Chapter 1

LUTTINGER LIQUIDS: THE BASIC CONCEPTS

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Abstract
This chapter reviews the theoretical description of interacting fermions in one dimension. The Luttinger liquid concept is elucidated using the Tomonaga-Luttinger model as well as integrable lattice models. Weakly coupled chains and the attempts to experimentally verify the theoretical predictions are discussed.

Keywords
Luttinger liquids, Tomonaga model, bosonization, anomalous power laws, breakdown of Fermi liquid theory, spin-charge separation, spectral functions, coupled chains, quasi-one-dimensional conductors

1. Introduction

In this chapter we attempt a simple selfcontained introduction to the main ideas and important computational tools for the description of interacting fermions in one spatial dimension. The reader is expected to have some knowledge of the method of second quantization. As in section 3 we describe a *constructive* approach to the important concept of bosonization, *no* quantum field-theoretical background is required. After mainly focusing on the Tomonaga-Luttinger

model in sections 2 and 3 we present results for integrable lattice models in section 4. In order to make contact to more realistic systems the coupling of strictly $1d$ systems as well as to the surrounding is addressed in section 5. The attempts to experimentally verify typical Luttinger liquid features like anomalous power laws in various correlation functions are only shortly discussed as this is treated in other chapters of this book.

2. Luttinger liquids - a short history of the ideas

As an introduction the basic steps towards the general concept of Luttinger liquids are presented in historical order. In this exposition the ideas are discussed without presenting all technical details. This is done in section 3 by disregarding the historical aspects aiming at a simple presentation of the important practical concepts like the " bosonization of field operators".

2.1 Bloch's method of "sound waves" (1934)

In a paper on incoherent x-ray diffraction Bloch [1] realized and used the fact that one-dimensional $(d = 1)$ noninteracting fermions have the same type of low energy excitations as a harmonic chain. The following discussion of this connection is very different from Bloch's own presentation.
The low energy excitations determine e.g. the low temperature specific heat. Debye's famous T^3-law for the lattice contribution of three dimensional solids reads in $d = 1$

$$c_L^{\rm Debye} = \frac{\pi}{3} k_B \left(\frac{k_B T}{\hbar c_s}\right), \tag{1.1}$$

where c_s is the sound velocity. At low temperatures the electronic contribution to the specific heat in the "Fermi gas" approximation of Pauli is also linear in T and involves the density of states of the noninteraction electrons at the Fermi energy. This yields for spinless fermions in $d = 1$

$$c_L^{\rm Pauli} = \frac{\pi}{3} k_B \left(\frac{k_B T}{\hbar v_F}\right), \tag{1.2}$$

where v_F is the Fermi velocity. With the replacement $c_s \leftrightarrow v_F$ the results are *identical*. This suggests that apart from a scale factor the (low energy) excitation energies and the degeneracies in the two types of systems are identical. For the harmonic chain the excited states are classified by the numbers n_k of phonons in the modes ω_k whith n_k taking integer values from zero to infinity. The excitation energy is given by $E(\{n_{k_m}\}) - E_0 = \sum_{k_m} \hbar\omega_{k_m} n_{k_m}$. For small wave numbers k_m the dispersion is *linear* $\omega_{k_m} \approx c_s|k_m|$. Therefore the excitations energies are multiples of $\hbar c_s(2\pi/L)$ for periodic boundary conditions and multiples of $\Delta_B \equiv \hbar c_s \pi/L$ for fixed boundary conditions. The calculation of the partition function is standard textbook material. This is also true for noninteracting electrons but there the calculation involves fermionic occupation numbers n_k^F

which take values zero and one. The two textbook calculations yield Eqs. (1.1) and (1.2), but through the "clever" use of the grand canonical ensemble in order to simplify the fermionic calculation the identity (apart from $c_s \leftrightarrow v_F$) remains mysterious. A deeper understanding involves two steps:

1) *Linearization* of the kinetic energy $\varepsilon_k = \hbar^2 k^2/(2m)$ of the free fermions around the Fermi point k_F for fixed boundary conditions or both Fermi points $\pm k_F$ for periodic boundary conditions. As the argument is simplest for fixed boundary conditions [2] which lead to $k_m = m\pi/L$ we discuss this case for the moment. Then the energies $\varepsilon_{k_n} - \varepsilon_F$ are integer multiples of $\Delta_F \equiv \hbar v_F \pi/L$ where v_F is the Fermi velocity.

2) *Classification* of any state of the Fermi system by the *number n_j of upward shifts by j units of Δ_F* with respect to the ground state. As the fermions are indistinguishable the construction of the $\{n_j\}$ shown in Fig. 1.1, where the highest occupied level in the excited state is connected with the highest occupied level in the ground state and so forth for the second, third ... highest levels, completely specifies the excited state. As the n_j can run from zero to infinity like bosonic quantum numbers and the excitation energy is given by $\sum_j (j\Delta_F) n_j$ the *canonical* partition function for the noninteracting fermions has the same form as the canonical partition function for the harmonic chain apart from $\Delta_F \leftrightarrow \Delta_B$ if one assumes the Fermi sea to be infinitely deep [3]. As we have linearized ω_k for small k as well as ε_k around k_F this equivalence only holds for the *low temperature* specific heats ($k_B T \ll \hbar\omega_{\max}, k_B T \ll \varepsilon_F$).

If we denote the creation (annihilation) operator of a fermion with $k_n = n\pi/L$ by $c_n^\dagger (c_n)$ and assume a *strictly linear* dispersion $\varepsilon_n^{\rm lin} = \hbar v_F k_n$ for *all* $k_n > 0$ a more technical formulation of the discussed equivalence can be given by the *exact operator identity*

$$\begin{aligned} T &= \sum_{n=1}^{\infty} \hbar v_F k_n c_n^\dagger c_n \\ &= \frac{\hbar v_F \pi}{L} \left[\sum_{l=1}^{\infty} l \; b_l^\dagger \; b_l + \frac{1}{2}\mathcal{N}(\mathcal{N}+1) \right], \end{aligned} \tag{1.3}$$

where the operators b_l with $l \geq 1$ are defined as

$$b_l \equiv \frac{1}{\sqrt{l}} \sum_{m=1}^{\infty} c_m^\dagger c_{m+l} \tag{1.4}$$

and $\mathcal{N} \equiv \sum_{n=1}^{\infty} c_n^\dagger c_n$ is the fermionic particle number operator. The proof of the "Kronig identity" (1.3) is simple (see Ref. [4]) . The operators b_l obey commutation relations $[b_l, b_{l'}] = 0$ and for $l \geq l'$

$$\left[b_l, b_{l'}^\dagger \right] = \frac{1}{\sqrt{l l'}} \sum_{m=1}^{l'} c_m^\dagger c_{m+l-l'}. \tag{1.5}$$

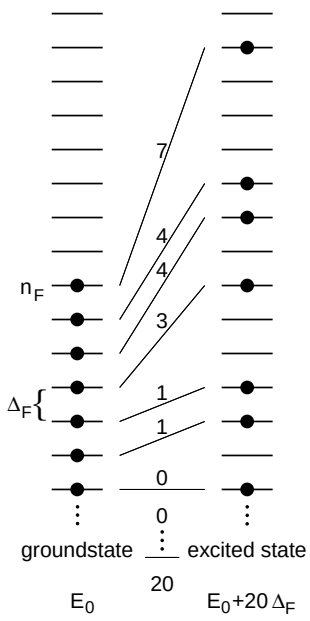

Figure 1.1. Example for the classification scheme for the excited states in terms of the numbers n_j of upward shifts by j units of Δ_F. In the example shown the nonzero n_j are $n_7 = 1, n_4 = 2, n_3 = 1$ and $n_1 = 2$.

For all N-particle states $|\phi_N^{(M)}\rangle \equiv \prod_{n=1}^{N} c_{i_n}^\dagger |\mathrm{Vac}\rangle$ in which the $M(< N)$ lowest one-particle levels are *all occupied* one obtains

$$\left[b_l, b_{l'}^\dagger\right] |\phi_N^{(M)}\rangle = \delta_{ll'} |\phi_N^{(M)}\rangle \tag{1.6}$$

for $l, l' \leq M$, i.e. these operators obey *boson commutation relations* $[b_l, b_{l'}^\dagger] = \delta_{ll'}\hat{1}$ in this *subspace* of all possible N-particle states.
Later it turns out to be useful to work with $\tilde{T} \equiv T - \langle T\rangle_0 - \mu_0\tilde{\mathcal{N}}$, where $\langle T\rangle_0 = \Delta_F n_F(n_F+1)/2$ is the ground-state energy, $\mu_0 = \Delta_F(n_F + 1/2)$ is the chemical potential of the noninteracting fermions and $\tilde{\mathcal{N}} \equiv \mathcal{N} - n_F\hat{1}$. Then $\tilde{T}$ is of the form as the rhs of Eq. (1.3) with $\mathcal{N}(\mathcal{N}+1)$ replaced by $\tilde{\mathcal{N}}^2$.

2.2 Tomonaga (1950): Bloch's method of sound waves applied to interacting fermions

When a two-body interaction between the fermions is switched on, the ground state is no longer the filled Fermi sea but it has admixtures of (multiple) particle-hole pair excitations. In order to simplify the problem Tomonaga studied the *high density limit* where the range of the interaction is much larger than the interparticle distance, using *periodic* boundary conditions [5]. Then the Fourier

transform $\tilde{v}(k)$ of the two-body interaction is nonzero only for values $|k| \leq k_c$ where the cut-off k_c is much smaller than the Fermi momentum $k_c \ll k_F$. This implies that for not too strong interaction the ground state and low energy excited states have negligible admixtures of holes deep in the Fermi sea and particles with momenta $|k| - k_F \gg k_c$. In the two intermediate regions around the two Fermi points $\pm k_F$, with particle-hole pairs present, the dispersion ε_k is *linearized* in order to apply Bloch's "sound wave method"

$$k \approx \pm k_F : \quad \varepsilon_k = \varepsilon_F \pm v_F(k \mp k_F). \tag{1.7}$$

Tomonaga realized that the Fourier components of the operator of the density

$$\begin{aligned} \hat{\rho}_n &= \int_{-L/2}^{L/2} \hat{\rho}(x) e^{-ik_n x} dx = \int_{-L/2}^{L/2} \psi^\dagger(x)\psi(x) e^{-ik_n x} dx \\ &= \sum_{n'} c^\dagger_{n'} c_{n'+n}, \end{aligned} \tag{1.8}$$

where $c^\dagger_n(c_n)$ creates (annihilates) a fermion in the state with momentum $k_n = \frac{2\pi}{L} n$, plays a central role in the interaction term, as well as the kinetic energy. Apart from an additional term linear in the particle number operator [4], which is usually neglected, the two-body interaction is given by

$$\hat{V} = \frac{1}{2L} \sum_{n \neq 0} \tilde{v}(k_n) \hat{\rho}_n \hat{\rho}_{-n} + \frac{1}{2L} \mathcal{N}^2 \tilde{v}(0) \tag{1.9}$$

Tomonaga's important step was to split $\hat{\rho}_n$ for $|k_n| \ll k_F$ into two parts, one containing operators of *"right movers"* i.e. involving fermions near the right Fermi point k_F with velocity v_F and *"left movers"* involving fermions near $-k_F$ with velocity $-v_F$

$$\hat{\rho}_n = \sum_{n' \geq 0} c^\dagger_{n'} c_{n'+n} + \sum_{n' < 0} c^\dagger_{n'} c_{n'+n} \equiv \hat{\rho}_{n,+} + \hat{\rho}_{n,-} \tag{1.10}$$

where the details of the splitting for small $|n'|$ are irrelevant. Apart from the square root factor the $\hat{\rho}_{n,\alpha}$ are similar to the b_l defined in Eq. (1.4) . Their commutation relations in the *low energy subspace* are

$$[\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\beta}] = \alpha m \delta_{\alpha\beta} \delta_{m,-n} \hat{1}. \tag{1.11}$$

