappa^{(ad)}} = \sqrt{2}$ of the adjoint representation of $SU(2)$.
As last step we need to specify our scalar fields entering the theory. Here we again choose the simplest possible realization which does not induce the trivial representation of $SU(2)$, namely a two-component field $\Phi = (\Phi^{(-)}, \Phi^{(+)})$. The reason why we do not choose only one component will become clear in the course of the discussion. The representation induced by Φ is the fundamental representation of $SU(2)$, i.e. the generators are given by $T^{(\Phi)}(\mathcal{T}_i) = \sigma_i/2$, $i = 1, 2, 3$. Finally, the Lagrange density reads

$$\mathcal{L} = -\frac{1}{32\pi q^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2}\left[(D_\mu\Phi, D^\mu\Phi) - m^2\left(\Phi, \Phi\right)\right] - W(\Phi(x)) \tag{10.21}$$

with the field tensor

$$F_{\mu\nu} = iq \sum_{k=0}^{3} T^{(ad)}(\mathcal{T}_k) F^{(k)}_{\mu\nu}(x) \tag{10.22a}$$

and the covariant derivative

$$D_\mu = \mathbb{1}\left(\partial_\mu + iqt_0 A^{(0)}_\mu(x)\right) + iq \sum_{k=1}^{3} \frac{\sigma_k}{2} A^{(k)}_\mu(x) \quad , \tag{10.22b}$$

where I defined $T^{(\Phi)}(\mathcal{T}_0) = t_0\sigma_0 = t_0\mathbb{1}$. The reason to include a factor t_0 here will become clear later.
Presently, there are three unknown parameters entering our theory: The coupling constant q, the mass m and the normalization t_0 . In addition, the potential energy $W(\Phi(x))$ of the scalar fields Φ has not been specified yet.
Without the scalar fields our theory describes four massless gauge bosons. It will turn out that it makes sense to replace two of them by new linear combinations

$$W^{(\pm)}_\mu := \frac{1}{\sqrt{2}}\left(A^{(1)}_\mu(x) \pm iA^{(2)}_\mu(x)\right) \quad . \tag{10.23}$$

For the generators this means that we have to introduce the combinations

$$\mathcal{T}_+ := \mathcal{T}_1 + i\mathcal{T}_2 \ , \quad \mathcal{T}_- := \mathcal{T}_1 - i\mathcal{T}_2$$

which leads to the explicit expressions

$$\sigma_+ := \frac{1}{2}\left(\sigma_1 + i\sigma_2\right) \ , \quad \sigma_- := \frac{1}{2}\left(\sigma_1 - i\sigma_2\right) \quad . \tag{10.24}$$

Finally, the parts containing $\mathcal{T}_1$ and $\mathcal{T}_2$ in the covariant derivative have to be replaced by

$$\frac{\sigma_1}{2}A^{(1)}_\mu(x) + \frac{\sigma_2}{2}A^{(2)}_\mu(x) = \frac{1}{\sqrt{2}}\left[\sigma_- W^{(+)}_\mu(x) + \sigma_+ W^{(-)}_\mu(x)\right] \quad . \tag{10.25}$$

From the point of view of quantum mechanics, the fields $W^{(\pm)}_\mu(x)$ describe hermitian conjugate partners, i.e. gauge bosons with opposite charges. Evidently, these two fields cannot be used to describe the anticipated photon, which must have a *genuine* $U(1)_{\rm em}$ structure group and stay massless (and of course should not have a charge!). This subgroup $U(1)_{\rm em}$ can, but by no means must be identical to the $U(1)$ subgroup of $U(2)$. Consequently, we will be cautious and form the linear combinations

$$\gamma_\mu(x) := A^{(0)}_\mu(x)\cos\Theta_W - A^{(3)}_\mu(x)\sin\Theta_W \tag{10.26a}$$

$$Z_\mu(x) := A^{(0)}_\mu(x)\cos\Theta_W + A^{(3)}_\mu(x)\sin\Theta_W \tag{10.26b}$$

from the remaining two gauge fields, which belong to the generators $\mathcal{T}_0$ and $\mathcal{T}_3$, respectively, and are the only possible candidates for the production of electromagnetic fields. We have thus introduced a further parameter Θ_W into our theory, called *Weinberg angle* after Steven Weinberg, who first introduced it into the theory of electroweak interactions.

