Göttingen and Quantum Mechanics

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1 Introduction

In this talk the central role played by Göttingen in the formulation of quantum mechanics is elucidated. It starts with a short historical account of the earlier important steps towards achieving this goal in the first twenty years of the twentieth century [1].
The first step in the formulation of quantum theory took place in the year 1900 with Max Planck's explanation of the spectrum of black-body radiation by treating the walls as a system of harmonic oscillators. He made the assumption that the energy quanta emitted and absorbed are integer multiples of $h\nu$, where ν is the frequency of the oscillator and h is Planck's constant, which he had introduced a year earlier using thermodynamic arguments.
In 1905 Albert Einstein postulated the existence of light quanta (only much later called photons). This allowed him to give a theoretical explanation o f the photoelectric effect.
Rutherford's scattering experiments of alpha particles from gold foils (1911) suggested a planetary model of atoms, with a heavy positively charged nucleus surrounded by electrons. As the electrons perform accelerated motion they radiate according to Maxwell's equations and therefore these atoms cannot be stable within a description by classical physics. This is in contrast to the the plum pudding model proposed J.J. Thomson in 1904. In this model the electrons are assumed to float freely in continuous positive charge background and stationary configurations exist in which no radiation is emitted [2]. This model was falsified by Rutherford's experiments. Despite this fact a similar "jellium model", was introduced much later in solid state physics to mimic properties of simple metals.
In order to understand the stability of atoms and to present a theoretical description for the spectral lines emitted by hydrogen at high temperatures Nils Bohr (1885-1962) *postulated* in 1913 that the electron does *not* radiate on *stationary orbits* for which the angular momentum takes values given by *integer multiples* of $\hbar \equiv h/2\pi$. This assumption leads to discrete energy values $E_n \equiv W(n) = -E_R/n^2$ whith integer n and the Rydberg energy $E_R = m_e e^4/2\hbar^2 \approx 13.6057$ eV. Further postulating that in the transition of the electron from orbit with quantum number m to n a light quantum of frequency $\nu_{mn} = (E_m - E_n)/h$ is emitted, Bohr was able to explain experimental hydrogen spectra like the Balmer series [3]. As in Bohr's later work the *correspondence principle* played an important role: Quantum theory should lead to the same results as classical physics when the *quantum*

numbers describing the systems are *large*. Despite the "ad hoc" character of his "rules" his work attracted enormous attention.
Bohr's approach was generalized by Arnold Sommerfeld (1868-1951) with the quantization of the classical action for multiple periodic systems in 1915 and in 1919 he presented the energy of a hydrogen-like atom with one electron and nuclear charge Ze within the framework of *relativistic* mechanics. In his formula the (small) *fine structure constant* $\alpha = e^2/\hbar c \approx 1/137$ provides a measure for the importance of relativistic effects. The status of the "Bohr-Sommerfeld theory" was presented by Sommerfeld in the same year in the first edition of his book *Atombau und Spektrallinien* (english translation: Atomic structure and Spectral Lines)[4]. The experimental spectral lines of the helium atom could not be properly described by the Bohr-Sommerfeld theory. In fact already the simplest molecule H_2^+ with a single electron presented a serious problem. It was treated by Wolfgang Pauli in his PhD thesis completed in July 1921 with Arnold Sommerfeld in Munich. Another student with Sommerfeld at that time was Werner Heisenberg who played an important role later in Göttingen.

2 Göttingen 1921-1924

The Institute of Theoretical Physics in Göttingen was founded in 1921 after Peter Debye who was hosted in an experimental institute had left Göttingen 1920 to become a professor at the ETH Zürich. The chair in the new Institute was filled by Max Born (1882-1970). James Franck (1882-1964) also came from Frankfurt on a new chair of experimental physics.