If one defines the operators

$$b_n \equiv \frac{1}{\sqrt{|n|}} \begin{cases} \hat{\rho}_{n,+} & \text{for } n > 0 \\ \hat{\rho}_{n,-} & \text{for } n < 0 \end{cases} \tag{1.12}$$

and the corresponding adjoint operators $b^\dagger_n$ this leads using $\rho^\dagger_{n,\alpha} = \rho_{-n,\alpha}$ to the bosonic commutation relations

$$[b_n, b_m] = 0, \quad [b_n, b^\dagger_m] = \delta_{mn} \hat{1}. \tag{1.13}$$

Now the kinetic energy of the right movers as well as that of the left movers can be "bosonized" as in Eq. (1.3). The interaction $\hat{V}$ is bilinear in the $\hat{\rho}_n$ as well as the $\hat{\rho}_{n,\alpha}$. Therefore apart from an additional term containing particle number operators *the Hamiltonian for the interacting fermions is a quadratic form in the boson operators*

$$\begin{aligned}\tilde{H} &= \sum_{n>0} \hbar k_n \left\{ \left(v_F + \frac{\tilde{v}(k_n)}{2\pi\hbar} \right) \left(b_n^\dagger b_n + b_{-n}^\dagger b_{-n} \right) \right. \\ &\quad \left. + \frac{\tilde{v}(k_n)}{2\pi\hbar} \left(b_n^\dagger b_{-n}^\dagger + b_{-n} b_n \right) \right\} + \frac{\hbar\pi}{2L} \left[v_N \tilde{\mathcal{N}}^2 + v_J \mathcal{J}^2 \right] \\ &\equiv H_B + H_{\tilde{\mathcal{N}},\mathcal{J}}, \end{aligned} \tag{1.14}$$

where $\tilde{\mathcal{N}} \equiv \tilde{\mathcal{N}}_+ + \tilde{\mathcal{N}}_-$ is the total particle number operator relative to the Fermi sea, $\mathcal{J} \equiv \tilde{\mathcal{N}}_+ - \tilde{\mathcal{N}}_-$ the "current operator", and the velocities are given by $v_N = v_F + \tilde{v}(0)/\pi\hbar$ and $v_J = v_F$. Here v_N determines the energy change for adding particles without generating bosons while v_J enters the energy change when the difference in the number of right and left movers is changed. Similar to the discussion at the end of section 1.1 we have defined $\tilde{H} \equiv H - E_0^H - (\mu_0 + \tilde{v}(0) n_0)\tilde{\mathcal{N}}$, where E_0^H is the Hartree energy and n_0 the particle density. As the particle number operators $\tilde{\mathcal{N}}_\alpha$ commute with the boson operators $b_m(b_m^\dagger)$ the two terms H_B and $H_{\tilde{\mathcal{N}},\mathcal{J}}$ in the Hamiltonian *commute* and can be treated separately. Because of the translational invariance the two-body interaction only couples the modes described by $b_n^\dagger$ and b_{-n}. With the Bogoliubov transformation $\alpha_n^\dagger = c_n b_n^\dagger - s_n b_{-n}$ the Hamiltonian H_B can be brought into the form

$$H_B = \sum_{n\neq 0} \hbar\omega_n \alpha_n^\dagger \alpha_n + \text{ const.}, \tag{1.15}$$

where the $\omega_n = v_F|k_n|\sqrt{1 + \tilde{v}(k_n)/\pi\hbar v_F}$ follow from 2×2 eigenvalue problems corresponding to the condition $[H_B, \alpha_n^\dagger] = \hbar\omega_n \alpha_n^\dagger$. For small k_n one obtains for smooth potentials $\tilde{v}(k)$ again a *linear* dispersion $\omega_n \approx v_c|k_n|$, with the "*charge velocity*" $v_c = \sqrt{v_N v_J}$, which is larger than v_F for $\tilde{v}(0) > 0$. The expression for the coefficients c_n and s_n with $c_n^2 - s_n^2 = 1$ will be presented later for the generalized model Eq. (1.17) . For fixed particle numbers N_+ and N_-, the excitation energies of the *interacting* Fermi system are given by $\sum_m \hbar\omega_m n_m$ with integer occupation numbers $0 \leq n_m < \infty$. For small enough excitation energies the only difference of the excitation spectrum for fixed particle numbers with respect to the noninteracting case is the replacement $v_F \leftrightarrow v_c$.

In his seminal paper Tomonaga did not realize the anomalous decay of correlation functions of the model because in his discussion of the density correlation function he missed the $2k_F$-contribution discussed in section 3.

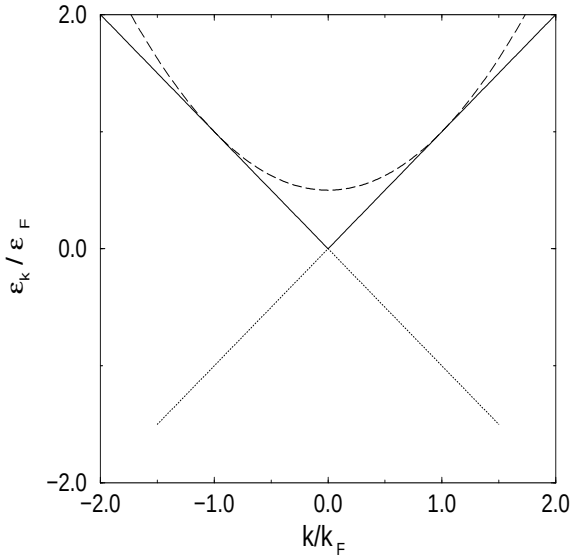

Figure 1.2. Energy dispersion as a function of momentum. The dashed curve shows the usual "nonrelativistic" dispersion and the full curve the linearized version used (apart from a constant shift) in Eq. (1.3) for $k > 0$ for fixed boundary conditions. The dot-dashed parts are the additional states for $k_0 = -1.5k_F$. The model discussed by Luttinger corresponds to $k_0 \to -\infty$.

2.3 Luttinger (1963): no discontinuity at the Fermi surface

Luttinger, apparently unaware of Tomonaga's work, treated spinless, massless fermions (in the relativistic sense, but $c \leftrightarrow v_F$) in one dimension, i.e. two *infinite* branches of right and left moving fermions with dispersion $\pm v_F k$ [6]. As Luttinger himself made an error with the fact that his Hamiltonian is not bounded from below, it is useful to switch from Tomonaga's to Luttinger's model keeping a *band cut-off* k_0 such that $k \geq k_0 = 2\pi m_0/L$ with $m_0 < 0$ for the right movers and correspondingly for the left movers (see Fig. 1.2). Fortunately Luttinger's error had no influence on his inquiry if a *sharp Fermi surface* exists in the exact ground state of the interacting model. After a rather complicated calculation using properties of so-called "Toeplitz determinants" Luttinger found that the average occupation $\langle n_{k,+} \rangle$ in the ground state for $k \approx k_F$ behaves as

$$\langle n_{k,+} \rangle - \frac{1}{2} \sim \left| \frac{k - k_F}{k_c} \right|^{\alpha_L} \text{sign}(k_F - k), \tag{1.16}$$

where α_L depends on the interaction strength (see below) [7]. "Thus, in this model, the smallest amount of interaction *always* destroys the discontinuity of $\langle n_k \rangle$ at the Fermi surface" [6]. This can be related to the fact that the equal

time correlation functions $\langle\psi_\alpha^\dagger(x)\psi_\alpha(0)\rangle$ decay as $1/|x|^{1+\alpha_L}$ in the *interacting* system in contrast to $\langle\psi_\alpha^\dagger(x)\psi_\alpha(0)\rangle \sim 1/|x|^d$ (with $d=1$) in the noninteracting case. Therefore α_L is called the *"anomalous dimension"*[8].

Apart from the different dispersion Luttinger also used a different interaction. In contrast to Tomonaga he only kept an interaction between the right and left movers but not the term $\sim \tilde{v}(k_n)(b_n^\dagger b_n + b_{-n}^\dagger b_{-n})$ in Eq. (1.14) . In the limit of a delta interaction of the right and left movers his model is identical to the *massless Thirring model* (1958) [9] at that time not well known in the solid state physics community.

2.4 Towards the "Luttinger liquid" concept

Luttinger's treatment of the Dirac sea was corrected in a paper by Mattis and Lieb (1965) [10] which also offered a simpler way to calculate $\langle n_{k,\alpha}\rangle$. The time dependent one-particle Green's function for the spinless Luttinger model was calculated by Theumann (1967) [11] by generalizing this method. She found *power law behaviour* in the corresponding spectral function $\rho(k,\omega)$, especially $\rho(k_F,\omega) \sim \alpha_L|\omega|^{\alpha_L-1}$, i.e. *no sharp quasiparticle for* $k=k_F$ consistent with Luttinger's result for the occupation numbers (Fig.1.3). For a delta interaction her results agreed with an earlier calculation for the massless Thirring model by Johnson (1961) [12]. Later the time dependent one-particle Green's function was calculated by various other methods, e.g. using Ward identities (Dzylaloshinski and Larkin (1974) [13]) as well as the important method of the *"bosonization of the field operator"* (Luther and Peschel (1974) [14]) which will be addressed in detail in section 3. It was first proposed in a different context by Schotte and Schotte (1969) [15].
What is now usually called the "Tomonaga-Luttinger (TL) model" is the following generalization of Eq. (1.14)

$$\begin{aligned}\tilde{H}_{TL} &= \frac{2\pi\hbar}{L}\sum_{n>0} n\left\{\left(v_F+\frac{g_4(k_n)}{2\pi\hbar}\right)\left(b_n^\dagger b_n + b_{-n}^\dagger b_{-n}\right)\right. \\ &+ \left.\frac{g_2(k_n)}{2\pi\hbar}\left(b_n^\dagger b_{-n}^\dagger + b_{-n}b_n\right)\right\} + \frac{\hbar\pi}{2L}\left\{v_N\tilde{\mathcal{N}}^2 + v_J\mathcal{J}^2\right\}, \quad (1.17)\end{aligned}$$

where $v_N = v_F + (g_4(0)+g_2(0))/2\pi\hbar$ and $v_J = v_F + (g_4(0)-g_2(0))/2\pi\hbar$. The interaction parameters $g_2(k_n)$ and $g_4(k_n)$ are allowed to be *different*. As Tomonaga's original model the TL model is exactly solvable, i.e. it can also be brought into the form of Eq. (1.15). The eigenvector components in $\alpha_n^\dagger = c_n b_n^\dagger - s_n b_{-n}$ are given by

$$c_n = \frac{1}{2}\left(\sqrt{K_n}+\frac{1}{\sqrt{K_n}}\right), \quad s_n = \frac{1}{2}\left(\sqrt{K_n}-\frac{1}{\sqrt{K_n}}\right) \qquad (1.18)$$

with $K_n = \sqrt{v_J(k_n)/v_N(k_n)}$, where $v_{J(N)}(k_n) \equiv v_F + [g_4(k_n) \mp g_2(k_n)]/2\pi\hbar$. The frequencies are given by $\omega_n = |k_n|\sqrt{v_J(k_n)v_N(k_n)} \equiv |k_n| v_c(k_n)$. The TL-Hamiltonian corresponds to a fermionic Hamiltonian that *conserves* the number of right and left movers.

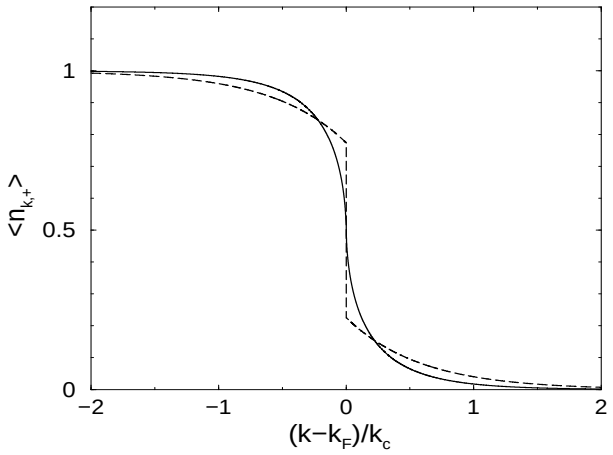

Figure 1.3. The full line shows the average occupation $\langle n_{k,+} \rangle$ for a TL model with $\alpha_L = 0.6$. The dashed line shows the expectation from Fermi liquid theory, where the discontinuity at k_F determines the quasi-particle weight Z_{k_F} in $\rho_+(k_F, \omega)$. As discussed following Eq. (1.48) this can also be realized in a TL model with $g_2(0) = 0$. There also the details of the interaction are specified.

A more general model including spin and terms changing right movers into left movers and vice versa is usually called the "g-ology model". An important step towards the general Luttinger liquid concept came from the *renormalization group* (RG) study of this model. It was shown that for *repulsive* interactions (see section 3) the renormalized interactions flow towards a fixed point Hamiltonian of the TL-type unless in lattice models for commensurate electron fillings strong enough interactions (for the half filled Hubbard model discussed in section 4 this happens for arbitrarily small on-site Coulomb interaction U) destroy the metallic state by opening a Mott-Hubbard gap. The RG approach is described in detail in reviews by Sólyom (1979) [16] and Shankar (1994) [17]. These results as well as insight from models which allow an exact solution by the Bethe ansatz led Haldane [18, 19] to propose the concept of *Luttinger liquids* (LL) as a replacement of Fermi liquid theory in one dimension, which "fails because of the infrared divergence of certain vertices it assumes to remain finite" [19] . At least for spinless fermions Haldane was able to show that "the Bogoliubov transformation technique that solves the Luttinger model provides a general method for resumming *all* infrared divergences present"[19]. Similar to

Fermi liquid theory in higher dimensions this new LL phenomenology allows to describe the low energy physics in terms of a few constants, *two* for the spinless case: the "stiffness constant" $K \equiv K_0 = \sqrt{v_J/v_N}$ (also called g in various publications) and the *"charge velocity"* $v_c = \sqrt{v_J v_N}$. In his seminal paper Haldane showed explicitly that the LL relations survive in not exactly soluble generalizations of the TL model with a non-linear fermion dispersion. He also gave a clear presentation how to calculate general correlation functions and e.g. the occupancies shown in Fig. 1.3 for the TL model. The technical details are addressed in section 3.

Before we do this two additional important aspects of LL-behaviour should be mentioned. The first concerns the strong influence of impurities on the low energy physics [20, 21, 22, 23, 24, 25], especially the peculiar modification of the electronic properties of a LL when a *single* impurity with an arbitrarily weak backscattering potential is present. For a spinless LL with a *repulsive* two-body interaction, i.e. $K < 1$ a perturbative bosonic RG calculation [25] shows that the backscattering strength V_B is a *relevant* perturbation which grows as Λ^{K-1} when the flow parameter Λ is sent to zero. This leads to a breakdown of the perturbative analysis in V_B. On the other hand a weak hopping between the open ends of two semi-infinite chains is *irrelevant* and scales to zero as $\Lambda^{K^{-1}-1}$. Assuming that the open chain presents the only stable fixed point it was argued that at low energy scales even for a weak impurity physical observables behave as if the system is split into two semi-infinite chains. This leads to a conductance which vanishes with a power law in T at low temperatures [25]. A more technical discussion is presented in section 3.