The theory we have just constructed is precisely what is called the standard model of electroweak interaction. The gauge fields $\gamma_\mu(x)$, $Z_\mu(x)$ and $W_\mu^{(\pm)}(x)$ represent the four bosons mediating the electromagnetic and weak interaction, respectively. However, from experiment we know that only one of these fields, namely the photon $\gamma_\mu(x)$, remains massless, the other three have masses. Let us therefore now see what the Higgs mechanism of spontaneous symmetry breaking can do for us to generate masses for three of the fields without explicitly breaking the gauge invariance of the Lagrange density (10.21). To this end we have to specify a form for the potential energy $W(\Phi(x))$. As already mentioned, we need a form that has an absolute minimum for a field configuration $\Phi_0 \neq 0$. The simplest potential showing this feature is given by

$$W(\Phi) = -\frac{\mu^2}{2}(\Phi,\Phi) + \frac{\lambda}{4}(\Phi,\Phi)^2 \tag{10.27}$$

with $\lambda > 0$. Note that this potential has precisely the same form as the one encountered in the Ginzburg-Landau theory of second order phase transitions. It has an absolute minimum at the value

$$(\Phi_0,\Phi_0) = \mu^2/\lambda =: v^2 \tag{10.28}$$

(see Fig. 10.1). As is apparent from Fig. 10.1, it is only the value of the scalar product (Φ_0,Φ_0) which is fixed, but not the field Φ_0. We thus still have the choice to fix a certain realization $\Phi_0^{(0)}$. Moreover, the latter is clearly not necessarily invariant under the full structure group. I will come back to this point later.

We can replace v^2 in (10.27) to obtain

$$W(\Phi) = \frac{\lambda}{4}\left((\Phi,\Phi) - v^2\right) - \frac{\lambda}{4}v^2 \quad .$$

For our gauge theory the important thing now is, that due to the appearance of an absolute minimum at $\Phi_0 \neq 0$ the actual dynamical field for the theory is not Φ, but rather $\theta(x) := \Phi(x) - \Phi_0$. This is similar to the situation in the case of a shifted harmonic oscillator, for which the proper dynamical variable is $z = x - x_0$ rather than the coordinate x.

If we insert $\Phi(x) = \theta(x) + \Phi_0$ into the Lagrange density (10.21), the term containing the covariant derivatives will lead among others to contributions of the

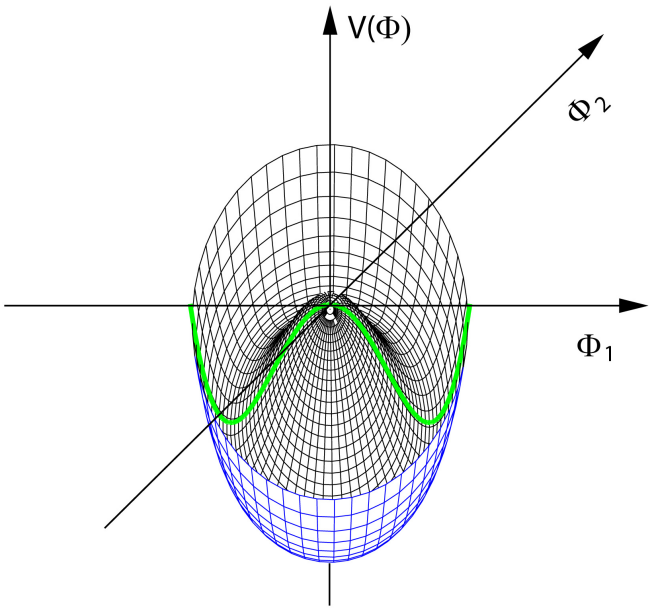

Figure 10.1: Sketch of $W(\phi)$ given by (10.27).

form

$$\frac{1}{2}\left(T^{(\Phi)}(A_\mu)\Phi_0, T^{(\Phi)}(A^\mu)\Phi_0\right) \tag{10.29}$$

which have the structure of a mass term for the gauge fields $A_\mu(x)$. However, gauge invariance of the Lagrange density is not violated at all, only the state of the system realized has a symmetry which is lower than the actual symmetry of the system, just like in solids with magnetic or other kinds of order.

As already mentioned, a particular value $\Phi_0^{(0)}$ for the actual field configuration is not necessarily invariant under operations of the structure group. More precisely, there will in general exist at least one element $g \in U(2)$ such that $g\Phi_0^{(0)} = \Phi_0^{(1)} \neq \Phi_0^{(0)}$. Since the elements g can be represented as[15]

$$g = \exp\left(i\sum_{k=1}^{N}\alpha_k \mathcal{T}_k\right)$$

the previous observation can be rephrased to: There exists at least one generator $\mathcal{T}_i$ such that

$$T^{(\Phi)}(\mathcal{T}_i)\Phi_0^{(0)} \neq 0 \ .$$

Let us now build new linear combinations

$$\mathcal{S}_i := \sum_{j=1}^{N} a_{ij}\mathcal{T}_j$$

[15]We will consider a general structure group here, to make clear that this discussion holds beyond the scope of the present theory.