Figure 1: Max Born in the 1920th

Max Born, born in Breslau, had studied mathematics in Heidelberg, Zürich and Göttingen. There he soon came in close contact to Felix Klein, David Hilbert and Herrmann Minkowski. For his PhD work (1905) on aspects of the stability of elastic media he received the prize of the Göttingen faculty. After spending time in England and Breslau Born returned to Göttingen in 1908 and collaborated with Minkowski (1864-1909) on Einstein's special theory of relativity. Unfortunately Minkowski died of appendicitis shortly later. Together with Theodore von Karman Born a few years later worked on the dynamical theory of crystal lattices, in order to calculate e.g. (independently of Peter Debye at about the same time) the specific heat (1912). This work led to Born's book "Dynamik der Kristallgitter (The Dynamics of Crystal lattices)" (1915). From 1915 to 1919 Born was associate professor in Berlin, where he came in contact with Max Planck and Albert Einstein. With the latter he had a friendly relationship until Einstein's death in 1955. From 1919-21 Born was full professor in Frankfurt (Main). His first two assistants in Göttingen were Wolfgang Pauli (1900-1958) and Friedrich Hund (1896-1997).
Wolfgang Pauli came to Göttingen in October 1921 and stayed until April 1922. His position was financed by Henry Goldman (Goldman and Sachs). In March 1922 he published his thesis work about the H_2^+ molecule mentioned earlier which he never cited himself. As Pauli did not like the life in a small provincial town as Göttingen he left for Hamburg to become assistant to Wilhelm Lenz (see also the comment at the end of reference [4]), known for the "(Laplace-Runge)-Lenz vector" which Lenz used in 1924 to describe the hydrogen atom in crossed external fields in the framework of "old quantum theory" [5].

Figure 2: Wolfgang Pauli in the 1920th

Friedrich Hund had studied mathematics, physics and geography and wanted to become a high school teacher. But he liked the scientific work and after getting his PhD in 1922 he became Born's assistant and participated with him in the attempts to generalize Hamilton mechanics to an "atom mechanics". He also intensely studied experimental atomic spectra and found the now famous "Hund's rules" in 1925.

Figure 3: Friedrich Hund in the 1920th

In June 1922 Nils Bohr gave the Wolfskehl lectures in Göttingen, later called the "Bohr-Festival" because it had started two weeks before the yearly Göttingen Händel-Opera-Festival. In seven lectures he presented the state of the art of the Bohr-Sommerfeld theory aiming at an understanding of atoms. In the later lectures he addressed in detail the construction of a theory of the periodic system of elements [6]. One should mention that Bohr emphasized "how incomplete and uncertain everything still is". Sommerfeld had come from Munich with his student Werner Heisenberg (1901-1976) and Pauli from Hamburg. Another young, mathematically very gifted, student who participated and played an important role later was Pascual Jordan (1902-1980). Also famous Göttingen mathematicians like David Hilbert, Felix Klein, Carl Runge and Richard Courant where present. Among the many other prominent physicists from outside Göttingen were Paul Ehrenfest and Oskar Klein. After Bohr's third lecture Werner Heisenberg asked a question which prompted a long discussion with him even after the lecture. At the end of the lectures David Hilbert thanked Bohr that he had allowed insight into the holy grail of his scientific personality [7].
Later in 1922 Bohr received the Nobel Prize in Physics "for his investigations of the structure of atoms and of the radiation emanating from them" [8]. Heisenberg spent the winter semester 1922/23 in Göttingen as Sommerfeld was in the US

during this time.

Figure 4: Werner Heisenberg in the 1920th

All physicist working hard to improve the understanding of the structure of atoms, including Bohr, were aware that many open questions remained. In Göttingen Werner Heisenberg joined the efforts to improve this state of affairs in 1923 after receiving his PhD in Munich with Sommerfeld with work about turbulence. In 1924 Pascual Jordan had finished his PhD work under Born's guidance on the quantum theory of radiation. One can summarize the situation by the end of 1924 by the statement that no real progress was visible.
A situation like this was later described in Thomas Kuhn's book "The structure of scientific revolutions" (1962), a very highly discussed (and also criticised) contribution to the history of science [9]. Kuhn distinguishes periods of "normal science", in which accepted models and theories are applied successfully, from periods of increasing "crisis", in which it becomes obvious that only a "revolution" changing basic assumptions or "paradigms", can bring a solution by a "paradigm shift".
In Kuhn's spirit Max Born's book "Vorlesungen über Atommechanik" is a presentation of the crisis [10]. In the preface of the book written in November 1924 Born explains that he wants to describe the limits of the present state of atomic theory. The book uses the Hamilton-Jacobi approach to mechanics and devotes a large part to atoms with a single valence electron. This is no surprise as for such systems Bohr-Sommerfeld theory is usually successful. But even in this area problems showed up. Born mentions that already the hydrogen atom in crossed magnetic and electric fields poses a problem for the theory. And for the simplest