Electrons are spin one-half particles and for their description it is necessary to include the spin degree of freedom in the model. For a fixed quantization axis the two spin states are denoted by $\sigma = \uparrow, \downarrow$. The fermionic creation (annihilation) operators $c^\dagger_{n,\pm,\sigma}(c_{n,\pm,\sigma})$ carry an additional spin label as well as the $\hat{\rho}_{n,\pm,\sigma}$ and the boson operators $b_{n,\sigma}$ which in a straightforward way generalize Eq. (1.12). The interactions $g_\nu(k)$ with $\nu = 2, 4$ in Eq. (1.17) become matrices $g_\nu^{\sigma\sigma'}$ in the spin labels. If they have the form $g_\nu^{\sigma\sigma'}(k) = \delta_{\sigma,\sigma'} g_{\nu\|}(k) + \delta_{\sigma,-\sigma'} g_{\nu\perp}(k)$ it is useful to switch to new boson operators $b_{n,a}$ with $a = c, s$

$$\begin{aligned} b_{n,c} &\equiv \frac{1}{\sqrt{2}} \left(b_{n,\uparrow} + b_{n,\downarrow} \right) \\ b_{n,s} &\equiv \frac{1}{\sqrt{2}} \left(b_{n\uparrow} - b_{n,\downarrow} \right) \ , \end{aligned} \tag{1.19}$$

which obey $[b_{a,n}, b_{a',n'}] = 0$ and $\left[b_{a,n}, b^\dagger_{a',n'} \right] = \delta_{aa'} \delta_{nn'} \hat{1}$. The kinetic energy can be expressed in terms of "charge" (c) and "spin" (s) boson operators using $b^\dagger_{n,\uparrow} b_{n,\uparrow} + b^\dagger_{n\downarrow} b_{n\downarrow} = b^\dagger_{n,c} b_{n,c} + b^\dagger_{n,s} b_{n,s}$. If one defines the interaction matrix

elements $g_{\nu,a}(q)$ via

$$\begin{aligned} g_{\nu,c}(q) &\equiv g_{\nu\|}(q) + g_{\nu\perp}(q) \\ g_{\nu,s}(q) &\equiv g_{\nu\|}(q) - g_{\nu\perp}(q) \ , \end{aligned} \tag{1.20}$$

and defines $\tilde{\mathcal{N}}_{\alpha,c(s)} \equiv (\tilde{\mathcal{N}}_{\alpha,\uparrow} \pm \tilde{\mathcal{N}}_{\alpha,\downarrow})/\sqrt{2}$ one can write the TL-Hamiltonian $\tilde{H}_{TL}^{(1/2)}$ for spin one-half fermions as

$$\tilde{H}_{TL}^{(1/2)} = \tilde{H}_{TL,c} + \tilde{H}_{TL,s} \ , \tag{1.21}$$

where the $\tilde{H}_{TL,a}$ are of the form Eq. (1.17) but the interaction matrix elements have the additional label a. The two terms on the rhs of Eq. (1.21) *commute*, i.e. the "charge" and "spin" excitation are completely independent. This is usually called "spin-charge separation". The "diagonalization" of the two separate parts proceeds exactly as before and the low energy excitations are "massless bosons" $\omega_{n,a} \approx v_a|k_n|$ with the *charge velocity* $v_c = (v_{J_c} v_{N_c})^{1/2}$ and the *spin velocity* $v_s = (v_{J_s} v_{N_s})^{1/2}$. The corresponding two stiffness constants are given by $K_c = (v_{J_c}/v_{N_c})^{1/2}$ and $K_s = (v_{J_s}/v_{N_s})^{1/2}$. Because of Eq. (1.21) the dependence of the velocities on the interaction strength (1.20) is obtained using the results for the spinless model following Eq. (1.18).

The low temperature thermodynamic properties of the TL model including spin, Eqs. (1.17) and (1.21), can be expressed in terms of the four velocities v_{N_c}, v_{J_c}, v_{N_s}, v_{J_s} or the four quantities v_c, K_c, v_s, K_s. Due to spin-charge separation the specific heat has two additive contributions of the same form as in Eqs. (1.1) and (1.2). If we denote, as usual, the proportionality factor in the linear T-term by γ one obtains

$$\frac{\gamma}{\gamma_0} = \frac{1}{2}\left(\frac{v_F}{v_c} + \frac{v_F}{v_s}\right), \tag{1.22}$$

where γ_0 is the value in the noninteracting limit. To calculate the spin susceptibility χ_s one adds a term $-h\tilde{\mathcal{N}}_s$ to $\tilde{H}_{TL}^{(1/2)}$. Then by minimizing the ground state energy with respect to N_s one obtains $\langle\tilde{\mathcal{N}}_s\rangle \sim h/v_{N_s}$, i.e. χ_s is inversely proportional to v_{N_s}. If one denotes the spin susceptibility of the noninteracting system by $\chi_{s,0}$, this yields for the zero temperature susceptibility

$$\frac{\chi_s}{\chi_{s,0}} = \frac{v_F}{v_{N_s}} = K_s \frac{v_F}{v_s}. \tag{1.23}$$

For spin rotational invariant systems one has $K_s = 1$ [26]. The zero temperature compressibilty κ is proportional to $(\partial^2 E_0/\partial N^2)_L^{-1}$ which using Eqs. (1.17) and (1.21) leads to

$$\frac{\kappa}{\kappa_0} = \frac{v_F}{v_{N_c}} = K_c \frac{v_F}{v_c}. \tag{1.24}$$

A simple manifestation of spin-charge separation occurs in the time evolution of a localized perturbation of e.g. the the spin-up density. The time evolution $\alpha_{n,a}(t) = \alpha_{n,a}e^{-i\omega_{n,a}t}$ for $a = c, s$ implies

$$b_{n,a}(t) = b_{n,a}\left(c_{n,a}^2 e^{-i\omega_{n,a}t} - s_{n,a}^2 e^{i\omega_{n,a}t}\right) - b_{-n,a}^{\dagger} c_{n,a} s_{n,a}\left(e^{-i\omega_{n,a}t} - e^{i\omega_{n,a}t}\right) \tag{1.25}$$

If the initial state of the system involves a perturbation of *right movers* only, i.e. $\langle b_{n,a}\rangle = 0$ for $n < 0$ and the perturbation is sufficiently smooth ($\langle b_{n,a}\rangle \neq 0$ for $0 < n \ll n_c$ only) the initial perturbation is split into four parts which move with velocities $\pm v_c$ and $\pm v_s$ without changing the initial *shape*. If only the initial expectation values of the $b_{n,\uparrow}$ are different from zero one obtains for $\delta\langle\rho_\uparrow(x,0)\rangle \equiv F(x)$ using Eq. (1.12)

$$\delta\langle\rho_\uparrow(x,t)\rangle = \sum_a \left[\frac{1+K_a}{4}F(x-v_a t) + \frac{1-K_a}{4}F(x+v_a t)\right]. \tag{1.26}$$

For the spin rotational invariant case $K_s = 1$ there is no contribution which moves to the left with the spin velocity. Already for the pure g_4-model with $K_c = 1$ but $v_c \neq v_s$ "spin-charge separation" of the distribution occurs. For the spinless model with $g_2 \neq 0$ the initial distribution splits into one right- and one left-moving part, which is often called "charge fractionalization" [27, 28]. Note that the splitting described in Eq. (1.26) is independent of the details of $F(x)$ like the corresponding total charge. An additional comment should be made: spin-charge separation is often described as the fact that when an electron is injected into the system *its* spin and charge move independently with different velocities. This is very misleading as it is a *collective* effect of the total system which produces expectation values like in Eq. (1.26).

The easiest way to understand the important manifestation of spin-charge separation in the momentum resolved one-particle spectral functions [29, 30] is to make use of the bosonization of the electronic field operators discussed in the next section.

3. Luttinger liquids - computational tools

In section 2 many of the important features of LL's like the absence of a discontinuity at the Fermi surface were presented without giving any details how these properties are actually determined. As the most important tool the bosonization of the field operators is presented in detail in this section. This method is then used to calculate correlation functions like the one-particle Green's function and the $2k_F$-density response function. In the second part of this section the TL model with additional interactions and (or) a one particle potential with a "backscattering" contribution is discussed. The model is no longer exactly solvable and RG arguments play an important role [16, 17, 25].

3.1 Bosonization of the field operator

In the following a selfcontained presentation of the bosonization of a *single* fermion operator including a straightforward construction of the particle number changing part ("Klein factor") is given. We present the bosonization of the field operator for the right movers described by the $c_{l,+}$ and just mention the corresponding result for the left movers.
The starting point are the commutation relations the $c_{l,+}$ obey for $m > 0$

$$[b_m, c_{l,+}] = -\frac{1}{\sqrt{m}} c_{l+m,+} \quad , \quad [b_m^\dagger, c_{l,+}] = -\frac{1}{\sqrt{m}} c_{l-m,+} \quad . \tag{1.27}$$

If (after taking the limit $m_0 \to -\infty$) one introduces the 2π-periodic auxiliary field operator $\tilde{\psi}_+(v)$, where v later will be taken as $2\pi x/L$

$$\tilde{\psi}_+(v) \equiv \sum_{l=-\infty}^{\infty} e^{ilv} c_{l,+} \ , \tag{1.28}$$

it obeys the simple commutation relations

$$[b_m, \tilde{\psi}_+(v)] = -\frac{1}{\sqrt{m}} e^{-imv} \tilde{\psi}_+(v) \ ; \ [b_m^\dagger, \tilde{\psi}_+(v)] = -\frac{1}{\sqrt{m}} e^{imv} \tilde{\psi}_+(v) \ . \tag{1.29}$$

Products of exponentials of operators *linear* in the boson operators

$$A_+ \equiv \sum_{n \neq 0} \lambda_n b_n^\dagger \quad ; \quad B_- \equiv \sum_{n \neq 0} \mu_n b_n \tag{1.30}$$

with arbitrary constants λ_n and μ_n obey similar commutation relations

$$[b_m, e^{A_+} e^{B_-}] = \lambda_m e^{A_+} e^{B_-} \quad ; \quad [b_m^\dagger, e^{A_+} e^{B_-}] = -\mu_m e^{A_+} e^{B_-} \ , \tag{1.31}$$

which follow from $[b_m, e^{\lambda b_m^\dagger}] = \lambda e^{\lambda b_m^\dagger}$. We therefore make the ansatz

$$\tilde{\psi}_+(v) = \hat{O}_+(v) e^{i\phi_+^\dagger(v)} e^{i\phi_+(v)}, \tag{1.32}$$

where the operator $i\phi_+(v)$ is given by [19]

$$i\phi_+(v) = \sum_{n=1}^{\infty} \frac{e^{inv}}{\sqrt{n}} b_n \ . \tag{1.33}$$

Then the operator $\hat{O}_+(v)$ *commutes* with all the b_m and $b_m^\dagger$. We next construct $\hat{O}_+(v)$ such that both sides of Eq. (1.32) yield identical matrix elements.
As $\tilde{\psi}_+(v)$ reduces the number of right movers by one, the operator $\hat{O}_+(v)$ also must have this property. In order to determine $\hat{O}_+(v)$ we work with the

eigenstates of the noninteracting system (the limit $m_0 \to -\infty$ is implied and n_F is an arbitrary positive integer later related to k_F)

$$|\{m_l\}_b, \tilde{N}_+, \tilde{N}_-\rangle \equiv \prod_l \frac{(b_l^\dagger)^{m_l}}{\sqrt{m_l!}} \left(\prod_{n=m_0}^{n_F+\tilde{N}_-} c_{-n,-}^\dagger \right) \left(\prod_{r=m_0}^{n_F+\tilde{N}_+} c_{r,+}^\dagger \right) |\mathrm{Vac}\rangle. \tag{1.34}$$

It is easy to see that $\hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle$ has no overlap to excited states

$$\begin{aligned} \langle \{m_l\}_b, \tilde{N}_+ - 1, \tilde{N}_-|\hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle \; = \\ \langle \{0\}_b, \tilde{N}_+ - 1, \tilde{N}_-| \prod_l \frac{(b_l)^{m_l}}{\sqrt{m_l!}} \hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle \; . \end{aligned} \tag{1.35}$$

As $\hat{O}_+(v)$ commutes with the b_l the rhs of Eq. (1.35) vanishes unless all m_l are zero. This implies

$$\hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle = c_+(\tilde{N}_+, \tilde{N}_-, v)|\{0\}_b, \tilde{N}_+ - 1, \tilde{N}_-\rangle \; , \tag{1.36}$$

where $c_+(\tilde{N}_+, \tilde{N}_-, v)$ is a c-number. In order to determine $c_+(\tilde{N}_+, \tilde{N}_-, v)$ we calculate $\langle \{0\}_b, \tilde{N}_+ - 1, \tilde{N}_-|\tilde{\psi}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle$ using Eq. (1.28) as well as Eq. (1.32). In the calculation of the matrix element with the fermionic form Eq. (1.28) we use Eq. (1.34) which yields

$$\langle \{0\}_b, \tilde{N}_+ - 1, \tilde{N}_-|c_{l,+}|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle = (-1)^{\tilde{N}_-} \delta_{l,n_F+\tilde{N}_+}. \tag{1.37}$$