of the generators, where the a_{ij} form a non-singular constant matrix such that for $\mathcal{S}_i$ for $i = 1, \ldots, F$ we have

$$T^{(\Phi)}(\mathcal{S}_i)\Phi_0^{(0)} = 0 \ , \tag{10.30}$$

while for the remaining $\mathcal{S}_i$ for $i = F+1, \ldots, N$

$$T^{(\Phi)}(\mathcal{S}_i)\Phi_0^{(0)} \neq 0$$

holds. Convince yourself that the elements obtained from (10.30) form a subgroup $H \subset G$ of G which consists of elements

$$H \ni h = \exp\left(i\sum_{k=1}^{F}\alpha_k \mathcal{S}_k\right)$$

with the property

$$h\Phi_0^{(0)} = \Phi_0^{(0)} \ .$$

This subgroup does not "move" the field configuration thus constitutes a genuine symmetry of the state the system has chosen. As a consequence, the gauge fields corresponding to H remain massless, which can also be seen from the mass term (10.29), where the contributions corresponding to the generators of H vanish due to the property (10.30). The subgroup H is also called *residual symmetry group* of the gauge theory.
The remaining generators $\mathcal{S}_i$ for $i = F+1, \ldots, N$ have the effect of "moving" around the state of the system in the minima valley. Obviously, their contributions to the mass term (10.29) stay finite, i.e. they lead to a finite mass for the corresponding gauge bosons. An important lemma now states that the dimension of the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ depends only on the structure group G and the residual group H, but not on the actual representation (i.e. dimension) of the multiplet Φ.
Let us now return to our $U(2)$ gauge theory. We will use the results of the previous discussion in the following way: We know that exactly one gauge field has to remain massless, i.e. we try to find a configuration $\Phi_0^{(0)}$ such that the symmetry will be broken spontaneously according to

$$U(2) \cong U(1) \times SU(2) \to H = U(1)_{\mathrm{em}} \ .$$

The action of the potentials on a general field configuration Φ_0 is given by

$$\begin{aligned} T^{(\Phi)}(A_\mu)\Phi_0 = iq\Big\{ & \frac{1}{\sqrt{2}}\left[W_\mu^{(-)}(x)\sigma_+ + W_\mu^{(+)}(x)\sigma_-\right] \\ & + Z_\mu(x)\left[\frac{\sigma_3}{2}\cos\Theta_W + t_0\sigma_0\sin\Theta_W\right] \\ & + \gamma_\mu(x)\left[-\frac{\sigma_3}{2}\sin\Theta_W + t_0\sigma_0\cos\Theta_W\right]\Big\}\Phi_0 \ . \end{aligned} \tag{10.31}$$

Inserting this expression into the mass term (10.29), we see that we have to make sure that e.g. the last term in (10.31) vanishes identically so that this field remains massless. This can be achieved by choosing the field configuration as $\Phi_0^{(0)} = (v, 0)$ (see Eq. (10.28) for the definition of v) and setting $2t_0 = \tan\Theta_W$. Then we obtain for the mass term

$$\begin{aligned}&\left(T^{(\Phi)}(A_\mu)\Phi_0^{(0)}, T^{(\Phi)}(A^\mu)\Phi_0^{(0)}\right)\\ &\quad= q^2\Bigg\{\frac{1}{2}\left(\Phi_0^{(0)}, \left[\sigma_+\sigma_i + \sigma_-\sigma_+\right]\Phi_0^{(0)}\right) W_\mu^{(-)}(x)W^{(+)\mu}(x)\\ &\quad+\left(\Phi_0^{(0)}, \left[\frac{\sigma_3}{2}\cos\Theta_W + \frac{1}{2}\sin\Theta_W\tan\Theta_W\right]^2\Phi_0^{(0)}\right) Z_\mu(x)Z^\mu(x)\Bigg\}\end{aligned} \tag{10.32}$$

where we used the fact that with respect to the scalar product $\sigma_+^\dagger = \sigma_-$ and $\sigma_3^\dagger = \sigma_3$ and terms containing $\sigma_\pm^2\Phi_0^{(0)} = 0$. Application of the matrices to $\Phi_0^{(0)}$ together with $(\Phi_0^{(0)}, \Phi_0^{(0)}) = v^2$ finally results in

$$\begin{aligned}\left(T^{(\Phi)}(A_\mu)\Phi_0^{(0)}, T^{(\Phi)}(A^\mu)\Phi_0^{(0)}\right) = q^2v^2\Big[\ &\frac{1}{2}W_\mu^{(-)}(x)W^{(+)\mu}(x)\\ &+\frac{1}{4\cos^2\Theta_W}Z_\mu(x)Z^\mu(x)\Big]\ .\end{aligned} \tag{10.33}$$