two center molecule, the H_2^+, the calculated ionization potential deviates considerably from the measured value. Towards the end of the book Born addresses the fact that quantizing the action integrals in multiples of Planck's constant fails in the treatment of the two electrons in the helium atom. As another failure of the usual Bohr-Sommerfeld theory Born mentions the anomalous Zeeman-effect. In order to show the preliminary character of this work he gives the book the subtitle "Volume 1". A later "Volume 2" should then be devoted to the "final atom mechanics". He calls this a daring attempt as rather little is known in this respect and it might take some years until Volume 2 will be written. Born acknowledges that many parts of Volume 1 were written by Friedrich Hund and only slightly revised by himself and the last chapter on the helium atom was outlined by Werner Heisenberg.
Born was right with the "some years" estimate for the appearance of Volume 2 (1930), but wrong about the time to the breakthrough. The paradigm shift to quantum mechanics, the most successful theory of physics, took place already within the next two years. It allows to calculate the properties of matter to an extremely high accuracy.
In the preface of the book Born points out the fact that for the radioactive decay only a probabilistic description is possible, a concept which should be used also for atomic transitions. This was first proposed by A. Einstein in his paper "On the quantum theory of radiation" [11], where he introduced probabilities per unit time for the transition between two atomic stationary states.

3 1925: Matrix Mechanics and beyond

Heisenberg spent the winter 1924/25 in Copenhagen working with Nils Bohr and Hendrik Kramers. In the letter in which he thanked Bohr for his hospitality Heisenberg wrote: "With respect to science the past half year was for me the most beautiful of my entire life as a student."

3.1 Göttingen

After a vacation Heisenberg was back in Göttingen end of April. End of May because of strong attacs of hay fever he retreated to the treeless island Helgoland in the North Sea for two weeks. He had started his attempt to formulate a completely new quantum mechanics. Instead of the hydrogen atom he addressed the anharmonic oscillator. On Helgoland he made progress with his new approach, especially concerning energy conservation he had initially problems with. Back in Göttingen he started to write his paper "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehung (Quantum theoretical re-interpretation of kinematic and mechanical relations)"[12].

Between June 21 and July 9 Heisenberg sent four letters to Pauli which clearly show the ups and downs of Heisenberg feelings about his achievements [13]. Unfortunately Pauli's response is not in the collection of his scientific correspondence.

June 21:

My attempts to formulate a quantum mechanics progress rather slowly...

June 24:

I don't want to write about my work because most of it is still unclear to me...

June 29:

I made some progress and in my heart I am now convinced that this quantum mechanics is quite correct...

And finally on July 9 Heisenbergs sends his manuscript to Pauli asking for critical remarks:

...I feel guilty to ask you to send back my draft within 2-3 days, as I want to finish or burn it before leaving. My opinion about my scribbling, about which I am not very happy is as follows: I am convinced about the negative critical part, but the positive one I judge as rather formal and meager, maybe more gifted people are able to make something reasonable out of it.

Now I request harsh critisism on your side and a speedy return of my paper.

As Pauli apparently had no objections Heisenberg gave the paper to Born in mid July and asked him to submit it to “Zeitschrift für Physik” in case its content would make sense to him and left for Munich and hiking tours in the Alps. A few days later Born read the paper and as he was impressed (see below) submitted it to *Zeitschrift für Physik*.

Heisenberg begins his paper by stating that he wants to lay the foundations of a quantum theoretical mechanics in which only relations between observable quantities occur. A similar statement can also be found in an earlier paper by Born and Jordan on the quantum theory of aperiodic processes, received by Zeitschrift für Physik June 11, 1925 [14]. As unobservable Heisenberg considers e.g. the position and the period of the electron motion. Considering the radiation emitted by an atom he points out the importance of the associated Einstein-Bohr transition frequencies

$$\nu(n, n-\alpha) = \frac{1}{h}[W(n) - W(n-\alpha)]$$

with integer α and the $W(m)$ are the energies of Bohr's stationary orbits. Using Newton's equations of motion Heisenberg treated the one-dimensional anharmonic oscillator where the electron undergoes a periodic motion labeled by quantum number n. The classical coordinate $x(n,t)$ can then be described by a Fourier series. In order to describe the radiative transitions Heisenberg proposed to replace the Fourier coefficients by quantities $X(n, n-\alpha)$ depending on the *two* quantum numbers n and $n-\alpha$, like the transition frequencies $\nu(n, n-\alpha)$. Also focussing the description on pairs of states and their transition amplitudes was already done in the paper of Born and Jordan mentioned above. The really bold step in Heisenberg's paper concerned the question: which quantum object $Y(t)$ corresponds the classical quantity $x(t)^2$? Arguing with the Ritz combination principle Heisenberg comes up with his multiplication rule for transition amplitudes