The factor $(-1)^{\tilde{N}_-}$ occurs because we have to commute $c_{l,+}$ through the product of $N_- = -m_0 + 1 + n_F + \tilde{N}_-$ fermionic operators of the left movers if we assume $-m_0 + n_F$ to be odd. We note that *no* such factor occurs for the corresponding matrix element of the left movers. The calculation of the ground state to ground state matrix element of $\tilde{\psi}_+(v)$ using Eq. (1.32) is simple as both exponentials involving the boson operators can be replaced by the unit operator and the matrix element is just $c_+(\tilde{N}_+, \tilde{N}_-, v)$. The comparison therefore yields

$$c_+(\tilde{N}_+, \tilde{N}_-, v) = (-1)^{\tilde{N}_-} e^{iv(n_F+\tilde{N}_+)} \tag{1.38}$$

and $c_-(\tilde{N}_+, \tilde{N}_-, v) = e^{-iv(n_F+\tilde{N}_-)}$. Together with Eq. (1.34) and the definition $\tilde{\mathcal{N}}_\alpha \equiv \mathcal{N}_\alpha - (-m_0 + 1 + n_F)\hat{1}$ this implies

$$\hat{O}_+(v)e^{-i(n_F+\tilde{\mathcal{N}}_+)v}(-1)^{\tilde{\mathcal{N}}_-}|\{0\}_b, \tilde{N}_+, \tilde{N}_-\rangle = |\{0\}_b, \tilde{N}_+ - 1, \tilde{N}_-\rangle \; . \tag{1.39}$$

If we apply the operator $\hat{O}_+(v)e^{-i(n_F+\tilde{\mathcal{N}}_+)v}(-1)^{\tilde{\mathcal{N}}_-}$ to the states in Eq. (1.34) and use again that $\hat{O}_+(v)$ commutes with the boson operators we obtain

$$\hat{O}_+(v)e^{-i(n_F+\tilde{\mathcal{N}}_+)v}(-1)^{\tilde{\mathcal{N}}_-}|\{m_l\}_b, \tilde{N}_+, \tilde{N}_-\rangle = |\{m_l\}_b, \tilde{N}_+ - 1, \tilde{N}_-\rangle \; . \tag{1.40}$$

This shows that the operator $U_+ \equiv \hat{O}_+(v)e^{-i(n_F+\tilde{\mathcal{N}}_+)v}(-1)^{\tilde{\mathcal{N}}_-}$ is *independent* of v and given by

$$U_+ = \sum_{\tilde{N}_+,\tilde{N}_-} \sum_{\{m_l\}} |\{m_l\}_b, \tilde{N}_+ - 1, \tilde{N}_-\rangle\langle\{m_l\}_b, \tilde{N}_+, \tilde{N}_-| \,. \tag{1.41}$$

It follows immediately that U_+ is *unitary*, i.e. $U_+U_+^\dagger = U_+^\dagger U_+ = \hat{1}$. From Eq. (1.41) one can infer that for arbitrary functions f of the number operator $\tilde{\mathcal{N}}_+$ one has $U_+ f(\tilde{\mathcal{N}}_+) = f(\tilde{\mathcal{N}}_+ + 1)U_+$.
To summarize we have shown that

$$\hat{O}_+(v) = U_+ e^{i(n_F+\tilde{\mathcal{N}}_+)v}(-1)^{\tilde{\mathcal{N}}_-} \,. \tag{1.42}$$

In $\hat{O}_-(u) = U_- e^{-i(n_F+\tilde{\mathcal{N}}_-)u}$ *no* factor $(-1)^{\tilde{\mathcal{N}}_+}$ appears and therefore $\hat{O}_+(v)$ and $\hat{O}_-(u)$ anticommute, which is necessary to enforce anticommutation relations between $\tilde{\psi}_+(v)$ and $\tilde{\psi}_-(u)$. It is an easy exercise to show that e.g. the anticommutation relations $[\tilde{\psi}_+(v), \tilde{\psi}_+(u)]_+ = 0$ are fulfilled. In the calculation the properties of $\hat{O}_+(v)$ as well as the factor in Eq. (1.32) involving the boson operators enter. If one replaces the operators $\hat{O}_\alpha(v)e^{-i\alpha v(\tilde{\mathcal{N}}_\alpha+n_F)}$ by "Majorana fermions" η_α which commute with the boson operators and obey the anticommutation relations $[\eta_\alpha, \eta_\beta]_+ = 2\delta_{\alpha\beta}\hat{1}$, as often done in the literature, this yields $[\tilde{\psi}_\alpha(v), \tilde{\psi}_\alpha(u)]_+ = [1 - \cos(u-v)]e^{i\alpha(u+v)(\tilde{\mathcal{N}}_\alpha+n_F)}$, i.e. a violation of the correct anticommutation relations. This implies that the U_α have to be properly treated. In many publications they are written as $U_\alpha = e^{i\hat{\theta}_\alpha}$, where the *phase operators* $\hat{\theta}_\alpha$ are assumed to obey the canonical commutation relations (CCR) $[\tilde{\mathcal{N}}_\alpha, \hat{\theta}_\alpha] = i\hat{1}$ [19]. We do not use this concept here because no phase operator can be constructed which obeys the CCR as an operator identity [4, 31, 32, 33, 34].

In the following we will always use the "normal ordered" form (all boson annihilation operators to the right of the creation operators) of the bosonization formula Eqs. (1.32, 1.33). Alternatively one introduces a convergence factor $e^{-n\lambda/2}$, whith $\lambda \to 0$ and works with the Hermitian Bose fields $\Phi_\alpha(v) \equiv \phi_\alpha(v) + \phi_\alpha^\dagger(v)$ as well as the fields $\Phi_+ \pm \Phi_-$. The derivatives of the latter fields are related to the total current and the deviation of the total charge density from its average value [35]. As we work with an interaction cut-off k_c the introduction of λ is not necessary and because of the space limitation this field-theoretic formulation is not used here.

3.2 Calculation of correlation functions for the TL model

In order to calculate correlation functions of the TL model with nonzero interactions it is necessary to express the field operator $\tilde{\psi}_+(v)$ Eq. (1.32) in terms of the $\alpha_n, \alpha_n^\dagger$ instead of the $b_n, b_n^\dagger$ because the former have a simple

time dependence and for the temperature dependent expectation values one has $\langle \alpha_m^\dagger \alpha_n \rangle = \delta_{mn} n_B(\omega_n)$, where $n_B(\omega) = 1/(e^{\beta\omega} - 1)$ is the Bose function. For the ground state calculation all one needs is $\alpha_n |\Phi_0\rangle = 0$ without using the explicit form of the interacting ground state $|\Phi_0\rangle$.

For periodic boundary conditions one has $b_m = c_m \alpha_m + s_m \alpha_{-m}^\dagger$ where the operators α_m and $\alpha_{-m}^\dagger$ commute. Therefore $e^{i\phi_+(v)}$ (and $e^{i\phi_+^\dagger(v)}$) in Eq. (1.32) can be written as a product of two exponentials with the annihilation operators to the right. After once using the Baker-Hausdorff formula, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ if the operators A and B commute with $[A, B]$, in order to complete the process of normal ordering one obtains for the physical field operator $\psi_\alpha(x) = \tilde{\psi}_\alpha(2\pi x/L)/\sqrt{L}$ for a system of finite length L with periodic boundary conditions [36]

$$\psi_\alpha(x) = \frac{A(L)}{\sqrt{L}} \hat{O}_\alpha \left(\frac{2\pi x}{L}\right) e^{i\chi_\alpha^\dagger(x)} e^{i\chi_\alpha(x)} \tag{1.43}$$

with

$$i\chi_\alpha(x) = \sum_{m\neq 0} \frac{\theta(\alpha m)}{\sqrt{|m|}} \left(c_m e^{ik_m x} \alpha_m - s_m e^{-ik_m x} \alpha_{-m} \right) , \tag{1.44}$$

$A(L) \equiv e^{-\sum_{n=1}^\infty s_n^2/n}$ and $\theta(x)$ is the unit step function.

This is a very useful formula for the calculation of properties of one-dimensional interacting fermions. For the special choice $s_n = s(0)e^{-n/n_c}$ where $n_c = k_c L/2\pi$ is determined by the interaction cut-off, $A(L)$ can be calculated analytically using $\sum_{n=1}^\infty z^n/n = -\log(1-z)$. For $n_c \gg 1$ this yields $A(L) = (4\pi/k_c L)^{s^2(0)}$ which shows that the prefactor in Eq.(1.44) has an anomalous power law proportional to $(1/L)^{\frac{1}{2}+s^2(0)}$. This implies that the $c_{n,\alpha}$ scale like $(1/L)^{s^2(0)}$.
The time dependent operator $\psi_+(x,t)$ follows from Eq. (1.44) by replacing α_m and α_{-m} by $\alpha_m e^{-i\omega_m t}$ and $\alpha_{-m} e^{-i\omega_m t}$ and U_+ in $\hat{O}_+$ by $U_+(t)$. Various kinds of time dependent correlation functions can quite simply be calculated using this result. Here we begin with $iG_+^<(x,t) \equiv \langle \psi_+^\dagger(0,0)\psi_+(x,t)\rangle$.
As U_+ commutes with the bosonic operator the particle number changing operators lead to a simple time dependent factor

$$U_+^\dagger U_+(t)|\Phi_0(\tilde{N}_+,\tilde{N}_-)\rangle = e^{-i\left[E_0(\tilde{N}_+,\tilde{N}_-)-E_0(\tilde{N}_+-1,\tilde{N}_-)\right]t}|\Phi_0(\tilde{N}_+,\tilde{N}_-)\rangle. \tag{1.45}$$

As $\psi_+(x)$ in Eq.(1.43) is normal ordered in the α's one has to use the Baker-Hausdorff formula only once to normal order $\psi_+^\dagger(0,0)\psi_+(x,t)$. This yields

with $k_F = 2\pi n_F/L$

$$\begin{aligned} ie^{i\mu t}G_+^<(x,t) &= \frac{A^2(L)}{L}e^{ik_F x}e^{[\chi_+(0,0),\chi_+^\dagger(x,t)]} \\ &= \frac{e^{ik_F x}}{L}e^{\sum_{n=1}^{\infty}\frac{1}{n}\left[e^{-i(k_n x-\omega_n t)}+2s_n^2\left(\cos(k_n x)e^{i\omega_n t}-1\right)\right]} \end{aligned} \tag{1.46}$$

where $\mu \equiv E_0(\tilde{N}_+,\tilde{N}_-) - E_0(\tilde{N}_+ - 1,\tilde{N}_-)$ is the chemical potential. The analytical evaluation of the sum (integral in the limit $L \to \infty$) in the exponent in Eq. (1.46) is not possible. An approximation which gives the correct large x and t behaviour [37] is to replace ω_n by $v_c k_n$. This yields for $L \to \infty$ with the exponential cut-off for the s_n used earlier [14]

$$ie^{i\mu t}G_+^<(x,t) = \frac{-i}{2\pi}\frac{e^{ik_F x}}{x - v_c t - i0}\left[\frac{r^2}{(x-v_c t - ir)(x+v_c t+ir)}\right]^{s^2(0)}, \tag{1.47}$$

where $r = 2/k_c$. As $\langle\psi_+^\dagger(0,0)\psi_+(x,0)\rangle$ for large x decays proportional to $(1/x)^{1+2s^2(0)}$ the *anomalous dimension* for the spinless model is given by

$$\alpha_L = 2s^2(0) = (K-1)^2/2K. \tag{1.48}$$

Luttinger's result for $\langle n_{k,+}\rangle$ follows by performing the Fourier transform with respect to x. The full line in Fig.1.3 was calculated with $s_n^2 = 0.3e^{-2k_n/k_c}$, while the dashed curve corresponds to $s_n^2 = 0.6(k_n/k_c)e^{-2k_n/k_c}$. The latter example corresponds to an interaction with $g_2(k \to 0) \to 0$ which leads to a vanishing anomalous dimension α_L. In this case the occupancies $\langle n_{k,+}\rangle$ have a discontinuity at k_F as in a Fermi liquid [38]. An efficient numerical algorithm to calculate $\langle n_{k,+}\rangle$ for arbitrary forms of s_n^2 is described in the appendix of reference [2].

The spectral function $\rho^<(k,\omega)$ relevant for describing angular resolved photoemission is obtained from Eq. (1.47) by a *double* Fourier transform

$$\begin{aligned} \rho_+^<(k,\omega) &= \langle c_{k,+}^\dagger\delta[\omega + (H - E_0(\tilde{N}_+ - 1,\tilde{N}_-))]c_{k,+}\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dt e^{i\omega t}\int_{-\infty}^{\infty}dx e^{-ikx}ie^{i\mu t}G_+^<(x,t). \end{aligned} \tag{1.49}$$

As Eq. (1.47) is reliable in the large x and t limit its use in Eq. (1.49) correctly describes the spectral function for $k \approx k_F$ and $\omega \approx 0$ [39]. Using the variable substitutions $u_\mp = x \mp v_c t$ the double integral factorizes and with the additional approximation $i0 \to ir$ on the rhs of Eq. (1.47) one obtains [11, 14]

$$\rho_+^<(k_F+\tilde{k},\omega) \sim \theta(-\omega - v_c|\tilde{k}|)(-\omega+\tilde{k}v_c)^{\frac{\alpha_L}{2}-1}(-\omega-\tilde{k}v_c)^{\frac{\alpha_L}{2}}e^{r\omega/v_c}. \tag{1.50}$$

Without the additional approximation there is an additional weak dependence on $\omega + \tilde{k}v_c$ [29]. The complete spectral function $\rho_+(k,\omega) = \rho_+^<(k,\omega) + \rho_+^>(k,\omega)$, where $\rho_+^>(k,\omega)$ is defined via $iG_+^>(x,t) \equiv \langle \psi_+(x,t)\psi_+^\dagger(0,0)\rangle$ can be obtained using $\rho_+^>(k_F+\tilde{k},\omega) = \rho_+^<(k_F-\tilde{k},-\omega)$ which follows from the particle-hole symmetry of the model. The absence of a sharp quasi-particle peak is manifest from $\rho_+(k_F,\omega) \sim \alpha_L|\omega|^{\alpha_L-1}e^{-r|\omega|/v_c}$.