From the relation (10.33) we can now readily read off the masses of the W and Z gauge bosons as[16] $2m_W^2 \propto q^2v^2/2$ and $m_Z^2 \propto q^2v^2/(4\cos^2\Theta_W)$. Note that the two fields $W^{(\pm)}$ are components of a doublet with respect to $SU(2)$, i.e. they *must* have the same mass. The proportionality factors in the equations for the masses are independent of the particular particle, and one can thus derive the important relation

$$\frac{m_W^2}{m_Z^2\cos^2\Theta_W} = 1\ , \tag{10.34}$$

independent of q and v, which can and has been verified experimentally.
Let me review the major findings of our theory: Starting from the Lagrange density (10.21), we found linear combinations of the gauge fields and a particular configuration of the scalar field (Higgs field) that lead to a system which shows a spontaneously broken symmetry $U(2) \to U(1)_{\rm em}$. The residual symmetry $U(1)_{\rm em}$ leads to a massless gauge boson which we can identify with the photon. The generator of its Lie algebra is $\mathcal{T}_{\rm em} = \mathcal{T}_0\cos\Theta_W - \mathcal{T}_3\sin\Theta_W$. Its counterpart (in the sense of group theory), the electrically neutral Z boson, acquires a mass as do the electrically charged doublet partners $W^{(\pm)}$.

[16] The prefactor of the W term in (10.33) has to be distributed among the two partners $W^{(\pm)}$, thus the factor 2.

One can show that the coupling to the photon (the electric charge) is given by $q \sin \Theta_W \equiv e$. The theory predicts that the masses of the massive gauge bosons are related by (10.34). It should also have become clear by now why it is crucial that the scalar field Φ has to be a multi-component field. If we had chosen a single-component field, the requirement of a residual symmetry group $U(1)_{\mathrm{em}}$ could not have been fulfilled. We also see now that the choice of $SU(2)$ as structure group would not have been sufficient: There is no way to construct a Higgs field that would leave the Z boson massless, thus employing it as photon in the game, while giving a mass to the W's, because the three gauge bosons would constitute a triplet under the $SU(2)$ and must consequently all have the same mass by symmetry.
A somewhat more elegant formulation can be achieved if one introduces a different generator

$$\mathcal{Y} := -2 \cot \Theta_W \mathcal{T}_0$$

of the $U(1)$ factor of the $U(2)$. The generator of the residual symmetry group $U(1)_{\mathrm{em}}$ then becomes

$$\mathcal{Q} := \mathcal{T}_3 + \frac{1}{2}\mathcal{Y}$$

and can be directly related to the electrical charge e, while the generator $\mathcal{Y}$ is called *weak hypercharge*.

10.4 Epilogue

This chapter should have given you an impression of the power of the concept of gauge theories to describe fundamental interactions in nature in terms of geometrical properties of suitably chosen field spaces. As a particular example, I presented the GSW model of the electroweak interaction among elementary particles.
You may have wondered where in our theory the actual physical particles (electrons, muons, ...) do remain. Obviously, they cannot be given by the scalar field, because they are fermions and must be described by Dirac fields. The scalar field (Higgs field) entering our theory had to be included to make some of the gauge bosons massive, but the particle corresponding to it has not yet been observed. Adding the known particles into the Lagrange density (10.21) is straightforward from a conceptual point of view – we already know the Lagrange density for Dirac fields from *Quantum Mechanics II* and the coupling to the fields is included through the covariant derivative. However, calculation of observable quantities like scattering cross sections requires the use of *quantum field theory*, which goes well beyond the scope of this lecture.

Let me just mention that the theory is indeed able to quantitatively reproduce results obtained in collider experiments.
A similar concept can be employed to study the strong interaction among the quarks as fundamental constituents of baryonic elementary particles. Here, the proper structure group turns out to be $SU(3)$, a non-Abelian group with eight generators, leading to (massive) gauge bosons called *gluons* mediating an interaction through a "charge" called *color*. Since these gluons carry a color-charge themselves, the theory – in particular its quantized version, the *quantum chromodynamics* – becomes very cumbersome and leads to effects known as *confinement* and *asymptotic freedom*.
Finally, the combination of electroweak and strong interactions is embedded in the *grand unified theory* (GUT), which has the $U(5) = U(1) \times SU(2) \times SU(3)$ as structure group.
Interested in more? **Then go ahead and start reading books ...** [13].