$$Y(n, n-\beta) = \sum_{\alpha} X(n, n-\alpha) X(n-\alpha, n-\beta) \ .$$

which he considers the "simplest and most natural assumption". A few steps later he points out that while classically $x(t)y(t)$ always equals $y(t)x(t)$ this is not necessarily the case in quantum theory. In order to find a quantitative result for the corresponding difference of position and momentum Heisenberg used the "old quantum theory" (Thomas-Reiche-Kuhn sum rule).
With the realization that quantum theory has to deal with possibly *noncommuting mathematical objects* quantum mechanics was born. But it took some time until this was formulated in the form students learn it today. No further details of Heisenbergs paper are discussed here. Comments on the readability of the paper are given later.

It took Born about a week to realize that Heisenberg's multiplication rule was nothing but the multiplication rule for *matrices* he had learned as a student in Breslau. The mathematical concept of matrices was not known to Heisenberg when he wrote his paper. In matrix language Heisenberg had only worked with the diagonal element of the commutator of position q and momentum p. Born "easily guessed" the off-diagonal elements and was the first to obtain the basic commutation relation [15]

$$pq - qp = \frac{h}{2\pi i} \ .$$

This is the form written on Born's tomb stone in Göttingen.

As Heisenberg was not in Göttingen, Born together with Pascual Jordan had a closer look at a proper "derivation" of his guess. Stimulated by Heisenberg's paper they began to formulate *matrix mechanics* for systems of a single degree of freedom. The fact that the infinite matrices for q and p are mathematically rather subtle objects was not taken very seriously. Their paper "Zur Quantenmechanik (On Quantum Mechanics)" was received by *Zeitschrift für Physik* September 27, 1925 [14]. In this paper the commutation relation of the matrices for position and momentum appeared in print for the first time and is called "the stronger quantum condition". All further conclusions are based on it.

Figure 5: Pascual Jordan in the 1920th

Heisenberg was in Copenhagen in September and therefore had been unavailable for discussions. Born had informed Heisenberg about his collaboration with Jordan. Heisenberg was excited about the achievements of his colleagues and began to work on matrix mechanics himself in Copenhagen after making himself familiar with the mathematical concept. By the end of September he had come up with the commutation relations for coordinates and momenta for systems with several degrees of freedom. Heisenberg raised an objection concerning the definition of the derivative of a product of several matrices with respect to one of its factors used by Born and Jordan and proposed a different definition which is closer to the usual differentiation procedure [17].
In a letter to Pauli Heisenberg pointed out that the most important thing was still missing, the solution for the hydrogen problem within matrix mechanics. It also is not discussed in the "three authors paper" (Dreimännerarbeit) of Born, Heisenberg and Jordan " Zur Quantenmechanik II (On Quantum Mechanics II)" which was received by Zeitschrift für Physik on November 25 [18]. It used Heisenberg's differentiation procedure and presented the state of the art of matrix mechanics. One can find e.g. a detailed discussion of the quantum mechanical properties of angular momentum.

3.2 Hamburg

The missing solution of the hydrogen problem was found by Pauli by the end of October after learning about the progress with the formalism of matrix mechanics from Heisenberg's letters. Proudly he reported to Heisenberg about his successful diagonalization of the Hamiltonian matrix of the hydrogen atom. As it did not seem possible to describe angular variables as matrices Pauli wrote the classical Laplace-Runge-Lenz vector $\vec{A}$ [5] in matrix form

$$\vec{\mathbf{A}} = \frac{1}{Ze^2 m_e} \cdot \frac{1}{2} \left(\vec{\mathbf{l}} \times \vec{\mathbf{p}} - \vec{\mathbf{p}} \times \vec{\mathbf{l}} \right) + \frac{\vec{\mathbf{r}}}{r}$$

and showed that its components are constants of motion in the Coulomb potential Ze^2/r, as in the classical case. With the help of the commutation relations of the components of the vector matrix $\vec{\mathbf{A}}$ among each other and the components of the angular momentum matrix $\vec{\mathbf{l}}$, Pauli obtained Bohr's energy values $W(n)$ after an algebraic tour de force. Therefore it is usually not presented in quantum mechanics textbooks. Pauli submitted this important missing piece for the success of of matrix mechanics only much later (received by *Zeitschrift für Physik* January 17, 1926) [19], because he wanted to include relativistic corrections. It was almost a bit too late, as discussed in the next section.