In order to calculate correlation functions of the spin one-half TL model the operators $b_{n,\sigma}$ which appear in the generalization of the bosonization formula Eqs. (1.32) and (1.33) have to be replaced by the spin and charge bosons $b_{n,\sigma} = (b_{n,c} + \sigma b_{n,s})/\sqrt{2}$. Because of the exponential occurence of the boson operators in Eq. (1.32) *and* spin-charge separation Eq.(1.21) the Green's function $G_{+,\sigma}^<(x,t)$ *factorizes* into a spin and a charge part, which both are of the form as the square root of the function on the rhs of Eq. (1.47) . This square root results from the factors $1/\sqrt{2}$ in the expression for the $b_{n,\sigma}$. In the spin factor the charge velocity v_c is replaced by the spin velocity v_s. For the average occupation numbers one again obtains Luttinger's result Eq. (1.16) with $\alpha_L = s_c^2(0) + s_s^2(0) \equiv \alpha_c + \alpha_s$. The individual contributions can be expressed in terms of the $K_a \equiv (v_{J,a}/v_{N,a})^{1/2}$ as $\alpha_a = (K_a-1)^2/(4K_a)$. As in the spinless model the fermionic (creation) annihilation operators $c_{n,\alpha,\sigma}^{(\dagger)}$ scale like $(1/L)^{\alpha_L/2}$. For spin rotational invariant systems one has $K_s = 1$, i.e. *no* contribution to the anomalous dimension α_L from the spin part of the Hamiltonian [26]. For the momentum integrated spectral functions one obtains $\rho_{\alpha,\sigma}(\omega) \sim |\omega|^{\alpha_L}$ as in the spinless model [40]. The k-resolved spectral functions $\rho_{\alpha,\sigma}(k,\omega)$ on the other hand show a drastic difference to the model without spin. The delta peaks of the noninteracting model are broadened into *one* power law threshold Eq. (1.47) in the model without spin and *two* power law singularities (see Fig. 1.4) in the model including spin [29, 30, 37] (for $\alpha_L < 1/2$ in the case of a spin independent interaction). The "peaks" disperse *linearly* with $k - k_F$.

It is also straightforward to calculate various response functions for the TL model. We discuss the density response function $R(q,z) \equiv -\langle\langle \hat{\rho}_q; \hat{\rho}_{-q}\rangle\rangle_z/L$ of the spinless model for $q \approx 0$ and $q \approx \pm 2k_F$, where

$$\langle\langle \hat{A}; \hat{B}\rangle\rangle_z \equiv -\frac{i}{\hbar}\int_0^\infty \langle [A(t), B]\rangle e^{izt}dt \tag{1.51}$$

involves the retarded commutator [41] and z is a frequency above the real axis. From the decomposition [42] $\psi(x) \approx \psi_+(x) + \psi_-(x)$ of the field operator $\psi(x)$ in the original Tomonaga model it is obvious that the operator $\hat{\rho}(x) = \psi^\dagger(x)\psi(x)$ of the density (see Eq. (1.8)) has two very different contributions

$$\hat{\rho}(x) \approx \hat{\rho}_+(x) + \hat{\rho}_-(x) + \left(\psi_+^\dagger(x)\psi_-(x) + h.c.\right) \tag{1.52}$$

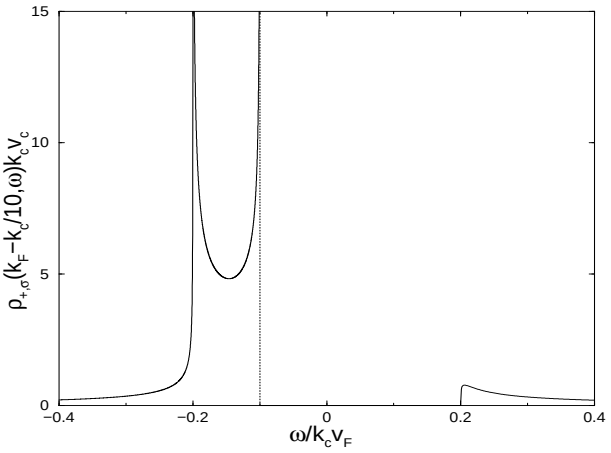

Figure 1.4. Spectral function $\rho_{+,\sigma}(k_F + \tilde{k}, \omega)$ as a function of normalized frequency for $\tilde{k} = -k_c/10$ for the TL- model with a spin independent interaction. The parameters are chosen such that $v_c = 2v_F$ and α_L=1/8.

$$\equiv \quad \hat{\rho}_0(x) + \hat{\rho}_{2k_F}(x).$$

The spatial Fourier transform of $\hat{\rho}_0$ is *linear* in the boson operators Eq. (1.12) and the $q \approx 0$ contribution of the density response function $[R(q,z)]_0$ defined with the operators $(\hat{\rho}_0)_q$ follows using the (linear) equations of motion for the $b_n(t)$ as

$$[R(q,z)]_0 = \frac{1}{\pi\hbar}\frac{q^2 v_J(q)}{[qv_c(q)]^2 - z^2} \tag{1.53}$$

This exact result for the $q \approx 0$ contribution agrees with the RPA result for the original Tomonaga model. This fact, not mentioned in Tomonaga's paper [5] as the RPA paper by Bohm and Pines [43] was not yet published, was "discovered" many times in the literature. For the spin $1/2$-model $[R(q,z)]_0$ has an additional factor 2 and one has to replace v_J by v_{J_c}.

The real part of the $(q = 0)$ frequency dependent conductivity $\sigma(\omega + i0)$ follows from $[R(q, \omega + i0)]_0$ by multiplication with $ie^2\omega/q^2$ and taking the limit $q \to 0$. This yields for the spinless model

$$(\hbar/e^2)\mathrm{Re}\sigma(\omega + i0) = v_J\delta(\omega) = Kv_c\delta(\omega) \tag{1.54}$$

For the Galilei invariant Tomonaga model Eq. (1.14) one has $v_J = v_F$, i.e. the weight D of the zero frequency "Drude peak" is independent of the interaction, as expected. As D apart from a constant is given by the second derivative of $E_0(\Phi)/L$ with respect to the magnetic flux through the $1d$ ring [44], K (or K_c) can be obtained from a *ground state calculation* for microscopic lattice

models using $K_{(c)} = (D\kappa/D_0\kappa_0)^{1/2}$, where κ is the compressibility discussed in Eq. (1.24). The anomalous decay of the correlation functions for these models, which are more difficult to calculate directly, can then be quantitatively predicted if Haldane's LL concept is taken for granted. For a *weak* two-body interaction the result for $K_{(c)} - 1$ *linear* in the interaction follows from first order perturbation theory for the ground-state energy, which involves the (non-selfconsistent) Hartree and Fock terms. As they are independent of the magnetic flux, D/D_0 has *no* term linear in $\tilde{v}$, i.e. $K_c \approx (\kappa/\kappa_0)^{1/2} = (v_F/v_{N_c})^{1/2}$, which holds exactly for Galilei invariant continuum models [45]. Performing the second derivative of $E_0^{(1)}(N)$ with respect to N yields [46]

$$K_c = 1 - \frac{2\tilde{v}(0) - \tilde{v}(2k_F)}{2\pi\hbar v_F} + O(\tilde{v}^2). \tag{1.55}$$

In the spinless case the factor 2 in front of $\tilde{v}(0)$ is missing in the result for K. Instead of D as the second input besides κ one can obtain v_c directly by calculating the lowest charge excitation energy (see section 4).

The easiest way to calculate the $q \approx \pm 2k_F$ contribution to the density response is to use the bosonization of the field operators [14]. The first step is to normal order $\psi_+^\dagger(x)\psi_-(x)$ using Eq. (1.43) . This gives a factor $e^{[\chi_+(x),\chi_-^\dagger(x)]}$ which using $[\chi_+(x), \chi_-^\dagger(x)] = -2\sum_{m>0} c_m s_m/m$ together with the factor $A^2(L)$ leads to

$$\psi_+^\dagger(x)\psi_-(x) = \frac{a_0}{L}\left(\frac{4\pi}{k_c L}\right)^{K-1} \hat{O}_+^\dagger(\frac{2\pi x}{L})\hat{O}_-(\frac{2\pi x}{L}) e^{-i\Delta\chi^\dagger(x)} e^{-i\Delta\chi(x)} \tag{1.56}$$

with

$$\Delta\chi(x) \equiv \chi_+(x) - \chi_-(x) = -i\sum_{m>0}\sqrt{\frac{K_m}{m}}\left(e^{ik_m x}\alpha_m - e^{-ik_m x}\alpha_{-m}\right).$$

Here a_0 is a dimensionless constant of order unity and the exponent $K-1$ of the second factor on the rhs follows using $2s_m^2 + 2c_m s_m = K_m - 1$. The importance of this factor for impurity scattering in Luttinger liquids was first pointed out by Mattis (1974) [21] and will be discussed later. The calculation of the two terms of the commutator $\langle[\psi_+^\dagger(x,t)\psi_-(x,t), \psi_-^\dagger(0,0)\psi_+(0,0)]\rangle$ is then straightforward and one obtains for the spectral function of the $q \approx \pm 2k_F$ response function the power law behaviour [14]

$$\mathrm{Im}[R(\pm 2k_F + Q, \omega)]_{2k_F} \sim \mathrm{sign}(\omega)\theta(\omega^2 - v_c^2Q^2)\left(\frac{\omega^2 - v_c^2Q^2}{v_c^2k_c^2}\right)^{K-1} \tag{1.57}$$

The *static* $\pm 2k_F + Q$ response diverges proportionally to $|Q|^{2(K-1)}$ which has to be contrasted with the logarithmic singularity in the noninteracting case. In the model including spin the exponent $2K - 2$ is replaced by $K_c + K_s - 2$.

The pair propagator $P(q, \omega)$ resulting from the response function for $\hat{A} = \psi_+^\dagger(x)\psi_-^\dagger(x)$ and $\hat{B} = \psi_-(0)\psi_+(0)$ was found by Luther and Peschel to be the same as the $2k_F$-density response, provided the *sign* of the interaction is reversed [14]. An *attractive* interaction leads to a power law divergence in $P(q = 0, \omega = 0)$ as the temperature is lowered, indicative of large *pairing fluctuations*.

3.3 The TL model with additional interactions and perturbations

The exact solution of the TL model essentially depends on the fact that the numbers of right and left movers are *conserved*. This symmetry can be destroyed by a one-particle potential with $\pm 2k_F$-Fourier components or by interaction terms which change the individual particle numbers, like $2k_F$-"backscattering" terms or Umklapp-terms for a half-filled band. With such additional terms the model is in general no longer exactly solvable. Important insights about the influence of such terms have come from a (perturbational) RG analysis [16, 25].

3.3.1 Impurity in a spinless TL model

We begin with the spinless model with an additional impurity which is described by

$$\hat{V}_I = \int [V_F(x)\hat{\rho}_0(x) + V_B(x)\hat{\rho}_{2k_F}(x)]\, dx \equiv \hat{V}_F + \hat{V}_B, \tag{1.58}$$

where $\hat{V}_F$ describes the forward and $\hat{V}_B$ the backward scattering due to the impurity and the two different operators for the densities are defined in Eq. (1.52). As the forward scattering term is *linear* in the boson operators it can be treated in an exact way. The backscattering term has the property $[\hat{V}_B, \tilde{\mathcal{N}}_\alpha] \neq 0$ and the model can no longer be solved exactly (except for $K = 1/2$ and a special assumption about V_B, as discussed below). For a zero range impurity it follows directly from Eq. (1.56) that $\hat{V}_B$ scales as $(1/L)^K$ while $\tilde{H}_{TL}$ in Eq. (1.17) scales as $1/L$. Therefore the influence of $\hat{V}_B$ depends crucially on the *sign* of the two-body interaction [21, 22]. For *repulsive* interactions one has $K < 1$ which shows that $\hat{V}_B$ is a *relevant* perturbation. For $K > 1$, i.e. an attractive interaction, $\hat{V}_B$ is irrelevant. A detailed RG analysis of the problem was presented in a seminal paper by Kane and Fisher [25]. For a

zero range backscattering potential and two-body interaction they mapped the problem to a *local* bosonic sine-Gordon model [25, 35, 47]. The subsequent RG analysis shows that the backscattering amplitude scales as Λ^{K-1} when the flow parameter Λ is sent to zero [48], as can be anticipated from Eq. (1.56) . This leads to the breakdown of the perturbational analysis in V_B for repulsive interactions. As already mentioned in section 2 this analysis was supplemented by a RG analysis of a weak hopping between two semi-infinite chains. The weak hopping scales to zero like Λ^{α_B} for repulsive interactions, where $\alpha_B = K^{-1} - 1$ is the *boundary exponent*. It describes e.g. the different scaling $\rho(x,\omega) \sim |\omega|^{\alpha_B}$ of the local spectral function near a hard wall boundary of a LL [25, 49, 50]. These scaling results together with the asumption mentioned in section 2 leads to the "split chain scenario" in which even for a *weak* impurity the observables at low energies behave as if the system is split into two chains with fixed boundaries at the ends. Within the bosonic field theory this assumption was verified by quantum Monte Carlo calculations [51] and the thermodynamic Bethe ansatz [52].