3.3 Cambridge

The only theoretical physicist who had no closer contact to the Göttingen physicists and made essential contributions to the algebraic formulation of quantum mechanics was Paul Adrien Maurice Dirac (1902-1984) in Cambridge. On July 28 Heisenberg had given a talk in Cambridge but only mentioned his new work to Ralph Fowler after the talk. Dirac received a copy of the proof-sheets of Heisenberg's paper from Ralph Fowler in the middle of August 1925, more than a month before it appeared in print. The paper made no easy reading for Dirac.

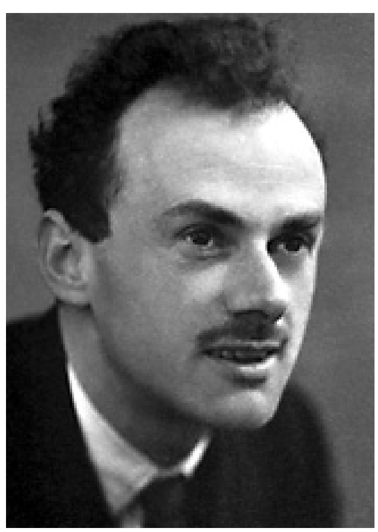

Figure 5: Paul Adrien Maurice Dirac in the 1920th

Probably by end of September he realized that most of the quantum mechanical equations can be written in a form similar to classical Hamilton mechanics using *Poisson brackets*. For differentiable functions of the canonical variables they are defined as

$$\{F,G\} \equiv \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i} \right) \ ,$$

which allow to write Hamilton's equations of motion as

$$\dot{q}_i = \{q_i, H\} \ , \qquad \dot{p}_i = \{p_i, H\}$$

with H the Hamilton function and

$$\{q_i, p_j\} = \delta_{ij}, \qquad \{q_i, q_j\} = 0 = \{p_i, p_j\}$$

holds. Dirac convinced himself of the correspondence of the quantum mechanical commutator $[A_q, B_q] \equiv A_q B_q - B_q A_q$ of the "q-numbers" (q for quantum) A_q, B_q (apart from a factor) to the Poisson bracket of the classical "c-numbers" A_c, B_c

$$[A_q, B_q] \leftrightarrow i\hbar\{A_c, B_c\} \ .$$

In his contribution "The fundamental equations of quantum mechanics", received by *Proceedings of the Royal Society* on November 7 [20] Dirac writes before presenting this correspondence: "We make the fundamental assumption that *the difference between the Heisenberg products of two quantum quantities is equal* $ih/2\pi$ *times their Poisson bracket expression.*" Later in the paper he points out the importance of the fact that "the mathematical operations on the two theories obey in many cases the same laws". Towards the end of the paper Dirac writes down what is usually called *Heisenberg's equation of motion.*
As in Dirac's approach algebraic relations between the quantum objects play the central role, independently of the special matrix form, it was the first step towards the abstract formulation of quantum mechanics students usually learn today.

Before switching to the year 1926 a few remarks about the later perception of Heisenberg's paper are appropriate. Some years ago two papers appeared in *American Journal of Physics* which address details of his 1925 paper [21,22]. They agree that Heisenberg's paper is "notoriously difficult to read". In one paper a reference is made to Nobel Prize winner Steven Weinberg. In his book *Dreams of a Final Theory* [23] he writes:
"I have tried several times to read the paper that Heisenberg wrote on returning from Helgoland, and, although I think I understand quantum mechanics, I have never understood Heisenberg's motivation for the mathematical steps in his paper....Heisenberg's paper was pure magic".
This assessment is useful for a better understanding of some of the later events.

4 1926: Wave Mechanics and beyond

4.1 Zurich

In Zurich Erwin Schrödinger (1887-1964), born in Vienna, arrived at a new quantum theory on a completely different way. His starting point was the 1924 PhD thesis of Louis de Broglie (1892-1987) extending the wave-particle dualism of light to particles with a nonzero rest mass m. He postulated that such a particle, like an electron, also has wave character, with the wave length $\lambda = h/p$ determined by the momentum p of the particle. The issue how Schrödinger arrived at his *wave equation* is not discussed here. In his first paper "Quantisierung als Eigenwertproblem (Quantization as an eigenvalue problem)" received by Annalen der Physik January 27, 1926 [24] he presented the (time independent) Schrödinger equation for the complex wave function ψ for a particle in an external field. This first in a series of papers presents the solution for the hydrogen atom, found in every quantum mechanics textbook today.