This implies e.g. for the local density of states $\rho(x,\omega) \sim |\omega|^{\alpha_B}$ for small $|\omega|$ and x *near the impurity* like in a LL near a hard wall. The transmission through the impurity vanishes near k_F proportional to $\sim |k - k_F|^{2\alpha_B}$ which leads to a conductance $G(T)$ which vanishes with temperature T in power law fashion $G(T) \sim T^{2\alpha_B}$ [25].

Additional insight comes from the analysis for the special value $K = 1/2$ [25, 35, 32] . For $V_B(x) = V_B\delta(x)$ the expression for $\Delta\chi(0)$ in Eq. (1.56) can be written in terms of new boson operators $\tilde{\alpha}_m \equiv (\alpha_m - \alpha_{-m})/\sqrt{2}$. If one neglects the momentum dependence of K_m in Eq. (1.56) and puts $K_m = 1/2$ one obtains $i\Delta\chi(0) = \sum_{m\geq 1} \tilde{\alpha}_m/\sqrt{m}$ as in the bosonization of a *single* field operator Eqs. (1.32) and (1.33) . It is then possible to *refermionize* the $K = 1/2$-TL model with a zero range impurity. Even the Klein factors can properly be handled [32] and one obtains a model of "shifted noninteracting Fermi oscillators" which can be solved exactly introducing an auxiliary *Majorana fermion* [35, 32]. Unfortunately the local densities of states *cannot* be calculated exactly because of the complicated nonlinear relationship between the original fermion operators and the fermion operators which diagonalize the shifted Fermi oscillator problem [32]. Additional results for the transport through a spinless LL containing one impurity were obtained by mapping the problem onto the boundary sine-Gordon model and using its integrability [53].

In order to bridge the two regimes treated by Kane and Fisher one can use a fermionic RG description bearing in mind that it is perturbational in the two-body interaction [54, 55]. It shows that the *long range* oscillatory effective impurity potential is responsible for the "splitting", for site impurities as well as for hopping impurities of arbitrary strength. For realistic parameters *very large* systems are needed to reach the asymptotic open chain regime [55]. Hence only

special mesoscopic systems, such as very long carbon nanotubes, are suitable for experimentally observing the impurity induced open boundary physics.

For a discussion of the impurity problem in the TL model including spin see also reference [56].

3.3.2 The TL- model with additional two-body interactions

Tomonaga was well aware of the limitations of his approach for more generic two-body interactions ("In the case of force of too short range this method fails" [5]). We therefore first discuss Tomonaga's continuum model in this short range limit $k_c \gg k_F$ opposite to the one considered in section 2. Then low energy scattering processes with momentum transfer $\approx \pm 2k_F$ are possible and have to be included in the theoretical description of the low energy physics.

In the "g-ology" approach one linearizes the nonrelativistic dispersion around the two Fermi points and goes over to right- and left-movers as in section 2. Then the "$2k_F$"-processes are described by the additional interaction term

$$H_{\rm int}^{(1)} = \sum_{\sigma,\sigma'} \int \left(g_{1\parallel} \delta_{\sigma,\sigma'} + g_{1\perp} \delta_{\sigma,-\sigma'} \right) \psi_{+,\sigma}^\dagger(x) \psi_{-,\sigma'}^\dagger(x) \psi_{+,\sigma'}(x) \psi_{-,\sigma}(x) dx. \tag{1.59}$$

For a spin-independent two particle interaction one has $g_{1\parallel} = g_{1\perp} = g_1$. For the zero range interaction assumed in Eq. (1.59) one has to introduce band cut-offs to regularize the interaction term. The RG flow equations for the cut-off dependent interactions on the one-loop level are quite simple [16]. If s runs from zero to infinity in the process of integrating out degrees of freedom one obtains for spin-independent interactions

$$\begin{aligned} \frac{dg_1(s)}{ds} &= -\frac{1}{\pi\hbar v_F} g_1^2(s) \\ \frac{dg_2(s)}{ds} &= -\frac{1}{2\pi\hbar v_F} g_1^2(s) \end{aligned} \tag{1.60}$$

and g_4 is not renormalized. Obviously $g_1(s)$ can be obtained from the first equation only

$$g_1(s) = \frac{g_1}{1 + s\frac{g_1}{\pi\hbar v_F}}, \tag{1.61}$$

where g_1 is the starting value. It is easy to see that $g_1(s) - 2g_2(s) = g_1 - 2g_2$ holds by subtracting twice the second equation from the first in Eq. (1.60) . In the following we use the notation $g_\nu^* \equiv g_\nu(s \to \infty)$. Now one has to distinguish two cases:

for $g_1 \geq 0$ one renormalizes to the *fixed line* $g_1^* = 0,\ g_2^* = g_2 - g_1/2$ and the *fixed point Hamiltonian is a TL model* which shows the generic importance of the TL model for repulsive interactions. In this case the g_1-interaction is

called *marginally irrelevant*. For the nonrelativistic continuum model with a spin independent interaction one has $g_{2c}^* = 2\tilde{v}(0) - \tilde{v}(2k_F)$ and $g_{2s}^* = 0$ and for the stiffness constant $K_c = [(2\pi v_F + g_{4c}^* - g_{2c}^*)/(2\pi v_F + g_{4c}^* + g_{2c}^*)]^{1/2} \approx 1 - [2\tilde{v}(0) - \tilde{v}(2k_F)]/(2\pi\hbar v_F)$ and $K_s = 1$. Due to the approximations made, also here only the result for $K_c - 1$ *linear* in $\tilde{v}$ is reliable. The agreement with the direct calculation Eq. (1.55) shows explicitly to leading order in the interaction that Haldane's Luttinger liquid concept is consistent.

For $g_1 < 0$ the solution (1.61) shows that $g_1(s)$ *diverges* at a finite value of s. Long before reaching this point the perturbational analysis breaks down and all one can say is that the flow is towards *strong coupling*. In this case the g_1-interaction is called *marginally relevant*. In order to to obtain an understanding of the strong coupling regime it is useful to bosonize the additional interaction $H_{\rm int}^{(1)}$ in Eq. (1.59) [57]. The term proportional to $g_{1\|}$ is of the form of a $g_{2\|}$-interaction and therefore bilinear in the boson operators Eq. (1.19) . For the $g_{1\perp}$-term one uses the bosonization of the field operators Eqs. (1.32) and (1.33) with additional spin labels. As the $g_{1\perp}$-term contains field operators $\psi_{\alpha\uparrow}^\dagger(x)\psi_{\alpha\downarrow}(x)$ of *opposite* spin it only involves "spin bosons" Eq. (1.19) , which implies "spin-charge separation" also for this model [58]. The charge part stays trivial with massless charge bosons as the elementary interactions. Luther and Emery showed that for a particular value of $g_{1\|}$ the $g_{1\perp}$-term can be written as a product of spinless fermion field operators and the exact solution for the spin part of the Hamiltonian is possible using *refermionization* [57], discussed earlier in connection with the backscattering impurity. The diagonalization of the resulting problem of noninteracting fermions is simple and shows that the spectrum for the spin excitations is *gapped*. It is generally believed that these properties of *Luther-Emery phases* are not restricted to the solvable parameter values.

Strong coupling phenomena which lead to deviations from LL-properties with gapped phases are discussed in detail in section 4 for *lattice models*. There in case of *commensurate* filling *Umklapp processes* can become important, e.g. for half filling where two left movers from the vicinity of the left Fermi point are scattered into two right movers near the right Fermi point or vice versa. As $G = 4k_F$ is a reciprocal lattice vector such a scattering process is a low energy process conserving quasi-momentum. In the g-ology model such processes are described by an additional term

$$H_{\rm int}^{(3)} = \frac{1}{2}\sum_{\sigma,\sigma'}\int g_3^{\sigma,\sigma'}(x-y)\left[\psi_{+,\sigma}^\dagger(x)\psi_{+,\sigma'}^\dagger(y)\psi_{-,\sigma'}(y)\psi_{-,\sigma}(x)e^{2ik_F(x+y)} + H.c.\right]dxdy \tag{1.62}$$

Umklapp processes for $\sigma = \sigma'$ are only possible for nonzero interaction range.

4. Results for integrable lattice models

As mentioned in subsection 2.4, results for integrable models which can be solved exactly by the Bethe ansatz played a central role in the emergence of the general "Luttinger liquid" concept [18]. It is therefore appropriate to shortly present results for the two most important lattice models of this type, the model of spinless fermions with nearest neighbour interaction and the $1d$-Hubbard model. (We put $\hbar = 1$ in this section.)

4.1 Spinless fermions with nearest neighbour interaction

The one-dimensional single band lattice model of spinless fermions with nearest neighbour hopping matrix element $t(> 0)$, and nearest neighbour interaction U (often called V in the literature) is given by

$$H = -t \sum_j \left(c_j^\dagger c_{j+1} + H.c. \right) + U \sum_j \hat{n}_j \hat{n}_{j+1} \equiv \hat{T} + \hat{U}, \qquad (1.63)$$

where j denotes the sites and the $\hat{n}_j = c_j^\dagger c_j$ are the local occupation number operators. In the noninteracting limit $U = 0$ one obtains for lattice constant $a = 1$ the well known dispersion $\epsilon_k = -2t \cos k$. For the following discussion of the interacting model ($U \neq 0$) we mainly focus on the *half filled* band case $k_F = \pi/2$ with $v_F = 2t$. In contrast to the (continuum) Tomonaga model Umklapp terms appear when the interaction term in Eq. (1.63) is written in the k-representation. As discussed below they are *irrelevant* at the noninteracting ($U = 0$) fixed point [17]. Therefore the system is a *Luttinger liquid* for small enough values of $|U|$. The large U limit of the model is easy to understand: For $U \gg t$ charge density wave (CDW) order develops in which only every other site is occupied thereby avoiding the "Coulomb penalty". For large but negative U the fermions want to be as close as possible and phase separation occurs. For the quantitative analysis it is useful that the model in Eq. (1.63) can be exactly mapped to a $S = 1/2$-Heisenberg chain with uniaxially anisotropic nearest neighbour exchange ("XXZ" model) in a magnetic field by use of the Jordan-Wigner transformation [59]. For $U > 0$ this model is also called the antiferromagnetic Heisenberg-Ising model. The point $U \equiv U_c = 2t$ corresponds to the *isotropic* Heisenberg model. For $U > 2t$ the Ising term dominates and the ground state is a well defined doublet separated by a gap from the continuum and long range antiferromagnetic order exists. For $-2t < U \leq 2t$ there is no long range magnetic order and the spin-excitation spectrum is a gapless continuum. The mapping to the XXZ-model therefore suggests that the spinlesss fermion model Eq. (1.63) in the half filled band case is a Luttinger liquid for $|U| < 2t$.

Before we present the exact results for the Luttinger liquid parameters K and v_c from the Bethe ansatz solution [18, 60], we shortly discuss the RG approach

to the model. A perturbative RG calculation around the free fermion fixed point is discussed in detail in Shankar's review article [17]. The first step is to write the four fermion matrix elements of the interaction $\hat{U}$ in Eq. (1.63) in the k-representation. This yields for a chain of N sites with periodic boundary condition and values $k_j = 2\pi j/N$ in the first Brillouin zone

$$\langle k_1, k_2|\hat{U}|k_3, k_4\rangle = \frac{2U\cos(k_1 - k_3)}{N} \sum_{m=0,\pm 1} \delta_{k_1+k_2,k_3+k_4+2\pi m} \qquad (1.64)$$

The $m = 0$ term on the rhs of Eq. (1.64) represents the direct scattering terms and the $m = \pm 1$ terms the Umklapp processes. The matrix element antisymmetrized in k_3 and k_4 is proportional to $\sin[(k_1 - k_2)/2] \sin[(k_3 - k_4)/2]$. Therefore the low energy Umklapp Hamiltonian scales like $(1/L)^3$ which shows that it is strongly irrelevant at the free field fixed point [17]. This analysis confirms the Luttinger liquid behaviour for small values of U, but gives no hint about the critical value U_c for the CDW transition. With the separation $\hat{U} \equiv \hat{U}_0 + \hat{U}_{\rm Umklapp}$ implied by Eq. (1.64) one can do better by first treating $\hat{T} + \hat{U}_0$ by bosonization and then perform the RG analysis around the corresponding TL fixed point to get information for which value of U the Umklapp term starts to be a relevant perturbation. For this analysis it is easier to work directly with the unsymmetrized matrix elements in Eq. (1.64). As $k_1 - k_3 \approx \pm\pi$ for the low energy Umklapp processes this leads after extending the (linearized) dispersion of the right and left movers from $-\infty$ to ∞ to a g_3-interaction with a range of order $r = a$. The scaling dimension of the corresponding $H^{(3)}_{\rm int}$ follows using bosonic normal ordering as in Eq. (1.56). For $x - y$ of order r or smaller one obtains

$$\begin{aligned} \psi^\dagger_+(x)\psi^\dagger_+(y)\psi_-(y)\psi_-(x)L^2 \quad &\sim \quad \left(\frac{x-y}{L}\right)^2 \left(\frac{r}{L}\right)^{4(K-1)} \qquad (1.65) \\ &\times (U^\dagger_+)^2 U^2_- e^{2ik_F(x+y)} e^{iB^\dagger(x,y)} e^{iB(x,y)} \,, \end{aligned}$$

where $B(x, y) = \chi_-(x) + \chi_-(y) - \chi_+(x) - \chi_+(y)$ with $\chi_\alpha(x)$ defined in Eq. (1.44). The first factor on the rhs is due to the Pauli principle and describes the same physics as the two sin-factors mentioned above for small arguments. Therefore the second factor has to provide more than two powers of L to make the Umklapp term a relevant perturbation, which happens for $K < 1/2$. As discussed below, the exact Bethe ansatz result for K yields $U_c = 2t$. If one uses the simple *linear* approximation for $K - 1$ in Eq. (1.55) one obtains with Eq. (1.64) $K^{\rm lin} = 1 - U/(\pi t)$ for the critical value $U^{\rm lin}_c/t = \pi/2$, not too bad an approximation.