Figure 5: Erwin Schrödinger in the 1920th

Early in 1926 there seemed to exist two different theoretical approaches to explain atoms: matrix mechanics and wave mechanics. But rather quickly Schrödinger (and others) showed the complete equivalence of the two approaches to (non-relativistic) quantum mechanics [25]. Schrödinger's wave mechanics quickly found more acceptance than matrix mechanics had. It was closer to the mathematics known to theoretical physicists from other fields of physics. Now external potentials different from the $1/r$ Coulomb potential could be successfully treated. This was not the case for matrix mechanics which was considered "difficult".

4.2 Göttingen

The “three man” Göttingen group was not happy about this, especially as the physical meaning of the wave function was not generally agreed on. Schrödinger had a “smearing” of the electron in mind. This question brought Max Born back into the game. As he was unable to describe the scattering of particles within matrix mechanics he successfully used the Schrödinger equation [26] for that purpose. He realized that Schrödinger’s quantum mechanics can describe scattering events, but not what definitely happens, only how *probable* the effect is. In a footnote of his paper he presented *his* interpretation of the wave function of a particle: $|\psi(\vec{x})|^2\Delta V$ determines the *probability* to find the particle in the (small) volume ΔV around the position $\vec{x}$.

Born reported about his switching to wave mechanics in a meeting of the Göttingen Academy of Sciences on January 14, 1927 [27]. In his introduction he states (my translation):

“While for periodic systems the wave mechanical description of the quantum laws according to *Schrödinger* provides nothing more and nothing less than the matrix representation of *Heisenberg, Jordan and myself*, it seems especially well suited for aperiodic processes. But it is *necessary* to drop completely Schrödinger’s ideas which are heading towards a revival of classical continuum theory. One only has to take the formalism and give it a new physical content. One has to assume the existence of a *guiding field* which determines the probability of discrete elementary acts. As shown recently one can get the laws for the scattering of point particles (electrons, α-particles) from atoms this way”.

In the last sentence Born referred to his own paper [24]. His statement nicely describes his mixed feelings about the events. Despite the fact quantum mechanics had started in Göttingen about half a year before Schrödinger, his approach was more openly accepted by the community.

By the end of 1926 Dirac and Jordan independently submitted papers on “transformation theory” which gave a general formal framework for (non-relativistic) quantum mechanics [28,29]. Using Dirac notation one can go from the wave function in position space $\psi(\vec{x}) \equiv \langle\vec{x}|\psi\rangle$ to the corresponding function in momentum space $\tilde{\psi}(\vec{p}) \equiv \langle\vec{p}|\psi\rangle$ to the probability density $|\tilde{\psi}(\vec{p})|^2$ for measurements of the momentum of the particle. This is in complete analogy to Born’s probability distribtution $|\psi(\vec{x})|^2$ for the position of the particle in the state $|\psi\rangle$ in Hilbert space.

The mathematical formulation of quantum laws was completed. Discussions about the interpretation of the formalism persists till today.

5 Interpretation, Applications and Nobel Prizes

5.1 Interpretation

While the formalism of non-relativistic quantum mechanics was ready by the end of 1926 its interpretation was still in its infancy. In March 1927 Heisenberg submitted a paper with his famous uncertainty relation [30]. First leaving aside the experimental implications it is a mathematical result for the square root of the variances $(\Delta x)_{|\psi\rangle}$ and $(\Delta p)_{|\psi\rangle}$ of the probability densities for the position and momentum of a particle in the state $|\psi\rangle$

$$(\Delta x)_{|\psi\rangle}(\Delta p)_{|\psi\rangle} \geq \hbar/2$$

which can be derived using the commutation relation for position and momentum. If one can repeatedly prepare (at least approximately) the same state $|\psi\rangle$ and the repeated position measurement shows that the wavefunction $\psi(\vec{x})$ is strongly localized in space, measurement of the momentum, when one again repeatedly prepares the state $|\psi\rangle$, shows a large variance and vice versa. Discussions of Heisenberg with Bohr and others about this inequality, Born's probability interpretation of the state and the proposal of the "collapse of the wave function" by a measurement with a macroscopic experimental device led to what is usually called the *Copenhagen interpretation* of quantum mechanics, despite the fact that there is no general agreement about its concise meaning. The discussion of the many alternative interpretations would be a topic for a talk itself.

5.2 Applications

Soon after the formulation of quantum mechanics it was successfully applied to various problems in atomic-, molecular-, nuclear- and solid state physics, which can only be touched on here. Again the focus is on the Göttingen players. Friedrich Hund discovered the phenomenon of "tunneling" in quantum mechanics and before Robert Mullikan showed the importance of what was later called "molecular orbitals" [31]. Also Born together with Robert J. Oppenheimer addressed problems in molecular physics. The much larger nuclear compared to electron mass led to the formulation of the "Born-Oppenheimer approximation" which lies at the heart of chemistry [32]. Other post-docs with Born at that time were Walter Heitler, Victor Weißkopf, Eugene Wigner, Vladimir Fock and Edward Teller. Born himself addressed nuclear decay, while Heisenberg meanwhile as professor in Leipzig addressed e.g. the helium atom which had resisted an explanation in Bohr-Sommerfeld theory and a theory of ferromagnetism.

5.3 Nobel Prizes

A very large number of Nobel Prizes was awarded for work related to quantum mechanis, especially for various applications. Three of the five authors who formulated quantum mechanics in 1925/26 were awarded the Nobel Prize for physics in 1933. Werner Heisenberg received the 1932 prize for "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen". Erwin Schrödinger and Paul Dirac shared the 1933 prize "for the discovery of new productive forms of atomic theory". Here we discussed Dirac's contribution to the formulation of nonrelatistic quantum mechanics. He had made an even more important contribution with his relativistic *"Dirac equation"* for spin 1/2 particles formulated in 1928 [33]. The other two of the five fathers of quantum mechanics, especially Max Born, were unhappy not to receive the prize. Born had to wait more than twenty years and received the 1954 Nobel Prize for physics, "for his fundamental research in quantum mechanics, especially for his statistical interpretation of the wave function". Pascual Jordan's contribution to the formulation of quantum mechanics was not honored by a Nobel Prize. This might be due to his nearness to Nazism.

Of the many Nobel Prizes for the application of quantum mechanics I mention only one, the 1966 chemistry prize, awarded to Robert Mullikan,"for his fundamental work concerning chemical bonds and the electronic structure of molecules by the molecular orbital method". In his Nobel lecture he wrote "Molecular orbital theory has long been known as Hund-Mullikan theory in recognition of the mayor contribution of Professor Hund in its early development". After receiving the prize Mullikan mentioned that he had preferred to share it with Hund.

6 Summary

Many of the modern quantum mechanics textbooks avoid a detailed discussion of its historical path. The first excellent textbook proceeding that way is Dirac's "The Principles of Quantum Mechanics"[34]. The first edition was published in 1930 and it treats the "Schrödinger representation" as well as the "Heisenberg representation". Its forth edition from 1958 has been reprinted many times and apart from the fact that e.g. the concept of *entanglement* which is very important in the context of quantum information is not discussed, students should have a look at it even today.

Dirac's book is in sharp contrast to Born's "Volume 2" promised in 1924 to appear probably several years later. The title of the book also published in 1930 with P. Jordan as coauthor is "Elementare Quantenmechanik" [35]. The book promises the treatment of wave mechanics in "Volume 3" which was never written. Therefore "Volume 2" contains no differential equation at all but only the algebraic methods of matrix mechanics. In a review of the book W. Pauli pointed out very clearly that he considered this restriction a bad idea [36]. Because of this one-sided approach the book could be of any use only for a very restricted readership. The only positive sentence in his review is the last one. In typical sarcastic Pauli style:

"The making of the book with respect to print and paper is excellent".

In hindsight one has to admit that this book definitely did not help to popularize the important first steps towards nonrelativistic quantum mechanics made in Göttingen.

7 Acknowledgements

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As little known, the "Sommerfeld fine structure formula" was first derived by Wilhelm Lenz after getting a copy from Sommerfeld of his first paper addressing relativistic mass corrections. This was ackowlegded by Sommerfeld in his later paper "Zur Quantentheorie der Spektrallinien" (Annalen der Physik **51**, 1 (1916)).
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