Exact analytical results for the Luttinger liquid parameters for the half filled model can be obtained from the Bethe Ansatz solution [18, 60, 61]. It is *not* necessary to address the anomalous decay of the correlation functions directly,

but one can use a ground state property and the lowest charge excitation to extract the parameters, as was dicussed in connection with Eq. (1.55). This yields for the stiffness constant $K = \pi/[2\arccos(-U/2t)]$ and for the charge velocity $v_c = \pi t\sqrt{1-(U/2t)^2}/[\pi - \arccos(-U/2t)]$. For repulsive interactions $U > 0$ the value of K decreases monotonously from the noninteracting value $K = 1$ to $K = 1/2$ for $U = 2t$, which corresponds to an anomalous dimension $\alpha_L = (K + 1/K)/2 - 1 = 1/4$. For attractive interactions K diverges when U approaches $-2t$, and the charge velocity v_c goes to zero. Results for the Luttinger liquid parameter K for less than half filled bands are shown in Fig. 1.5 [62]. The limit $a \to 0$ and $n \to 0$ corresponds to the continuum limit.

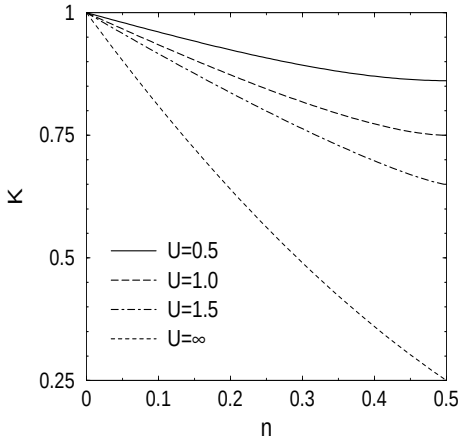

Figure 1.5. Luttinger liquid parameter K from the Bethe ansatz solution as a function of the band filling n for different values of U $(t = 1)$. The short dashed curve shows the infinite U result $(1/2 + |n - 1/2|)^2$.

As the interaction goes over to a contact interaction its effect vanishes because of the Pauli principle and K goes to 1. For small enough values of U the linear approximation Eq. (1.55) $K^{\rm lin} = 1 - U\sin(n\pi)/\pi t$ provides a good approximation for *all* values of n, in contrast to the Hubbard model discussed below. In the infinite U limit the Bethe Ansatz equations simplify considerably and the ground-state energy as well as low lying excited states can be calculated analytically [61]. With these results it is easy to show that $K = (1-n)^2$ holds for $0 < n < 1/2$, i.e. $K = 1/4$, is the lower bound for K in the LL regime of the model [18]. The corresponding upper bound of the anomalous dimension is $\alpha_L = 9/8$. In order to achieve larger values of α_L the model in Eq. (1.63) has to be generalized to include longer range interactions [63].

4.2 The Hubbard model

As there exists an excellent review on the LL behaviour in the $1d$-Hubbard model [64], the following discussion will be rather short. As the model includes spin the on-site interaction between electrons of opposite spins is not forbidden by the Pauli principle. This is taken as the only interaction in the model. The $1d$ Hubbard Hamiltonian reads

$$H = -t \sum_{j,\sigma} \left(c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + H.c. \right) + U \sum_{j} \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow}. \tag{1.66}$$

In the *extended* Hubbard model a next nearest interaction term $V \sum_j \hat{n}_j \hat{n}_{j+1}$ with $\hat{n}_j \equiv \hat{n}_{j,\uparrow} + \hat{n}_{j,\downarrow}$ is added [65]. In order to show the important difference to the spinless model Eq. (1.63) we again first discuss the *half-filled* band case, which is metallic for $U = 0$. For $U \gg t$ the "Coulomb penalty" is avoided when each site is *singly* occpied. Then only the spin degrees of freedom matter. In this limit the Hubbard model can be mapped to a spin-$1/2$ Heisenberg antiferromagnet with an exchange coupling $J = 4t^2/U$. In the charge sector there is a large gap $\Delta_c \sim U$ while the spin excitations are gapless. The $1d$ Hubbard model can also be solved exactly using the Bethe ansatz [66] and properties like the charge gap or the ground-state energy can be obtained by solving Lieb and Wu's integral equation. In contrast to the spinless model described in the previous subsection the charge gap in the Hubbard model is finite for *all* $U > 0$. While for $U \gg t$ it is asymptotically given by U it is exponentially small, $\Delta_c \approx (8t/\pi)\sqrt{U/t} \exp(-2\pi t/U)$, for $0 < U \ll t$. This shows that the Umklapp term is no longer irrelevant at the free field fixed point. The Pauli principle factor of Eq. (1.65) is missing as the interaction is between electrons of *opposite* spin. The Umklapp term is therefore a marginal perturbation. The RG analysis [16] shows that the Umklapp term is marginally *relevant* while the $2k_F$-backscattering ("g_1") interaction is marginally *irrelevant* for $U > 0$ as discussed following Eq. (1.60).

When the band is *not* half filled Umklapp is not a low energy process and the Hubbard model is a Luttinger liquid with $K_s = 1$. The LL parameters K_c and v_a can be obtained by (numerically) solving Lieb an Wu's integral equation [67]. Even for $0 < U \ll t$ the perturbative result Eq. (1.55) works well only for intermediate filling $n \equiv N_{el}/N \approx 0.5$, where N_{el} is the number of electrons (half filling corresponds to $n = 1$) . In the limit $n \to 0$ the Fermi velocity $v_F = 2t \sin(\pi n/2)$ goes to zero but $2\tilde{v}(0) - \tilde{v}(2k_F) = U$ stays finite and the correction term increases with decreasing n in contrast to the spinless model. The Bethe ansatz results show that $K_c \to 1/2$ for $n \to 0$ as well as $n \to 1$ for *all* $U > 0$. For $U \to \infty$ it leads to $K_c \to 1/2$ for *all fillings* n different from 1. In this limit the velocities are given by $v_c = 2t \sin(\pi n)$ and $v_s = (2\pi t^2/U)[1 - \sin(2\pi n)/(2\pi n)]$, i.e. the spin velocity goes to zero [64, 67]. The $U = \infty$ results for v_c and K_c can be

understood *without* the Bethe ansatz solution. Double occupancies of the lattice sites are forbidden and the system behaves like a system of *noninteracting spinless fermions* with k_F replaced by $2k_F$ [64]. The spin degrees of freedom play no role and any configuration of the spins gives an eigenfunction of the same energy. This immediately explains the result for v_c mentioned above. For a TL model with spin one obtains (for fixed $N_\uparrow - N_\downarrow$) from Eqs. (1.17) and (1.21) $L(\partial^2 E_0/\partial N^2)_L = \pi v_{N_c}/2$, while the factor $1/2$ is missing in the spinless case. The formula for the spinless case can be used to calculate $L(\partial^2 E_0/\partial N^2)_L$ for $U = \infty$ with v_N replaced by $v_F(2k_F)$, using the spinless fermion analogy. This yields $v_{N_c} = 2v_c$ i.e. $K_c = 1/2$.

As the calculation of correlation functions not only requires excitation energies but also many electron matrix elements which are difficult to evaluate using the Bethe ansatz, various numerical methods have been used to study e.g. the manifestation of spin-charge separation in the one-particle spectral function [68, 69]. The Bethe ansatz approach simplifies in the infinite U limit [70]. After earlier work [71, 72] the frequency dependent optical conductivity of the $1d$ Hubbard model was also studied using Bethe ansatz methods [73, 74], as well as the dynamical density-matrix renormalization group [74].

5. Weakly coupled chains: the Luttinger to Fermi liquid transition

Strictly one-dimensional systems are a theoretical idealization. Apart from this even the coupling to an experimental probe presents a nontrivial disturbance of a Luttinger liquid. Unfortunately the weak coupling of a $1d$ system to such a probe as well as the coupling between several LLs is theoretically not completely understood [26]. The coupling between the chains in a very anisotropic $3d$ compound generally, at low enough temperatures, leads to true *long-range order*. The order develops in the phase for which the algebraic decay of the correponding correlation function of the single chain LL is slowest [64]. This can lead e.g. to charge-density wave (CDW), spin-density wave (SDW) order or superconductivity.

In the following we shortly address some important issues of the coupled chain problem, which are a prerequisite for the theoretical descriptions of the attempts to experimentally verify LL behaviour. In the first part of this section theoretical aspects of the problem of an infinite number of coupled chains are addressed. This is followed by a short discussion of the (approximate) experimental realizations of LLs. As there are other chapters in this book addressing this question the discussion will be rather short.

5.1 Theoretical models

We consider a systems of $N_\perp$ coupled chains described by the Hamiltonian

$$H = \sum_{i=1}^{N_\perp} H_i + \sum_{i \neq j} H_{ij}^{(ee)} + \sum_{n,(\sigma)} \sum_{i,j} t_{\perp,ij} c^\dagger_{n,(\sigma),i}, c_{n,(\sigma),j} \tag{1.67}$$

where the H_i are the Hamiltonians of the individual chains, the $H_{ij}^{(ee)}$ represent the two-body (Coulomb) interaction of electrons on different chains and the last term $H^{(t_\perp)}$ describes the hopping between the chains with $t_{\perp,ij}$ the transverse hopping matrix elements and the $c^{(\dagger)}_{n,(\sigma),i}$ the (creation) annihilation operators of one-particle states with quasi-momentum k_n along the chain i and spin σ (if spin is included in the model). The individual H_i can be TL-Hamiltonians Eq. (1.17) or lattice Hamiltonians like in Eqs. (1.63) or (1.66).

We address the question if LL physics survives in such a model. The second and the third term on the rhs of Eq. (1.67) describe different types of couplings between the chains. If the transverse hopping is neglected ($t_\perp \equiv 0$) the model can be solved exactly for special assumptions about the two-body interaction and the H_i. If the individual chains are described by TL-Hamiltonians Eq. (1.17) and the interaction $H_{ij}^{(ee)}$ can be expressed in terms of the densities $\hat{\rho}_{n,(a),\alpha,i}$ the exact solution is possible by bosonization [75, 76]. This is important when the long range Coulomb interaction is taken into account. For a *single* chain the corresponding one-dimensional Fourier transform $\tilde{v}(q)$ (which has to be regularized at short distances) has a logarithmic singularity for $q \to 0$. This leads to $K_{(c)} = 0$ and the divergence of the anomalous dimension, i.e. the system is *not* a LL. The $4k_F$ harmonic of the density-density correlation function shows a very slow decay almost like in a Wigner crystal [77]. The Coulomb coupling *between* the chains removes this singularity and a three-dimensional extended system of coupled chains *is* a LL [75]. The corresponding anomalous dimension can be calculated and leads to values of order unity for realistic values of the coupling constant $e^2/(\pi\hbar v_F)$ [76]. If $2k_F$-scattering terms of the interaction are kept the model can no longer be solved exactly and a more complicated scenario emerges in the parquet approximation [78].

The inclusion of the transverse hopping presents a difficult problem even if the inter-chain two-body interactions are *neglected*. This is related to the fact that the transverse hopping is a *relevant* perturbation for $\alpha_L < 1$ [79, 80, 81]. This can easily be seen if the individual chains are described by TL-Hamiltonions Eq. (1.17), scaling like $1/L$. As discussed in section 3 the $c^{(\dagger)}_{n,(\sigma),i}$ scale like $(1/L)^{\alpha_L/2}$. As $H^{(t_\perp)}$ involves products of creation and annihilation operators on *different* chains no further boson normal ordering is necessary and $H^{(t_\perp)}$ scales as $(1/L)^{\alpha_L}$. This suggests "confinement" for $\alpha_L > 1$: if an extra electron is put on the j-th chain it stays there with probability close to 1 even

in the long time limit. This conclusion can be questioned as RG calculations perturbative in $t_\perp$ demonstrate that the hopping term generates new and relevant interchain two-particle hoppings. These calculations show that the system flows to a strong-coupling fixed point which cannot be determined within the approach [81, 82].

If inter-chain two-body interactions are *included* the relevance of hopping terms can be different. When only density-density and current-current interactions between the wires are included, as discussed above [75, 76], the possible relevance around this Gaussian model, recently called *sliding* LL [83, 84, 85], can be different. If the single chains are in the spin-gapped Luther-Emery regime [57] single-particle hopping between the chains is irrelevant and the coupled system can show power-law correlations characteristic of a $1d$-LL [83, 85]. For the spinless model single particle and pair hoppings can be irrelevant for strong enough forward interactions [84].

In the following we concentrate on the Luttinger to Fermi liquid crossover. In order to get a *quantitative* picture it is desirable to study models which allow *controlled approximations*. The simple perturbative calculation in $t_\perp$ for the calculation of the one-particle Green's function by Wen [80] discussed below is unfortunately only controlled in the rather unphysical limit when the transverse hopping is independent of the distances of the chains ($t_{\perp,ij} \equiv t_\perp$) [86]. The (retarded) one-particle Green's function G is expressed in terms of the selfenergy Σ

$$G(k_\parallel, \vec{k}_\perp, z; t_\perp) = \frac{1}{z - \epsilon_{k_\parallel, \vec{k}_\perp} - \Sigma(k_\parallel, \vec{k}_\perp, z; t_\perp)}, \tag{1.68}$$

where $\epsilon_{k_\parallel, \vec{k}_\perp}$ denotes the energy dispersion for the noninteracting model and $z = \omega + i0$ is the frequency above the real axis. For small $t_\perp$ the dispersion can be linearized around $k_\parallel = \pm k_F$ near the *open* noninteracting Fermi surface. This yields $\epsilon_{k_\parallel, \vec{k}_\perp} \approx \pm v_F(k_\parallel \mp k_F) + t_\perp(\vec{k}_\perp)$. In the context of Fermi liquid theory the selfenergy is studied in (all orders) perturbation theory in the two-body interaction v around the noninteracting limit. This can be done using standard Feynman diagrams. In the present context one wants to study how the LL behaviour for *finite* two body interaction and finite anomalous dimension is modified by the transverse hopping. Similar to perturbation theory for the Hubbard model around the atomic limit nonstandard techniques have to be used [87]. The simplest approximation, which corresponds to the "Hubbard I" approximation for the Hubbard model, is to replace Σ in Eq. (1.68) in zeroth order in $t_\perp$ by the selfenergy $\Sigma^{(\mathrm{chain})}(k_\parallel, z)$ of a single chain [80]. This

approximation first used by Wen reads for $k_{\|} \approx k_F$

$$G(k_{\|}, \vec{k}_{\perp}, z; t_{\perp})_{\rm Wen} = \frac{1}{\Big[G_+(k_{\|}, z)\Big]^{-1} - t_{\perp}(\vec{k}_{\perp})}, \tag{1.69}$$

where G_+ is determined by the spectral function ρ_+ discussed following Eq. (1.50) via a Hilbert transform. In the asymptotic low-energy regime this yields $G_+(k_F + \tilde{k}_{\|}, z) = A_0[(\tilde{k}_{\|}/k_c)^2 - (z/\omega_c)^2]^{\alpha_L/2}/(z - v_c\tilde{k}_{\|})$ for spinless fermions, with $\omega_c \equiv k_c v_c$ and $A_0 = \pi\alpha_L/[2\sin(\pi\alpha_L/2)]$. Wen's approximate Green's function leads to a spectral function with the *same* range of continua as $\rho_+(k_{\|}, \omega)$. In *addition* there can be *poles* at $\omega_{k_{\|},\vec{k}_{\perp}}$, determined by setting the denominator in Eq. (1.69) equal to zero. The poles located at $\omega_{k_{\|},\vec{k}_{\perp}} = 0$ determine the Fermi surface $\tilde{k}_{\|}(\vec{k}_{\perp})$ of the *interacting* coupled system. From Eq. (1.69) and the simple form of G_+ one obtains $A_0(\tilde{k}_{\|}/k_c)^{(1-\alpha_L)} = t_{\perp}(\vec{k}_{\perp})$, which shows that the reduction of warping of the Fermi surface (FS) by the interaction is proportional to $[t_{\perp}(\vec{k}_{\perp})/\omega_c]^{\alpha_L/(1-\alpha_L)}$. This is shown in Fig. 1.6 for a two dimensional system of coupled chains. If one writes $t_{\perp}(\vec{k}_{\perp}) \equiv t_{\perp}c(\vec{k}_{\perp})$,

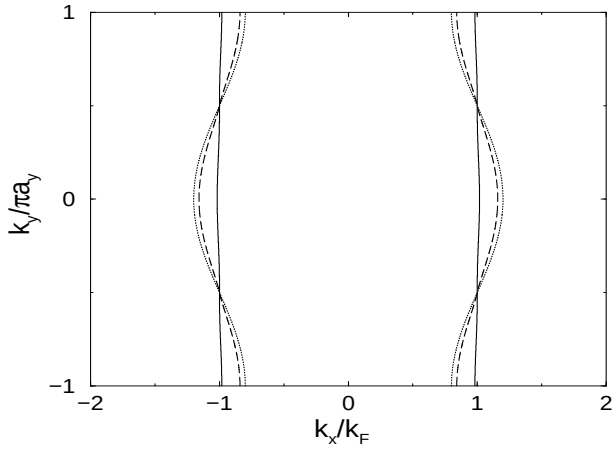

Figure 1.6. Fermi surface "flattening" in Wen's approximation for coupled chains for different values of the the anomalous dimension α_L for a single chain. The dotted lines show the noninteracting FS, the long dashed curves correspond to $\alpha_L = 0.125$ and the full ones to $\alpha_L = 0.6$. At $\alpha_L = 1$ the FS degenerates to two parallel lines as without interchain coupling, called the "confinement transition".

with $c(\vec{k}_{\perp})$ a dimensionless function, the new effective low energy scale is given by $t_{\rm eff} = \omega_c(t_{\perp}/\omega_c)^{1/(1-\alpha_L)}$. The weights $Z_{\vec{k}_{\perp}}$ of the poles for $\vec{k}$ values on the Fermi surface are also proportinal to $[t_{\perp}(\vec{k}_{\perp})/\omega_c]^{\alpha_L/(1-\alpha_L)}$. Wen's approximate solution has the Fermi liquid type property of quasi-particle poles with

nonzero weight on the Fermi surface, except at the special points where $t_\perp(\vec{k}_\perp)$ vanishes. The improved treatment by Arrigoni [88] shows that this peculiar vanishing of the quasi-particle weights is an artefact of Wen's approximation. The new idea involved is to let the number of "perpendicular" dimensions "$d-1$" go to infinity. This extends the original idea of the "dynamical mean field theory" (DMFT) [89], where one treats the Hubbard model in infinite dimensions as an effective *impurity* problem to the case of a *chain* embedded in an effective medium. Results are obtained by carrying out a resummation of *all* diagrams in the $t_\perp$-expansion which contribute in this large dimension limit [88]. This approach shows explicitly how the leading order Wen approximation is uncontrolled at low energies. For the case of weakly coupled one-dimensional Mott insulators one expects the approximation to be better controlled [90].

Despite the Fermi liquid like properties at energy scales much smaller than $t_{\rm eff}$ the coupled chain system can nevertheless show LL like properties for energy scales larger than $t_{\rm eff}$ if there is a large enough energy window to the high energy cutoff $\tilde{\omega}_c$ which describes the regime where the asymptotic LL power laws hold for a single chain. Then for temperatures lower than $\tilde{\omega}_c$ but higher than $t_{\rm eff}$ the system behaves like a LL. The integrated spectral functions $\rho^{<}_{\alpha,(\sigma)}(\omega)$ probed by angular integrated photoemission, for example, show approximate power law behaviour $\sim (-\omega)^{\alpha_L}$ for temperatures larger than $t_{\rm eff}$ in the energy window $k_B T < -\omega < \tilde{\omega}_c$. Unfortunately little is known about the value of $\tilde{\omega}_c$ for microscopic models. An exception is the Tomonaga model Eq. (1.14) with a constant $\tilde{v}(k)$ up to the cutoff k_c, where the high energy cutoff $\tilde{\omega}_c$ equals $\omega_c = \min(v_c, v_s)k_c$ [4]. This implies for the integrated spectral function for the very large U Hubbard model with periodic boundary conditions that the power law $|\omega|^{\alpha_L}$ only holds in a narrow energy window $\sim v_s$, which vanishes proportional to $1/U$ in the $U \to \infty$ limit [65]. Another example is the Hubbard model at boundaries where $\tilde{\omega}_c$ can be very small for small U [50].

As an alternative way to treat the "anisotropic large dimension model" [88] one can try to solve the resulting chain-DMFT equations numerically, using e.g. a quantum Monte Carlo algorithm [91]. In this reference the H_i were chosen as Hubbard Hamiltonians (1.66) with chain lengths of 16 and 32 sites. The results for a partly filled band as a function of temperature indicate in fact a crossover from a LL to a FL at the estimated crossover scale as the temperature is lowered. In agreement with Arrigoni [88] the authors find that the quasi-particle weight is more uniform along the Fermi surface than suggested by Wen's approximation Eq. (1.69). At half filling and low but finite temperatures the crossover from the Mott insulator to FL crossover was examined (the intermediate LL regime was too narrow to be visible). In the future it is to be expected that this method applied to longer chains and additional nearest neighbour interaction will provide important results which allow a more realistic comparison with experimental work.

Because of space limitations the interesting field of a *finite* number of coupled chains cannot be discussed here [92].

5.2 On the experimental verification of LL behaviour

There exist several types of experimental systems where a predominantly $1d$ character can be hoped to lead to an (approximate) verification of the physics of Luttinger liquids. In the following we present a short list of the most promising systems and discuss some of the experimental techniques which have been used. As these topics are also discussed in other chapters of this book we do not attempt a complete list of references but only refer to most recent papers or to review articles on the subject.
The following systems look promising:

- Highly anisotropic "quasi-one-dimensional" conductors

  There has been extensive work on organic conductors, like the Bechgaard salts [93, 94], as well as inorganic materials [95, 96].

- Artificial quantum wires

  Two important types of realizations are quantum wires in *semiconductor heterostructures* [97, 98] or quantum wires on *surface substrates* [99, 100].

- Carbon nanotubes

  The long cylindrical fullerenes called quantum nanotubes are also quantum wires but have been listed separately because of their special importance in future applications like "molecular electronics" [101, 102]. Using the peculiar band structure of the π-electrons of a single graphite plane it was shown that single wall "armchair" nanotubes should show LL behaviour with $K_c \sim 0.2 - 0.3$ down to very low temperatures [103, 104], despite the fact that *two* low energy channels are present.

- Fractional quantum Hall fluids

  Electrons at the edges of a two-dimensional fractional quantum Hall system can be described as a *chiral Luttinger liquid* [105]. The power law tunneling density of states observable in the tunneling current-voltage characteristics shows power laws of extraordinary quality [106]. The theoretical predictions for general filling factors between the Laughlin states $\nu = 1$ and $\nu = 1/3$ [107, 108] are not borne out by experiment [109]. As in these chiral LLs the right- and left-movers are *spatially separated* the edge state transport is quite different from the case of quantum wires and FQH fluids are not further discussed in the following.

Promising experimental techniques to verify LL behaviour are:

- High resolution photoemission

  One of the earliest claims of possible verification of Luttinger liquid behaviour was from angular integrated photoemission of the Bechgaard salt $(\mathrm{TMTSF})_2\mathrm{PF}_6$, which showed a power law supression at the chemical potential with an exponent of order 1 over an energy range of almost one eV [110]. There are serious doubts that this suppression can be simply explained by the LL power law behaviour [94]. Therefore a large number of other quasi-one-dimensional conductors were examined [94, 95, 96, 111]. In addition periodic arrays of quantum wires on surface substrates were studied by angular resolved photoemisssion (ARPES), but the interpretation of a two peak structure as spin-charge separation [99] was questioned [100]. Spin-charge separation was shown to occur in the $1d$ Hubbard model also at higher energies on the scale of the conduction band width [69, 70, 73]. Recent ARPES spectra of TTF-TCNQ were interpreted with the $1d$ Hubbard model at finite doping to show signatures of spin-charge separation over an energy scale of the conduction band width. As for the Hubbard model $K_c > 1/2$ for $n \neq 1$ which implies $\alpha_L < 1/8$ for the anomalous dimension the experimentally found nearly linear spectral onset at low energies cannot be explained within the same model. ARPES data for the "Li purple bronze" seem to compare favorably to the LL lineshape [96]. For the quasi-one-dimensional antiferromagnetic insulators SrCuO_2 and $\mathrm{Sr}_2\mathrm{CuO}_3$ ARPES spectra have been interpreted to show evidence of spin-charge separation [112]. For a more in depth discussion see the chapter by Grioni in this book.

- Transport

  As discussed in section 3 even a single impurity has a drastic effect on the conductance of a LL, which vanishes as a power law with temperature. Another issue is the "conductance puzzle" of a clean LL. There has been an extended discussion whether the quantized value e^2/h for noninteracting electrons in a single channel is modified by the interaction to $K_c e^2/h$ [113, 114]. Apparently the answer depends sensitively on the assumptions made about the contacts, a very delicate theoretical as well as experimental problem [115]. Experimental results are available for cleaved edge overgrowth quantum wires [97] as well as carbon nanotubes [116, 117, 118]. In the nanotubes the authors observe approximate power laws of the conductance which seem to be consistent with LL behaviour. A detailed dicussion of transport through quantum wires is presented in the chapter by Yacoby. For a recent theoretical discussion of experimental results on the interchain transport in the Bechgaard salts see references [119, 120]. There the question of energy scales and the importance of the proximity of the incipient Mott insulator are addressed.

- Optical properties

  Optical properties have long been used to investigate electronic properties of quasi-one-dimensional systems [121]. The optical behaviour of different Bechgaard salts was analyzed recently using LL concepts [122]. At low energies, smaller than about ten times the Mott gap, the importance of dimerization and interchain hopping was pointed out [123]. As there is a separate chapter about the optical response in chains and ladders it will not be discussed further here.

Obviously neither the list of systems nor that of methods is coming close to being complete. They were presented to show that there are intensive experimental activities in the attempt to verify the elegant LL concept put forward by theoreticians. Further work on both sides is necessary to come to unambiguous conclusions.

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